Vector Line Integrals

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In previous sections, we found multiple ways to integrate scalar-valued functions: first over a region, then over a curve. Now, we will compute vector line integrals by using the dot product. First, we will explore the physical applications of vector line integrals. Then we will formally derive and define vector line integrals. We will conclude by discussing different types of curves, allowing us to determine if a curve is suitable for integration, or if it must be first broken up into piecewise components.

Physical Interpretation of a Vector Line Integral

Before we formally define the vector line integral, let us interpret what a vector line integral means physically. In order to do that, we first need to define and describe some relevant terms in physics. A **Force**, put simply, is the interaction, whether it be push or pull, applied on an object. Thus, a force could either help or deter the motion of an object in space. **Work** occurs when a force causes displacement in an object, and is given by $\mathbf{F} \cdot \Delta\mathbf{x}$, where $\Delta\mathbf{x}$ is the change in position of the object. Using this definition, we can see that work is a scalar that is positive when force and displacement point in the same general direction, negative when they point in the general opposite direction, and zero when they are perpendicular.

But how do we calculate work when the path is curved? When the path is curved, we can still calculate work by finding the summation of infinitesimal portions of $\mathbf{F} \cdot \Delta\mathbf{x}$. This summation is what the vector line integral represents. Thus, we can interpret the vector line integral as the amount of work the field applies to the particle traveling on the path **C**.

Let $\mathbf{F}$ represent the force field of $\mathbb{R}^n$ and the path of the particle be **C**, parameterized by $\mathbf{x}$: $[a, b] \to \mathbb{R}^n$. Then, $\int_x \mathbf{F} \cdot d\mathbf{s}$ is the **Net Work** over **C**.

With this interpretation, let's look at some examples.

Example 1

Consider figure 1. Let **C** be the circle, **D** be the leftmost path, and **E** be the rightmost path. Using only the graph, determine if the net work for each path is negative, positive, or zero.

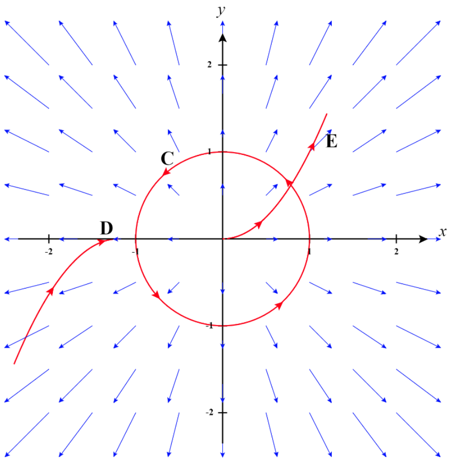

Figure 1

For the path $\mathbf{C}$, we can see that the path is always perpendicular to the vector field, therefore $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$. For $\mathbf{D}$, We can see that the path and the vector field point in the general opposite direction, therefore $\int_D \mathbf{F} \cdot d\mathbf{s}$ is negative. Finally, for $\mathbf{E}$, We can see that the path and the vector field point in a similar direction, therefore $\int_D \mathbf{F} \cdot d\mathbf{s}$ is positive.

Derivation

Now that we have a more intuitive understanding of vector line integrals, we are ready to derive the formula. Let $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ and let $\mathbf{C}$ be a path in the xy-plane. If $\mathbf{x}(t)$ is a parameterization of $\mathbf{C}$ where $t \in [a, b]$, then we can use Riemann sums to compute the net work done by $\mathbf{F}$ in the direction of $\mathbf{x}$, given by $\mathbf{F} \cdot \Delta\mathbf{x}$.

To calculate the total work $\mathbf{F}$ does on $\mathbf{C}$, we can divide our path into n subdivisions. Let $\Delta t = \frac{b-a}{n}$, and let t^* be any point within Δt. Then $\mathbf{x}(t^*)$ represents any given point on $\mathbf{C}$, so $\mathbf{F}(\mathbf{x}(t^*))$ represents the force vector at $\mathbf{x}(t^*)$. Therefore $\mathbf{F}(\mathbf{x}(t^*)) \cdot \Delta\mathbf{x}$ represents the work done by $\mathbf{F}$ on $\mathbf{x}(t^*)$. By adding up $\mathbf{F}(\mathbf{x}(t^*)) \cdot \Delta\mathbf{x}$ for every subdivision of $\mathbf{x}$, we get an approximation of the net work that $\mathbf{F}$ does on $\mathbf{C}$. Just like in single-variable calculus, as we approach infinite subdivisions, the approximation gets better and better. Therefore, we can define a **vector line integral** as

$$\int_{\mathbf{C}} \mathbf{F} \cdot \; d\mathbf{s} \quad = \quad \lim_{\Delta t \to \infty} \sum_{i=1}^{n} \mathbf{F}(\mathbf{x}(t^*)) \cdot \Delta\mathbf{x}. \tag{1}$$

Riemann sums are fairly impractical given the complexity of the functions we will be using throughout this course. We will therefore manipulate this definition to include an integral on the right hand side. Take careful note of the differential term notation. We use $d\mathbf{s}$ instead of ds to indicate that we are integrating over a position along a path, as opposed to arclength. We use a dot product because of our definition of work. Looking back to equation (1), we can rewrite $\Delta\mathbf{x}$ as

$$\Delta\mathbf{x} = \mathbf{x}(t^* + \Delta t) - \mathbf{x}(t^*).$$

Using the Mean Value Theorem to replace the right-hand side of the equation, we get $\Delta\mathbf{x} = \mathbf{x}'(t)\Delta t$. Plugging this back into the Riemann sum, we have

$$\int_{\mathbf{C}} \mathbf{F} \cdot \; d\mathbf{s} = \lim_{\Delta t \to \infty} \sum_{i=1}^{n} \mathbf{F}(\mathbf{x}(t^*)) \cdot \mathbf{x}'(t)\Delta t.$$

Because Δt is along $\mathbf{C}$, we can use our knowledge of scalar line integrals to modify this sum to

$$\int_{\mathbf{C}} \mathbf{F} \cdot \; d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \; dt.$$

This leads us to our formal definition of vector line integrals.

The **vector line integral** of $\mathbf{F}$ along $\mathbf{x}\colon [a, b] \to \mathbb{R}^n$ is
$\int_{\mathbf{x}} \mathbf{F} \cdot \; d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \; dt$

Note that the dot product allows us to compute vector line integrals in as many dimensions as we like, provided that the path on which we are integrating is in the same dimension as the vector field.

Example 2

Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (-y, x, z)$ and let $\mathbf{x}$ be the path $\mathbf{x} = (e^t, 3, t)$, where $t \in [0, 1]$. Evaluate $\int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s}$.

We have $\mathbf{x}'(t) = (e^t, 0, 1)$ and $\mathbf{F}(\mathbf{x}(t)) = (-3, e^t, t)$. Using our formal definition of vector line integrals,

$$\begin{aligned}
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (-3, e^t, t) \cdot (e^t, 0, 1) \; dt \\
&= \int_0^1 -3e^t + t \; dt \\
&= \left[-3e^t + \frac{1}{2}t^2 \right]_0^1 \\
&= -3e + \frac{7}{2}.
\end{aligned}$$

We can also compute vector line integrals along certain piecewise curves. (We'll look at specific requirements later on in this section.)

Example 3

Let $\mathbf{F}$ the force field given by $\mathbf{F}(x, y) = (-y, x)$ and let $\mathbf{x}$ be the piecewise path defined by

$$\mathbf{x}(t) = \begin{cases} (t, t^2), & 0 \leq t \leq 1 \\ (2t - 1, 2 - t), & 1 < t \leq 2. \end{cases}$$

Evaluate the work that $\mathbf{F}$ does on $\mathbf{x}$.

The work that $\mathbf{F}$ does on $\mathbf{x}$ is equivalent to $\int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s}$. We can evaluate this integral simply by splitting it up along each element of the piecewise curve. For $0 \leq t \leq 1$, $\mathbf{F}(\mathbf{x}(t)) = (-t^2, t)$ and $\mathbf{x}'(t) = (1, 2t)$. For $1 < t \leq 2$, $\mathbf{F}(\mathbf{x}(t)) = (t - 2, 2t - 1)$ and $\mathbf{x}'(t) = (2, -1)$. So

$$\begin{aligned}
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (-t^2, t) \cdot (1, 2t) \; dt + \int_1^2 (t - 2, 2t - 1) \cdot (2, -1) \; dt \\
&= \int_0^1 t^2 \; dt + \int_1^2 -3 \; dt \\
&= \left[\frac{1}{3}t^3 \right]_0^1 + [-3t]_1^2 \\
&= \frac{-8}{3}.
\end{aligned}$$

Not all curves are easy to deal with. In order to understand which curves are easily integrable and which ones are not, we first need a basic understanding of the different types of curves.

Simple and Closed Curves

When working with curves, it is important to have classifications that aid in the understanding of their properties. Two such classifications are simple and closed curves. To be simple, a curve must not intersect with itself or be discontinuous. To be closed, a curve must start and end at the same point without being discontinuous. Without delving unnecessarily into the numerous useful theorems that rely on these properties, it should be rather intuitive that curves that are simple and closed are often easier to parameterize, reparameterize, and evaluate line integrals of than curves that are not simple and not closed. Refer to figure 2 below for examples of curves that are simple, closed, both, and neither.

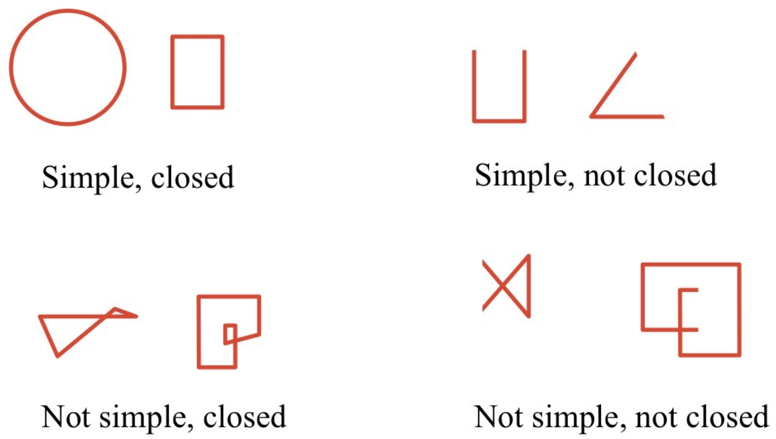

Figure 2

Example 4

Consider the following parametric equations for curves in the xy-plane. Identify if each curve is simple and if it is closed.

1. $\mathbf{x}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$
2. $\mathbf{x}(t) = (2\cos t, 2\sin t), 0 \leq t \leq 4\pi$
3. $\mathbf{x}(t) = (\cos t, \sin t), 0 \leq t \leq \frac{3\pi}{2}$
4. $\mathbf{x}(t) = (1/t, t), 0 \leq t \leq 1$

1. Both. This equation describes the unit circle, which is both simple and closed.

2. Closed, not simple. Due to the bounds on t, a particle travelling on this path would cross each point twice, meaning that the curve is not simple.

3. Simple, not closed. This equation describes an incomplete portion of the unit circle that, while simple, does not start and end at the same point. Thus, it is not closed.

4. Neither. This curve is discontinuous when $t = 0$ and is thus neither simple nor closed.

Reparameterization

When computing vector line integrals, a situation often arises in which reparameterizing the curve would lend itself to a simpler integral. A question logically follows this realization: Does reparameterizing a curve change its vector line integral? The answer is no, on the condition that the reparameterization is oriented the same way as the original parameterization. For a more formal explanation, see Theorem 1.1 below.

Theorem 1.1: Let $\mathbf{x}$: $[a, b] \to \mathbb{R}^n$ be a continuous vector field whose domain $\mathbf{X}$ contains the image of $\mathbf{x}$. Let $\mathbf{y}$: $[c, d] \to \mathbb{R}^n$ be any reparameterization of $\mathbf{x}$. Then

1. If y is orientation-preserving, then $\int_{\mathbf{y}} \mathbf{F} \cdot \, d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s}$.
2. If y is orientation-reversing, then $\int_{\mathbf{y}} \mathbf{F} \cdot \, d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s}$.

In simple terms, this theorem states two things:

1. If the orientation of the reparameterization is that same as that of the original parameterization, the resulting vector line integral will equate to the same value as the vector line integral of the original parameterization.
2. If the orientation of the reparameterization is opposite that of the original parameterization, the resulting vector line integral will equate to negative one times the value of the vector line integral of the original parameterization.

To illustrate this concept, consider Example 5 below, which demonstrates the relationship between a vector line integral of a curve and its orientation reversing reparameterization.

Example 5

Let $\mathbf{F}(x, y) = (-y, x)$ and $\mathbf{x}(t) = (\cos t, \sin t)$, $t \in [0, \pi]$. Calculate the Integral $\int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s}$.
We find that

$$\begin{aligned}
\int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s} &= \int_0^{\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) \, d\mathbf{s} \\
&= \int_0^{\pi} \sin^2(t) + \cos^2(t) \, dt \\
&= \pi
\end{aligned}$$

Now that we have solved this vector line integral, consider the following reparameterization $\mathbf{y}(t) = (\sin t, \cos t)$, for $t \in [-\pi/2, \pi/2]$. Because this reparameterization traces out the same curve as $\mathbf{x}(t)$ in reverse orientation, Theorem 1.1 tells us that $\int_{\mathbf{y}} \mathbf{F} \cdot \, d\mathbf{s} = -\pi$. Rather than take the theorem at face value, let's evaluate $\int_{\mathbf{y}} \mathbf{F} \cdot \, d\mathbf{s}$ to be certain.

$$\begin{aligned}
\int_{\mathbf{y}} \mathbf{F} \cdot \, d\mathbf{s} &= \int_{\pi/2}^{\pi/2} (-\cos t, \sin t) \cdot (\cos t, -\sin t) \, dt \\
&= \int_{-\pi/2}^{\pi/2} -\cos^2 t - \sin^2 t \, dt \\
&= \int_{-\pi/2}^{\pi/2} -1 \, dt \\
&= \pi
\end{aligned}$$

As Theorem 1.1 stated, $\int_{\mathbf{y}} \mathbf{F} \cdot \, d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F} \cdot \, d\mathbf{s}$.

Conclusion

In this section, we explored how to calculate vector line integrals, as well as their applications in physics. In addition, we discussed how simple and closed curves can be used to integrate over piecewise smooth curves, and how reparameterization affects the sign of the vector line integral.

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This source provided the background information necessary to derive the formal definition of vector line integrals.

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