Math 246A Final

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Abstract

The Fourier transform is one of the most powerful tools from mathematics with far reaching applications in areas such as differential equations, signal processing, physics. Roughly speaking, it is the continuous analogue to Fourier coefficients. The purpose of this paper is explore the relationship between functions and their Fourier transforms, focusing on analytic extensions of Fourier transforms and the Paley-Wiener theorems.

Contents

1 Introduction to Fourier Series and Transform

In this section, we will establish the basic definitions necessary to explore the analyticity of a Fourier Transform. We will begin with a brief introduction to Fourier series and its relation to the Fourier transform.

We will assume that all functions to be absolutely integrable unless otherwise stated.

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a L-periodic function for some $L \in \mathbb{R}$. The n^{th} *complex Fourier coefficient* defined as

$$c_n := \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-2\pi i n x/L} \, dx, \qquad n \in \mathbb{Z} \tag{1}$$

and the *Fourier series of* $f(x)$ is

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L} \tag{2}$$

The convergence properties of Fourier series are well-known and not in the scope of this paper. As such, we will take the following result for granted.

Theorem 1.1. *If f is Riemann integrable, then its Fourier series converges pointwise everywhere except at points of discontinuity, where it converges to the average between the left and right hand limits.*

We now move on to the the Fourier transform, which can be thought of as the generalization of the Fourier coefficient in two ways.

1. We no longer assume f to be periodic
2. Replace discrete variables with continuous variables.

The basic idea for the derivation is that since we no longer assume f to be periodic, taking a definite integral no longer makes sense. We fix this issue by interpreting non-periodic functions to be "∞-periodic", thus we take the integral along all of $\mathbb{R}$ by sending $L \to \infty$. Note that the derivation presented will not be rigorous as it is meant to provide intuition behind the motivation for the Fourier transform.

We begin by applying Theorem 1.1 to equation (2) to get

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L} \tag{3}$$

for all but finitely many points $x_1, \ldots, x_k$. Next, we substitute (1) into (3) and let $\xi_n := \frac{2\pi n}{L}$ so $\Delta\xi = \xi_n - \xi_{n-1} = \frac{2\pi}{L}$

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-2\pi i n x/L} \, dx \right) e^{2\pi i n x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\xi}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-2\pi i \xi_n x} \, dx \right) e^{2\pi i \xi_n x} \end{aligned} \tag{4}$$

We then take $L \to \infty$ inside of the sum because we want the integral to blow up first in order to get the correct "Fourier coefficients". This turns ξ_n into the continuous variable $\xi \in \mathbb{R}$ and the equation becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx \right) e^{2\pi i \xi x} d\xi \tag{5}$$

which we can interpret as the Riemann integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \tag{6}$$

where

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx \tag{7}$$

This derivation is the motivation for the definition of the Fourier transform:

Definition 1.2. For a function $f : \mathbb{R} \to \mathbb{R}$, the *Fourier transform* of f is equation (7) and the *Inverse Fourier transform* is equation (6).

Remark 1.1. Notice that in our derivation, the Inverse Fourier transform recovers the entirety of f except at finitely many points. This is due to the application of Theorem 1.1 in the first step.

Remark 1.2. There are several conventions for the notation of the Fourier transform due to the ambiguity of the placement of $\frac{1}{2\pi}$. For example, it is not uncommon to see it the Fourier transform defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx$$

and the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

Furthermore, the positive/negative exponential is also ambiguous. One can also define the Fourier transform with a positive exponential while the Inverse Fourier transform has the negative exponential.

Theorem 1.2 (Plancherel's theorem)**.** *Let $f \in L^2$ and $\hat{f}$ be the Fourier transform of f. Then*

$$||f||_{L^2} = ||\hat{f}||_{L^2}$$

This is a basic result from Fourier analysis that we will not prove

2 Analyticity of Fourier Transform

In the previous section, we defined the Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ as a function $\hat{f}(\xi) : \mathbb{R} \to \mathbb{C}$. Notice then that it is possible to extend $\hat{f}(\xi)$ to a complex function. We will write the extension of $\hat{f}(\xi)$ to the complex plane as $\hat{f}(z)$ and we emphasize that ξ is a real variable and z is a complex variable.

This begs the question: *for which functions f can we expect $\hat{f}(z)$ to be analytic on some open $\Omega \subset \mathbb{C}$?* Before we propose some possible solutions, let us first begin by exploring some examples to gauge the problem. The following is a table of some common functions and their Fourier transforms.

Function name	$f(x)$	$\hat{f}(z)$
Constant	a	$a \cdot \delta(z)$
Dirac delta	$\delta(x)$	1
Cosine	$\cos(2\pi a x)$	$\frac{\delta(z-a)+\delta(z+a)}{2}$
Sine	$\sin(2\pi a x)$	$\frac{\delta(z-a)+\delta(z+a)}{2i}$
Step function	$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$	$\frac{2}{2\pi i z}$
Decaying exponential	$e^{ax}u(x)$	$\frac{1}{a+2\pi i z}$
Rectangle function	$\mathrm{rect}(x) = \begin{cases} 0 & \|t\| > \frac{1}{2} \\ 1 & \|t\| \leq \frac{1}{2} \end{cases}$	$\frac{\sin(\pi z)}{\pi z}$
Guassian	e^{ax^2}	$\frac{\pi}{a} e^{-\frac{(\pi z)^2}{a}}$

Table 1: Fourier transforms of some common functions

We observe that the functions with Fourier transforms that can be analytically extended to some open subset of $\mathbb{C}$ are the Dirac delta, step function, decaying exponential, rectangle function, and the Gaussian. And if we ignore the Gaussian for a moment, all of these functions are nonzero on some interval that is not all of $\mathbb{R}$. This hints towards the idea that perhaps, if $f(x) = 0$ everywhere except on some interval like $[a, b]$ or $[a, \infty)$, then $\hat{f}(x)$ will be analytic. If we restrict our attention to $L^2(\mathbb{R})$, this hypothesis gives us two potential candidates:

1. $L^2[0, \infty) = \{f \in L^2(\mathbb{R}) \mid f(x) = 0,\ x < 0\}$
2. $L^2[-a, a] = \{f \in L^2(\mathbb{R}) \mid f(x) = 0, |x| > a\}$.

Indeed, we can prove that given any function in either of these spaces, then its Fourier transform is analytic on some open set of $\mathbb{C}$.

Theorem 2.1. *If $f \in L^2[0, \infty)$. Then the Fourier transform of f is analytic in $\mathbb{C}^- = \{z \in \mathbb{C} \mid \Im(z) < 0\}$. Furthermore, the restriction of $\hat{f}$ along any horizontal line in $\mathbb{C}^-$ is in $L^2(\mathbb{R})$.*

Proof. We will prove that $\hat{f}$ is analytic by applying Morera's theorem. To do this, we first show that $\hat{f}(z)$ is continuous. Let $z_0 \in \mathbb{C}^-$ be fixed and let $z_n \in \mathbb{C}^-$ be a sequence such that $z_n \to z_0$. It suffices to show that as $n \to \infty$, $|\hat{f}(z_0) - \hat{f}(z_n)|^2 \to 0$.

$$\begin{aligned} |\hat{f}(z_0) - \hat{f}(z_n)|^2 &= \left| \int_0^\infty f(x) \left(e^{-2\pi i z_0 x} - e^{-2\pi i z_n x} \right) dx \right|^2 \\ &\leq ||f||_{L^2}^2 \int_0^\infty \left| e^{-2\pi i z_0 x} - e^{-2\pi i z_n x} \right|^2 dx \end{aligned}$$

Since $\mathbb{C}^-$ is open, it is possible to find some $\delta > 0$ such that $\Im(z_n) < -\delta$ and $-\Im(z_0) > \delta$. Then

$$\begin{aligned} |e^{-2\pi i z_n x}| &= |e^{2\pi \Im(z_n) x}| \leq |e^{-2\pi \delta x}| \\ |e^{-2\pi i z_0 x}| &= |e^{2\pi \Im(z_0) x}| \leq |e^{-2\pi \delta x}| \end{aligned}$$

By the triangle inequality,

$$\left| e^{-2\pi i z_0 x} - e^{-2\pi i z_n x} \right|^2 \leq 4e^{-2\pi \delta x}$$

Thus by dominated convergence,

$$\lim_{n\to\infty} |\hat{f}(z_0) - \hat{f}(z_n)|^2 = \lim_{n\to\infty} \left| \int_0^\infty f(x) \left(e^{-2\pi i z_0 x} - e^{-2\pi i z_n x} \right) dx \right|^2 = 0$$

Finally, to show that $\hat{f}(z)$ is conservative, it suffices to show that $\int_\gamma \hat{f}(z)\,dz = 0$ for any triangular curve γ in C^-. By the Fubini-Tonelli theorem, we find that

$$\int_\gamma \hat{f}(z)\,dz = \int_\gamma \int_0^\infty f(x)e^{-2\pi izx}\,dx\,dz = \int_0^\infty f(x)\int_\gamma e^{-2\pi izx}\,dz\,dx$$

Since $e^{-2\pi izx}$ is holomorphic for fixed $x \in \mathbb{R}$, Cauchy's theorem gives us

$$\int_\gamma \hat{f}(z)\,dz = 0$$

Now we will prove that the restriction $\hat{f}(\xi - iy)$ for any $y > 0$ fixed is $L^2(\mathbb{R})$. First, notice that

$$\hat{f}(\xi - iy) = \int_0^\infty \left(f(x)e^{-2\pi yx}\right) e^{-2\pi i\xi x}\,dt = \hat{g}(\xi)$$

where $g(x) = f(x)e^{-2\pi yx}$. which gives us

$$\begin{aligned} ||\hat{f}(\xi - iy)||_{L^2}^2 &= \int_{-\infty}^\infty |\hat{f}(\xi - iy)|^2\,d\xi \\ &= \int_{-\infty}^\infty |\hat{g}(\xi)|^2\,d\xi \end{aligned}$$

Then by Plancherel's theorem,

$$\begin{aligned} \int_{-\infty}^\infty |\hat{g}(\xi)|^2\,d\xi &= \int_{-\infty}^\infty |g(x)|^2\,dx \\ &= \int_{-\infty}^\infty |f(x)e^{-2\pi yx}|^2\,dx \\ &\leq \int_0^\infty |f(x)|^2 e^{-4\pi yx}\,dx \\ &\leq \int_0^\infty |f(x)|^2\,dx \end{aligned}$$

Hence the restriction of $\hat{f}$ to any horizontal line in $\mathbb{C}^-$ is $L^2(\mathbb{R})$ $\square$

Theorem 2.2. *If $f \in L^2[-a,a]$ for some $0 < a < \infty$, then $\hat{f}(x)$ is entire and $|\hat{f}(z)| \leq Ce^{a|z|}$, where*

$$C = \int_{-a}^a |f(x)|\,dx$$

Proof. The proof is the same as the one given in Theorem 2.1. □

These results are just as we predicted in the beginning of the section: in order for the Fourier transform of a function to be analytic, the function itself must vanish at least on half of the real line. Furthermore, thanks to Plancherel's theorem, we are able to discover some additional properties of $\hat{f}$ such as the upper bounds on its growth.

3 The Paley-Wiener Theorems

As with any if-then statement, it is always good to check whether or not the converse is true as well. Remarkably, the converses of both Theorem 2.1 and 2.2 are true, and any theorem that assumes the analyticity of a Fourier transform to get properties of the original function is known as a Paley-Wiener Theorem. The Paley-Wiener theorems that correspond to theorems 2.1 and 2.2 are as follows:

Theorem 3.1. *Suppose that f is holomorphic on $\mathbb{C}^-$ and*

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(\xi - iy)|^2 \, d\xi = C < \infty$$

Then there exists a function $F \in L^2[0, \infty)$ such that

$$f(z) = \int_0^{\infty} F(x) e^{-2\pi i z x} \, dx, \quad z \in \mathbb{C}^-$$

and

$$\int_0^{\infty} |F(x)|^2 \, dx = C$$

Theorem 3.2. *Let $C, a > 0$ and f be an entire function such that*

$$|f(z)| < Ce^{a|z|} \quad \text{and} \quad \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

Then there exists $F \in L^2[-a, a]$ such that

$$f(z) = \int_{-a}^{a} F(x) e^{-2\pi i z x} \, dx$$

for all $z \in \mathbb{C}$

Before we begin the proof for theorem 3.1, some intuition will be provided. We want a function F such that it's Fourier transform is f. If we look back at the proof for theorem 2.1, we expect $f(\xi - iy)$ to be the Fourier transform of $F(x)e^{-2\pi yx}$ for all fixed $y > 0$. This means that by the Inverse Fourier transform equation, we should have

$$F(x) = \int_{-\infty}^{\infty} f(\xi - iy)e^{2\pi xy}e^{2\pi ix\xi}\, d\xi$$

which should hold regardless of the value of y. This hints that, eventually in the proof, we will have to show that this integral remains the same for differing values of y.

Proof of Theorem 3.1. For clarity, we will denote $f_y(x) = f(x - iy)$ for $y > 0$ fixed. For simplicity, let

$$F(x) := \int_{\Im(z)=-1} f(z)e^{2\pi ixz} = e^{2\pi x}\int_{-\infty}^{\infty} f_1(\xi)e^{2\pi ix\xi}\, d\xi$$

We need to show that $F(x) = 0$ for all $x < 0$ and

$$f_y(\xi) = \int_{-\infty}^{\infty} F(x)e^{-2\pi y\xi}e^{-2\pi ix\xi}\, dx$$

We will first try to prove that the choice of $y > 1$ does not matter in our definition for F, in other words: $F(x) = e^{2\pi xy}\int_{-\infty}^{\infty} f_y(\xi)e^{2\pi ix\xi}\, d\xi$ for all $y > 0$. We will accomplish this by doing a very familiar technique from complex analysis: which is to define a rectangular contour and stretch some sides of it to ∞.

WLOG, let $y > 1$ (if $y < 1$ we would simply need to reverse some contours). Let $k > 0$ and γ_k be the rectangular contour with corners at $\pm k - iy, \pm k - i$. By Cauchy's theorem we have

$$\int_{\gamma_k} f(z)e^{2\pi ixz}\, dz = 0$$

In order to show that our choice of $y > 1$ does not matter, we need to show that the vertical components of the integral goes to 0 as we send $k \to \infty$. However, it is only possible to show that there is a sequence of k's such that the vertical components go to 0, which is good enough.

Lemma. *Let*

$$V(k) = \int_{k-iy}^{k-i} f(z)e^{2\pi ixz}\, dz$$

There exists a sequence $k_n \to \infty$ such that $V(\pm k_n) \to 0$

Proof. By Cauchy-Schwartz and the fact that $|e^{iz}| = e^{\Im(z)}$,

$$|V(k)|^2 \leq \int_{k-iy}^{k-i} |f(z)|^2 \, dz \int_{k-iy}^{k-i} e^{4\pi x \Im(z)} \, dz = \int_{-y}^{-1} |f(k+iu)|^2 \, du \int_{-y}^{-1} e^{4\pi x u} \, du$$

The second integral does not depend on k, so we focus on the first integral only. Recall that for $u < 0$,

$$\int_{-\infty}^{\infty} |f(k+iu)|^2 \, dk = C < \infty$$

hence

$$\int_{-y}^{-1} \int_{-\infty}^{\infty} |f(k+iu)|^2 \, dk du = C(y-1)$$

By Fubini's theorem,

$$\int_{-\infty}^{\infty} \int_{-y}^{-1} |f(k+iu)|^2 = C(y-1)$$

Thus there must exist some sequence $k_n \to \infty$ such that $\int_{-y}^{-1} |f(k+iu)|^2 \to 0$ because otherwise, the double integral should always diverge. We then also have $V(\pm k_n) \to 0$. $\square$

(From this point forward, the proof could contain inaccuracies due to the lack of understanding of measure theory of the author) As a result of this lemma, we have that for all $y > 0$,

$$\lim_{n\to\infty} \left(e^{2\pi x} \int_{-\infty}^{\infty} f_1(\xi) e^{2\pi i x \xi} \, d\xi - e^{2\pi x y} \int_{-\infty}^{\infty} f_y(\xi) e^{2\pi i x \xi} \, d\xi \right) = 0 \tag{8}$$

We now move on to prove the main result

$$f_y(\xi) = \int_{-\infty}^{\infty} F(x) e^{-2\pi y \xi} e^{-2\pi i x \xi} \, dx$$

Firstly, we know that by Plancherel's theorem that

$$\int_{-k_n}^{k_n} f(\xi - iy) e^{2\pi i t x} \xrightarrow{L^2} \int_{-\infty}^{\infty} f(\xi - iy) e^{2\pi i t x}$$

Theorem 3.12 in Rudin's *Real and Complex Analysis* then gives us a subsequence k_{n_j} such that

$$\int_{-k_{n_j}}^{k_{n_j}} f(\xi - iy) e^{2\pi i t x} \xrightarrow{\text{pointwise}} \int_{-\infty}^{\infty} f(\xi - iy) e^{2\pi i t x}$$

almost everywhere. For the sake of notation let $\hat{f}_y(x) = \int_{-\infty}^{\infty} f_y(\xi) e^{2\pi i x \xi}\, d\xi$. Then, by equation (8), if we define

$$F(x) = e^{2\pi x} \hat{f}_1(x)$$

we also have

$$F(x) = e^{2\pi x y} \hat{f}_y(x) \tag{9}$$

for all $y > 0$. Since $\hat{f}_y(x)$ is the Inverse Fourier transform of f_y, we can apply Plancherel's theorem to (9) and get

$$\int_{-\infty}^{\infty} e^{-4\pi x y} |F(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}_y(x)|^2\, d\xi = \int_{-\infty}^{\infty} |f(\xi)|^2\, d\xi \leq C \tag{10}$$

By taking limits of y, we can find out two things:

1. If we send $y \to \infty$ we find that $F(x) = 0$ almost everywhere for $x < 0$. If it did not, then we will have a positive exponential $e^{4\pi y}$ over a positive measure, which will cause the first integral in (10) to diverge as $y \to \infty$.

2. If we send $y \to 0$ by monotone convergence theorem we find that $\int_0^{\infty} |F(x)|^2\, dx \leq C$. Knowing this, we can apply Cauchy-Schwarz and (9) to get

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \hat{f}_y(x)\, dx \right)^2 &= \left(\int_0^{\infty} F(x) e^{-2\pi x y}\, dx \right)^2 \\ &\leq \left(\int_0^{\infty} |F(x)|^2\, dx \right) \left(\int_0^{\infty} e^{-4\pi x y}\, dx \right) \\ &\leq \frac{C}{4\pi y} \end{aligned}$$

   hence $\hat{f}_y \in L^1$ for all $y > 0$

Therefore, we can finally conclude

$$f(\xi - iy) = \int_{-\infty}^{\infty} \hat{f}_y(x) e^{-2\pi i x \xi}\, dx = \int_0^{\infty} F(x) e^{-2\pi x y} e^{-2\pi i x \xi}\, dx$$

for all $y > 0$. □

The proof for theorem 3.2 is far more complex than that for theorem 3.1 and hence will be omitted. However, one can expect that the technique to prove it is similar to before: by integrating along some contour and eventually applying Cauchy's theorem. The biggest difference, which consequently causes a lot more difficulties, is that we instead choose our contours to be rays coming out of the origin.

These theorems, along with our results from section 2, provide us some interesting relationships between entire classes of functions that appear to have little connection. Theorems 2.1 and 3.1 gives us a relationship between functions in $L^2[0,\infty)$ and analytic functions on the $\mathbb{C}^-$, and theorems 2.2 and 3.2 for functions in $L^2[-a,a]$ to entire functions with exponential growth. From an informal standpoint, these results are already immensely helpful. The classes of functions stated is already perhaps more than enough for most applications. For example, if we look back at the Fourier transforms table, suddenly it is no longer mysterious as to why only some of the functions have smooth Fourier transforms.

The second class, $L^2[-a,a]$ also provides a potential problem to explore in the future. One would expect that some form of the Paley-Wiener theorem to be true for general $L^2[a,b]$. And if that was to be the case, means that we can take two functions $f \in L^2[a,b], g \in L^2[b,c]$ and "glue" them together by defining

$$h(x) = \begin{cases} 0 & x < a \text{ or } x > b \\ f(x) & x \in [a,b] \\ g(x) & x \in [b,c] \end{cases}$$

which would be in $L^2[a,c]$. It should then be possible and interesting to explore the relationships between $\hat{f}, \hat{g}$, and $\hat{h}$. If such relationships exist, they could be quite useful since this suggests it is likely possible to estimate the Fourier transforms of complicated L^2 functions using analytic Fourier transforms of piecewise estimations in $L^2[a,b]$.

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