ENGG1040 Foundations in Engineering Mathematics

Chapter 1: Functions

1.1 Review on Algebra

We first review some useful formulas.

Let a, b be real numbers and n is a positive integer. Then

$$
\begin{aligned}
(a \pm b)^2 &= a^2 \pm 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
a^2 - b^2 &= (a + b)(a - b) \\
a^3 + b^3 &= (a + b)\left(a^2 - ab + b^2\right) \\
a^3 - b^3 &= (a - b)\left(a^2 + ab + b^2\right) \\
a^n - b^n &= (a - b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right) \\
\sqrt{a^2} = |a| &= \begin{cases} a & \text{if } a \geqslant 0; \\ -a & \text{if } a < 0. \end{cases}
\end{aligned}
$$

Next are some formulas for powers. Recall that if n is even, then $a^{\frac{1}{n}} = \sqrt[n]{a}$ is defined only when $a \geq 0$.

Let a, p, q be real numbers and m, n be integers. Then the followings are identities whenever defined.

$$
\begin{array}{rclcrcl}
a^0 & = & 1 & \qquad & a^1 & = & a \\
a^{\frac{m}{n}} & = & \sqrt[n]{a^m} & & a^{-p} & = & 1/a^p \\
a^p \cdot a^q & = & a^{p+q} & & a^p/a^q & = & a^{p-q} \\
a^p \cdot b^p & = & (ab)^p & & a^p/b^p & = & (a/b)^p \\
(a^p)^q & = & a^{pq} & & & &
\end{array}
$$

Rationalization

We also want to review the technique of rationalization.

Example 1 (Rationalize Denominator)**.**

$$\begin{aligned}
\frac{\sqrt{2}}{\sqrt{3}} &= \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \\
\frac{4-\sqrt{3}}{5+\sqrt{3}} &= \frac{4-\sqrt{3}}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{20-9\sqrt{3}+3}{5^2-(\sqrt{3})^2} = \frac{23-9\sqrt{3}}{22}
\end{aligned}$$

Example 2 (Rationalize Numerator)**.**

$$\begin{aligned}
\frac{\sqrt{2}}{\sqrt{3}} &= \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{\sqrt{6}} \\
\frac{1+\sqrt{x}}{1-\sqrt{x}} &= \frac{1+\sqrt{x}}{1-\sqrt{x}} \cdot \frac{1-\sqrt{x}}{1-\sqrt{x}} = \frac{1-(\sqrt{x})^2}{(1-\sqrt{x})(1-\sqrt{x})} = \frac{1-x}{1+x-2\sqrt{x}} \\
\sqrt{x^2+4x}-x &= \frac{\sqrt{x^2+4x}-x}{1} \cdot \frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x} = \frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x} = \frac{4x}{\sqrt{x^2+4x}+x}
\end{aligned}$$

1.2 Difference Quotient (for calculus later)

In this section, we will consider expressions of the form

$$\frac{f(x)-f(a)}{x-a} \quad \text{or} \quad \frac{f(a+h)-f(a)}{h}$$

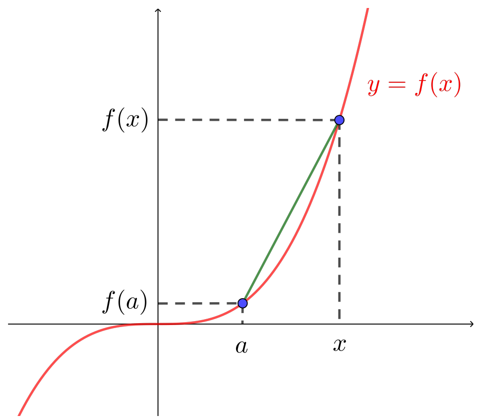
Note that they are equal to the slope of the line joining the point $(a, f(a))$ to $(x, f(x))$ or $(a+h, f(a+h))$ on the graph of f. They are called **difference quotients** and will be considered in calculus.

Example 3. Simplify the following difference quotients.

1. $\dfrac{f(x)-f(a)}{x-a}$, where $f(x) = x^3$;
2. $\dfrac{g(a+h)-g(a)}{h}$, where $g(x) = \dfrac{1}{\sqrt{x+1}}$.

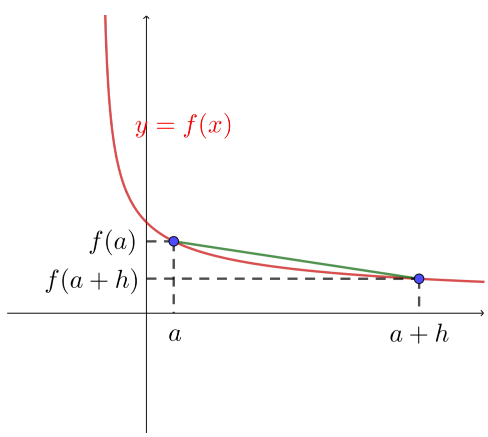
Solution. 1.

$$\begin{aligned}\frac{f(x)-f(a)}{x-a} &= \frac{x^3-a^3}{x-a} \\ &= \frac{(x-a)(x^2+ax+a^2)}{x-a} \\ &= x^2+ax+a^2\end{aligned}$$



2.

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{\frac{1}{\sqrt{a+h+1}}-\frac{1}{\sqrt{a+1}}}{h} \\ &= \frac{\sqrt{a+1}-\sqrt{a+h+1}}{h\cdot\sqrt{a+h+1}\cdot\sqrt{a+1}}\cdot\frac{\sqrt{a+1}+\sqrt{a+h+1}}{\sqrt{a+1}+\sqrt{a+h+1}} \\ &= \frac{(a+1)-(a+h+1)}{h\cdot\sqrt{a+h+1}\cdot\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})} \\ &= \frac{-h}{h\cdot\sqrt{a+h+1}\cdot\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})} \\ &= \frac{-1}{\sqrt{a+h+1}\cdot\sqrt{a+1}(\sqrt{a+1}+\sqrt{a+h+1})}\end{aligned}$$



Remark. The expression of the last line is “simpler” than that of the first one from the point of view of taking limit $h \to 0$. It will be more apparent when we discuss derivatives in calculus later.

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1.3 Sets and Functions

In mathematics, a set is a collection of objects, called elements. The objects can be numbers, symbols, words, points or other things. Sets can be described by listing their elements inside $\{\ \}$. For examples:

- $A = \{2, 4, 6, 8\}$ is the set of the 4 smallest positive even numbers. The numbers 2, 4, 6 and 8 are elements of the set A.
- $B = \{$a,e,i,o,u$\}$ is the all set of all vowel letters.
- $C = \{0, 1, 2, 3, 4, 5, \ldots\}$ is the set of all non-negative integers.
- $D = \{(1, 0), (3, 2), (4, 7)\}$ is a set containing three particular points on the xy-plane.

We list out the elements in a set above. We can also describe a set using condition(s). For example, we may describe the set A above by

$$A = \{2, 4, 6, 8\} = \{x : x \text{ is even, } 0 < x < 10\},$$

which is understood as the set of all x such that x is even and $0 < x < 10$.

If x is an element of A, we write $x \in A$.

Let A, B be sets. A is called a subset of B if every element of A is also an element of B. It is denoted by $A \subset B$ or $A \subseteq B$.

Example 4. Let $A = \{2, 4, 6, 8\}, B = \{2, 8\}, C = \{2, 4\}$. Then $8 \in A, B$ and $8 \notin C$. Also, $B \subset A, C \subset A, B \not\subset A$

Here are some common notations for subsets of real numbers.

$\mathbb{R}$	The set of all real numbers.
$\mathbb{Q}$	The set of all rational numbers.
$\mathbb{Z}$	The set of all integers.
$\mathbb{N}$	The set of all natural numbers $\{1, 2, 3, \ldots\}$.

Let $a, b \in \mathbb{R}$ or $\pm\infty$. Then

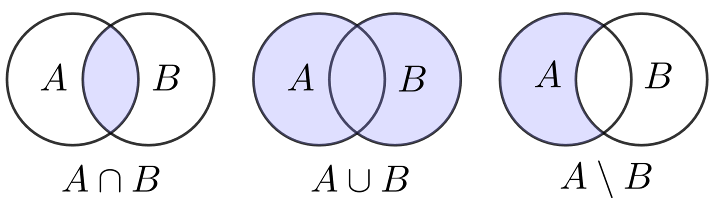
$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} : a < x < b\} && \text{(Open interval)} \\
[a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} && \text{(Closed interval)} \\
[a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} : a < x \leq b\}
\end{aligned}$$

Operations on Sets

Let A, B be sets. Define

$$\begin{aligned}
A \cap B &= \{x : x \in A \text{ and } x \in B\} && \text{(Intersection)} \\
A \cup B &= \{x : x \in A \text{ or } x \in B\} && \text{(Union)} \\
A \setminus B &= \{x : x \in A \text{ and } x \notin B\} && \text{(Relative complement of } B \text{ in } A\text{)}
\end{aligned}$$

They can be illustrated using ”Venn Diagrams”.



Example 5. Let $A = \{2, 4, 6\}$ and $B = 3, 6, 9$. Then

$$A \cap B = \{6\}, \quad A \cup B = \{2, 3, 4, 6, 9\}, \quad A \setminus B = \{2, 4\}$$

Functions

Let A, B be sets. A function

$$f : A \to B$$

is a rule of assigning each element $a \in A$ to an element $f(a) \in B$. The set A and B is called the **domain** and **codomain** of f respectively. We will often denote the domain of f by D_f.

The **range** of f is defined to be

$$R_f = \{f(a) : a \in A\},$$

which is the set of all values of f.

We will focus on functions whose domain and codomain are subsets of real numbers.

Example 6. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 - 1$. Both the domain and codomain of f is $\mathbb{R}$.

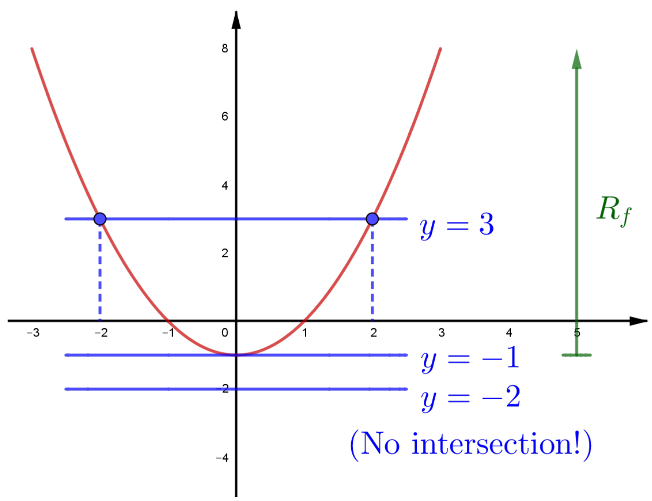
$$f(0) = -1 \Rightarrow -1 \in R_f.$$
$$f(2) = 3 \Rightarrow 3 \in R_f.$$

However, $-2 \notin R_f$. It is because for any $x \in D_f = \mathbb{R}$,

$$f(x) = x^2 - 1 \geq 0 - 1 = -1$$

Since $-2 < -1$, that means $f(x) \neq -2$ and so $-2 \notin R_f$.

From the graph of f, we can see that $R_f = [-1, \infty)$.



Implied domain

If a function $f(x)$ is given by an expression without specifying its domain, then its domain will be assumed to be the largest subset of $\mathbb{R}$ such that the expression is defined. This domain is called the **implied domain** or **natural domain**.

Some useful rules for finding implied domains:

1. Denominator cannot be zero.
2. For $log(g(x))$ to be defined, we need $g(x) > 0$.

3. Let m be a positive even number. Then for $\sqrt[m]{h(x)} = h(x)^{1/m}$ to be defined, we need $h(x) \geq 0$.

Example 7. Find the implied domains of the following functions.

1. $\log(x^2 - 3x - 10)$
2. $\dfrac{x-3}{\sqrt[4]{3-x}}$
3. $(x+2)^{5/3}$
4. $f(x) - g(x)$, where $f(x) = \dfrac{1}{1+x}$ and $g(x) = \dfrac{1}{1-x}$.

Solution.

1. For $\log(x^2 - 3x - 10)$ to be defined, we need

$$\begin{aligned} x^2 - 3x - 10 &> 0 \\ (x-5)(x+2) &> 0 \\ x > 5 \quad &\text{or} \quad x < -2 \end{aligned}$$

Hence, the implied domain is $(-\infty, -2) \cup (5, \infty)$.

2. For the fourth root to be defined, we need

$$3 - x \geq 0.$$

Moreover, we need the denominator

$$\sqrt[4]{3-x} \neq 0.$$

Hence, for the given function to be defined, we need

$$3 - x > 0 \Rightarrow x < 3$$

Hence, the implied domain is $(-\infty, 3)$.

3. Note that $(x+2)^{5/3} = \sqrt[3]{(x+2)^5}$. Since 3 is odd, the root is defined for any values of $(x+2)^5$. Hence the implied domain is $\mathbb{R} = (-\infty, \infty)$.

4. For $f(x) - g(x)$ to be defined, we need both $f(x)$ and $g(x)$ to be defined. Hence, the implied domain of $f - g$ is

$$\begin{aligned} D_{f-g} &= D_f \cap D_g \\ &= (\mathbb{R} \setminus \{-1\}) \cap (\mathbb{R} \setminus \{1\}) \\ &= \mathbb{R} \setminus \{\pm 1\} \\ &= (-\infty, -1) \cup (-1, 1) \cup (1, \infty). \end{aligned}$$

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Operations on Functions

Let $f(x), g(x)$ be functions. Define

$$\begin{aligned} (f \pm g)(x) &= f(x) \pm g(x) && \text{(Sum / Difference)} \\ (fg)(x) &= f(x)g(x) && \text{(Product)} \\ (f/g)(x) &= f(x)/g(x) && \text{(Quotient)} \\ (g \circ f)(x) &= g(f(x)) && \text{(Composition)} \end{aligned}$$

In the composition $g \circ f$, the output of f becomes the input of g:

$$x \xmapsto{f} f(x) \xmapsto{g} g(f(x))$$

The domains of these functions are

$$\boxed{\begin{aligned} D_{f \pm g} &= D_f \cap D_g \\ D_{fg} &= D_f \cap D_g \\ D_{f/g} &= (D_f \cap D_g) \setminus \{x \in D_g : g(x) = 0\} \\ D_{g \circ f} &= \{x \in D_f : f(x) \in D_g\} \end{aligned}}$$

Example 8. Let $f(x) = x^2 - x$ and $g : (2, \infty) \to \mathbb{R}$ be functions.

1. Find $(f \circ f)(3)$.
2. Find the implied domain of $g \circ f$.

Solution. 1. $(f \circ f)(3) = f(f(3)) = f(6) = 30$.

2. $$(g \circ f)(x) = g(f(x)) = g(x^2 - x).$$

   For this to be defined, we need $x^2 - x \in D_g = (2, \infty)$. Hence,

$$\begin{aligned} & x^2 - x > 2 \\ \Rightarrow\quad & x^2 - x - 2 > 0 \\ \Rightarrow\quad & (x-2)(x+1) > 0 \\ \Rightarrow\quad & x > 2 \quad \text{or} \quad x < -1. \end{aligned}$$

   $D_{g \circ f} = (-\infty, -1) \cup (2, \infty)$.

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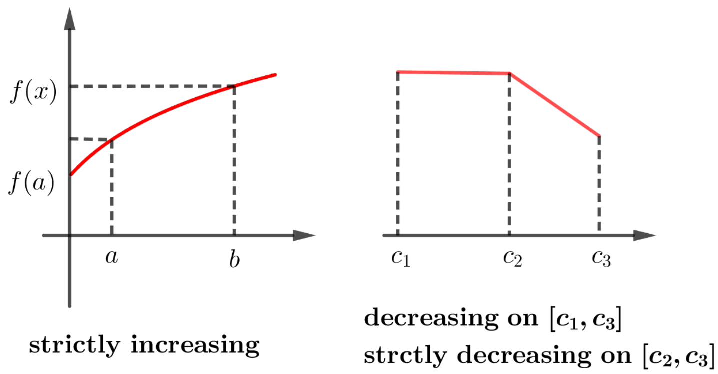
Increasing and Decreasing Functions

Let I be an interval. A function $f(x)$ is said to be increasing (or strictly increasing) on I, if $f(a) \leq f(b)$ (or $f(a) < f(b)$) for any $a < b$ on I.

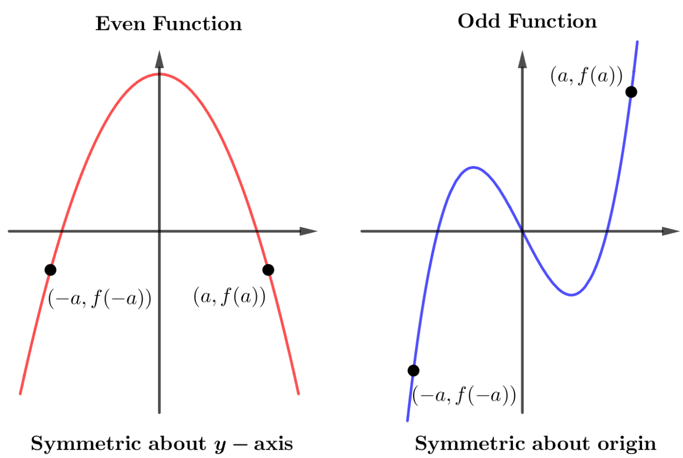
Similarly, $f(x)$ is said to be decreasing (or strictly decreasing) on I, if $f(a) \geq f(b)$ (or $f(a) > f(b)$) for any $a < b$ on I.



Even and Odd Functions

Definition 1. If $f(-x) = f(x)$ for any $x \in D_f$, then $f(x)$ is called an even function.

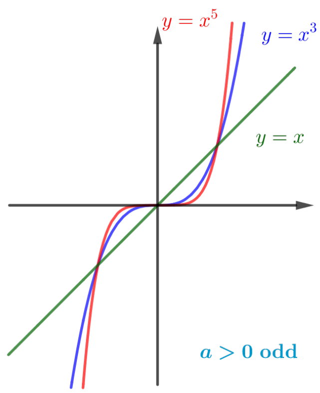
If $f(-x) = -f(x)$ for any $x \in D_f$, then $f(x)$ is called an odd function.



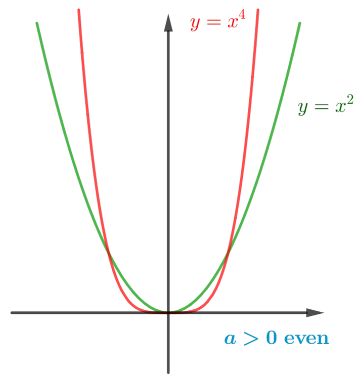
1.4 More Examples of Functions

Power functions

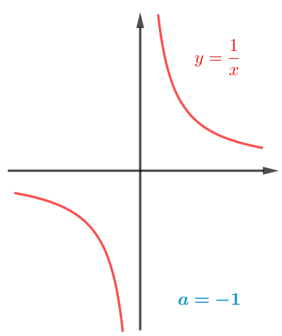
$$\boxed{f(x) = x^a}$$

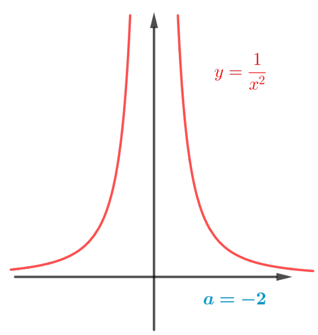


Domain $= \mathbb{R} = (-\infty, \infty)$
Odd function
Strictly increasing on $(-\infty, \infty)$

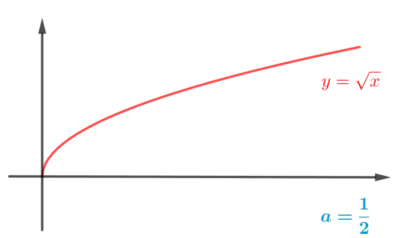


Domain $= \mathbb{R} = (-\infty, \infty)$
Even function
Strictly increasing on $[0, \infty)$
Strictly decreasing on $(-\infty, 0]$





Domain $= \mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$



Domain $= [0, \infty)$

Piecewise functions

A piecewise function is defined by more than one formula, with each individual formula defined on a subset of the domain.

Example 9. $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 2x, & \text{if } x \geq 0. \end{cases}$$

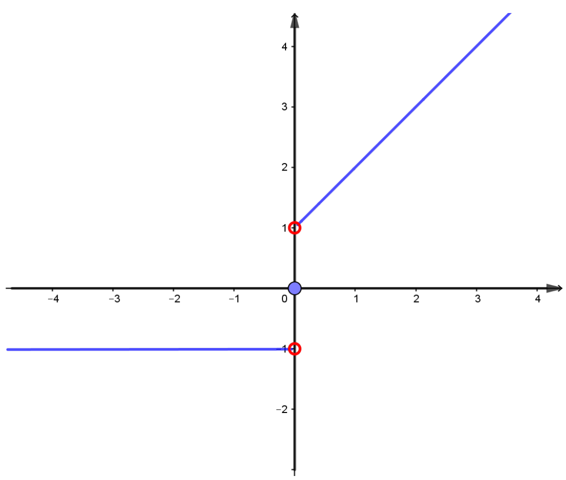
Then $f(-1) = 1$, $f(0) = 0$ and $f(1) = 2$.

Remark. Even though this piecewise function is defined using two formulas, it is a single function whose domain is the entire set of real numbers.

Example 10. $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x+1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

is a piecewise function.



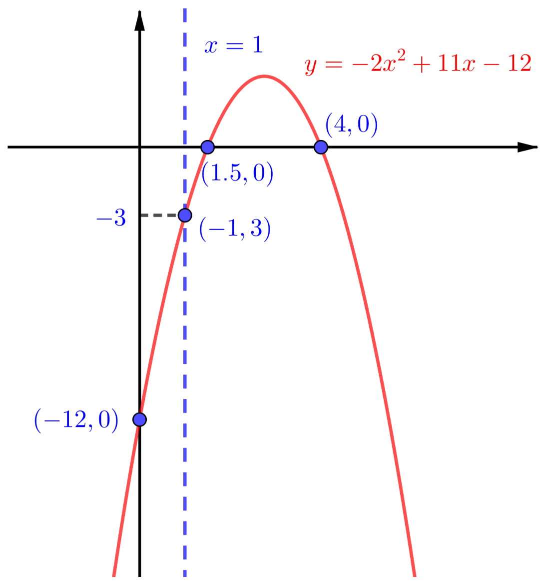
Example 11. Graph

$$h(x) = \begin{cases} 2x+3, & \text{if } x < 1, \\ -2x^2 + 11x - 12, & \text{if } x \geq 1. \end{cases}$$

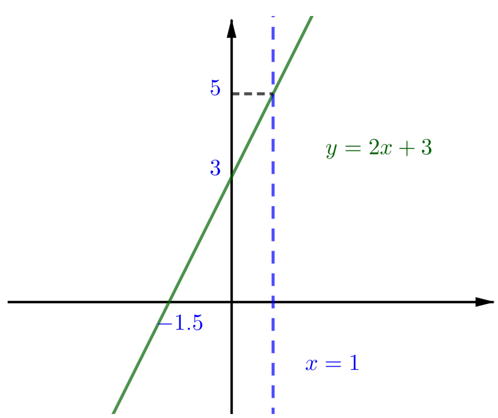
Solution. Note that if $-2x^2 + 11x - 12 = 0$, then

$$
\begin{aligned}
x &= \frac{-11 \pm \sqrt{11^2 - 4(-2)(-12)}}{2(-2)} \\
&= \frac{-11 \pm \sqrt{25}}{-4} \\
&= \frac{3}{2} \text{ or } 4
\end{aligned}
$$

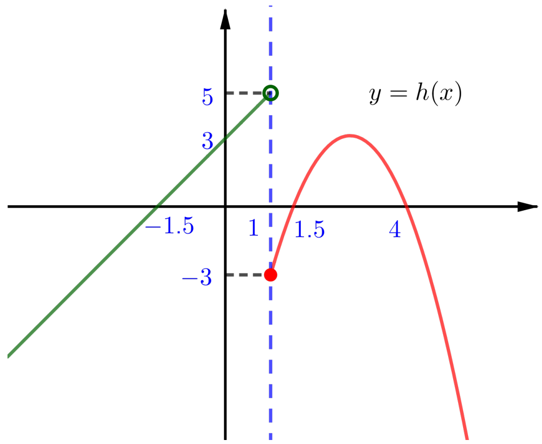
Also, the leading coefficient is $-2 < 0$, so the parabola opens downwards.



For the graph of $y = 2x + 3$, it is a straight line with slope $= 2$, $x-$intercept $= -\frac{3}{2}$, and $y-$intercept $= 3$.



Combining the two graphs, we obtain the graph of $h(x)$.



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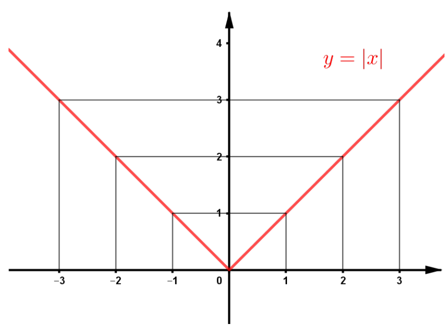
Absolute Value Function

The absolute value function $|x|$ can be expressed as a piecewise function:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



Proposition 1. For $x, y \in \mathbb{R}$,

$$\begin{aligned} |x| &\geq 0 & |x| &= |-x| \\ |x| &= \sqrt{x^2} & |x|^2 &= |x^2| \\ |xy| &= |x||y| & \left|\frac{x}{y}\right| &= \frac{|x|}{|y|} \\ |x+y| &\leq |x| + |y| \end{aligned}$$

Proposition 2. Let $c > 0$. Then

1. $|f(x)| < c \quad \Leftrightarrow \quad -c < f(x) < c$.
2. $|f(x)| > c \quad \Leftrightarrow \quad f(x) > c \quad$ or $\quad f(x) < -c$.
3. Similar statements hold for $\leq$ and $\geq$.

Example 12. Solve

1. $|2x - 3| \leq 7$
2. $|3x + 2| > 4$

Solution. 1.

$$\begin{aligned} & |2x-3| \leq 7 \\ \Rightarrow \quad & -7 \leq 2x - 3 \leq 7 \\ \Rightarrow \quad & -4 \leq 2x \leq 10 \\ \Rightarrow \quad & -2 \leq x \leq 5 \end{aligned}$$

2.

$$\begin{aligned} & |3x+2| > 4 \\ \Rightarrow \quad & 3x + 2 > 4 \quad \text{or} \quad 3x + 2 < -4 \\ \Rightarrow \quad & 3x > 2 \quad \text{or} \quad 3x < -6 \\ \Rightarrow \quad & x > \tfrac{2}{3} \quad \text{or} \quad x < -2 \end{aligned}$$

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Example 13. Graph $f(x) = |x - 2| + |x + 2|$.

Solution. Consider 3 cases:

Case I. If $x \geq 2$, then $x - 2 \geq 0$, $x + 2 \geq 0$.

$$\begin{aligned} f(x) \quad & = |x-2| + |x+2| \\ & = (x-2) + (x+2) \\ & = 2x \end{aligned}$$

Case II. If $-2 \leq x < 2$, then $x - 2 < 0$, $x + 2 \geq 0$.

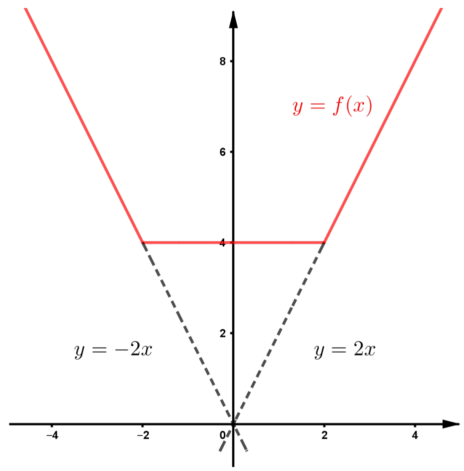
$$\begin{aligned} f(x) \quad & = |x-2| + |x+2| \\ & = -(x-2) + (x+2) \\ & = 4 \end{aligned}$$

Case III. If $x < -2$, then $x - 2 < 0$, $x + 2 < 0$.

$$\begin{aligned} f(x) &= |x-2| + |x+2| \\ &= -(x-2) - (x+2) \\ &= -2x \end{aligned}$$

Hence,

$$f(x) = \begin{cases} 2x, & \text{if } x \geq 2, \\ 4, & \text{if } -2 \leq x < 2, \\ -2x, & \text{if } x < -2. \end{cases}$$



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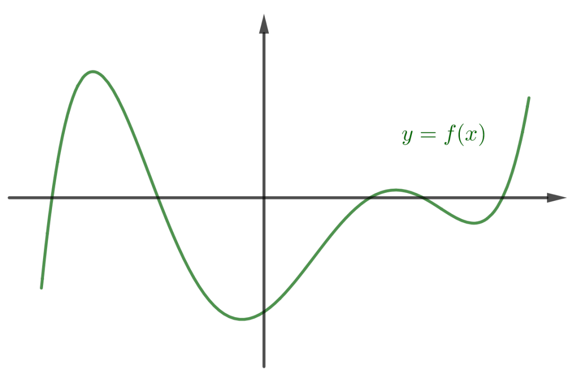
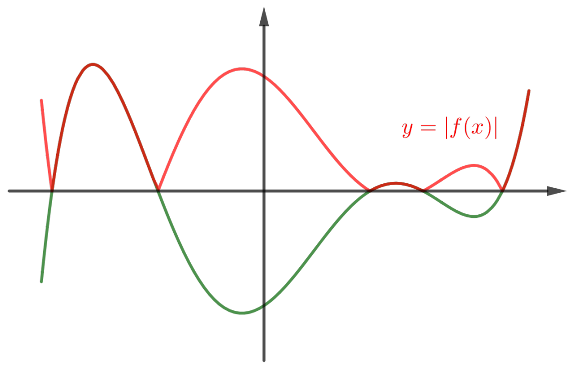
Given



Question: How to get the graph of $|f(x)|$ and $f(|x|)$?

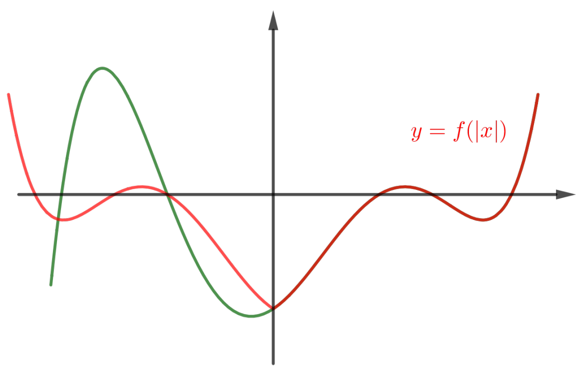
To obtain the graph of $|f(x)|$ from $f(x)$,

- keep the graph for $f(x) \geq 0$
- reflect the graph in the $x-$axis for $f(x) < 0$, discarding what was there
- points on the $x-$axis are invariant.



To obtain the graph of $f(|x|)$ from $f(x)$,

- discard the graph for $x < 0$
- reflect the graph for $x \geq 0$ in the $y-$axis, keeping what was there
- points on the $y-$axis are invariant.



Note that $f(|x|)$ is an even function and $f(|x|) = f(x)$ for $x \geq 0$.

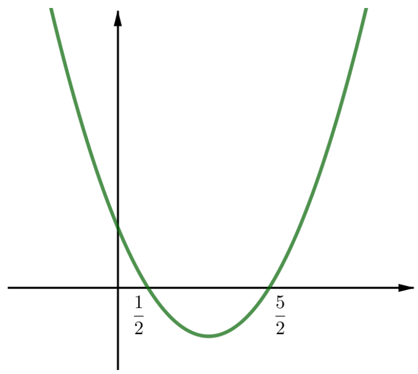
Exercise 1.4.1. How about the graph of $-|f(x)|$ and $f(-|x|)$?

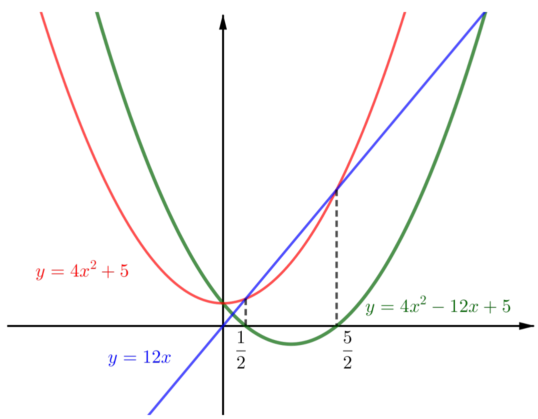
1.5 Inequality

Example 14. $4x^2 + 5 \geq 12x$

Solution.

$$4x^2 - 12x + 5 \geq 0$$
$$(2x-1)(2x-5) \geq 0$$
$$\therefore x \leq \tfrac{1}{2} \quad \text{or} \quad x \geq \tfrac{5}{2}$$





■

Example 15. $\dfrac{2x-1}{x+1} < 1$

Wrong Approach!

$$\frac{2x-1}{x+1} < 1$$

Multiply both sides by $x+1 \quad (\oplus) \quad \Rightarrow \quad 2x - 1 < x + 1 \quad \Rightarrow \quad x < 2.$

Why is the above approach wrong? It is because if $x+1<0$, the step $(\oplus)$ reverses the inequality.

Proposition 3. Let $a, b, c \in \mathbb{R}$, $a > b$,

1. If $c > 0$, then $ca > cb$
2. If $c < 0$, then $ca < cb$
3. Similar statements hold for $a \geq b$

Correct approach 1.

$$\frac{2x-1}{x+1} < 1$$

Note that $x+1 \neq 0 \quad \Rightarrow \quad (x+1)^2 > 0$. Multiply both sides by $(x+1)^2$.

$$\begin{aligned} \Rightarrow \quad (2x-1)(x+1) &< (x+1)^2 \\ (2x-1)(x+1)-(x+1)^2 &< 0 \\ (2x-1-x-1)(x+1) &< 0 \\ (x-2)(x+1) &< 0 \\ -1 < x &< 2 \end{aligned}$$

Correct approach 2.

$$\begin{aligned} \frac{2x-1}{x+1} &< 1 \\ \frac{2x-1}{x+1} - 1 &< 0 \\ \frac{2x-1-(x+1)}{x+1} &< 0 \\ \frac{x-2}{x+1} &< 0 \end{aligned}$$

Consider 3 cases:

	$x < -1$	$-1 < x < 2$	$x > 2$
$x-2$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
$\frac{x-2}{x+1}$	$+$	$-$	$+$

$\therefore \quad -1 < x < 2$.

Example 16. Find the implied domain of $f(x) = \sqrt[4]{x - \dfrac{3}{x} - 2}$.

Solution.

$$\begin{aligned} \text{Need} \quad g(x) = x - \frac{3}{x} - 2 &\geq 0 \\ \frac{x^2 - 3 - 2x}{x} &\geq 0 \\ \frac{(x-3)(x+1)}{x} &\geq 0 \end{aligned}$$

The points $-1, 0, 3$ divide $(-\infty, \infty)$ into 4 intervals.

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$x - 3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x + 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$\frac{(x-3)(x+1)}{x}$	$-$	0	$+$	undefined	$-$	0	$+$

$\therefore \quad -1 \leq x < 0$ or $x \geq 3$, $D_f = [-1, 0) \cup [3, \infty)$. ■

Remark. One may also determine the sign on each interval by testing with a point on that interval:

For example,

$$g(-2) = (-2) - \frac{3}{-2} - 2 = -\tfrac{5}{2} < 0 \Rightarrow x - \tfrac{3}{x} - 2 < 0 \quad \text{on } (-\infty, -1)$$
$$g\left(-\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right) - \frac{3}{-\frac{1}{2}} - 2 = \tfrac{7}{2} > 0 \Rightarrow x - \tfrac{3}{x} - 2 > 0 \quad \text{on } (-1, 0)$$