ENGG1040 Foundations in Engineering Mathematics

Chapter 2: Polynomials

2.1 Summation

The summation notation Σ is used to represent a sum of terms indexed by integers.

For integers $m \leq n$,

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots a_n$$

Here i represents the index of summation and a_i represents each successive term in the sum. The $i = m$ under the summation symbol and n above it mean that i starts with $i = m$ and stops when $i = n$. m and n is called the lower and upper bound of summation respectively.

Example 1.

$$\sum_{i=3}^{7} i = 3 + 4 + 5 + 6 + 7$$

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2$$

Note that the sums above remain unchanged even if the index of summation i is replaced by another variable. For example,

$$\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = \sum_{i=1}^{4} i^2$$

Here are some properties of summation.

For integers m, n, p and real numbers a_i, b_i and p,

1. $$\sum_{i=m}^{n}(a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$$

2. $$\sum_{i=m}^{n} ra_i = r\sum_{i=m}^{n} a_i$$

3. $$\sum_{i=m}^{n} r = r(n-m+1)$$

4. $$\sum_{i=m+p}^{n+p} a_i = \sum_{i=m}^{n} a_{i+p}$$

These formulas can be seen to be true easily by writing out the terms in the summation. For example,

$$\begin{aligned}\sum_{i=m}^{n} ra_i &= ra_m + ra_{m+1} + ra_{m+2} + \ldots + ra_n \\ &= r(a_m + a_{m+1} + a_{m+2} + \ldots + a_n) \\ &= r\sum_{i=m}^{n} a_i\end{aligned}$$

2.2 Polynomials

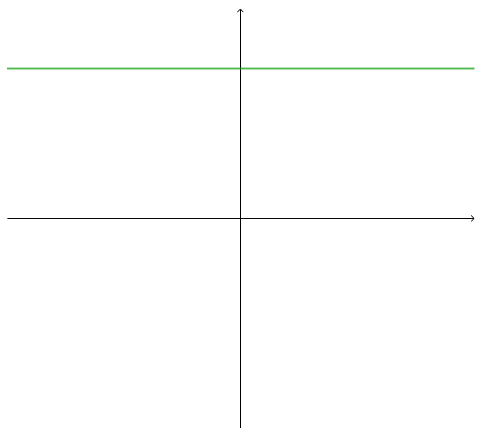
A polynomial with variable x can be written as in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0 = \sum_{i=0}^{n} a_i x^i$$

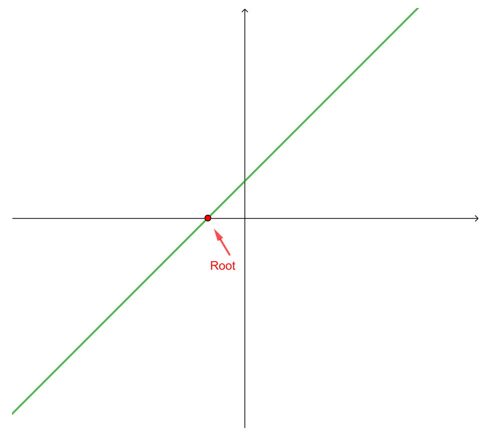
The numbers $a_0, a_1, \ldots, a_n$ are called the coefficients of $f(x)$. In this chapter, we assume that all coefficients are real numbers. Such polynomials are called real polynomials. If $a_n \neq 0$, then this polynomial f has degree n, denoted by $\deg f = n$. a_n is called the leading coefficient.

A real number α is called a real root of $f(x)$ if $f(\alpha) = 0$. It can be showed that a non-zero polynomial of degree n has at most n real roots.

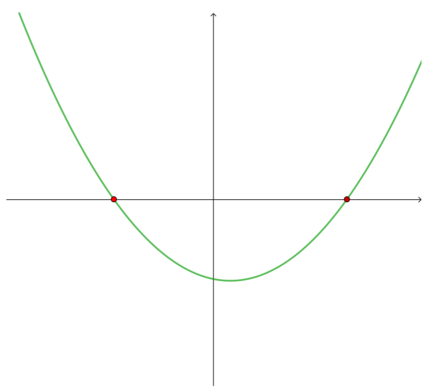
Below are the graphs of some polynomials of different degrees.



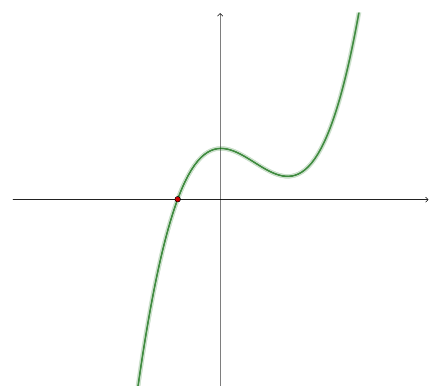
$\deg f = 0$ (constant)



$\deg f = 1$ (linear)



$\deg f = 2$ (quadratic)

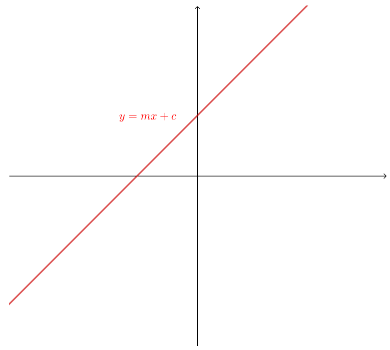


$\deg f = 3$ (cubic)

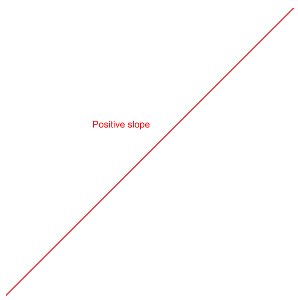
Let us quickly review some properties of linear and quadratic polynomials.

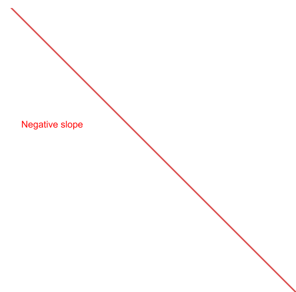
Linear polynomial ($\deg f = 1$)

A linear polynomial is a polynomial of degree 1. It has the form $f(x) = mx + c$ with $m \neq 0$. Its graph is a straight line.



The coefficients m and c are the slope and the y-intercept respectively. The following shows some straight lines with positive and negative slopes.





$m > 0$: Positive slope

$m < 0$: Negative slope

Quadratic polynomial ($\deg f = 2$)

A quadratic polynomial is a polynomial of degree 2. It has the form

$$f(x) = ax^2 + bx + c, \quad \text{with} \quad a \neq 0.$$

Its graph is a parabola. The roots of it are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant of f, denoted by Δ, is $b^2 - 4ac$. Its sign is related to the nature of the roots of f as follows:

Sign of Δ	$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
Nature of the roots	No real root	One double real root	Two distinct real roots

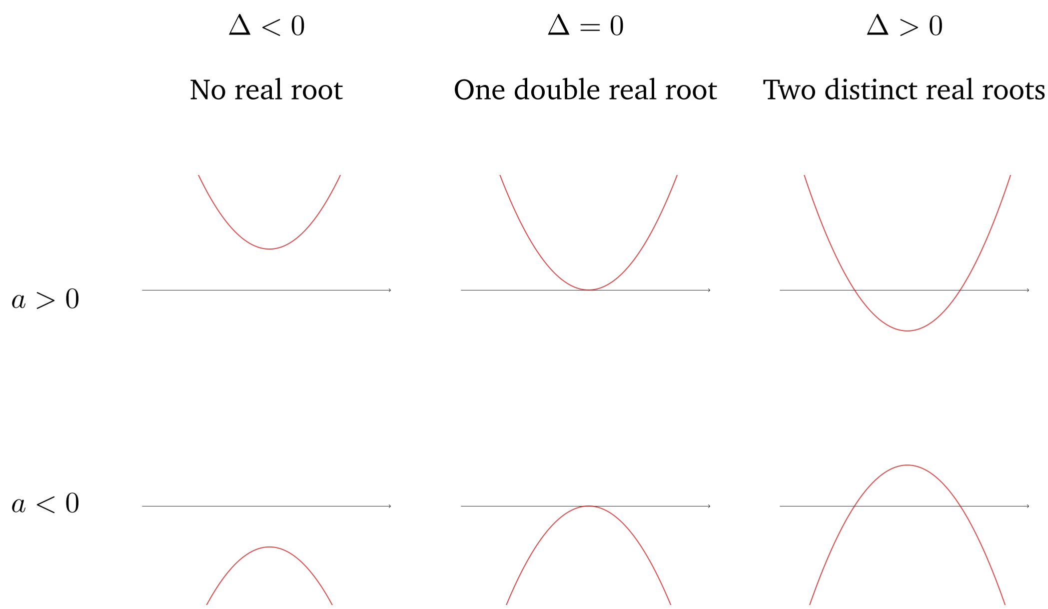
A polynomial is said to be reducible if it can be factorized into a product of polynomials of lower degrees. Otherwise, it is said to be irreducible.

A quadratic polynomial is irreducible if $\Delta < 0$ and reducible if $\Delta \geq 0$.

Example 2. $f(x) = x^2 + 3x + 2$ has discriminant $\Delta = 3^2 - 4(1)(2) = 1 > 0$. That means $f(x)$ has two distinct real roots and so is reducible. Indeed, the roots can be easily computed to be -1 and -2 and $f(x)$ can be factorized as $f(x) = (x+1)(x+2)$.

Example 3. $f(x) = x^2 + 3x + 4$ has discriminant $\Delta = 3^2 - 4(1)(4) = -7 < 0$. That means $f(x)$ has no real root and is irreducible.

The relationship between the graph of f and the signs of a and Δ is showed below.

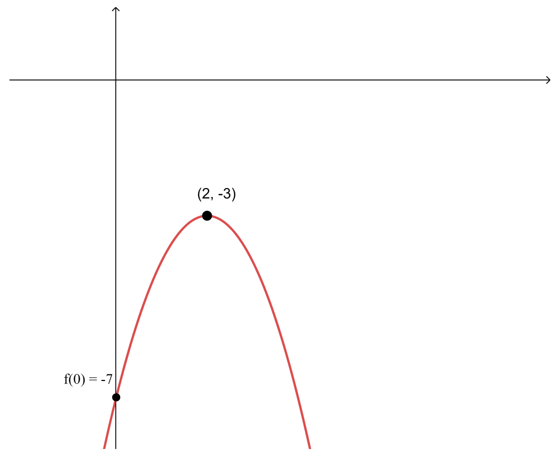


Example 4. Find the maximum/minimum value of $f(x) = -x^2 + 4x - 7$.

Solution.

$$
\begin{aligned}
f(x) &= -x^2 + 4x - 7 \\
&= -(x^2 - 4x + 7) \\
&= -(x^2 - 4x + 4 + 3) \\
&= -[(x-2)^2 + 3] \\
&= -(x-2)^2 - 3 \leq -3.
\end{aligned}
$$

Therefore, the maximum value of f is -3 at $x = 2$, and f has no minimum value.



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2.3 Factorization of Polynomials

> **Theorem 1** (Remainder Theorem)**.**
>
> *When a polynomial $f(x)$ is divided by $x - c$, the remainder is $f(c)$.*

Example 5. Consider that $f(x) = 2x^2 + x - 1$ is divided by $x + 2$. By remainder theorem, the remainder is $f(-2) = 5$. It can also be verified using long division: The quotient and reminder can be found to be is $2x - 3$ and 5 respectively and so

$$f(x) = (x + 2)(2x - 3) + 5.$$

If $f(c) = 0$, we have the following special case.

> **Theorem 2** (Factor Theorem)**.**
>
> $x - c$ *is a factor of a polynomial* $f(x) \iff f(c) = 0.$

Example 6. Factorize $f(x) = -2x^3 + 4x^2 - 6$

Solution. We try to find a root first. Consider the following trials:

$$f(0) = -6, \quad f(1) = -4, \quad f(2) = -6\ldots \quad \text{They are not zero.}$$
$$f(-1) = 0 \implies x - (-1) = x + 1 \text{ is a factor of } f.$$

By long division,

$$f(x) = (x + 1)(-2x^2 + 6x - 6),$$

which cannot be further factorized because the quadratic factor $-2x^2+6x-6$, with discriminant $\Delta = 6^2 - 4(-2)(-6) = -12 < 0$, is irreducbile. ■

Example 7. Factorize $g(x) = x^3 - x^2 - 8x + 12$.

Solution. $g(2) = 0 \implies x - 2$ is a factor. By long division,

$$\begin{aligned} g(x) &= (x - 2)(x^2 + x - 6) \\ &= (x - 2)(x + 3)(x - 2) \\ &= (x - 2)^2(x + 3)^1. \end{aligned}$$

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In the factorization of $g(x)$ above, the power of the factor $x - 2$ and $x + 3$ is 2 and 1 respectively. We say that 2 is a root of multiplicity 2 and -3 is a root of multiplicity 1.

Example 8. Let $h(x) = (x+1)^2(x-5)^6(x^2+x+1000)^9$. Note that the factor $x^2+x+1000$ is irreducible. Therefore, $h(x)$ has only two real roots: -1 with multiplicity 2, and 5 with multiplicity 6.

Example 9. $x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$.

Regarding factorization of real polynomials, we have the following result.

> **Proposition 3.** Every non-constant polynomial can be factorized as a product of linear and irreducible quadratic polynomials.

Here is a question: Can we factorize $x^4 + 1$? It may seem to be no because $x^4 + 1$ does not have any real root. However, by the fact above, the answer is indeed yes.

Example 10. $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.

The next question would be: How to find this factorization? It can be done by considering the complex roots of $x^4 + 1$.