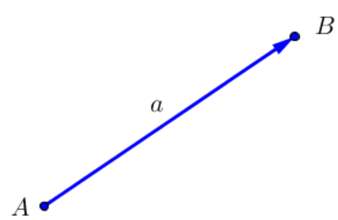
ENGG1040 Foundations in Engineering Mathematics

Chapter 4: Vectors

4.1 What is a Vector?

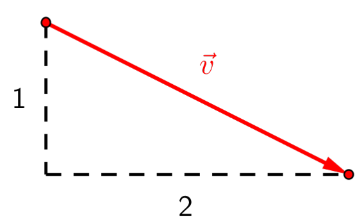
Vectors were first introduced in geometry and physics. They are used to represent quantities with both **magnitude** and **direction**, for example, displacement and force. The concept of vector has been generalized and widely used in mathematics, computer science and other areas of engineering.

A vector can be represented geometrically by a directed line segment, or arrow diagram, that shows both the magnitude and direction. Consider the directed line segment with **initial point** A (also known as the *tail*) and **end point** B (also known as the *terminal point* or *head*) shown. This vector is denoted by $\overrightarrow{AB}$.

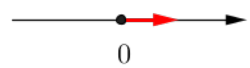


The length of the arrow represents the magnitude of the quantity, and the arrowhead shows its direction. We can also use arrow or bold letter to denote a vector, for example, $\vec{a}$, or $\mathbf{a}$. The magnitude or length of these vectors could be represented by $|\overrightarrow{AB}|$, $|\vec{a}|$, or $|\mathbf{a}|$.

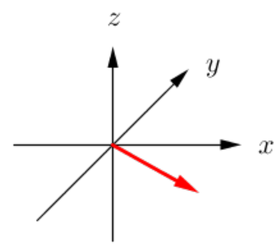
Algebraically, a vector $\vec{v} = (2, -1)$ in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (2D) means a vector that goes to the right by 2 units and goes up by -1 unit, i.e., down by 1 unit.



We can talk about vector in other dimensions too. For example,



(2) in 1D



(1, 3, −4) in 3D

Remark. There are several common notations for vectors. For example, all the notations below represent the same vector.

$$(2,-1), \quad \langle 2,-1\rangle, \quad [2,-1], \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad 2\vec{i}-\vec{j}$$

4.2 Basic Vector Operations

The following operations can be defined for a scalar k and 2D vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$. These operations can be similarly defined for vectors of other dimensions.

Vector Addition	$\vec{a}+\vec{b} = (a_1+b_1, a_2+b_2)$
Vector Subtraction	$\vec{a}-\vec{b} = (a_1-b_1, a_2-b_2)$
Scalar Multiplication	$k\vec{a} = (ka_1, ka_2)$

Remark. These operations can be defined similarly in other dimensions.

Example 1. Find each of the following for $\vec{v} = (1,1)$, $\vec{w} = (-1,2)$.

1. Addition

$$\begin{aligned} \vec{v}+\vec{w} &= (1,1)+(-1,2) \\ &= (1+(-1), 1+2) \\ &= (0,3) \end{aligned}$$

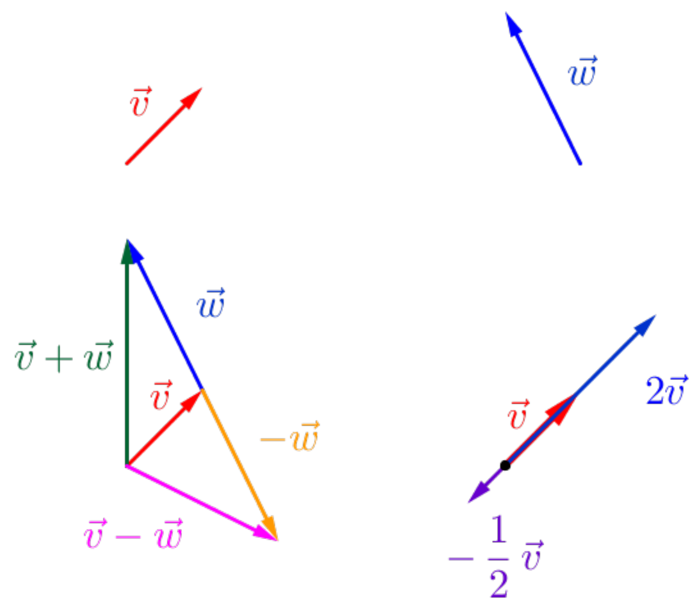
2. Subtraction

$$\begin{aligned} \vec{v}-\vec{w} &= (1,1)-(-1,2) \\ &= (1-(-1), 1-2) \\ &= (2,-1) \end{aligned}$$

3. <u>Scalar multiplication</u>

$$\begin{aligned} 2\vec{v} &= 2(1,1) = (2(1),2(1)) = (2,2) \\ -\frac{1}{2}\vec{v} &= -\frac{1}{2}(1,1) = (-\frac{1}{2},-\frac{1}{2}) \end{aligned}$$

These operations can be represented graphically:



In particular, if A, B, C are points, then

- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\overrightarrow{BA} = -\overrightarrow{AB}$

Similar to numbers, there is also a zero vector in each dimension.

$$\textbf{zero vector} = \vec{0} = \begin{cases} (0,0) & \text{in } \mathbb{R}^2; \\ (0,0,0) & \text{in } \mathbb{R}^3. \end{cases}$$

These operations on vectors have many properties similar to those on numbers.

Proposition 1. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, $\alpha, \beta \in \mathbb{R}$.

1. $0\vec{v} = \vec{0}$
2. $1\vec{v} = \vec{v}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative rule)
4. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (Commutative rule)
5. $\vec{v} + \vec{0} = \vec{v}$
6. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ (Distributive rule)
7. $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$ (Distributive rule)
8. $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$

A **position vector** is a vector with initial point at the origin. Such a vector is uniquely described by the coordinates of its terminal point. If $P(x, y)$ is the terminal point, then the position vector $\overrightarrow{OP} = (x, y)$.

More generally, consider the vector with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$. To move from the initial point to the terminal point, the vector goes to the right by $x_2 - x_1$ and up by $y_2 - y_1$. Hence,

$$\boxed{\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1).}$$

Remark. It is not uncommon for people to use the notations $\langle x, y, z \rangle$ for vectors and (x, y, z) for points. We will not follow this and write (x, y, z) for both vectors and points.

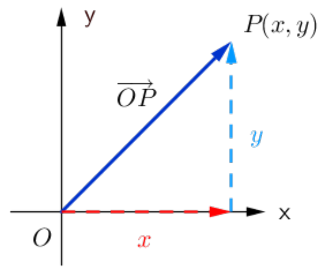
Example 2. $A = (1, 0), B = (3, 3), C = (2, 4), D = (0, 1)$. Show that $ABCD$ is a parallelogram.

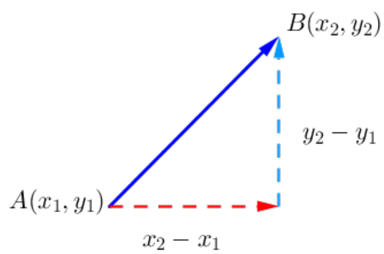
Solution.

$$\begin{aligned} \overrightarrow{AB} &= (3, 3) - (1, 0) = (2, 3) \\ \overrightarrow{DC} &= (2, 4) - (0, 1) = (2, 3) = \overrightarrow{AB} \end{aligned}$$

Hence, $ABCD$ is a parallelogram. ■

Remark. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are considered equal as they have the same magnitude and direction even though with different initial points.





4.3 Length and Dot Product

We can define the followings for vectors in 2D:

For vectors $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$ in $\mathbb{R}^2$,

Length	$\|\|\vec{a}\|\| = \sqrt{a_1^2 + a_2^2}$
Dot Product	$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

We will show that dot product can be used to compute angles and projection. These concepts can be generalized to other dimensions. For example, in $\mathbb{R}^3$, we have

For vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$,

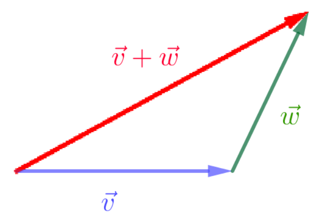
Length	$\|\|\vec{a}\|\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
Dot Product	$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

In $\mathbb{R}$, $||\vec{a}|| = \sqrt{a^2} = |a|$ is the absolute value!

We will first focus on length.

Proposition 2. (Properties of $||\vec{v}||$)

1. $||\vec{v}|| \geq 0$
2. $||\vec{v}|| = 0 \quad \Leftrightarrow \vec{v} = \vec{0}$
3. $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$
4. $||\alpha\vec{v}|| = |\alpha|||\vec{v}||$, α is a scalar.
5. $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$ (Triangle Inequality!)



A **unit vector** is a vector with length 1. It is common to denote a unit vector by the hat notation $\hat{v}$, instead of an arrow, to emphasize its unit length.

Example 3. Let $\vec{v} = (1, 2, -2)$.

1. Find $||\vec{v}||$.
2. Find the unit vector $\hat{u}$ in the opposite direction of $\vec{v}$.

Solution.

1. $$||\vec{v}|| = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

2. Unit vector in the *same* direction of $\vec{v}$:
$$\frac{\vec{v}}{||\vec{v}||} = \frac{1}{3}\vec{v} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$
Unit vector in the *opposite* direction of $\vec{v}$:
$$\hat{u} = -\frac{\vec{v}}{||\vec{v}||} = -\frac{1}{3}\vec{v} = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

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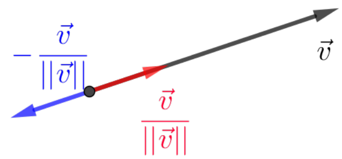
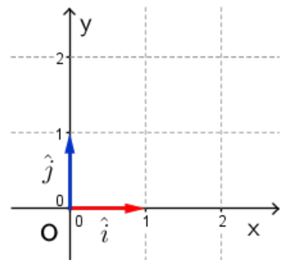
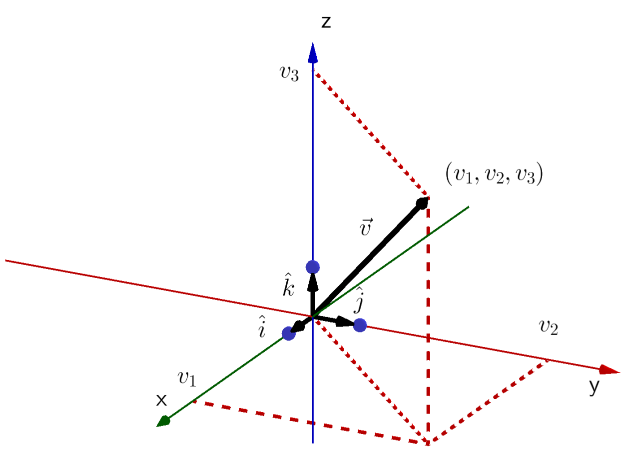


In $\mathbb{R}^2$, the unit vector in the direction of the positive x-axis and positive y-axis is denoted by $\hat{\imath} = (1, 0)$ and $\hat{\jmath} = (0, 1)$, respectively. They are called **standard unit vectors**.



Similarly, in $\mathbb{R}^3$, the standard unit vectors are $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$.



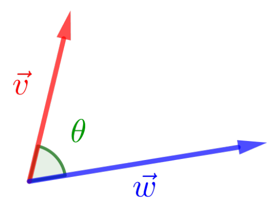
Remark. Standard unit vectors form a "building unit" of other vectors. For example:

$$(1, -2, 3) = (1, 0, 0) - (0, 2, 0) + (0, 0, 3) = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$$

Next, we will focus on dot product.

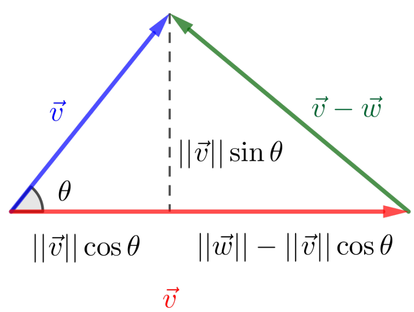
Proposition 3. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, $\alpha, \beta \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\alpha\vec{u} + \beta\vec{v}) \cdot \vec{w} = \alpha\vec{u} \cdot \vec{w} + \beta\vec{v} \cdot \vec{w}$
3. $\vec{0} \cdot \vec{v} = 0$
4. $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$
5. $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta$. where θ is the angle between $\vec{v}$ and $\vec{w}$.



Proof. We will prove property 5. The proof is essentially the proof of cosine law.

Consider the following triangle.



Note

$$\begin{aligned} ||\vec{v} - \vec{w}||^2 &= (||\vec{v}|| \sin\theta)^2 + (||\vec{w}|| - ||\vec{v}|| \cos\theta)^2 \\ &= ||\vec{v}||^2 \sin^2\theta + ||\vec{w}||^2 - 2||\vec{w}|| \, ||\vec{v}|| \cos\theta + ||\vec{v}||^2 \cos^2\theta \\ &= ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{w}|| \, ||\vec{v}|| \cos\theta \qquad (1) \end{aligned}$$

Also,

$$\begin{aligned} ||\vec{v} - \vec{w}||^2 &= (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) \\ &= \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \\ &= ||\vec{v}||^2 + ||\vec{w}||^2 - 2\vec{v} \cdot \vec{w} \qquad (2) \end{aligned}$$

Compare (1) and (2), we have

$$\vec{v}\cdot\vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta.$$

$\square$

Suppose $\vec{v}, \vec{w} \neq \vec{0}$ are non-zero vectors. Then $||\vec{v}||, ||\vec{w}|| > 0$. It follows from the formula $\vec{v}\cdot\vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta$ that

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||\,||\vec{w}||} \Rightarrow \begin{cases} \vec{v}\cdot\vec{w} > 0 & \Leftrightarrow \quad 0^\circ \leq \theta < 90^\circ \qquad \text{acute angle} \\ \vec{v}\cdot\vec{w} = 0 & \Leftrightarrow \quad \theta = 90^\circ \qquad \text{right angle} \\ \vec{v}\cdot\vec{w} < 0 & \Leftrightarrow \quad 90^\circ < \theta \leq 180^\circ \qquad \text{obtuse angle} \end{cases}$$

An important case is that:

$$\boxed{\vec{v} \perp \vec{w} \quad \Leftrightarrow \quad \vec{v}\cdot\vec{w} = 0}$$

Example 4. Direct computation gives that

$$\boxed{\begin{array}{lll} \hat{\imath}\cdot\hat{\imath} = 1 & & \\ \hat{\imath}\cdot\hat{\jmath} = 0 & \hat{\jmath}\cdot\hat{\jmath} = 1 & \\ \hat{\imath}\cdot\hat{k} = 0 & \hat{\jmath}\cdot\hat{k} = 0 & \hat{k}\cdot\hat{k} = 1 \end{array}}$$

The results agree with the fact that $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are pairwisely perpendicular unit vectors.

Example 5. Find the angle θ between $\vec{v} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\vec{w} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$.

Solution.

$$\begin{aligned} \vec{v}\cdot\vec{w} &= (\hat{\imath} + \hat{\jmath} + 3\hat{k})\cdot(2\hat{\imath} - \hat{\jmath} - 2\hat{k}) \\ &= (1)(2) + (1)(-1) + (3)(-2) \\ &= -5 \end{aligned}$$

$$\begin{aligned} ||\vec{v}|| &= \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11} \\ ||\vec{w}|| &= \sqrt{\vec{w}\cdot\vec{w}} = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3 \end{aligned}$$

$$\therefore \quad \cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||\,||\vec{w}||} = \frac{-5}{\sqrt{11}\cdot 3}$$

$$\Rightarrow \quad \theta = \arccos\left(\frac{-5}{3\sqrt{11}}\right) \approx 120.17^\circ$$

$\blacksquare$

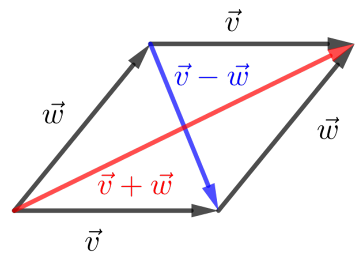
Example 6. Let $\vec{v}$, $\vec{w}$ have the same length. Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w})=0$.

Solution.

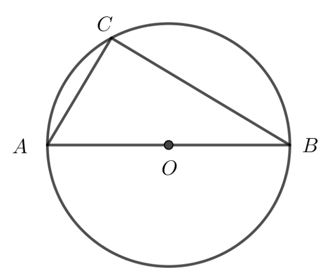
$$\begin{aligned}(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) &= \vec{v}\cdot\vec{v}-\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{v}-\vec{w}\cdot\vec{w}\\ &= \|v\|^2-\vec{v}\cdot\vec{w}+\vec{v}\cdot\vec{w}-\|w\|^2\\ &= \|v\|^2-\|v\|^2\\ &= 0\end{aligned}$$

■

Remark. The assumption $||\vec{v}||=||\vec{w}||$ means that the parallelogram spanned by $\vec{v}$ and $\vec{w}$ is a rhombus. The computation above shows that the diagonals of a rhombus are perpendicular.



Example 7. Consider a circle centered at O. AB is diameter. Show that $\angle ACB = 90°$.



Solution.

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO}+\overrightarrow{OC}\\ \overrightarrow{BC} &= \overrightarrow{BO}+\overrightarrow{OC} = -\overrightarrow{AO}+\overrightarrow{OC}\\ \overrightarrow{AC}\cdot\overrightarrow{BC} &= (\overrightarrow{AO}+\overrightarrow{OC})\cdot(-\overrightarrow{AO}+\overrightarrow{OC})\\ &= -\overrightarrow{AO}\cdot\overrightarrow{AO}+\overrightarrow{AO}\cdot\overrightarrow{OC}-\overrightarrow{OC}\cdot\overrightarrow{AO}+\overrightarrow{OC}\cdot\overrightarrow{OC}\\ &= -||\overrightarrow{AO}||^2+||\overrightarrow{OC}||^2 \qquad (||\overrightarrow{AO}||=||\overrightarrow{OC}|| \text{ are radius})\\ &= 0\end{aligned}$$

$$\therefore \overrightarrow{AC}\perp\overrightarrow{BC} \quad \Rightarrow \quad \angle ACB = 90°$$

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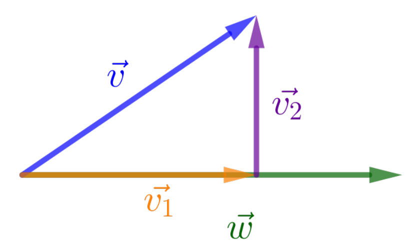
4.4 Projection Vector

Given vectors $\vec{v}, \vec{w}$ with $\vec{w} \neq 0$. Consider the diagram:



Decompose $\vec{v}$ as $\vec{v} = \vec{v_1} + \vec{v_2}$ such that

- $\vec{v_1} // \vec{w}$
- $\vec{v_2} \perp \vec{w}$

The vector $\vec{v_1}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$ and is denoted by $\text{proj}_{\vec{w}}\,\vec{v}$. To find a formula for it, note that

1. $$\vec{v_1} // \vec{w} \quad \Rightarrow \quad \vec{v_1} = k\vec{w} \quad \text{for some } k \in \mathbb{R}$$

2. $$\begin{aligned} \vec{v_2} \perp \vec{w} \quad & \Rightarrow \quad \vec{v_2} \cdot \vec{w} = 0 \\ & \Rightarrow \quad (\vec{v} - \vec{v_1}) \cdot \vec{w} = 0 \\ & \Rightarrow \quad \vec{v} \cdot \vec{w} = \vec{v_1} \cdot \vec{w} = (k\vec{w}) \cdot \vec{w} = k(\vec{w} \cdot \vec{w}) \\ & \Rightarrow \quad k = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \end{aligned}$$

Therefore,

Proposition 4. The projection of $\vec{v}$ onto a non-zero $\vec{w}$ is given by

$$\text{proj}_{\vec{w}}\,\vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w} = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2}\vec{w}$$

Remark. We cannot cancel the common factor $\vec{w}$ in a quotient of dot products.

Example 8. Find the projection of $\vec{v} = (3, 2)$ onto $\vec{w} = (5, -5)$. Then write $\vec{v}$ as the sum of two orthogonal vectors, one of which is the projection of $\vec{v}$ onto $\vec{w}$.

$$\xcancel{\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}} \neq \frac{\vec{v}}{\vec{w}}}$$

Solution.

1. Find the projection of $\vec{v}$ onto $\vec{w}$.

$$\begin{aligned}
\operatorname{proj}_{\vec{w}}\vec{v} &= \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w} \\
&= \frac{(3,2)\cdot(5,-5)}{||(5,-5)||^2}(5,-5) \\
&= \frac{5}{50}(5,-5) \\
&= \left(\frac{1}{2},-\frac{1}{2}\right)
\end{aligned}$$

2. Find $\vec{v_2}$.

$$\begin{aligned}
\vec{v_2} &= \vec{v}-\vec{v_1} \\
&= \vec{v}-\operatorname{proj}_{\vec{w}}\vec{v} \\
&= (3,2)-\left(\frac{1}{2},-\frac{1}{2}\right) \\
&= \left(\frac{5}{2},\frac{5}{2}\right)
\end{aligned}$$

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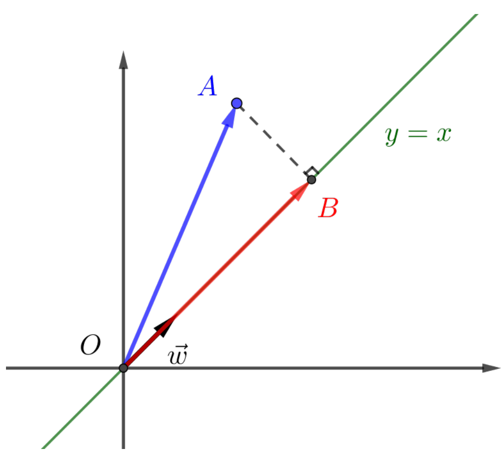
Example 9. Find the point B on the line $L : y = x$ which is closest to $A = (3,7)$.



Solution. Let $\vec{w} = (1, 1)$, $\vec{w} // L$.

$$\begin{aligned}
\overrightarrow{OB} &= \operatorname{proj}_{\vec{w}} \overrightarrow{OA} \\
&= \Big(\frac{\overrightarrow{OA} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\Big)\vec{w} \\
&= \frac{(3, 7) \cdot (1, 1)}{(1, 1) \cdot (1, 1)}(1, 1) \\
&= (5, 5)
\end{aligned}$$

$$\therefore \quad B = (5, 5)$$

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4.5 Cross Product

Besides dot product, there is another type of product, called cross product, for 3D vectors. It can be defined using determinant, which is a useful tool for matrices.

Definition 1. For a 2×2 matrix, its determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 matrix, its determinant is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example 10.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= (1)\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - (2)\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (3)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= (1)[(5)(9) - (6)(8)] - (2)[(4)(9) - (6)(7)] + (3)[(4)(8) - (5)(7)] \\
&= -3 + 12 - 9 \\
&= 0
\end{aligned}$$

Definition 2 (Cross product, defined only in 3D)**.** Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. Define their cross product by

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \\ &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)\end{aligned}$$

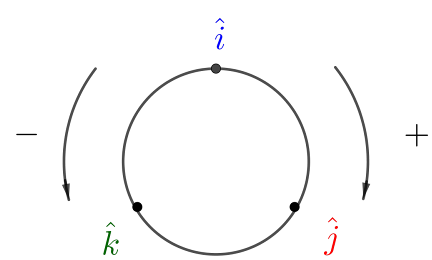
Example 11.

$$\begin{aligned}\hat{\imath} \times \hat{\jmath} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{k} \\ &= 0\hat{\imath} - 0\hat{\jmath} + 1\hat{k} = \hat{k}\end{aligned}$$

Similarly, we can compute the cross products of other standard unit vectors:

$$\begin{array}{lll} \hat{\imath} \times \hat{\imath} = \vec{0} & \hat{\imath} \times \hat{\jmath} = \hat{k} & \hat{\imath} \times \hat{k} = -\hat{\jmath} \\ \hat{\jmath} \times \hat{\imath} = -\hat{k} & \hat{\jmath} \times \hat{\jmath} = \vec{0} & \hat{\jmath} \times \hat{k} = \hat{\imath} \\ \hat{k} \times \hat{\imath} = \hat{\jmath} & \hat{k} \times \hat{\jmath} = -\hat{\imath} & \hat{k} \times \hat{k} = \vec{0} \end{array}$$

The diagram below helps you to remember the cross products of standard unit vectors.



Example 12. Let $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Solution.

$$\begin{aligned}\vec{a}\times\vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\hat{k} \\ &= -\hat{i} - \hat{j} + \hat{k}\end{aligned}$$

■

Exercise 4.5.1. Find $\vec{b}\times\vec{a}$ and $\vec{b}\times\vec{b}$.

Cross product has the following properties.

Proposition 5. Let $\vec{a},\vec{b},\vec{c}$ be vectors in $\mathbb{R}^3$, $\alpha,\beta\in\mathbb{R}$. Then

1. $\vec{a}\times\vec{a}=\vec{0}$
2. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
3. $(\alpha\vec{a}+\beta\vec{b})\times\vec{c}=\alpha\vec{a}\times\vec{c}+\beta\vec{b}\times\vec{c}$
4. Let θ be the angle between $\vec{a}$, $\vec{b}$.

   $$||\vec{a}\times\vec{b}|| = ||\vec{a}||\,||\vec{b}||\sin\theta = \text{Area of the parallelogram spanned by } \vec{a} \text{ and } \vec{b}.$$

5. $(\vec{a}\times\vec{b})\cdot\vec{a}=(\vec{a}\times\vec{b})\cdot\vec{b}=0$.

From property 4 above,

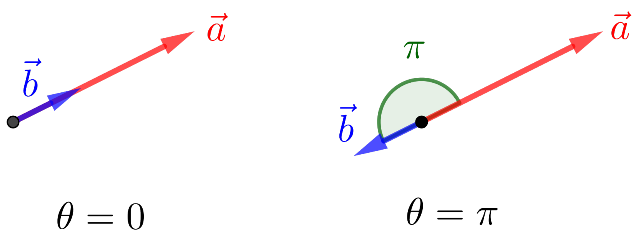
$$\begin{aligned}\vec{a}\times\vec{b}=\vec{0} &\Leftrightarrow ||\vec{a}\times\vec{b}||=0 \\ &\Leftrightarrow ||\vec{a}|| \text{ or } ||\vec{b}|| \text{ or } \sin\theta=0 \\ &\Leftrightarrow \vec{a}=\vec{0} \text{ or } \vec{b}=\vec{0} \text{ or } \theta=0 \text{ or } \theta=\pi.\end{aligned}$$

Hence,

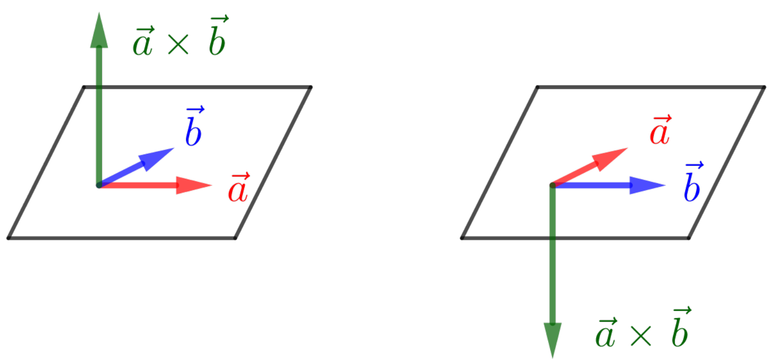
Two non-zero vectors have zero cross product if and only if they are pointing the same or opposite directions.

On the other hand, suppose $\vec{a}\times\vec{b}$ is non-zero. Then $\vec{a}$ and $\vec{b}$ are both non-zero. From property 5 above,

$\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

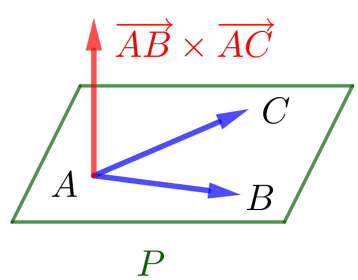


Also, it can be showed that $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy the **right hand rule**.



Example 13. Let $A = (1, 2, 1), B = (1, -1, 0), C = (2, 3, 2)$ be points on a plane P. Find a vector which is perpendicular to P.

Solution. The line segments AB and AC both lie on P. Hence, the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to P.



$$\begin{aligned}
\overrightarrow{AB} &= (1,-1,0)-(1,2,1) = (0,-3,-1) \\
\overrightarrow{AC} &= (2,3,2)-(1,2,1) = (1,1,1) \\
\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} \hat{k} \\
&= [(-3)(1)-(-1)(1)]\,\hat{\imath} - [(0)(1)-(-1)(1)]\,\hat{\jmath} + [(0)(1)-(-3)(1)]\,\hat{k} \\
&= -2\hat{\imath} - \hat{\jmath} + 3\hat{k}
\end{aligned}$$

Therefore, $(-2,-1,3) \perp P$. ■

Another product closely related to cross product is also defined for vectors in 3D.

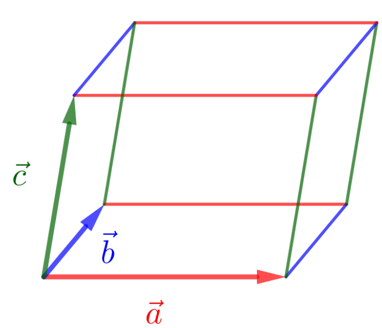
Definition 3 (Triple Product)**.** The triple product of $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ is defined to be

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Remark. For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = -\vec{a} \cdot (\vec{c} \times \vec{b}) = -\vec{b} \cdot (\vec{a} \times \vec{c}) = -\vec{c} \cdot (\vec{b} \times \vec{a})$$

Given 3D vectors $\vec{a}, \vec{b}$ and $\vec{c}$, they can form a parallelepiped.



Its volume can be computed using triple product.

$$|\vec{a} \cdot (\vec{b} \times \vec{c})| = \text{Volume of parallelepiped spanned by } \vec{a}, \vec{b}, \vec{c}.$$

Example 14. Let A, B, C, D, E be points in $\mathbb{R}^3$ with $A = (2,2,1), B = (1,1,1), C = (1,2,0), E = (2,4,-5)$. Suppose $ABCD$ is a parallelogram. Find

1. the coordinates of D;
2. the area of the parallelogram $ABCD$;
3. a unit vector which is perpendicular to $ABCD$;
4. the equation of the plane containing A, B, C;
5. the volume of the parallelepiped with adjacent sides $\overrightarrow{BA}, \overrightarrow{BC}$ and $\overrightarrow{BE}$.

Solution.

1. Since $ABCD$ is a parallelogram, we have

$$\overrightarrow{CD} = \overrightarrow{BA} = (2,2,1) - (1,1,1) = (1,1,0)$$

So,

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = (1,2,0) + (1,1,0) = (2,3,0)$$

2.

$$\begin{aligned} \overrightarrow{BC} &= (1,2,0) - (1,1,1) = (0,1,-1) \\ \Rightarrow \quad \overrightarrow{BA} \times \overrightarrow{BC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = -\hat{i} + \hat{j} + \hat{k} \\ \Rightarrow \quad \text{Area of } ABCD &= ||\overrightarrow{BA} \times \overrightarrow{BC}|| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3} \end{aligned}$$

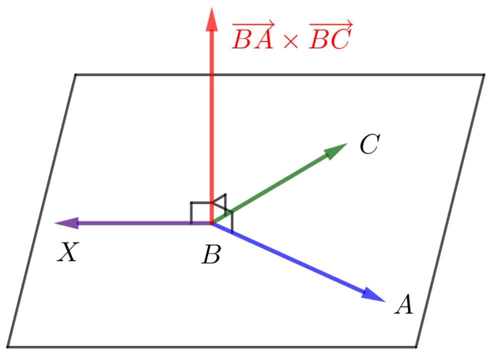
3. Since $\overrightarrow{BA} \times \overrightarrow{BC}$ is perpendicular to the parallelogram $ABCD$, the required vector can be taken to be

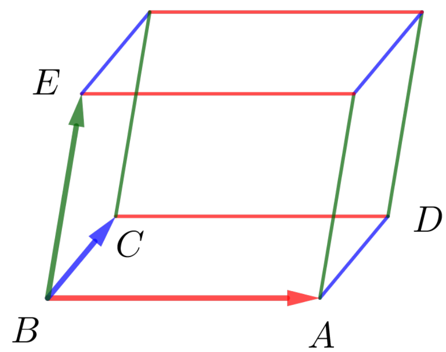
$$\frac{\overrightarrow{BA} \times \overrightarrow{BC}}{||\overrightarrow{BA} \times \overrightarrow{BC}||} = \frac{1}{\sqrt{3}}(-i + j + k)$$

4. Suppose $X = (x, y, z)$ is on the plane.

$$\begin{aligned} \overrightarrow{BA} \times \overrightarrow{BC} \perp \text{the plane} \Rightarrow \quad & \overrightarrow{BA} \times \overrightarrow{BC} \perp \overrightarrow{BX} \\ \Rightarrow \quad & (\overrightarrow{BA} \times \overrightarrow{BC}) \cdot \overrightarrow{BX} = 0 \end{aligned}$$

$$
\begin{aligned}
(-1,1,1)\cdot[(x,y,z)-(1,1,1)] &= 0\\
(-1,1,1)\cdot(x-1,y-1,z-1) &= 0\\
\Rightarrow \qquad -x+y+z &= 1
\end{aligned}
$$





5.

$$
\begin{aligned}
\overrightarrow{BE} &= (2,4,-5)-(1,1,1) = (1,3,-6)\\
\text{Triple Product} &= \overrightarrow{BE}\cdot(\overrightarrow{BA}\times\overrightarrow{BC})\\
&= (1,3,-6)\cdot(-1,1,1)\\
&= -4
\end{aligned}
$$

$$\Rightarrow \quad \text{Volume of the parallelepiped} = |-4| = 4$$

■

4.6 Vector-valued Function (Parametric Equation)

Most functions you saw are real-valued:

For example,

$$f(x) = x^2 + 2$$

is a real-valued function. When $x = 0$, $f(0) = 2$. The output 2 is a real number.

One can also consider functions which are vector-valued. For example,

$$\vec{r}(t) = (t^2, 2t - 1) = t^2\hat{\imath} + (2t - 1)\hat{\jmath}$$

is a vector-valued function with outputs in $\mathbb{R}^2$. When $t = 2$, $\vec{r}(2) = (4, 3) = 4\hat{\imath} + 3\hat{\jmath}$. The output is a vector.

Remark.

1. In the above example, we can write

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath},$$

where $x(t) = t^2$ and $y(t) = 2t - 1$ are called the **component functions** of $\vec{r}(t)$. Here t is called the **parameter**.

2. To understand $\vec{r}$, sometimes it is useful to regard t as time,

$$\vec{r}(t) = \text{ displacement of an object at time } t$$

and graph it on the xy-plane or xyz-space.

Example 15.

$$\begin{aligned}\vec{r}(t) &= (e^t, t, \sqrt{1 + t^2}) \\ &= e^t\hat{\imath} + t\hat{\jmath} + \sqrt{1 + t^2}\hat{k}\end{aligned}$$

is a 3D vector-valued function with component functions

$$x(t) = e^t \quad y(t) = t \quad z(t) = \sqrt{1 + t^2}$$

Example 16. To plot the function

$$\vec{r}(t) = t^2\hat{\imath} + (2t - 1)\hat{\jmath}$$

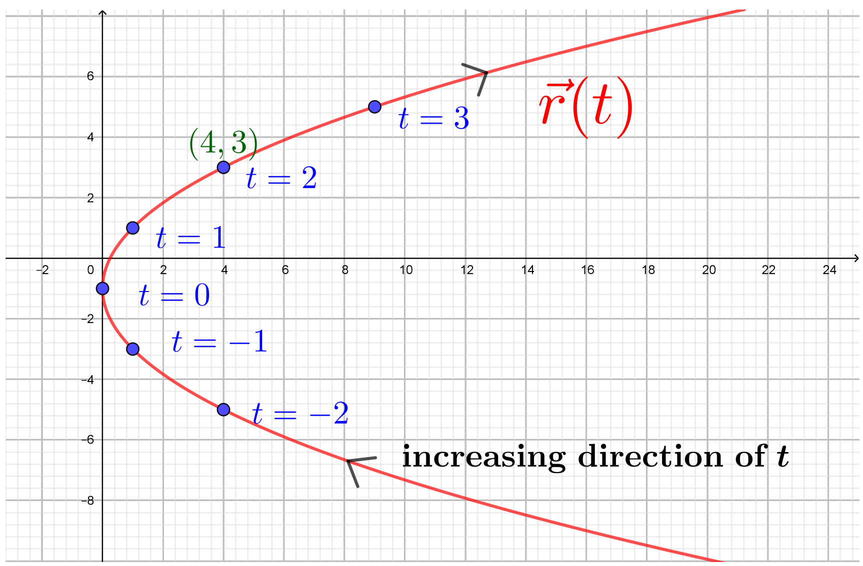
on the xy-plane, one approach is to plug in different values of t to find a few points on $\vec{r}(t)$.

t	-2	-1	0	1	2
$x(t)$	4	1	0	1	4
$y(t)$	-5	-3	-1	1	3

However, it does not give an accurate plot unless many points are used.

Another approach to plot the graph is to find a relation between x and y by eliminating t:

$$\begin{aligned} x &= x(t) = t^2 \\ y &= y(t) = 2t - 1 \quad \Rightarrow \quad t = \frac{y+1}{2} \\ \Rightarrow \quad x &= \left(\frac{y+1}{2}\right)^2 = \frac{1}{4}(y+1)^2 \quad \text{(Parabola!)} \end{aligned}$$



Example 17. Graph $\vec{r}(t) = (2\cos t^\circ)\hat{i} + (2\sin t^\circ)\hat{j}$ for $0 \le t \le 180$.

Solution.

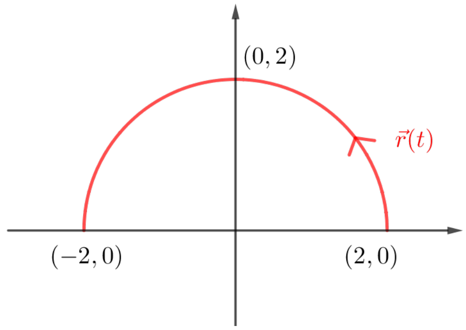
$$x = 2\cos t^\circ \quad y = 2\sin t^\circ$$

$$\begin{aligned} \therefore \quad x^2 + y^2 &= (2\cos t^\circ)^2 + (2\sin t^\circ)^2 \\ &= 4(\cos^2 t^\circ + \sin^2 t^\circ) \\ &= 4 \end{aligned}$$

$$\therefore \quad \vec{r}(t) \text{ lies on the circle } x^2 + y^2 = 4$$

Also, as t increases from 0 to 180, $x(t)$ decreases from 2 to -2, $y(t)$ increases from 0 to 2 and then decreases from 2 to 0. Therefore, the graph is

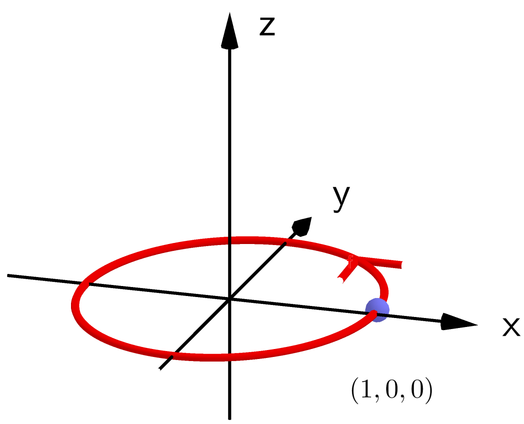
■



Example 18. Let $\vec{r}(t) = (\cos 360t^\circ)\hat{i} + (\sin 360t^\circ)\hat{j} + t\hat{k}$. Plot the 3D graph of $\vec{r}(t)$ for $0 \leq t \leq 1$.

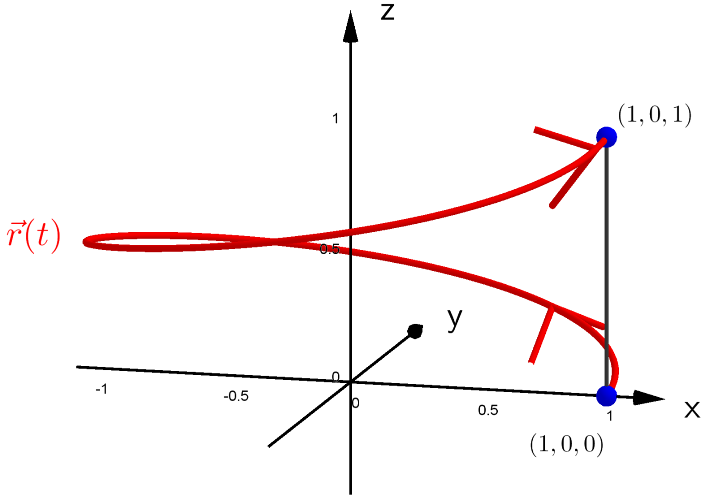
Solution. Note that as t increases from 0 to 1,

- $(\cos 360t^\circ)\hat{i}+(\sin 360t^\circ)\hat{j}$ rotates around the origin along the unit circle on the $xy-$plane.



- $z(t) = t$ increases from 0 to 1.

Hence $\vec{r}(t)$ rotates around the z-axis while going up from $z = 0$ to $z = 1$. Its graph is a "helix".



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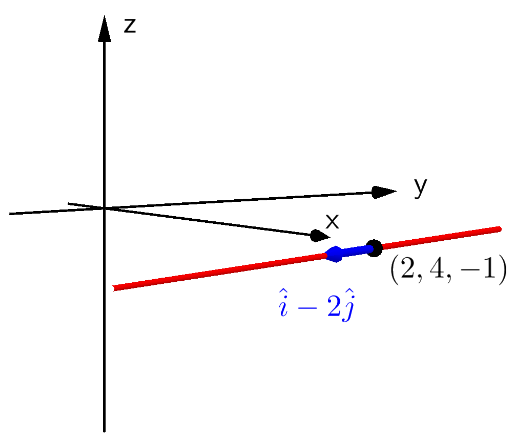
Example 19. Plot $\vec{r}(t) = (2+t)\hat{\imath} + (4-2t)\hat{\jmath} - \hat{k}$.

Solution.

$$\vec{r}(t) = (2\hat{\imath} + 4\hat{\jmath} - \hat{k}) + t(\hat{\imath} - 2\hat{\jmath})$$

$\Rightarrow \quad \vec{r}(t)$ is the straight line parallel to $\hat{\imath} - 2\hat{\jmath}$ and passes through $(2,4,-1)$.



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