Analysis Notes

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Contents

0.1 Authors Note and Miscellaneous Content

This is a very rough amalgamation of my notes from 131ABH. There are parts that are missing or skipped completely. It was quite difficult to organize the content: I orginally wanted to order everything in agreement with how Greene taught his classes, but soon found out that it was messy and made no sense. As such, I decided to do some organization on my own in a way that makes the most sense to me. With that said, there are still a few areas where organization can be improved. For example, the metric space topology section relies on knowledge about continuous functions, which is not introduced until the next chapter. Finally, in order to get something out of these notes, there are some very basic facts that I take for granted. Things such as the Least Upper Bound property of the Reals and Cauchy sequences were omitted as preliminary knowledge. So, in order to get the most out of these notes, I recommend understanding and constructing the real numbers first.

Theorem. $[0,1] \subset \mathbb{R}$ *is uncountable.*

Proof. Suppose that $[0,1]$ is countable. Then we can list off every number in $[0,1]$. So let the set $\{x_1, x_2, ...\}$ be all the numbers in $[0,1]$. Partition $[0,1]$ into 3 equally sized chunks. Call the chunk $x_1 \notin I_1$. Then we can partition I_1 the same way to get I_2. If we continue the process, we get that for all $i \in \mathbb{N}$,

$$x_i \notin \bigcap_{n=1}^{\infty} I_n$$

But it's pretty clear that $\bigcap_{n=1}^{\infty} I_n$ is nonempty. So $[0,1]$ must be uncountable. *(There's an alternative proof using Baire Category Theorem)* $\square$

Theorem (Interchanging of Limits)**.** *Suppose* $\{f_n\} \to f$ *uniformly and let* $\lim_{x\to a} f_n(x) = A_n$*. Then* $\{A_n\}$ *converges and*

$$\lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to a} f_n(x)$$

Proof. Let $A_n = \lim_{x\to a} f_n(x)$ and $\epsilon > 0$ be given. Then by uniform convergence, there exists N such that for $n, m > N$, we have $|f_n(x) - f_m(x)| < \epsilon$ for all x. If we let $x \to a$, we get $|A_n(x) - A_m(x)| < \epsilon$ so A_n is cauchy and converges to some A. Then we have the 3 point estimate:

$$|f(x) - A| \leq |f(x) - f_n(x)| + |f_n(x) - A_n| + |A_n - A|$$

We can pick n such that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for all x by uniform convergence. We can choose n to be bigger to satisfy $|A_n - A| < \frac{\epsilon}{3}$ if necessary. Finally, since $A_n = \lim_{x \to a} f_n(x)$, we can always choose a ball $B(a, \delta)$ such that $|f_n(x) - A_n| < \frac{\epsilon}{3}$. $\square$

Of course, this theorem works for double sequences as well.

Chapter 1

Metric Spaces and Topology

1.1 Metric Spaces and Convergence

Before discussing convergence of mathematical objects, we must first be able to describe the "distance" between two objects in a set because convergence is the idea that a sequence of objects gets "closer and closer" until we converge to something. Luckily, all we need to do is have the following definition.

Definition. Let X be a set and $x, y, z \in X$. A *Metric* is a function $d : X \times X \to \mathbb{R}$ with the following properties:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff y = x$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x + y) + d(y, z)$

If such d is defined, we call (X, d) a metric space.

Example. Define $d : \mathbb{R}^2 \to \mathbb{R}$ such that $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is a metric space.

The intuitive idea of the metric is that it is a function associated to X that measures the "distance" between any two elements in X, much like the familiar Euclidean space $\mathbb{R}^n$.

Now that we have established a way to determine distance between 2 objects, we can now look into how convergence works. Just as in standard euclidean spaces, the intuition of convergence remains the same: a sequence converges if their distances get smaller and smaller and eventually approach a limit point. The following definitions formalize this idea.

Definition. Let (X, d) be a metric space.

1. A sequence $\{x_n\}$ in X *converges* to x_∞ if
$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n > N,\ d(x_n, x_\infty) < \epsilon$$

2. The sequence $\{x_n\}$ is *cauchy* if
$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n, m > N,\ d(x_n, x_m) < \epsilon$$

3. If $(X, d_X), (Y, d_Y)$ are metric spaces. The function $F : X \to Y$ is *continuous* at x_0 if
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } d_X(x, x_0) < \delta \implies d_Y(F(x), F(x_0)) < \epsilon$$

Just as Cauchy sequences converge in $\mathbb{R}$ but not converge in $\mathbb{Q}$, it's also possible that Cauchy sequences converge in some metric spaces and not in others. Therefore, we need to be able to distinguish between the metric spaces that are like $\mathbb{R}$ and the ones that are like $\mathbb{Q}$.

Definition. A metric space (X, d) is *Complete* if all Cauchy sequences converge in X.

This definition of completeness also extends to general topological spaces and is important because it tells us that the limits that ought to converge, indeed converge and that we aren't "missing" things in the space. With this idea in mind, the following 2 theorems should make some intuitive sense.

Theorem. *Every complete metric space is closed.*

Proof. Let (X, d) be a metric space and $Y \subset X$ be complete. Let $\{y_n\}$ be a sequence in Y that converges to some $y_\infty \in X$. Since $\{y_n\}$ converges, it must be Cauchy. Since Y is complete, $\{y_n\}$ converges to some point in Y □

Theorem. *Let (X, d) be a complete metric space and $Y \subset X$ be closed. Then (Y, d) is complete.*

Proof. Let $\{y_n\}$ be a Cauchy sequence in Y. Then $\{y_n\}$ is also in X. Since X is complete, $\{y_n\}$ converges to some $y_\infty \in X$. Since Y is closed, it contains its limit points, so $y_\infty \in Y$ hence Y is complete. □

1.2 Metric Space Topology

1.2.1 Open and Closed Sets

First, we begin with some basic definitons and theorems. Note that we will be using X from now on to denote metric spaces.

Definition. Let $x \in X$. The *open ball* around x with radius r is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}$$

Definition. A set $U \subset X$ is *open* if for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Furthermore, we define the complement of an open set, $X - U$ to be *closed.*

Theorem (DeMorgan's Laws)**.** *The following are always true*

1. $X - \bigcup_{\lambda \in \Lambda} S_\lambda = \bigcap_{\lambda \in \Lambda}(X - S_\lambda)$
2. $X - \bigcap_{\lambda \in \Lambda} S_\lambda = \bigcup_{\lambda \in \Lambda}(X - S_\lambda)$

Proof. The proof of this is left to the ambitious reader. $\square$

Theorem (Properties of Open and Closed sets)**.** *Let U_λ be open and C_λ be closed.*

1. $\bigcup_{\lambda \in \Lambda} U_\lambda$ *is open*
2. $\bigcap_{\lambda \in \Lambda} U_\lambda$ *is open if* Λ *is finite*
3. $\bigcup_{\lambda \in \Lambda} C_\lambda$ *is closed if* Λ *is finite*
4. $\bigcap_{\lambda \in \Lambda} C_\lambda$ *is closed*

Proof. The proofs are as follows:

1. Trivial
2. Consider $U_n = (-\frac{1}{n}, \frac{1}{n})$
3. Consider $C_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$
4. Follows from (1) and DeMorgan's Laws

$\square$

Now that these preliminary definitions are out of the way, we should check that they are indeed well-defined by confirming that the open ball is indeed, open.

Theorem. *An open ball is open.*

Proof. Let $B(x, r) \subset X$. For any $y \in B(x, r)$, choose $\epsilon < r - d(x, y)$. Then for any $z \in B(y, \epsilon)$, $d(z, x) \leq d(z, y) + d(y, x) < r$. Hence $B(y, \epsilon) \subset B(x, r)$ and the open ball is open. $\square$

With that out of the way, we should now establish some relationships between these definitions with what we already know about sequences.

Definition. A set $C \subset X$ is *sequencially closed* if every convergent sequence in C converges inside C.

Theorem. *C is closed $\iff$ C is sequencially closed*

Proof. Let C be closed. Suppose that there exists $\{x_n\} \subset C$ such that $\{x_n\} \to x \in X - C$. Since $X - C$ is open, there exists some $r > 0$ such that $B(x, r) \subset X - C$. This contradicts $\{x_n\} \subset C$.

Now let C be sequencially closed. Suppose that C is not closed so $X - C$ not open. Then there exists $x \in X - C$ such that for all $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \not\subset X - C$. This means that every $\frac{1}{n}$ ball contains elements in C. Therefore, we can define a sequence $\{x_n\} \subset C$ that converges x. Since C is sequencially closed, $x \in C$. Contradiction. $\square$

1.2.2 Compactness

We now arrive at one of the most important ideas and topics in Topology. The following definitions provide powerful results and allows us to generalize lots of ideas we have previously seen.

Definition. $S \subset X$ is *sequencially compact* if every sequence in X has a convergent subsequence.

Definition. An *open cover* of a set $S \subset X$ is a collection of open sets $\{U_\lambda \subset X \mid \lambda \in \Lambda\}$ such that $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$.

Definition. $S \subset X$ is *(covering) compact* if every open cover has a finite subcover.

Example. $[0, 1]$ is compact.

Proof. Let $\{U_\lambda\}$ be an open cover for $[0,1]$ and let

$$S = \{x \in [0,1] \text{ such that } [0,x] \text{ has finite subcover}\}$$

Then S is bounded so there is a LUB $c = \sup S$. Suppose $c < 1$. Then there is a finite cover $U_{\lambda_1}, ..., U_{\lambda_n}$ and $c \in U_{\lambda_i}$. Since U_{λ_i} is open, there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_{\lambda_i}$. Thus c is not the LUB of S. Contradiction.

(An alternate proof is really easy with Heine-Borel) $\square$

Theorem. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. Let $\{U_\lambda\}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\lambda)\}$ is an open cover of X. Then by compactness of X, there is a finite subcover $f^{-1}(U_{\lambda_1}), ..., f^{-1}(U_{\lambda_n})$. Of course, this means $U_{\lambda_1}, ..., U_{\lambda_n}$ covers $f(X)$ $\square$

Definition. $S \subset X$ is *dense* in X if for all $\epsilon > 0, x \in X$ there exists some $s \in S$ such that $d(s,x) < \epsilon$. An equivalent definition is $\bar{S} = \text{closure}(S) = X$.

Definition. X is *separable* if there exists a countable, dense subset $S \subset X$.

Example. $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

Theorem. *If X is separable, then $Y \subset X$ is also separable.*

Proof. Let $S = s_k$ be countable and dense in X. Let $p_{n,k} \in Y$ such that

$$d(p_{n,j}, s_k) < \inf_{y \in Y} d(s_k, y) + \frac{1}{n}$$

Claim: $P = p_{n,k}$ is countable and dense in Y.

To see this, let $\epsilon > 0$ and choose $\frac{1}{N} < \frac{\epsilon}{3}$. Since S is dense in X, for all $y \in Y$, there exists some $s_{k_0} \in S$ such that $d(s_{k_0}, y) < \frac{\epsilon}{3}$. Then

$$\begin{aligned} d(p_{N,k_0}, y) &\leq d(p_{N,k_0}, s_{k_0} + d(s_{k_0}, y) \\ &< \inf_{y \in Y} d(s_{k_0}, y) + \frac{1}{n} + d(s_{k_0}, y) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned}$$

Therefore, P is countable and dense in Y $\square$

1.2.3 Compactness = Sequencial Compactness

Our goal in this section is to prove that in a metric space, compact and sequencially compact are equivalent. The proof for compact implies sequencial compact is easy, the other not so much. So, the proof for sequencial compact implies compact is broken into many parts. The outline of the proof is:

$$\text{Seq. Compact} \implies \text{Separable} \implies \text{Countable Subcover} \implies \text{Compact}$$

Theorem. *Sequencial Compact $\implies$ Separable*

Proof. Let X be sequencially compact. Let $S_n = \{s_k \in X\}$ and s_k be chosen as follows:

$$\begin{aligned}
&s_1 \in X \\
&s_2 \in X - B(s_1, \frac{1}{n}) \\
&s_3 \in X - [B(s_1, \frac{1}{n}) \cup B(s_2, \frac{1}{n})] \\
&\vdots
\end{aligned}$$

The set S_n must be finite. If it wasn't then the sequence $\{s_k\}$ has no convergent subsequence since $d(s_i, s_j) \geq \frac{1}{n}$ for all $i \neq j$. Then, its not hard to see that

$$\bigcup_{n \in \mathbb{N}} S_n \text{ is countable and dense in X}$$

because for any $\epsilon > 0$, there is some $\frac{1}{N} < \epsilon$, and for any $x \in X$, there is some $s \in S_N$ such that $d(x, s) < \frac{1}{n} < \epsilon$. Hence X is separable. □

Lemma. *If X is separable, then every open $U \subset X$ is a countable union of open balls.*

Proof. Let S countable and dense in X and U be open in X. Since U is open, there exists an $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Furthermore, since S is dense in X, there exists some $s \in S$ such that $d(s, p) < \frac{\epsilon}{2}$. Therefore, $B(s, \frac{\epsilon}{2}) \subset B(p, \epsilon) \subset U$. Hence, every $p \in U$ belongs to some open ball centered at some $s \in S$. Since S is countable, U is a countable union of open balls. □

Theorem. *Separable $\implies$ Countable Subcover*

Proof. Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of X. By the lemma

$$U_\lambda = \bigcup_{s\in S, n\in\mathbb{N}} B(s,\epsilon)$$

Then, for each $s \in S$ and $n \in \mathbb{N}$, choose a $U_{\lambda_{s,n}}$ such that $B(s, \frac{1}{n}) \subset U_{\lambda_{s,n}}$ Then

$$X = \bigcup_{s\in S,\, n\in\mathbb{N}} U_{\lambda_{s,n}}$$

is a countable subcover. □

Theorem. *Sequencially Compact and Countable Subcover* $\implies$ *Compact*

Proof. Suppose that no finite subcover exists. Then for every $N \in \mathbb{N}$, there exists some $x \in X - \bigcup_{k=1}^{N} U_k = \bigcap_{k=1}^{N}(X - U_k)$. Therefore, we can define the sequence $\{x_n\}$ such that

$$x_n \in \bigcap_{k=1}^{n}(X - U_k)$$

Since X is sequencially compact, there is a subsequence

$$\{x_{n_k}\} \to x_\infty \in \bigcap_{k=1}^{\infty}(X - U_k)$$

However, this is a contradiction because we must have $x_\infty \in X$ but we have $\bigcap_{k=1}^{\infty}(X - U_k) = \emptyset$. □

Theorem. *Compact* $\implies$ *Sequencially Compact*

Proof. Suppose that $\{x_n\}$ has no convergent subsequence. Then for every $p \in X$, there exists $\epsilon_p > 0$ such that the ball $B(p, \epsilon_p)$ contains finitely many x_n's (if the ball contains infinitely many x_n's, then we can define a convergent subsequence to p). Then,

$$\bigcup_{p\in X} B(p,\epsilon_p) = X$$

is an open cover for X, so there's a finite subcover such that

$$\bigcup_{k=1}^{N} B(p_k, \epsilon_{p_k}) = X$$

However, this means that X contains finitely many x_n's. Contradiction. □

With that, we can finally conclude that in a metric space,

$$\text{Sequencially Compact} \iff \text{Compact.}$$

This result, combined with the following definition, gives us a powerful corollary:

Definition. X is totally bounded if for all $\epsilon > 0$, there is a finite cover $B(x_1, \epsilon) \cup ... \cup B(x_1, \epsilon) = X$.

Corollary. *If X is complete and totally bounded, then X is compact.*

Proof. The proof for this is left to the reader. Hint: prove that X is sequencially compact by showing every sequence has a cauchy subsequence. □

Note that totally bounded and compact are not the same thing! Total boundedness only cares about finite covers of open balls, while compactness considers finite subcovers for **all** open covers.

1.2.4 Two Big Theorems

In this section, we will see two major theorems: Heine-Borel theorem and Baire Category theorem. The Heine-Borel theorem provides us with an easy way to check of compactness for sets in $\mathbb{R}^n$, while the Baire Category theorem is flexible and can be used for many purposes.

Theorem (Heine-Borel Theorem)**.** *Let $X \in \mathbb{R}^n$. Then*

$$X \text{ is compact} \iff X \text{ is closed and bounded.}$$

Proof. As established in the previous section, compact and sequencially compact are the same in metric spaces. So, suppose X is sequencially compact. Then X is closed because convergent subsequences converge in X. Now suppose that X is not bounded. Fix $x_0 \in X$. Then there is a sequence $\{x_n\}$ such that $d(x_0, x_n) > n$ for all $n \in \mathbb{N}$. We have a subsequence $\{x_{n_k}\} \to x_\infty$, so $d(x_{n_k}, x_\infty) < \epsilon$. Since x_∞ is fixed, $d(x_0, x_\infty) = M$ for some $M > 0$. Hence

$$d(x_0, x_{n_k}) \leq d(x_0, x_\infty) + d(x_\infty, x_{n_k}) < M + \epsilon$$

Contradiction.
Now suppose that $X \in \mathbb{R}^n$ is closed and bounded and $\{(x_1, ..., x_n)_k\}$ be a sequence in X. By Bolzano-Weierstrass, we can pick out a subsequence such that the 1st column converges. Then by the same logic, we can take a subsequence again so that the 2nd column converges, etc. Repeat this n times and we have a convergent subsequence. □

Corollary. $[0,1]$ *is compact.*

Definition. Let $S \subset X$. We say S is *nowhere dense* if interior$(S) = \emptyset$. I.E. for any $s \in S$ and $\epsilon > 0$, there exists some $x \in B(s,\epsilon)$ such that $x \notin S$.

Theorem (Baire Category Theorem)**.** *If X is complete, then*

1. $X \neq \bigcup_{n=1}^{\infty} S_n$ *where $S_n \subset X$ is nowhere dense.*

2. *If U_n is open and dense in X, then $\bigcap_{n=1}^{\infty} U_n$ is also dense in X*

Proof. Both proofs will be presented but only one needs to be known (2 is just the complement set of 1).

1. Suppose $X = \bigcup_{n=1}^{\infty} S_n$ where $S_n \subset X$ is nowhere dense. Then clearly $X = \bigcup_{n=1}^{\infty} \overline{S_n}$ and $\overline{S_n}$ is still nowhere dense. So WOLOG, let S_n be closed. Define $\{x_n\}$ in the following way: choose $x_1 \in X - S_1$. Since S_n is closed, $X - S_1$ is open, hence there is some $B(x_1,\epsilon_1) \subset X - S_1$. So we can inductively define $x_{n+1} \in B(x_n,\epsilon_n) - S_{n+1}$. Clearly, this sequence is cauchy and hence converges to some $x \in X$. However, since $x_N \notin \bigcup_{n=1}^{N} S_n$, we must have that $x \in X - \bigcup_{n=1}^{\infty} S_n = \emptyset$. Contradiction.

2. Let $x \in X$ and $\epsilon > 0$. It suffices to find a $y \in B(x,\epsilon)$ and $y \in \bigcap_{n=1}^{\infty} U_n$.

   Since U_1 is dense in X, there exists $y_1 \in U$ such that $y_1 \in B(x,\epsilon)$. Since U_1 is open, there exists some $r_1 > 0$ such that $B(y_1,r_1) \subset U_1 \cap B(x,\epsilon)$. If we shrink r_1 a little, we can get the closed ball $\overline{B(y_1,r_1)} \subset U_1 \cap B(x,\epsilon)$. By the same argument, we can get some $y_2 \in U_2$ such that $\overline{B(y_2,r_2)} \subset U_1 \cap B(y_1,r_1)$.

   Repeating this, we get a sequence $\{y_n\}$ that is Cauchy. Hence, since X is complete, $\{y_n\} \to y_\infty \in \bigcap_{n=1}^{\infty} U_n$ and $y_\infty \in B(x,\epsilon)$.

□

Corollary. $[0,1]$ *is uncountable*

Proof. Since $[0,1]$ is compact, it is also complete. Consider any singleton set $\{x_n\}$. This set is closed and has empty interior. Therefore by Baire Category, $[0,1] \neq \bigcup_{n=1}^{\infty}\{x_n\}$. Hence $[0,1]$ must be uncountable. □

1.2.5 Connectedness TO BE FINISHED

Definition. X is *disconnected* if there exists open sets $U, V \neq \emptyset$ such that $U \cup V = X$, $U \cap V = \emptyset$. We say X is *connected* if it is not disconnected.

Corollary. $[0,1]$ *is connected.*

Proof. Suppose U, V is a disconnection of $[0,1]$. WOLOG, let $0 \in U$. Let $S = \{x \in [0,1]$ such that $[0,x]$ is connected$\}$. By the LUB principle, $c = \sup S$ exists. Suppose that $c < 1$. If $c \in U$, then since U is open, $c + \epsilon \in U$ so c is not the LUB. If $c \in V$, then since V is open, $(c - \epsilon, c] \subset V$ and so there cannot be a sequence in U that converges to c. Thus $c = 1$ and $[0,1]$ is connected. □

Definition. $Y \subset X$ is *path connected* if for any $x, y \in Y$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$ and $f(t) \in Y$ for all $t \in [0,1]$

Theorem. *If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.*

Proof. Suppose $f(X)$ is disconnected. Then there is a disconnection U, V. By continuity of f, $f^{-1}(U), f^{-1}(V)$ are open. Furthermore, we have

$$X = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

$$\emptyset = f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

Thus $f^{-1}(U), f^{-1}(V)$ is a disconnection of X. Contradiction. □

Theorem. *If $Y \subset X$ is path connected, then Y is connected*

Proof. Suppose Y is not connected. Let U, V be a disconnection. Choose $x \in U$ and $y \in V$. Since Y is path connected, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$ and $f([0,1]) \subset Y$. Then $f([0,1]) \cap U$ and $f([0,1]) \cap V$ is a disconnection of $f([0,1])$. By continuity of f we have $[0,1] \cap f^{-1}(U)$ and $[0,1] \cap f^{-1}(V)$ is a disconnection of $[0,1]$ but $[0,1]$ connected. Contradiction. □

Note that the converse if not true. To see this consider the topologist's sine curve:

Example (Topologist's sine curve)**.** The set

$$T = \{(x, \sin\frac{1}{x}) \mid x \in (0,1]\} \cup \{(0,0)\}$$

is connected but not path connected.

Proof. **T is connected:** Suppose U, V is a disconnection of T. WOLOG, suppose $(0,0) \in U$. Then there exists some $\epsilon > 0$ such that $B(0,\epsilon) \cap T \subset U$. Clearly $\{(x, \sin\frac{1}{x}) \mid x \in (0,1]\}$ is path connected hence connected. Since A contains an element in $\{(x, \sin\frac{1}{x}) \mid x \in (0,1]\}$ we must have that $A = T$.

T is not path connected: PROOF IS CRINGE AND INCOMPLETE

Let $f = (f_1, f_2) : [0,1] \to T$ with $f(0) = (0,0)$. Claim: $f(t) = (0,0)$ for all t. Suppose not. Suppose that $f(t) \neq (0,0)$. By continuity of f_2, there is a $\delta > 0$ such that $|f(t)| < 1$ for all $t < \delta$ □

1.3 The ℓ_2 space

Finite dimensional spaces are not very interesting because $\mathbb{R}^n$ is essentially the same as $\mathbb{R}$. And so we will take a closer look at infinite dimensional metric spaces instead. We begin with a countable space and then move on to a very similar uncountable space.

Definition. The set ℓ_2 is a vector space defined to be

$$\ell_2 = \left\{ \{x_n\} \text{ such that } \sum_{n=1}^{\infty} x_n^2 < \infty \right\}$$

A natural definition for the norm of this space is then

$$||\{x_n\}||_2 = \left(\sum_{n=1}^{\infty} (x_n)^2 \right)$$

and so the induced metric is $d(x_n, y_n) = ||x_n - y_n||_2$

1.3.1 Completeness

Theorem. (ℓ_2, d) *is complete.*

Proof. Denote elements of ℓ_2 to be of the form $\{x_n\}$ and denote sequences to be $\{x_n\}^j$.

Let $\{x_n\}^j$ be Cauchy under the ℓ_2 metric. Then for each fixed k, $\{x_k\}^j$ is Cauchy and hence converges to some $\{x_k\}^\infty$ by completeness of $\mathbb{R}$. Thus, the sequence converges pointwise to $\{x_n\}^\infty$. We must now confirm that $\{x_n\}^\infty \in \ell_2$ and that $||\{x_n\}^j - \{x_n\}^\infty||_2 \to 0$.

To see that $\{x_n\}^\infty \in \ell_2$, we know that $||\{x_n\}^j||_2 \leq M$ for any j. Thus

$$||\{x_n\}^\infty||_2 = \lim_{j\to\infty} \sum_{k=1}^{N} (x_k^j)^2 \leq M$$

Since this holds for any N, $||\{x_n\}^\infty||_2 \leq M$

To see that the series converges in ℓ_2, we find that

$$||\{x_n\}^j - \{x_n\}^\infty||_2 = \sum_{k=1}^{N} (x_k^j - x_k^\infty)^2 \ \to 0 \text{ as } j \to \infty$$

Since this is true for all N, we have $||\{x_n\}^j - \{x_n\}^\infty||_2 \to 0$ $\square$

Actually, every ℓ^p space is complete. The proof for is similar since $\mathbb{R}$ is complete under all p-norms.

1.3.2 Compactness

We are familiar with Heine-Borel, which states that compactness $\iff$ closed and bounded in $\mathbb{R}^n$. We will observe that this is not necessarily true for infinite dimensional spaces.

Theorem. *The closed ball*

$$V = \left\{ \{x_n\} \text{ such that } \sum_{n=1}^{\infty} x_n^2 \leq 1 \right\}$$

*is **NOT** compact.*

Proof. Consider the sequence

$$\begin{aligned} x_1 &= (1,0,0,0,...) \\ x_2 &= (0,1,0,0,...) \\ x_3 &= (0,0,1,0,...) \\ &\vdots \end{aligned}$$

and so on. There is no cauchy subsequence under ℓ_2 norm because given any $n \neq m$:

$$||x_n - x_m||_2 = \sqrt{2}$$

Since sequence converge implies cauchy, by contrapositive, there is no convergent subsequence for this sequence. □

Theorem. *The subspace*

$$V =$$

Proof. This is a Cantor diagonalization argument. Let $\{x_n\}$ be a sequence in V. Note that given $x_n \in V$, $x_k \leq \sqrt{M}$ hence V is uniformly bounded.

Then by Bolzano-Weierstrass, we can make a subsequence $\{x_n\}_1$ such that the first term converges pointwise. Inductively, we can then take subsequences so that the first nth terms converge pointwise. Then, define the subsequence $\{y_n\}$ where you take

$$y_k = k\text{th element of the } k\text{th subsequence}$$

By construction, $\{y_n\}$ converges pointwise. □

Chapter 2

Continuous Functions

2.1 Continuity

Recall the definition of a continuous function $f : X \to Y$ at x_0 where $(X, d_X), (Y, d_Y)$ are metric spaces:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(x, x_0) < \delta \implies d_Y(F(x), F(x_0)) < \epsilon$$

The following is an equivalent (and perhaps more intuitive) definition:

$$\{x_n\} \to x_0 \implies \{f(x_n)\} \to f(x_0)$$

which is analogous to the well known definition from calculus:

$$f(x_0) = \lim_{x \to x_0} f(x)$$

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then*

1. *f is bounded*
2. *f attains its maxima and minima*
3. *for all $y \in [f(x_{min}), f(x_{max})]$, there exists $x \in [a, b]$ such that $f(x) = y$*

Proof. The proofs are as follows:

1. Suppose f is not bounded. Then for all $n \in \mathbb{N}$, there exists some $x_n \in [a, b]$ such that $f(x_n) > n$. By Bolzano-Weierstrass, there is a convergent subsequence $\{x_{n_k}\} \to x_\infty \in [a, b]$. Let $M = f(x_\infty)$. Then by continuity of f, $\{f(x_{n_k})\} \to M$ which is a contradiction because we should have $f(x_{n_k}) \to \infty$.

2. Part 1 tells us that f is bounded hence there is a LUB. Let $f(x_0) = \max_{x\in[a,b]} f(x)$. Then there is a sequence $\{f(x_n)\} \to f(x_0)$. By Bolzano-Weierstrass, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \to x_\infty \in [a,b]$. By continuity of f, $\lim f(x_{n_k}) = x_\infty = \lim f(x_n) = x_0$. Hence $x_0 \in [a,b]$

3. WOLOG, suppose $x_{\min} < x_{\max}$. Given $y \in [f(x_{\min}), f(x_{\max})]$, let $S = \{x \in [a, x_{\max}] \mid f(x) < y\}$. WOLOG, $y \neq f(x_{\min})$ or $f(x_{\max})$.

   Note that S is nonempty because $x_{\min} \in S$. Then by the LUB property, $c = \sup S$ exists. So there is a sequence $\{x_n\}$ in S such that $\{x_n\} \to c$. By continuity of f, we have $\{f(x_n)\} \to f(c)$. Since $f(x_n) < y$ for all n, we must have $f(c) \leq y$

   Furthermore, consider the sequence $\{c+\frac{1}{n}\}$. Note that since $c = \sup S$, we must have that this sequence is not in S. Thus $f(c+\frac{1}{n}) \geq y$ for all n. Thus by continuity of f, we have $f(c) \geq y$. As such, we must have $f(c) = y$

□

Theorem. *The following are equivalent for $f : X \to Y$ where $(X, d_X), (Y, d_Y)$ are metric spaces*

1. *f is continuous*
2. *for each $x \in X$ and $\forall \epsilon > 0, \exists \delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon$*
3. *$U \in Y$ open in $Y \implies f^{-1}(U)$ open in X*

Proof. We will only prove (2) $\iff$ (3) since (1) $\iff$ (2) is trivial.

Suppose (2) is true and let U be open in Y. Let $x \in f^{-1}(U)$. Since U is open, there exists some ϵ such that $B(x,\epsilon) \subset U$. Then by (2), there exists some δ such that $B(x,\delta) \in f^{-1}(U)$. Hence (2) $\implies$ (3)

Suppose (3) is true and let $x \in X$, $\epsilon > 0$. Then $f^{-1}(B(f(x),\epsilon))$ is open in X. Since $x \in f^{-1}(B(f(x),\epsilon))$, there exists some $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(B(f(x),\epsilon))$ and $f(B(x,\delta)) \subset B(f(x),\epsilon)$. So, for any $y \in B(x,\delta)$, $f(y) \in B(f(x),\epsilon)$. Which is exactly $d_X(y,x) < \delta \implies d_Y(F(y), F(x)) < \epsilon$. □

2.2 Uniform Continuity

Definition. A function $f : X \to Y$ is *uniformly continuous* if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon \quad \forall x, y \in X$$

Since the definitions are so similar, it can be hard to distinguish the differences between continuity and *uniform* continuity.

The difference is that given an $\epsilon > 0$, continuity only tells us that at every point in the domain, there is a δ that works, but this δ can be different for each point. So some points could work with a very large δ, while others need very small δ

Uniform continuity tells us something stronger. It tells us that given an $\epsilon > 0$, there is a δ that works for *every* point in the domain all at the same time! So it tells us that somehow, all the points in the image behaves similarly, so the function is far more controlled than a continuous function.

Example. Consider the function $f(x) = \frac{1}{1-x}$, $x \in [0, 1)$. Then f is continuous but not uniformly continuous. Why?

This is because as $x \to 1$, the growth of f becomes bigger and bigger. So for any fixed $\epsilon > 0$, no matter what x_0 we choose, we can always choose $x_0 < x_1 < 1$ and the delta for x_1 will be smaller than that for x_0.

Example. Now consider $g(x) = x^2$, $x \in [0, 1]$. the growth of g also gets bigger as $x \to 1$, but there's a maximum to the growth. Therefore, given an ϵ there is a smallest δ and this δ works for all points. However, if we consider $x \in \mathbb{R}$, there would be no smallest δ due to unrestricted growth.

Pictorially, continuity is when given a point p and a height ϵ, we can always draw a box around p with width δ and the line of the function never touches the top or bottom edges of the box. Uniform continuity occurs when there is some box with height ϵ that we can move around and the line of the function still never touches the top or bottom edges of the box.

Theorem. *Let $f : X \to Y$ be uniformly continuous. Then $\{x_n\}$ cauchy $\implies$ $\{f(x_n)\}$ cauchy.*

Proof. Let $\{x_n\}$ be cauchy. By uniform continuity, given $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$. Since $\{x_n\}$ is cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $d_X(x_n, x_m) < \delta$. Hence we have $d_Y(f(x_n), f(x_m)) < \epsilon$ so $\{f(x_n)\}$ is cauchy. $\square$

Note how this is not true for normal continuity! To see why, consider the function $f(x) = \frac{1}{x}$ and any sequence $\{x_n\} \to 0$. Then $\{x_n\}$ is cauchy but $\{f(x_n)\} \to \pm\infty$ so it is not cauchy.

Theorem. *Let X, Y be metric spaces and X compact. Then every continuous function $f : X \to Y$ is uniformly continuous.*

Proof. Suppose for sake of contradiction that $f : X \to Y$ is continuous but not uniformly continuous. So

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0, d_X(x, y) < \delta \text{ and } d_Y(f(x), f(y)) \geq \epsilon$$

Then this must be true for $\delta = \frac{1}{n}, \forall n \in \mathbb{N}$. Define sequences $\{x_n\}, \{y_n\}$ such that $d_X(x_n, y_n) < \frac{1}{n}$. Since X is compact, there is a subsequence $\{x_{n_k}\} \to x_\infty$, $\{y_{n_k}\} \to y_\infty$ As $n \to \infty$, $d_X(x_n, y_n) < \frac{1}{n} \to 0$. Hence we must also have that $d_X(x_{n_k}, y_{n_k}) \to 0$. Thus we must have $x_\infty = y_\infty$. This means that

$$d_Y(f(x_{n_k}), f(y_{n_k})) \leq d_Y(f(x_{n_k}), f(x_\infty)) + d_Y(f(x_\infty), f(y_{n_k})) < \epsilon$$

Contradiction. □

Theorem. *$f : (a, b) \to \mathbb{R}$ is uniformly continuous $\iff$ $\lim_{x \to a+} f(x)$ and $\lim_{x \to b-} f(x)$ exist and*

$$F(x) = \begin{cases} \lim_{x \to a+} f(x) & \text{if } x = a \\ f(x) & \text{if } a < x < b \\ \lim_{x \to b-} f(x) & \text{if } x = b \end{cases}$$

is continuous.

Proof. The proof is as follows

1. Consider the sequences $a_n = a + \frac{1}{n}$, $b_n = b - \frac{1}{n}$ which converges to a and b, respectively. Since f is uniformly continuous, $\{f(a_n)\}$ and $\{f(b_n)\}$ are cauchy and hence converges by completeness of $\mathbb{R}$. Thus $\lim_{x \to a+} f(x)$ and $\lim_{x \to b-} f(x)$ exist.

   Now we need to show that F is continuous. We already know that F is continuous for all $x \in (a, b)$ so we only need to check the edge cases. Doing so is a standard $\epsilon - \delta$ proof.

2. Suppose $F(x)$ is continuous. Since $[a, b]$ is compact, F is uniformly continuous. Since f is just a restriction of F, we must also have f uniformly continuous.

□

Chapter 3

Sequences of Functions

Just as for sequences of sequences defined on $\mathbb{R}$, we can define how a sequences of functions defined on metric spaces converge as well. However, there are now multiple different ways a function can converge. The first of which is pointwise convergence which was in the proof for ℓ_2 completeness. Of course, this also means that there is norm convergence as well, which will be examined in a later section. The last one is as follows,

Definition. Let $\{f_n\}$ be a sequence of functions $f_n : S \to X$ where S is any set and X is a metric space. Then

1. $\{f_n\}$ is *uniformly cauchy* if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n, m > N, \, d(f_n(s), f_m(s)) < \epsilon \quad \forall s \in S$$

2. (f_n) *uniformly converges* to f_∞ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N, \, d(f_n(s), f_\infty(s)) < \epsilon \quad \forall s \in S$$

Theorem. *Let S be any set, X be a complete metric space, and $f_n : S \to X$. If $\{f_n\}$ is uniformly cauchy, then $\{f_n\}$ converges uniformly to some $f_\infty : S \to X$.*

Proof. Since $\{f_n\}$ is uniformly cauchy, the sequence $\{f_n(s)\}$ where $s \in S$ fixed must be a cauchy sequence in X. Since X is complete, $(f_n(s)) \to p \in X$. Define $f_\infty(s) = p$ for each $s \in S$. Then, given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n, m \geq N$, $s \in S$,

$$d(f_n(s), f_m(s)) < \epsilon$$

as $m \to \infty$, $f_m(s) \to f_\infty(s)$ for each $s \in S$, so

$$d(f_n(s), f_\infty(s)) < \epsilon$$

$\square$

Therefore, we can see that completeness also applies to uniform convergence as well. This implies a powerful result that illustrates the powerful concept of uniform-ness: the idea that the "closeness" happens at every point of the function all at the same time.

Theorem (Weierstrass M-Test)**.** *Let $\{f_n\}$ be a sequence of continuous functions $f_n : X \to \mathbb{R}$ such that $|f_n| \leq M_n$. Then*

$$\sum_{n=0}^{\infty} M_n \textit{ converges} \implies \sum_{n=0}^{\infty} f_n \textit{ converges uniformly}$$

Proof. Given $\epsilon > 0$, for all x we have

$$\sum_{N=n}^{m} f_N(x) \leq \sum_{N=n}^{m} M_N \leq \epsilon$$

Then by the previous theorem, we have uniform convergence. $\square$

Note that the converse is not true. With that, we now move on to an equally important result of uniform convergence:

Theorem. *If $\{f_n(x)\}$ is continuous for each n and the sequence converges uniformly to $f(x)$, then $f(x)$ is continuous.*

Proof. Define $f_n(x)$ and $f(x)$ as stated and $\epsilon > 0$. Then by uniform convergence, there exists some $N > 0$ such that for all x and $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$. By continuity, there exists a $\delta > 0$ such that for all n and $|x - y| < \delta$, $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Hence, by the triangle inequality,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon$$

Thus f is continuous. $\square$

This result is extremely powerful because it implies the space of continuous functions is closed under uniform convergence. When combined with previous theorems, we have completeness of the space of continuous functions. The only problem now is that we need to somehow force the idea of uniform-ness onto the space.

3.1 Completeness of $C([0,1])$

We move on now to examine our second infinite dimensional metric space:

Definition. Define $C([0,1])$ to be the set of all continuous functions on the interval $[0,1]$. Naturally, since $[0,1]$ is compact, every function in the set is bounded. We will see that in this case, our choice of the metric does matter immensely.

Theorem. *For all $p \in \mathbb{N}$, $C([0,1])$ is **not** complete under $||\cdot||_p$ norm.*

Proof. For any $p \in \mathbb{N}$, consider the functions

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}(x - (\frac{1}{2} - \frac{1}{n})) & \text{if } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$

This sequence is cauchy but converges to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

which clearly does not belong to $C([0,1])$. $\square$

From our previous theorems, we've established that this space is complete if we can somehow force uniform-ness into cauchy-ness and convergence. So somehow, $||\cdot||_p$ does not force uniform-ness in sequences. However, the following norm does:

Theorem. *$C([0,1])$ is complete under $||\cdot||_\infty$ norm, which is defined*

$$||f||_\infty = \sup_{x \in [0,1]} |f(x)|$$

Proof. Let $\{f_n\}$ be cauchy with respect to the sup norm. It's not hard to see that $\{f_n(x)\}$ is cauchy at every fixed x. By completeness of $\mathbb{R}$, we can find the pointwise limit f_∞. Now, just as before in the ℓ_2 proof, we need to show that $f_\infty \in C([0,1])$ and $||f_n - f_\infty||_\infty \to 0$.

Since $\{f_n\}$ is cauchy with respect to the sup norm, it is uniformly cauchy. Hence, the sequence converges uniformly to f_∞ hence f_∞ is continuous.

Since $\{f_n\}$ is cauchy, for any fixed $\epsilon > 0$, there exists N such that for all $n, m > N$, $||f_n - f_m||_\infty < \epsilon$. Thus if we fix n, we must have

$$||f_n - \lim_{m\to\infty} f_m||_\infty = ||f_n - f_\infty||_\infty < \epsilon$$

hence $||f_n - f_\infty||_\infty \to 0$. $\square$

I encourage the reader to deeply think and try to understand why the $||\cdot||_\infty$ norm forces cauchy-ness and convergence to be uniform.

3.2 Equicontinuity and Arzelà–Ascoli

Of course, now that we've established conditions for completeness in the space of continuous functions, we are also interested in the conditions for compactness as well. As such, we have one last important concept. We can think of uniformly continuous as given an ϵ, there exists of some δ that works everywhere all at the same time. A similar concept also applies to functional spaces.

Definition. A family of functions $\{f_n\}$ is *equicontinuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x_1, x_2) < \delta \implies d(f_n(x_1), f_n(x_2)) < \epsilon$ for all n.

Theorem (Arzelà–Ascoli)**.** *If a sequence of functions $\{f_n\}$ in $C(X)$ is uniformly bounded, equicontinuous and X is compact, then there exists a uniformly convergent subsequence.*

Proof. Firstly, we show that X compact implies X countable by doing the following: For all $n \in \mathbb{N}$, there is a finite cover $B(x_1, \frac{1}{n}), ..., B(x_k, \frac{1}{n})$. Let $S_n = \{x_1, ..., x_k\}$. It's not hard to see then that $S = \bigcup S_n$ is countable and dense in X.

Next, we find a subsequence of $\{f_n\}$ that converges pointwise on S. This is done much similarly to the ℓ^2 space: You make a subsequence such that the first position converges pointwise, then inductively make the $n+1$th subsequence from the nth one. Then you take the ith element from the ith subsequence and let that be your final sequence $\{g_n\}$, which converges pointwise on S.

Finally, we claim is that this subsequence actually converges uniformly. By equicontinuity, given any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$, we have $d(g_n(x_1), g_n(x_2)) < \frac{\epsilon}{3}$ for all n. Since the sequence converges pointwise on S, there exists an N such that for all $n, m > N$, we have $d(g_n(s), g_m(s))$ where $s \in S$. Hence for any $x \in X, s \in S$ we have

$$d(g_n(x), g_m(x)) \leq d(g_n(x), g_n(s)) + d(g_n(s), g_m(s)) + d(g_m(s), g_m(x)) \leq \epsilon$$

Thus the subsequence is uniformly cauchy and by completeness of X, converges uniformly. ☐

Corollary. *A subset $F \subset C(X)$ is compact iff it is closed, bounded, and equicontinuous*

Proof. Suppose F is compact. Then F is sequencially compact, and so we get closed (by completeness of compact spaces) and bounded (by a simple contradiction when assuming unbounded). By compactness, for any $\epsilon > 0$, there is a finite cover $B(f_1, \epsilon), ..., B(f_n, \epsilon)$. By the sup norm, for any $g \in B(f_k, \epsilon)$ and any $x \in X$, $d(f_k(x) - g(x)) < 2\epsilon$. By continuity of f there exists a $\delta_k > 0$ such that for any $d(x, y) < \delta$, we have $d(f_k(x) - f_k(y)) < \epsilon$. Hence by the triangle inequality, if we keep the same δ_k,

$$|g(x) - g(y)| \leq |g(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - g(y)| \leq 3\epsilon \leq \epsilon$$

So each $B(f_k, \epsilon)$ is equicontinuous. Then, it is not hard to see that if we choose $\delta = \min_{k \in 1,...,n} \delta_k$, we get equicontinuity on all of X.

The reverse direction is simple: Since F is bounded and equicontinuous (and X is compact), by Arzelà–Ascoli there is always a convergent subsequence. Since F is closed, this subsequence always converges in F. Hence F is sequencially compact and we are done. ☐

Chapter 4

Normed Linear Spaces

4.1 Introduction

Definition. Let V be a vector space over a field F. A *norm* is a function $||\cdot|| : V \to \mathbb{R}$ that satisfies the following properties for all $x, y \in V$, $c \in F$:

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||cx|| = |c|\,||x||$
4. $||x+y|| \leq ||x|| + ||y||$

It's no coincidence that the definition of norms look very similar to that for metrics. The following definition helps establish the relationships between norms and metrics.

Definition. Let V be a vector space and $||\cdot||$ be a norm on V. Then

$$d(x,y) = ||x-y||$$

is a metric on V. We call d the metric induced by $||\cdot||$.

Theorem. *Norms are continuous functions*

Proof. Suppose that $\{x_n\} \to x$. Then for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $||x_n - x|| < \epsilon$. Then by the triangle inequality,

$$\big|\, ||x_n|| - ||x|| \,\big| < \epsilon$$

Therefore $||x_n|| \to ||x||$ which makes the norm continuous. By the same reason, all metrics are continuous. $\square$

Since V is a vector space, a metric space under the induced norm is a lot more interesting and important than arbitrary metric spaces. In fact, if these spaces are complete, they get their own special name.

Definition. A normed vector space that is complete under the induced metric is called a *Banach Space*.

Example (Nonexamples)**.** The following are all not Banach:

1. Polynomials $P[x]$ on [0,1] with the sup norm is not complete. *Reason*:
$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$
2. The set of {eventually 0 sequences} are not complete in ℓ^2 norm. *Reason*: the sequence of sequences $\{x_n\}$ where all terms up to n are 1 and rest are all 0 converges to the sequence of all 1's.

4.2 Equivalence of norms

Definition. Let $||\cdot||_1, ||\cdot||_2$ be two different norms in a vector space V. The norms are *equivalent* if there exists some $C_1, C_2 > 0$ such that for all $x \in V$,
$$||x||_1 \leq C_1||x||_2$$
$$||x||_2 \leq C_2||x||_1$$

The intuitive idea behind the inequalities is that if we have $||x||_1 \leq C_1||x||_2$, then $||\cdot||_2$, in a way, "controls" $||\cdot||_1$. So any sequence that converges in $||\cdot||_2$ must also converge in $||\cdot||_1$.

The equivalence of norms occurs then when the norms "control" each other. So we get kind of like an "if and only if" statement: A sequence converges in one norm if and only if it also converges in the other norm.

Theorem. *All norms on $\mathbb{R}^n$ are equivalent.*

Proof. It suffices to prove that any arbitrary norm $||\cdot||$ is equivalent to the 1-norm $||\cdot||_1$. Consider any $(x_1, ..., x_n) = \sum_{i=1}^{n} x_i e_i \in \mathbb{R}^n$. By the triangle inequality,

$$\begin{aligned} ||x|| &= ||\sum_{i=1}^{n} x_i e_i|| \\ &\leq \sup_{1\leq j\leq n} ||e_j|| \sum_{i=1}^{n} |x_j| \\ &\leq C||x||_1 \end{aligned}$$

So $||\cdot||$ is controlled by $||\cdot||_1$. Therefore, the set $S = \{x \in \mathbb{R}^n \mid ||x||_1 \leq 1\}$ is compact in $||\cdot||_1$ but also in $||\cdot||$. Since norms are continuous functions, $||S|| = [a, b]$. Hence, for some a, b,

$$a||x||_1 \leq ||x|| \leq b||x||_1$$

Thus the two norms are equivalent. □

Corollary. $\mathbb{R}^n$ *is complete in every norm.*

4.2.1 General Inner Product Spaces over $\mathbb{R}$

Establishing the equivalence of all norms on $\mathbb{R}^n$ has some powerful implications. We can establish some properties about any finite dimensional inner product space (over $\mathbb{R}$).

Theorem. *All norms are equivalent in a finite dimensional normed vector space.*

Proof. Any finite dimensional V is isomorphic to $\mathbb{R}^n$ □

Theorem. *If $(V, ||\cdot||)$ is a normed vector space with infinite dimensions and W is a finite dimensional subspace, then W is closed in V.*

Proof. Since W is isomorphic to $\mathbb{R}^n$, $||\cdot||$ is some norm in $\mathbb{R}^n$. $\mathbb{R}^n$ is complete in this norm since all norms are equal, so W is complete and hence closed in $||\cdot||$. □

Theorem. *If V is infinite dimensional and Banach, then*

$$V \neq \bigcup_{n=1}^{\infty} W_n$$

where W_n are finite dimensional subspaces of V.

Proof. This is a direct consequence of the previous corollary and the Baire Category Theorem. □

4.3 Bounded Linear Operators

Definition. A linear operator $T : X \to Y$ is *bounded* if

$$\sup\{||T(x)|| : x \in X, \, ||x|| = 1\} < \infty$$

If T is bounded, then the *operator norm* is defined as

$$||T||_{\text{op}} = \sup\{||T(x)|| : x \in X, \, ||x|| = 1\}$$

The intuitive idea here is that the operator norm is essentially just the "biggest" transformation any vector can have in under T. Hence, another way to think about bounded linear operators is:

Theorem. *If $T : X \to Y$ is a bounded linear operator, then for all $x \in X$,*

$$||T(x)|| \leq ||T||_{op} \, ||x||$$

Now that we have an intuitive understanding of this norm, let's move on to some actual theorems.

Theorem. *Every finite dimensional linear operator is bounded*

Proof. The proof is left to the reader. The idea is that in finite dimensions, we can always represent the linear operator as a matrix, and there is no way to have ∞ as an entry in a matrix. □

Theorem. *The set $\mathcal{B}(X, Y) = \{T : X \to Y \mid T$ bounded$\}$ is a normed space under the operator norm.*

Proof. The checking the vector space and norm axioms are left to reader. □

Theorem. *Let X, Y be normed spaces and $T : X \to Y$ be a linear operator. Then the following are equivalent:*

1. *T is continuous*
2. *T is continuous at 0*
3. *T is bounded*

Proof. (1) $\implies$ (2) is clear.

Suppose (2). By continuity at 0, there exists δ such that $||x|| \leq \delta \implies ||T(x)|| \leq 1$. Then since T is linear,

$$||x|| \leq 1 \implies ||T(x)|| \leq \frac{1}{\delta}$$

Hence T is bounded.

Suppose (3). Since T is bounded, $||T(x) - T(x_0)|| \leq ||T||_{\text{op}}||x - x_0||$. Hence T is continuous at every $x_0 \in X$. □

4.3.1 Completeness Linear Operator Space

Theorem. *Let X, Y be metric spaces and Y complete. Then the space of linear operators $B(X, Y)$ is complete under $||\cdot||_{op}$.*

Proof. Let $\{A_n\}$ be cauchy with respect to $||\cdot||_{\text{op}}$. Then given $\epsilon > 0$, there exists N such that for all $n, m > N$, $||A_n - A_m||_{\text{op}} < \epsilon$. Then for any $x \in X$,

$$\epsilon > ||A_n - A_m||_{\text{op}} \geq ||A_n(x) - A_m(x)|| \geq \Big| ||A_n(x)|| - ||A_m(x)|| \Big|$$

Thus the $\{A_n(x)\}$ is cauchy for all $x \in X$. Then define A such that

$$A(x) = \lim_{y \to x} A_n(y)$$

then clearly, $A_n \to A$ with respect to $||\cdot||_{\text{op}}$.

An alternative proof if we are looking at $M_{n\times n}(\mathbb{R})$ is to just state isomorphism to $\mathbb{R}^{n\times n}$ and equivalence of all norms. □

4.4 The Contraction Principle

Definition. Let X be a metric space and $\Phi : X \to X$. Then $x_0 \in X$ is a *fixed point* if $\Phi(x_0) = x_0$.

Definition. Let X be a metric space and $\Phi : X \to X$. Then Φ is a *contraction mapping* is there exists some $C \in (0, 1)$ such that $d(\Phi(x), \Phi(y)) \leq Cd(x, y)$

Theorem (Contraction Mapping Theorem)**.** *A contraction mapping on a complete metric space X has exactly one fixed point.*

Proof. Let X be a complete metric space, $\Phi : X \to X$ be a contraction with $C \in (0, 1)$ and fix some $x \in X$. Then for all $m \in \mathbb{N}$

$$d(\Phi^m(x), \Phi^{m+1}(x)) \leq ... \leq C^m d(x, \Phi(x))$$

We have a fixed point if $\{\Phi^m(x)\}$ converges. Since X is complete, we need to show that $\{\Phi^m(x)\}$ Cauchy. Let $n < m$, then

$$\begin{aligned} d(\Phi^n(x), \Phi^m(x)) &\leq C^{m-n} d(x, \Phi(x)) + ... + C^n d(x, \Phi(x)) \\ &\leq \frac{C^m}{1 - C} d(x, \Phi(x)) \end{aligned}$$

Hence $\{\Phi^m(x)\}$ Cauchy and converges to some x_∞ and

$$d(x_\infty, \Phi(x_\infty)) = \lim_{m\to\infty} d(x_\infty, \Phi^m(x)) = 0$$

□

4.4.1 Application to Linear Algebra

It turns out that the Contraction Mapping theorem has many applications, one of which is in Linear Algebra. Suppose we have a Banach space X and a function $T : X \to X$ that's a contraction. Now, given some $u \in X$, we wish to find some $x_\infty \in X$ such that

$$(I - T)(x_\infty) = u$$

We can solve this by first seeing that

$$(I - T)(x_\infty) = u \iff x - T(x_\infty) = u \iff x = T(x_\infty) + u$$

Define $\Phi(x_\infty) = T(x_\infty) + u$, then

$$x_\infty \text{ is a solution } \iff x_\infty \text{ is a fixed point of} \Phi$$

Since $||\Phi(x) - \Phi(y)|| = ||T(x) + u - T(y) - u|| \leq ||T||_{\text{op}}||x - y||$, Φ must be a contraction. So, by the contraction mapping theorem, Φ has a fixed point. Furthermore, we can find x_∞ by iteration.

Fix $x_0 = 0$ and let $x_{n+1} = \Phi(x_n)$, then just as in the proof for the contraction mapping theorem, as $n \to \infty$

$$||x_n - x_\infty|| \leq \frac{||T^m||_{\text{op}}}{1 - ||T||_{\text{op}}}||u|| \to \infty$$

Hence, we can get a good idea of what x_∞ is:

$$\begin{aligned}
&x_0 = 0 \\
&\Phi(x) = u \\
&\Phi^2(x) = T(u) + u \\
&\Phi^3(x) = T^2(u) + T(u) + u \\
&\vdots \\
&x_\infty = \lim_{n \to \infty} (I + T + T^2 + ... + T^n)(u)
\end{aligned}$$

We know that this limit must exist because $||T||_{\text{op}} < 1$.

Notice that we just proved that for any $u \in X$, there is exactly one $x_\infty \in X$ such that $x_\infty = T(x_\infty) + u$ which is the same as

$$x_\infty - T(x_\infty) = (I - T)(x_\infty) = u$$

This means that $(I - T)$ is bijective so $(I - T)^{-1}$ exists! Therefore, we now know that $x_\infty = (I - T)^{-1}(u)$. But this is exactly

$$x_\infty = \lim_{n \to \infty} (I + T + T^2 + ... + T^n)(u)$$

So we have also discovered something very interesting... If $0 < ||T||_{\text{op}} < 1$, then

$$(I - T)^{-1} = \lim_{n \to \infty} (I + T + T^2 + ... + T^n)$$

Chapter 5

Differentiation

5.1 The Fréchet Derivative

Definition. Let X, Y be Banach, $U \subseteq X$ be open, and $F : U \to Y$ be linear. Then F is *differentiable* at $p \in U$ if there exists a (continuous $\iff$ continuous at 0 $\iff$ bounded) linear transformation $A : X \to Y$ such that

$$\lim_{h \to 0} \frac{||F(p+h) - F(p) - Ah||_Y}{||h||_X} = 0$$

Theorem. *If F if differentiable at p, then F is continuous at p.*

Proof. Suppose F is differentiable with derivative A, so $||Av|| \leq C||v||$ for some $C \in \mathbb{R}$. Since F is differentiable at p, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||h||_X \leq \delta \implies ||F(p+h) - F(p) - Ah||_Y \leq \epsilon ||h||_X$. Then,

$$\begin{aligned} ||F(p+h) - F(p)||_Y &\leq ||F(p+h) - F(p) - Ah||_Y + ||Ah|| \\ &\leq \epsilon ||h||_X + C||h||_X \\ &\leq (\epsilon + C)||h||_X \\ &< \epsilon \end{aligned}$$

hence F is continuous at p. $\square$

Theorem (Sub Mean Value)**.** *Let X, Y be Banach and $U \subset X$ be open. Let $F : X \to Y$ be differentiable on*

$$\ell(t) = \{(1-t)\,a + tb \mid t \in [0,1]\} \subset U \text{ and } \sup_{t \in [0,1]} ||DF|_{\ell(t)}|| \leq M$$

then $||T(a) - T(b)|| \leq M||a - b||$

Proof. The proof for this is actually really tedious so I won't even bother. But the idea is that the change must be bounded by the biggest derivative. □

5.2 Inverse Function Theorem

Theorem (Inverse Function Theorem)**.** *Let X, Y be Banach, $B(0,\delta) \subset X$, and $F : B(0,\delta) \to Y$ such that F is continuously differentiable. If $DF|_0 = I$, then there exists some $\epsilon \in (0,\delta]$ such that F is bijective over $B(0,\epsilon)$.*

Proof. (Injective)
Let $G = I - F$ and $v_1, v_2 \in B(0,\epsilon)$. One can easily check that

$$\ell(t) = \{(1-t)v_1 + tv_2 \mid t \in [0,1]\} \subset B(0,\epsilon)$$

Thus, we can use the sub mean value theorem to get

$$||G(v_1) - G(v_2)|| \leq \sup_{t\in[0,1]} ||DG|_{\ell(t)}||\ ||v_1 - v_2||$$

And since $G'(0) = I - I = 0$ and G' is continuous, we can choose some $\epsilon > 0$ so that $\sup_{t\in[0,1]} ||DG|_{\ell(t)}|| = \frac{1}{2}$. Thus forcing

$$\begin{aligned} ||G(v_1) - G(v_2)|| &\leq \frac{1}{2}||v_1 - v_2|| \\ ||v_1 - v_2|| &\leq \frac{1}{2}||v_1 - v_2|| \end{aligned}$$

hence $||v_1 - v_2|| = 0$ and $v_1 = v_2$. So $F : B(0,\epsilon) \to B(0, \frac{\epsilon}{2}$ is injective. □

Proof. (Surjective)
Fix $u \in B(0, \frac{\epsilon}{2}) \subset Y$ and let $\Phi : B(0,\epsilon) \to B(0, \frac{\epsilon}{2})$ be defined such that $\Phi(v) = v - F(v) + u$. Then

$$\begin{aligned} ||\Phi(v_1) - \Phi(v_2)|| &= ||v_1 - F(v_1) + u - v_2 + F(v_2) - u|| \\ &= ||G(v_1) - G(v_2)|| \\ &\leq \frac{1}{2}||v_1 - v_2|| \end{aligned}$$

hence Φ is a contraction. So by the contraction mapping theorem, there exists some unique $x \in B(0,\epsilon)$ such that

$$\Phi(x) = x \iff F(x) = u$$

for each $u \in B(0, \frac{\epsilon}{2})$. Hence F is surjective. □

Corollary. *The inverse function theorem is also true if $DF|_0$ is invertible. To see this, simply apply the inverse function theorem to $F^* = (DF^{-1}|_0)F$, which has $DF^*|_0 = I$.*

5.3 Mixed Partials

5.3.1 Equality of Mixed Partials

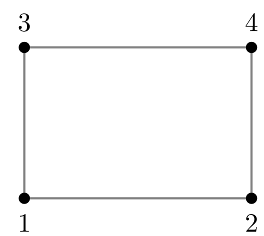
It is a well known fact that given a "nice" function $f : \mathbb{R}^2 \to \mathbb{R}$ that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But how do we know that this is true? What if this function or domain is not "nice"? We will begin this section with a satisfactory proof of the equality of mixed partials.

Theorem. *Let $U \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ such that f and DF are continuous. Then $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ exists and is continuous everywhere in U.*

Proof. Fix some $(x_0, y_0) \in U$. Since U is open, we can always find a rectangle that is in U. Suppose that the following image is in U:



where $\mathbf{1} = (x_0, y_0)$ and $\mathbf{4} = (x_0 + h, y_0 + k)$. Define

$$\Delta_{h,k} = f(\mathbf{4}) + f(\mathbf{3}) - f(\mathbf{2}) - f(\mathbf{1})$$

Notice that

$$\begin{aligned}&\text{Vertical change in } f \text{ at } x_0 + h = f(\mathbf{4}) - f(\mathbf{2})\\ &\text{Vertical change in } f \text{ at } x_0 = f(\mathbf{3}) - f(\mathbf{1})\end{aligned}$$

Now define $G(x) =$ vertical change of f at $x = f(x, y_0 + k) - f(x, y_0)$. So $\Delta_{h,k} = G(x_0 + h) - G(x_0)$. But this is a 1D equation, so by the Mean Value Theorem,

$$\Delta_{h,k} = G(x_0 + h) - G(x_0) = G'(c)h$$

for some $c \in [x_0, x_0 + h]$. We can find

$$\frac{dG(c)}{dx} = \frac{\partial f}{\partial x}\Big|_{(c,y_0+k)} - \frac{\partial f}{\partial x}\Big|_{(c,y_0+k)}$$

So now fix $x = c$. If we vary over y we can apply MVT again:

$$\begin{aligned}\frac{dG(c)}{dx} &= \frac{\partial f}{\partial x}\Big|_{(c,y_0+k)} - \frac{\partial f}{\partial x}\Big|_{(c,y_0+k)} \\ &= k\left(\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\Big|_{(c,d)}\right)\right)\end{aligned}$$

for some $d \in [y_0, y_0 + k]$. Hence

$$\Delta_{h,k} = hk\left(\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\Big|_{(c,d)}\right)\right)$$

. If we repeat this process but horizontally, we get that

$$\left(\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\Big|_{(c,d)}\right)\right) = \left(\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\Big|_{(a,b)}\right)\right)$$

for some $a \in [x_0, x_0 + h], b \in [y_0, y_0 + k]$. So if we send $h, k \to 0$, we get that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ at (x_0, y_0). $\square$

5.3.2 Converse of Mixed Partials Equality

Now that we have established what it takes for the mixed partials to be equal, what about the converse? If there are some p, q such that $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ everywhere, then is there some f such that $\frac{\partial f}{\partial x} = p$, $\frac{\partial f}{\partial y} = q$? This question inspires our next theorem:

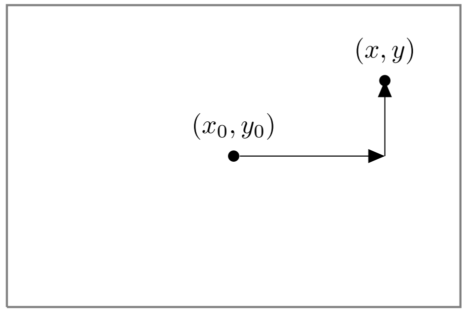
Theorem. *Suppose that we have some p, q defined on an open rectangle U and*

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$

Then there exists some $f : U \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = p,\ \frac{\partial f}{\partial y} = q$$

Proof. Suppose we have some f that works. Then for any constant C, $f + C$ works as well. WOLOG, let $f(x_0, y_0) = 0$ where (x_0, y_0) is the center of the rectangle. Now we just need to explicitly define f such that $\frac{\partial f}{\partial x} = p$, and $\frac{\partial f}{\partial y} = q$. Consider the following picture of U:



We want

$$\frac{\partial^2 f}{\partial x \partial y}\Big|_{(x,y)} = \frac{\partial^2 f}{\partial y \partial x}\Big|_{(x,y)}$$

for all $(x, y) \in U$. It's clear by the fundamental theorem of calculus that

$$\Delta f_{\text{horizontal}} = \int_{x_0}^{x} \frac{\partial f}{\partial x}\Big|_{(t,y_0)} dt = \int_{x_0}^{x} p(t, y_0) dt$$

$$\Delta f_{\text{vertical}} = \int_{y_0}^{y} \frac{\partial f}{\partial y}\Big|_{(x,s)} ds = \int_{y_0}^{y} q(x, s) ds$$

Pay special attention to where we are integrating. When we are at doing the vertical integral, we integrate along the arbitrary x, and not the fixed x_0. Since we have the Δf's, if f works, it better be

$$f(x, y) = \Delta f_{\text{horizontal}} + \Delta f_{\text{vertical}}$$

Indeed,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\int_{x_0}^{x} p(t, y_0) dt + \int_{y_0}^{y} q(x, s) ds\right) \\
&= p(x, y_0) + \frac{\partial}{\partial x}\int_{y_0}^{y} q(x, s) ds \\
&= p(x, y_0) + \int_{y_0}^{y} \frac{\partial q}{\partial x}\Big|_{(x,s)} ds
\end{aligned}$$

Note that this interchanging of the derivative and integral is only possible because the limit for the derivative converges uniformly. The proof of which is left for later. For the next step, recall that $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ which makes

$$\begin{aligned}
\frac{\partial f}{\partial x} &= p(x, y_0) + \int_{y_0}^{y} \frac{\partial p}{\partial x}\Big|_{(x,s)} ds \\
&= p(x, y_0) + p(x, y) - p(x, y_0) \\
&= p(x, y)
\end{aligned}$$

The partial for y is far easier:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\int_{x_0}^{x} p(t, y_0)dt + \int_{y_0}^{y} q(x, s)ds\right) \\ &= 0 + \frac{\partial}{\partial y}\int_{y_0}^{y} q(x, s)ds \\ &= q(x, y)\end{aligned}$$

Thus we have successfully defined a function such that the mixed partials are equal. $\square$

5.3.3 Important Things to Remember

Example. An example of p, q that is not the gradient of any $f : \mathbb{R}^2 \to \mathbb{R}$ is

$$p = \frac{-y}{x^2 + y^2}, \; q = \frac{x}{x^2 + y^2}$$

One can check if we integrate over the unit circle,

$$\oint (p, q) \cdot dr = 2\pi \neq 0$$

But it is a fact that for any $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\oint \nabla f \cdot dr = 0$$

for any closed curve.

Example. An example of a vector space V with $\text{div}(V) = 0$ but is not the curl of any vector space W is

$$V = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Stokes' Theorem tells us that

$$\iint_S \text{curl } F \cdot dS = \int_{\partial S} F \cdot dr$$

However, One can check that

$$\iint_C V = 4\pi, \; \iint_D V = 0$$

where C is the upper half of the unit ball, and D is the unit disk in the xy plane. But $\partial C = \partial D$ which is the unit circle. Hence if V is the curl of some F, there would be a contradiction of Stokes' Theorem.

Chapter 6

Integration

6.1 Riemann Integrals

Chapter 7

Fourier Analysis

7.1 Hilbert Spaces