ENGG1040 Foundations in Engineering Mathematics

Chapter 5: Sequences

5.1 Sequences

A sequence is an ordered list of numbers written in a definite order. Each number in the sequence is called a term. For examples,

$$1, 4, 9, 16, 25, \dots$$

is a sequence with n-th term equals n^2.

We can denote a sequence by $\{a_n\}$, where the index n is an integer, usually begins from $n = 1$.

Example 1. Let $a_n = 1/n$ with $n \geq 1$. It is called the harmonic sequence:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$$

It is also possible for $\{a_n\}$ to begin from integers other than 1 or have finitely many terms.

Example 2. Let $a_n = 2n + 1$ with $n = 0, 1, \dots, 10$. It is the finite sequence:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21.$$

Recursive Sequence

A sequence can be defined explicitly by a formula for a_n as above. It can also be defined recursively by expressing a_n in terms of previous terms. Such a sequence is called a recursive sequence.

Example 3. The Fibonacci sequence is defined by:

$$a_0 = 1, \quad a_1 = 1, \quad a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2$$

The first few terms are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

Arithmetic Sequence

A sequence $\{a_n\}$ is said to be an arithmetic sequence if the difference between consecutive terms is constant. That is, for any n,

$$a_n - a_{n-1} = d$$

for some constant d. By applying the formula repeatedly, we have

$$\begin{aligned} a_n &= a_{n-1} + d \\ &= (a_{n-2} + d) + d \\ &= a_{n-2} + 2d \\ &= (a_{n-3} + d) + 2d \\ &= a_{n-3} + 3d \\ &\vdots \\ a_n &= a_1 + (n-1)d. \end{aligned}$$

Example 4. Let $\{a_n\}$ be an arithmetic sequence with first few terms $2, 5, 8, 11, \ldots$. Then $a_1 = 2$ and $d = 3$. The general term is given by $a_n = 2 + 3(n-1)$.

Geometric Sequence

A sequence $\{a_n\}$ is said to be a geometric sequence if there exists a constant r such that

$$a_n = r a_{n-1}$$

for any $n \geq 1$. In other words, the ratio between consecutive terms is constant. Inductively, we have $a_2 = a_1 r,\ a_3 = a_1 r^2,\ a_4 = a_1 r^3$ and inductively

$$a_n = a_1 r^{n-1}.$$

Example 5. Let $\{a_n\}$ be a geometric sequence with first few terms $3, 6, 12, 24, \ldots$. Then $a_1 = 3$ and $r = 2$. The general term is given by $a_n = 3 \cdot 2^{n-1}$.

5.2 Convergence of Sequences

A sequence $\{a_n\}$ is said to converge to a limit L if the terms a_n get arbitrarily close to L as n becomes large. It is denoted by

$$\lim_{n \to \infty} a_n = L$$

The sequence $\{a_n\}$ is also said to be convergent and tends to L as n tends to infinity.

If no such L exists, the sequence $\{a_n\}$ is said to be divergent and the limit does not exist.

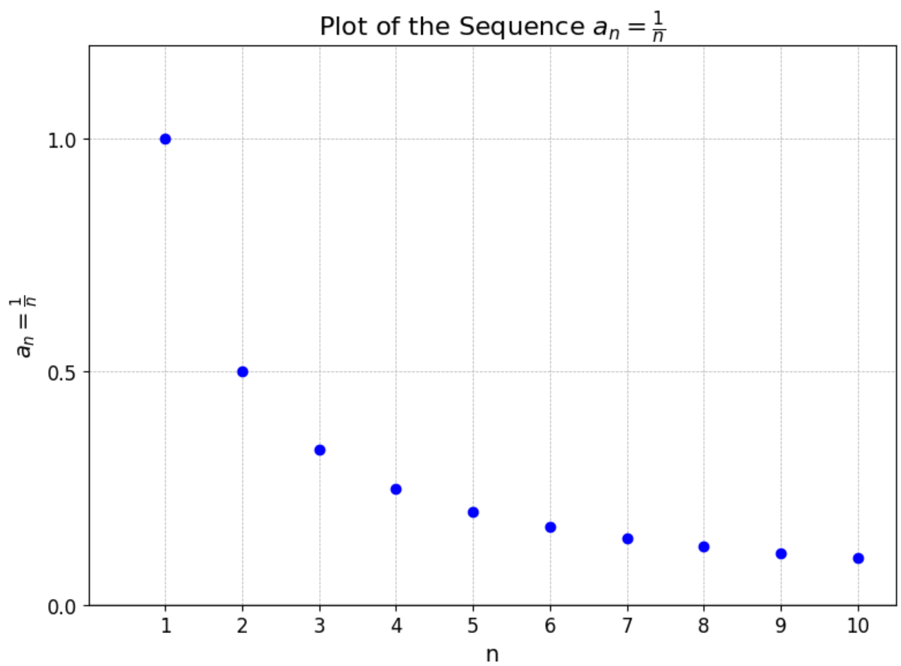
The sequence $\{a_n\}$ is said to diverge to infinity if the terms a_n get arbitrarily large as n becomes large. It is denoted by

$$\lim_{n\to\infty} a_n = \infty$$

Similarly, $\{a_n\}$ is said to diverge to negative infinity if the terms a_n get arbitrarily large in the negative direction as n becomes large. It is denoted by

$$\lim_{n\to\infty} a_n = -\infty$$

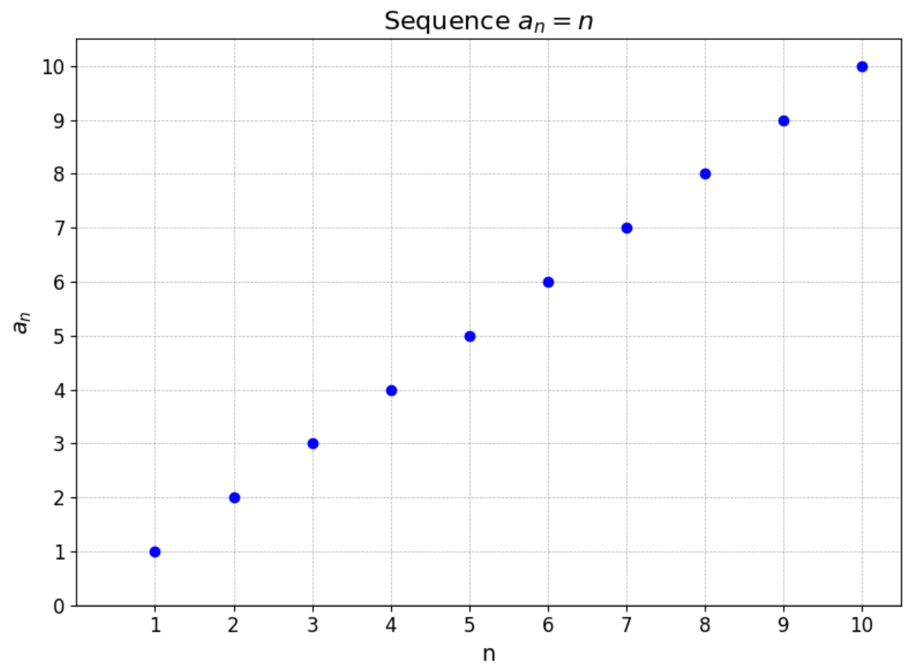
Example 6. Let $a_n = \frac{1}{n}$. When n gets large, $\frac{1}{n}$ gets arbitrarily close to 0. We can also visualize it from the graph below: The points represent the values of a_n for $n = 1, 2, 3, \ldots$. The points get arbitrarily close to the y-axis as going further to the right.



The sequence $\{a_n\}$ converges to 0 and we write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

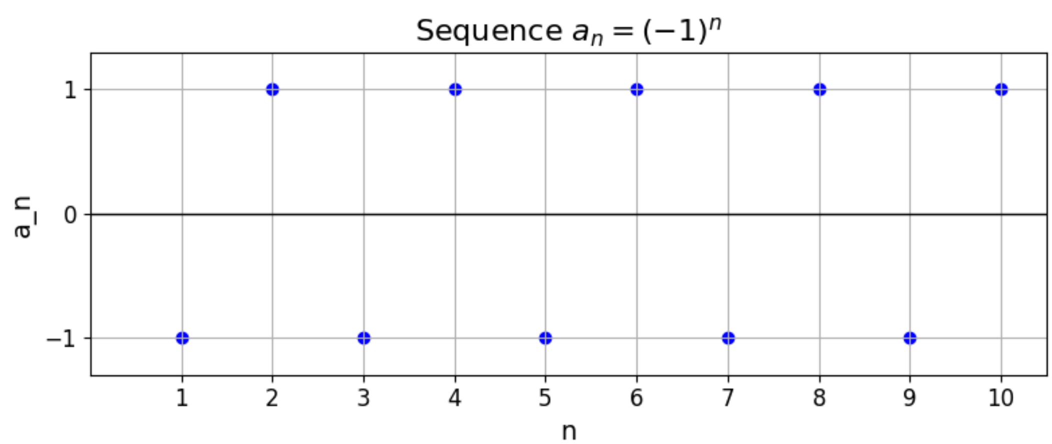
Example 7. Let $a_n = n$. When n gets large, a_n gets arbitrarily large too. It can be visualized from its graph: The points go arbitrarily up as going further to the right.



The sequence $\{a_n\}$ diverges to infinity and we write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty.$$

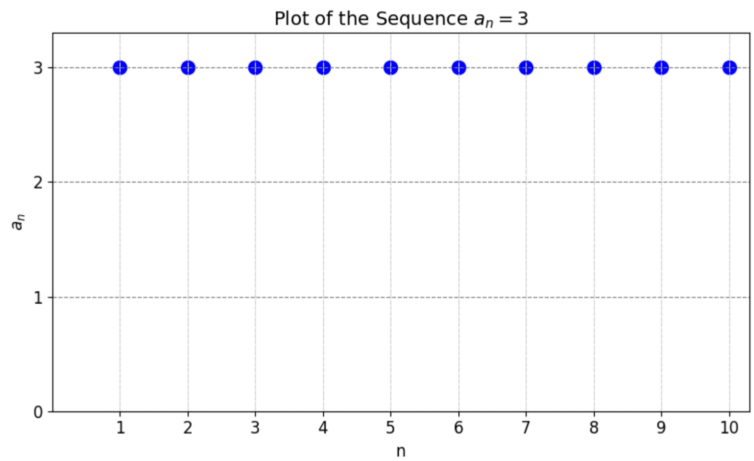
Example 8. Let $a_n = (-1)^n$. It equals 1 for even n and -1 for odd n.



The sequence is oscillating between the value ± 1 and does not approach to a single value. Hence the sequence $\{a_n\}$ diverges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \text{ does not exist (DNE).}$$

Example 9. Let $a_n = 3$ for any positive integer n.



The sequence is constantly equal to 3. Hence the sequence converges to 3.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3 = 3.$$

Here are two basic results for limit of sequences.

Proposition 1. Let a be a constant.

$$\lim_{n\to\infty} a^n \begin{cases} = \infty & \text{if } a > 1 \\ = 1 & \text{if } a = 1 \\ = 0 & \text{if } -1 < a < 1 \\ \text{DNE} & \text{if } a \leq -1 \end{cases}$$

Proposition 2. Suppose $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Then for any $k \in \mathbb{R}$,

$$\lim_{n\to\infty} (a_n \pm b_n) = \left(\lim_{n\to\infty} a_n\right) \pm \left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad (\text{ If } \lim_{n\to\infty} b_n \neq 0)$$

$$\lim_{n\to\infty} k a_n = k\left(\lim_{n\to\infty} a_n\right)$$

$$\lim_{n\to\infty} a_n^k = \left(\lim_{n\to\infty} a_n\right)^k \qquad (\text{ If } \lim_{n\to\infty} a_n > 0)$$

Example 10.

$$\lim_{n\to\infty} 2^n = \infty.$$
$$\lim_{n\to\infty} (-0.5)^n = 0.$$

Example 11. Evaluate $\lim_{n\to\infty} \left(2 + \frac{1}{n}\right)$.

$$\begin{aligned}\lim_{n\to\infty} \left(2 + \frac{1}{n}\right) &= \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} \\ &= 2 + 0 \\ &= 2\end{aligned}$$

Example 12. Evaluate $\lim_{n\to\infty} \left(\frac{2}{1+(0.5)^n}\right)^3$.

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{2}{1+(0.5)^n}\right)^3 &= \left(\lim_{n\to\infty} \frac{2}{1+(0.5)^n}\right)^3 \\ &= \left(\frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} (1+(0.5)^n)}\right)^3 \\ &= \left(\frac{2}{1+0}\right)^3 \\ &= 2^3 = 8\end{aligned}$$

Example 13. Evaluate $\lim_{n\to\infty} \left(\frac{1}{n} - n^2\right)$.

Observe that $\frac{1}{n}$ tends to 0 and n^2 tends to infinity as n tends to infinity. Hence, $\frac{1}{n} - n^2$ tends to negative infinity.

$$\lim_{n\to\infty} \left(\frac{1}{n} - n^2\right) = -\infty.$$

Indeterminate forms

While computing the limit of a sequence, one might encounter situations such as

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\cdot(\pm\infty),\ \infty-\infty$$

These are called **indeterminate forms**. In this case, we try to simplify or alter the sequence into another form.

Example 14. Evaluate $\lim_{n\to\infty} \frac{-2n+7}{4n^2-1}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{-2n+7}{4n^2-1} &= \lim_{n\to\infty} \frac{-\frac{2}{n}+\frac{7}{n^2}}{4-\frac{1}{n^2}} \\ &= \frac{0+0}{4-0} = 0\end{aligned}$$

In this example above, both the numerator and denominator approach to infinity. The computation shows that their ratio approaches to zero, meaning that the denominator approach to infinity much faster.

Example 15. Evaluate $\lim_{n\to\infty} \frac{2n^2+3n+2}{3n^2-n}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{2n^2+3n+2}{3n^2-n} &= \lim_{n\to\infty} \frac{2+\frac{3}{n}+\frac{2}{n^2}}{3-\frac{1}{n}} \\ &= \frac{2+0+0}{3-0} = \frac{2}{3}\end{aligned}$$

Example 16. Evaluate $\lim_{n\to\infty} \frac{4n^3+2n^2-n+1}{n^2+5n+3}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{4n^3+2n^2-n+1}{n^2+5n+3} &= \lim_{n\to\infty} \frac{4n+2-\frac{1}{n}+\frac{1}{n^2}}{1+\frac{5}{n}+\frac{3}{n^2}} \\ &= \infty\end{aligned}$$

Example 17. Evaluate $\lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$.

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} &= \lim_{n\to\infty} \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0\end{aligned}$$

Example 18. Evaluate $\lim_{n\to\infty} \left(\sqrt{n^2+n}-n\right)$.

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n^2+n}-n &= \lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{(n^2+n)-n^2}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}\end{aligned}$$