An Introduction on Topics in Enumerative Combinatorics

Zhaomeng Chen

Chapter 1

Introduction

1.1 Set Theory

A *set* is simply a collection of mathematical objects. These objects could be anything: numbers, geometric shapes, or even other sets. Objects in a set are called *elements*. To notate a set, we put $\{\}$ around objects. For example, a set with the elements x, y, and z is $\{x, y, z\}$.

To say that an object x is an element in the set X, we use the notation $x \in X$. Similarly, to say that x is not an element in the set X, we use $x \notin X$.

The following are the notations of commonly used sets relevant to today:

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$
$$\mathbb{Z} = \{..., -1, -2, 0, 1, 2, ...\}$$
$$[n] = \{1, 2, 3, ..., n\}$$
$$\varnothing = \{\}$$

Since listing everything in an set is tedious, we can just say that all objects that meet some requirement are in this set. For example

$$A = \{x \in \mathbb{Z} \mid x \geq 0\} = \mathbb{N}$$

We read the "$|$" as "such that". We would read the previous example as

$$A \text{ is the set of integers } x \text{ such that } x \geq 0$$

The left side of the $|$ defines what type of objects are in the set and the right side defines some restrictions to it.

If a set A is finite, then $|A|$ is the number of elements in the set.

Now suppose A and B are sets. then

$$A \cup B = \{x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x \in A \text{ and } x \in B\}$$
$$A \setminus B = \{x \in A \mid x \notin B\}$$

1.2 Functions

Given two sets A and B, a function from A to B assigns every element of A to some element of B. The set A is called the *domain* of the function, and B is called the *codomain* (or sometimes the *range*). If the function is called f, then we write $f : A \to B$ to indicate that A is the domain and B is the codomain.

$$\text{For example, } f(x) = (x+1)(x-1) \text{ is a function } f : \mathbb{R} \to \mathbb{R}$$

The image of a function $f : A \to B$ is the subset of B that contains all the outputs of f:

$$\text{im}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$$

There are some special functions with special names. A function is called one-to-one or injective if every unique input generates a unique output. This means $f(x)$ is not allowed to be the same as $f(y)$ if x and y are different. Written technically,

$$x \neq y \Rightarrow f(x) \neq f(y)$$

A function is called onto or surjective if the image is the entire codomain. This means every element of the codomain occurs as an output. Technically, this is written as

$$\forall b \in B, \exists a \in A \text{ such that } f(a) = b$$

A function is bijective if it is both injective and surjective. **Bijective functions have inverses.**

Chapter 2

Counting Finite Sets

2.1 Special Sets

Let X have n elements.

$$\mathcal{P}(X) = \{A | A \subseteq X\} = \text{set of all subsets of } X$$

$$\binom{X}{k} = \{A \subseteq X \| A| = k\} = \text{set of subsets of } X \text{ with size } k \textbf{ (combinations)}$$

$$\text{Sym}(X) = \{f : X \to X \mid f \text{is bijective}\} = \text{set of } \textbf{permutations} \text{ of } X$$

$$S_n = \text{Sym}(X) = \text{special notation when } |X| = n$$

2.2 Counting Principles

Sum Principle: Let A and B be disjoint finite sets, then $|A \cup B| = |A| + |B|$.

***Product Principle*:** Let A and B be finite sets, then $|A \times B| = |A| \cdot |B|$

Bijection Principle: Let A and B be finite sets and suppose there is a bijection $f\colon A \to B$, then $|A| = |B|$

2.3 Counting Basic Sets

Let $|X| = n$ for all of this section.

Preposition 2.3.1:

$$|\mathcal{P}(X)| = 2^{|X|} = 2^n$$

Proof: Let $B = \{0, 1\}$. By the product principle, $|B^n| = 2^n$. Next, we choose an ordering of the elements of x, given as $x_1, x_2, ..., x_n$. Define the function $f\colon \mathcal{P}(X) \to B^n$, where given a subset $(A \in \mathcal{P}(X)) \subseteq X$, we define an element $(b_1, b_2, ..., b_n) \in B^n$ as follows. If $x_i \in A$, then $b_i = 1$, and otherwise $b_i = 0$.

To see this is injective, suppose that sets A and A' give the same $(b_1, ..., b_n)$. Then for each i, since $b_i = b_i'$, this means $x_i \in A$ if and only if $x_i \in A'$, and so $A = A'$.

To see this is surjective, take any $(b_1, ..., b_n) \in B_n$. Then form the set A which has $x_i \in A$ for all i where $b_i = 1$. Then A maps back to the original $(b_1, ..., b_n)$.

We can also prove by finding an inverse function of f

Preposition 2.3.2:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: Consider the power set, which we know has size 2^n. The power set is all subsets of X, so $\mathcal{P}(X) = \binom{X}{0} \cup \binom{X}{1} \cup ... \cup \binom{X}{n}$. The sets $\binom{X}{k}$ are all disjoint, so

$$|\mathcal{P}(X)| = \left|\binom{X}{0}\right| + \left|\binom{X}{1}\right| + ... + \left|\binom{X}{n}\right| = \sum_{k=0}^{n} \left|\binom{X}{k}\right|$$

Using the fact that $|\binom{X}{k}| = \binom{n}{k}$, the proof is complete.

Preposition 2.3.2:

$$\sum_{k=1}^{n} k = \binom{n+1}{2}$$

Proof: Consider the set $X = \binom{[n+1]}{2}$, which consists of all pairs of elements from $[n+1] = \{1, 2, 3, ..., n+1\}$. We will break down X into smaller, disjoint sets so that $X = A_1 \cup A_2 \cup ... \cup A_n$. Let A_1 be the set whose smaller element is n, then A_1 only contains $\{n, n+1\}$. Let A_2 be the set whose smaller element is $n-1$, then A_2 contains $\{n-1, n\}$ and $\{n-1, n+1\}$. In general, we define A_k to be the set of pairs whose smaller element is $n-k+1$. Clearly, $|A_k| = k$ and thus the proof is complete.

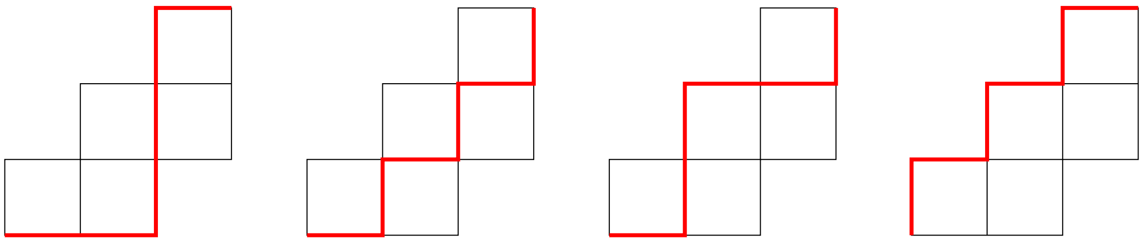
Chapter 3

Famous Combinatorial Sequences

3.1 The Fibonacci Numbers

We all know what the Fibonacci Numbers are, so instead, we will look at a unique case were the Fibonacci Numbers "randomly" show up and examine why it is that they do.

Example 3.1.1: Consider n squares arranged in a "staircase" shape going right, up, right, up, etc. Let A_n be the set of paths (using the edges of the squares) from the bottom-left corner to the top-right corner, which are only allowed to go right or up (the paths cannot go left or down). Some examples are shown below for $n = 5$ (the paths are in red):



This pattern follows the Fibonacci numbers: $|A_n| = f_{n+2}$ (the Fibonacci numbers).
Why? We will see once we prove that the previous statement is true.

Proof by induction: For the base case, we will need to show that $A_0 = f_2$ and $A_1 = f_3$ (Alternatively, we could do A_2 and A_3). We can say that $|A_0| = 1$ because there is 1 way to draw a path when there's 0 squares: by doing nothing. Next, we can simply draw out all the paths for one square to confirm that $|A_1| = 2$. Thus the base case is done.

For the induction case, we can see that for A_n, the nth box always touches the top right corner of boxes $n-1$ and $n-2$. Thus, all paths of A_n must go through the top right corner of boxes $n-1$ or $n-2$. We know that A_{n-1} are all the paths that go to the top right corner of box $n-1$, and A_{n-2} of box $n-2$. Thus, A_n consists of extensions of paths in A_{n-1}, and paths in A_{n-2} that doesn't become paths in A_{n-1}. Hence $|A_n| = |A_{n-1}| + |A_{n-2}| = f_{n+2}$

3.2 The Catalan Numbers

The Catalan Numbers are quite interesting, like the Fibonacci, they appear in very unexpected places. They are so unexpectedly common that Richard Stanley, a professor at MIT, wrote a book with around 100 or so things that corresponds to the Catalan Numbers. However, the equation for the Catalan is not nearly as "nice" as the Fibonacci:

$$C_{n+1} = \sum_{k=0}^{n} C_n C_{n-k}$$

with initial conditions $C_0 = C_1 = 0$.

Because of this, the proofs for Catalan numbers is quite tedious, so we will simply just look at some very interesting things that corresponds to the Catalan Numbers and not prove them. If you really want to see a proof, there is an example of one that is at the Appendix.

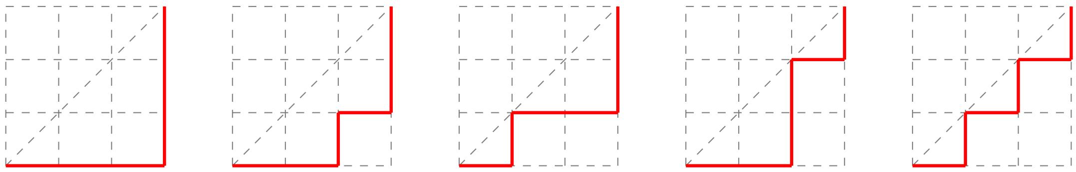
Example 3.2.1: Consider multiplying with $n+1$ factors:

$$x_0x_1...x_n$$

The number of ways we can evaluate this expression, one pair at a time, without changing the order of factors, is the Catalan number C_n

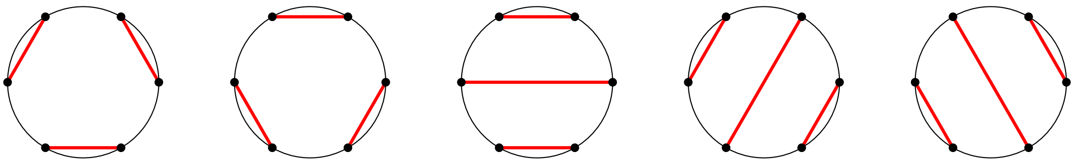
We can also represent this as binary trees

Example 3.2.2: Consider *lattice paths*: picture a square n-by-n grid, whose bottom-left corner has coordinate $(0,0)$ and top-right corner has coordinate (n,n). A lattice path is a path from $(0,0)$ to (n,n) which only takes steps to the right or up. Let L_n be the set of lattice paths with the additional property that they do not go above the diagonal line $y=x$. All of the elements in L_3 are shown below:



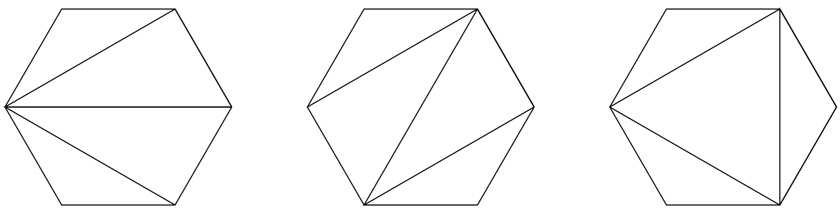
The number of lattice paths for an n-by-n corresponds to the Catalan numbers: $|L_n| = C_n$

Example 3.2.3: Draw a circle and label $2n$ points on it. A "*non-crossing matching*" is a way of drawing n lines connecting pairs of points so that none of the lines cross, and all the points are an endpoint of exactly one of the lines. All the possibilities for $n=3$ are shown below.



The number of non-crossing matchings of a circle with $2n$ points is the Catalan number C_n.

Example 3.2.4: Consider *triangulations*[1]: a way to draw diagonals which cut the polygon completely into triangles. Let T_n be the set of all triangulations of a convex n-gon. For example, some triangulations of a hexagon are shown below (there are 14 total triangulation of a hexagon, so these are not all of them):



The number of triangulations of an n-gon corresponds to the Catalan numbers: $|T_n| = C_{n-2}$.

[1]The proof for this example is extremely interesting, view the Appendix if time allows

Appendix A

Catalan Number Proofs

A.1 Example 3.2.3

Proof By Induction
Base Case: Let $C_0 = 1$ since there's one way to connect 0 points. When $n = 1$, there is 1 possibility for connecting the line. When $n = 2$ we find that there is 2 ways to connect the lines without overlap. Thus the base case follows the recurrence $C_2 = \sum_{k=0}^{2} C_k C_{2-k}$

Induction Case: Assume that for $n = i$, there are C_i ways to connect the points. As an extension of this assumption, we can also assume that this is true for $n < i$ as well. We will now prove that for $n = i + 1$, there are C_{i+1} possible ways to connect the points.

Let $n = i + 1$, then there are $2i + 2$ points on the circle. Designate a point as point 1 and number the rest of the points counterclockwise in increasing order. Any line we draw must be between an adjacent point or after skipping an even number of points. Then we have $i+1$ points to choose from.

Choose the point $j = 2x$ where $x \in [2n]$, then one side of the circle there are $2x - 2$ empty points and $2i + 2 - 2x$ empty points on the other side. Thus, the total ways we can connect the points given that we connect points 1 and j is equal to $C_{x-1}C_{i+1-x}$. We repeat this for all $i + 1$ options to find that there are $\sum_{k=0}^{i+1} C_k C_{i+1-k} = C_{i+1}$ possible ways to connect the points.

A.2 Example 3.2.4

Bijective Proof
Consider the function f: $Tn \to \text{Tree}(n-2)$. The function will work like this: start by choosing one edge of the polygon, and "erase" it (draw it as a dotted line in the pictures). Now, starting to the left of the dotted edge, and going counter-clockwise, label the sides (not vertices) of the polygon $0, 1, 2, ..., n-2$.

We will construct a tree. The nodes of the tree will be drawn on the edges of the triangulation. Start by drawing the "root node" on the dotted line. Then draw branches of the tree from this node to the two other sides of the same triangle. From each of these nodes, draw branches to the other sides of those triangles, etc. At the end, you will have a tree whose "leaves" are the boundary sides labelled $0, 1, ..., n-2$.