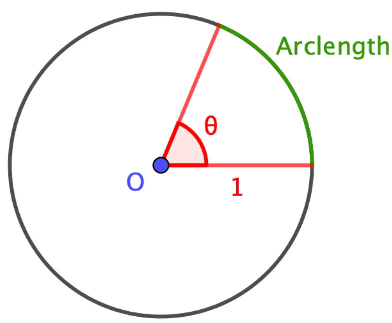
ENGG1040 Foundations in Engineering Mathematics

Chapter 3: Trigonometry

3.1 Radian

Besides degree, **radian** is another commonly used unit for measuring angles. It is important in calculus and many other areas of mathematics.

Consider a unit circle (circle with radius 1) and a sector with central angle θ. If the arclength of the sector is x, the angle θ is defined to be x rad, where rad stands for radian.



Since the circumference of a unit circle is 2π,

$$\begin{aligned} 360^\circ &= \text{full circle} = 2\pi \text{ rad}; \\ 180^\circ &= \text{half circle} = \pi \text{ rad}; \\ 90^\circ &= \text{right angle} = \frac{\pi}{2} \text{ rad}. \end{aligned}$$

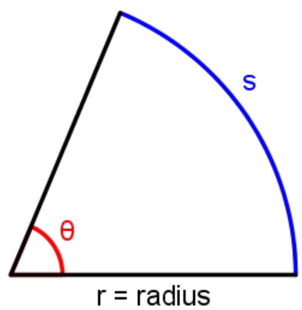
Generally, the conversion between degree and radian units is given by

$$\text{Degree} \underset{\times\frac{180}{\pi}}{\overset{\times\frac{\pi}{180}}{\rightleftharpoons}} \text{Radian}$$

$$x^\circ = \frac{\pi x}{180} \text{ rad} \qquad y \text{ rad} = \frac{180y}{\pi}^\circ$$

Very often, if the unit of an angle is omitted, it is assumed to be radian. For example, we write $\theta = \frac{\pi}{2}$ without any unit for a right angle θ.

Using radian, the formulas for the arclength and area of a sector are simple. For a sector with central angle θ and radius r,

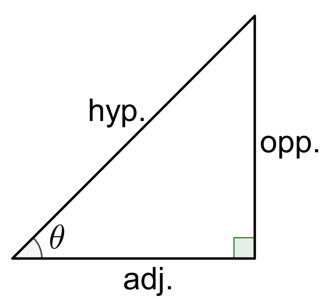


Arclength	$s = r\theta$
Area	$A = \pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta$

3.2 Definitions of Trigonometric Functions

First Definition

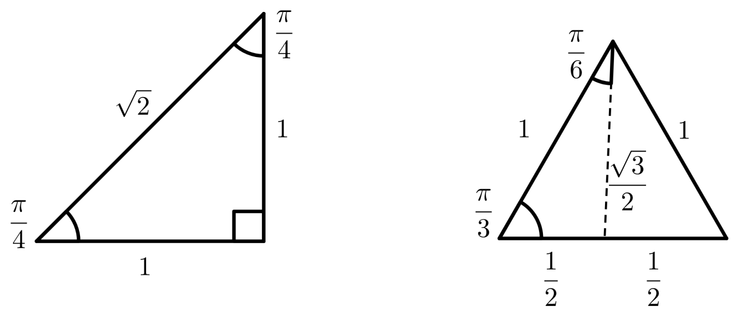
Let $0 < \theta < \frac{\pi}{2}$. Consider the following right-angled triangle.



Here hyp., opp. and adj. stands for the hypotenuse, opposite side and adjacent side of the angle θ respectively. The sine, cosine and tangent functions can be defined by

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}}$$

From the two triangles below, we obtain the values of sine, cosine and tangent at $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

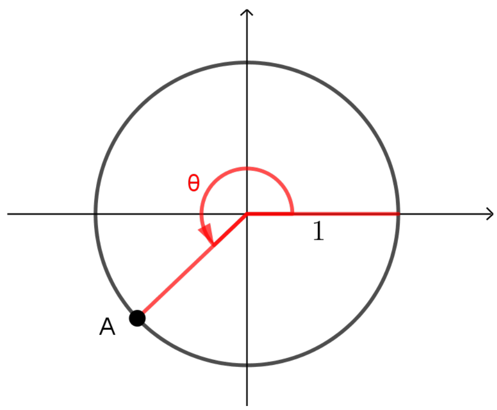


θ in degree	30°	45°	60°
θ in radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Extension of the Domains of Trigonometric Functions

The domains of sine and cosine can be extended to all real numbers.

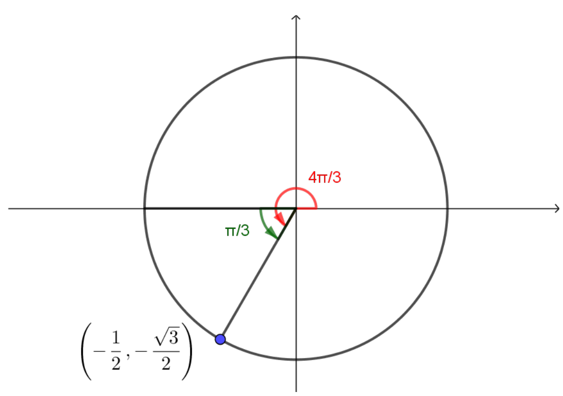
Definition 1. Consider the unit circle centered at the origin O. Let A be the point on it such that the angle from the positive x-axis to OA, measured anti-clockwisely, is equal to θ.



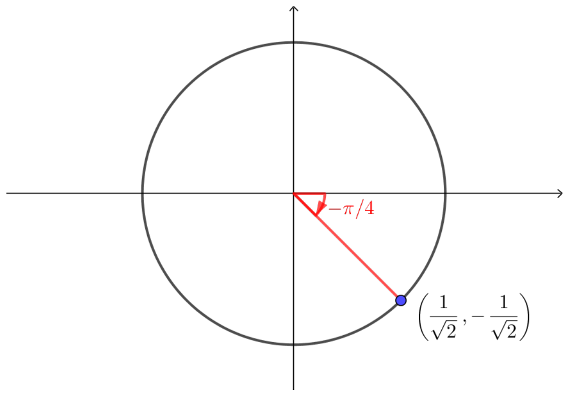
Define

$$A = (\cos\theta, \sin\theta).$$

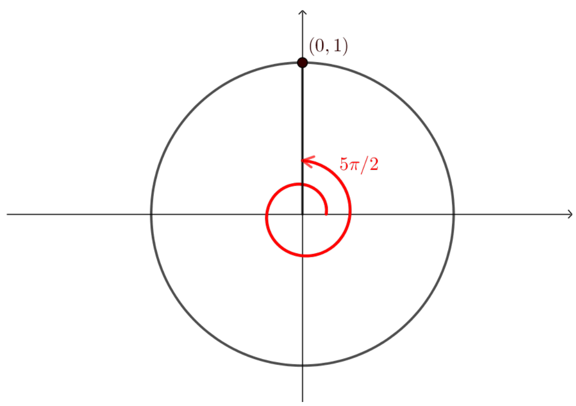
Example 1. Consider the following angles:



$$\cos\frac{4\pi}{3} = -\frac{1}{2} \quad \text{and} \quad \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$



$$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$



$$\cos\frac{5\pi}{2} = 0 \quad \text{and} \quad \sin\frac{5\pi}{2} = 1$$

Four other trigonometric functions are defined as ratios with sine and cosine.

Definition 2. For a real number θ, define

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

1. For domains, note that

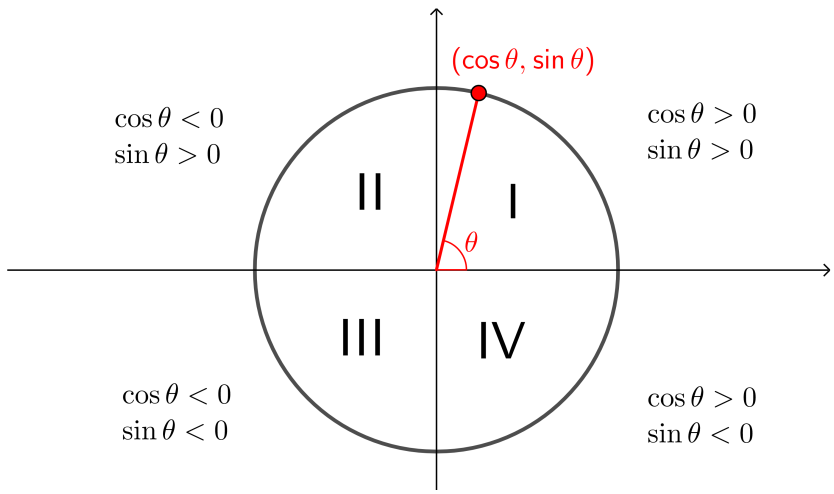
$$\sin\theta = 0 \text{ if } \theta = 0, \pm\pi, \pm 2\pi, \ldots$$
$$\cos\theta = 0 \text{ if } \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Therefore,

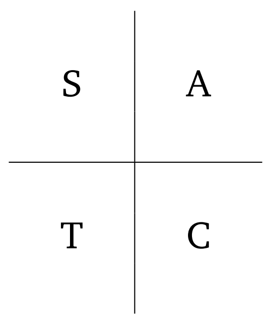
$$D_{\cot\theta} = D_{\csc\theta} = \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$$
$$D_{\tan\theta} = D_{\sec\theta} = \mathbb{R} \setminus \{(k + \tfrac{1}{2})\pi : k \in \mathbb{Z}\}$$

2. $\cot\theta = \dfrac{1}{\tan\theta}$ when $\theta \neq \dfrac{k\pi}{2}, k \in \mathbb{Z}$

CAST Diagram



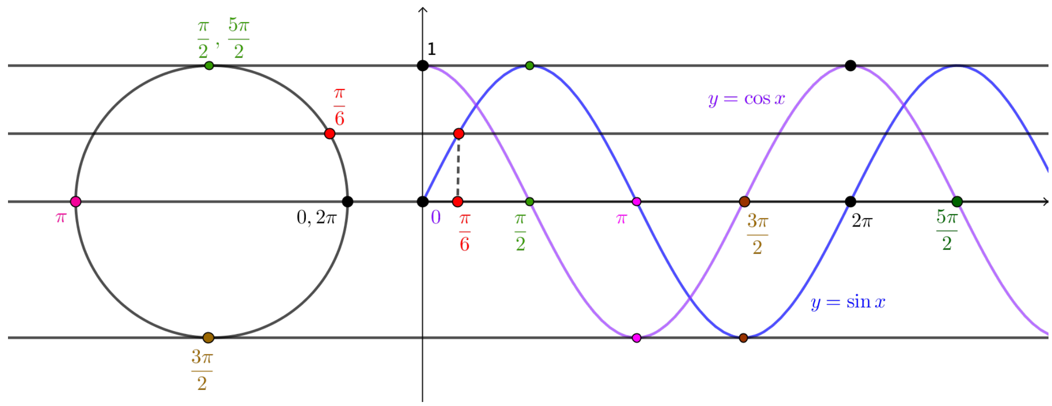
The "CAST" diagram, as shown below, can be used for remembering the signs of sine, cosine and tangent functions in each quadrant.



Here "A" means all the three functions are positive in the quadrant I. "S", "T" and "C" means only sine, tangent and cosine function is positive in quadrant II,III and IV respectively.

3.3 Graphs of Trigonometric Functions

The figure on the right below shows the graphs of $\cos x$ and $\sin x$ for $x \geq 0$. Note that they are equal to the x and y coordinates of the point $(\cos x, \sin x)$ on the unit circle.

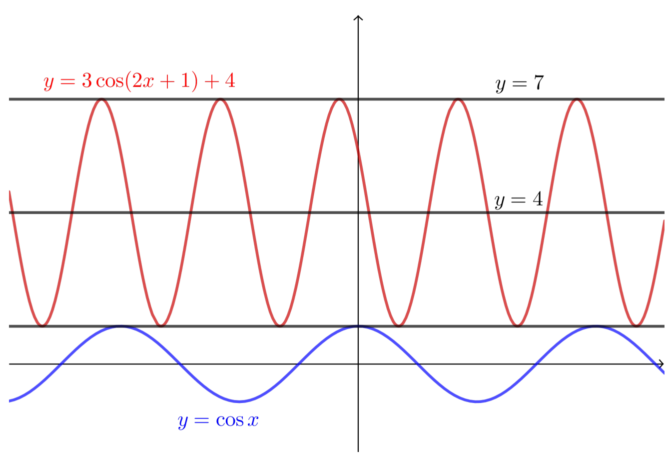


We can graph functions of the form $a\cos(bx+c)+d$ or $a\sin(bx+c)+d$ by transformation.

Example 2. Graph $y = 3\cos(2x+1)+4$.

Solution. The graph $y = 3\cos(2x+1)+4$ can be obtained by the following series of transformations from $y = \cos x$:

$$\begin{aligned}\cos x &\xrightarrow{\text{move to left by 1 unit}} \cos(x+1)\\ &\xrightarrow{\text{constrict horizontally by half}} \cos(2x+1)\\ &\xrightarrow{\text{enlarge vertically 3 times}} 3\cos(2x+1)\\ &\xrightarrow{\text{move up by 4 units}} 3\cos(2x+1)+4\end{aligned}$$



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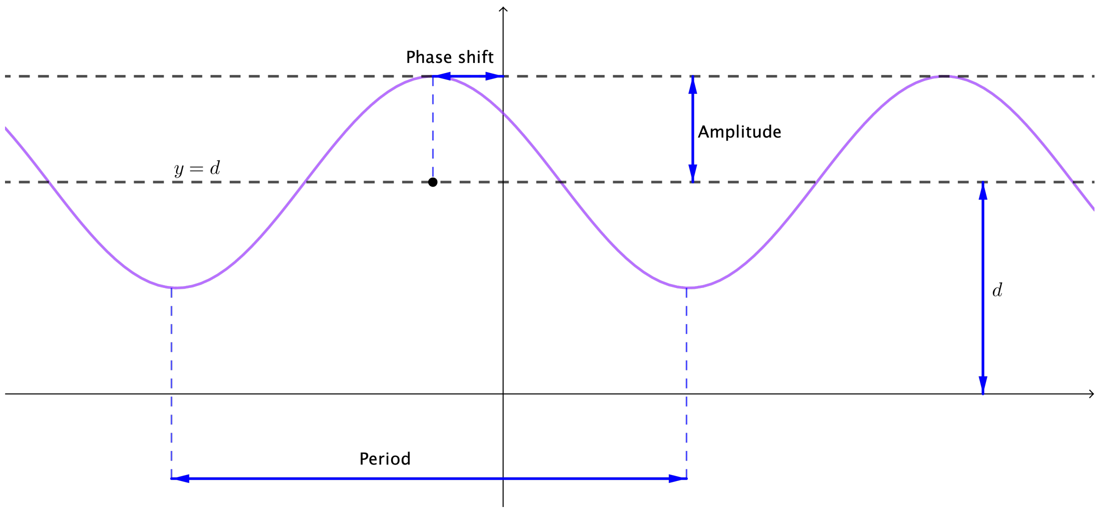
More generally, let a, b, c, d be real numbers with $a, b > 0$. The figure below is the graph

$$y = a\cos(bx + c) + d = a\cos\left[b\left(x + \frac{c}{b}\right)\right] + d$$



Some features from the graph:

Center line: $y = d$

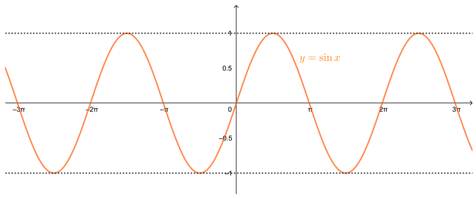
Period: $\frac{2\pi}{b}$

Amplitude: a

Phase shift: $-\frac{c}{b}$

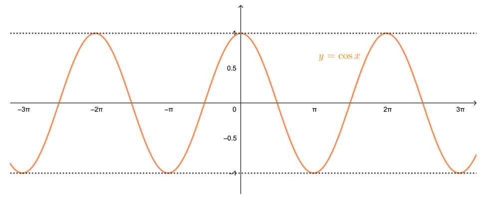
Graphs of the form $y = a\sin(bx + c) + d$ can be obtained similarly from that of $y = \sin x$.

Below are the graphs of the six basic trigonometric functions.



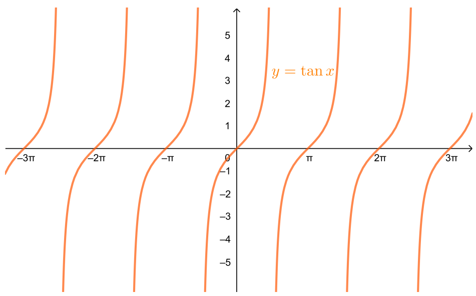
$y = \sin x$

Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π



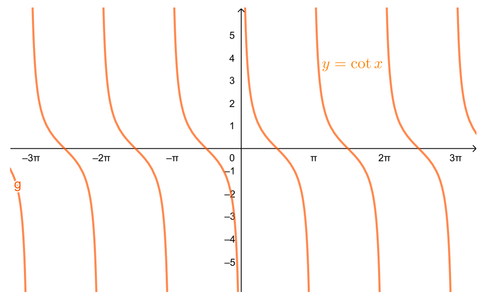
$y = \cos x$

Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π



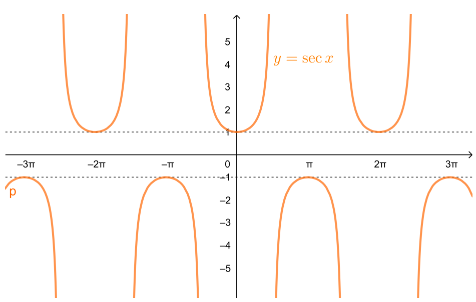
$y = \tan x$

Domain: $\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$
Range: $(-\infty, \infty)$
Period: π



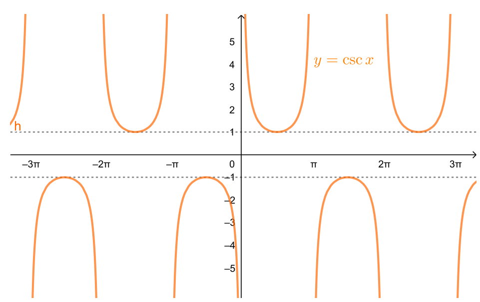
$y = \cot x$

Domain: $\{x \in \mathbb{R} : x \neq n\pi, n \in \mathbb{Z}\}$
Range: $(-\infty, \infty)$
Period: π



$y = \sec x$

Domain: $\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: 2π



$y = \csc x$

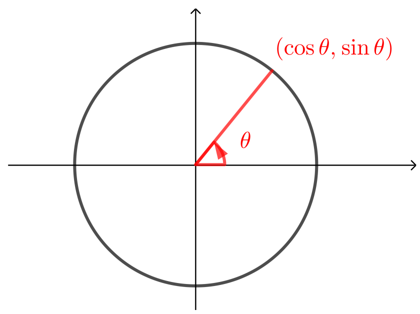
Domain: $\{x \in \mathbb{R} : x \neq n\pi, n \in \mathbb{Z}\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: 2π

3.4 Trigonometric Identities

In this section, we introduce some standard trigonometric identities.

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$



Proof. By definition, $(\cos\theta, \sin\theta)$ is on the unit circle. Hence $\sin^2\theta + \cos^2\theta = 1$.

For the second one,

$$\text{L.H.S.} = \tan^2\theta + 1 = \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2 = \sec^2\theta = \text{R.H.S.}$$

The third one is proved similarly. □

Example 3. Let θ be in quadrant II and $\sin\theta = \frac{1}{3}$. Find $\cos\theta$ and $\cot\theta$.

Solution.

$$\sin^2\theta + \cos^2\theta = 1 \implies \left(\frac{1}{3}\right)^2 + \cos^2\theta = 1 \implies \cos^2\theta = \frac{8}{9}.$$

Since θ is in quadrant II, $\cos\theta < 0$ from the CAST diagram. Therefore

$$\cos\theta = -\frac{\sqrt{8}}{3} = -\frac{2\sqrt{2}}{3} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = -2\sqrt{2}.$$

■

Example 4. Show that $\dfrac{\sec\theta - 1}{\tan\theta} = \dfrac{\tan\theta}{\sec\theta + 1}$

Solution.

$$\text{L.H.S.} = \frac{\sec\theta - 1}{\tan\theta}\cdot\frac{\sec\theta+1}{\sec\theta+1} = \frac{\sec^2\theta - 1}{\tan\theta(\sec\theta+1)} = \frac{\tan^2\theta}{\tan\theta(\sec\theta+1)} = \frac{\tan\theta}{\sec\theta+1} = \text{R.H.S.}$$

■

Recall that a function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all x in its domain.

Even/Odd Formulae

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Cofunction Formulae

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$$

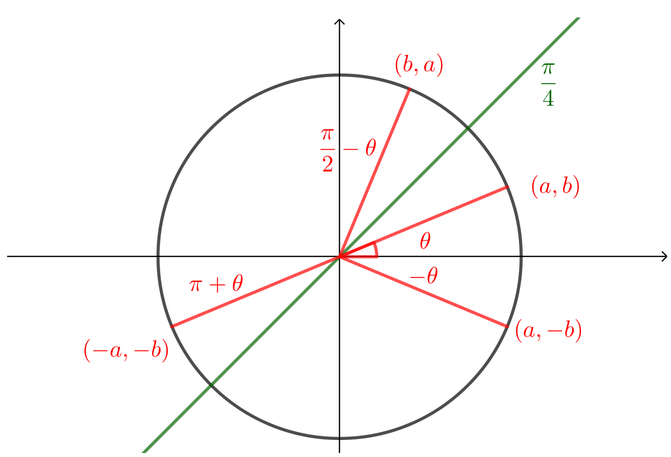
$$\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$

Periodicity

If n is an integer,

$$\sin(\theta+2n\pi) = \sin\theta \qquad \cos(\theta+2n\pi) = \cos\theta \qquad \tan(\theta+n\pi) = \tan\theta$$
$$\csc(\theta+2n\pi) = \csc\theta \qquad \sec(\theta+2n\pi) = \sec\theta \qquad \cot(\theta+n\pi) = \cot\theta$$
$$\sin(\theta+\pi) = -\sin\theta \qquad \cos(\theta+\pi) = -\cos\theta$$

These formulae can be obtained by relating the x and y coordinates of points on the unit circle with different central angles. For example, let $a = \cos\theta$ and $b = \sin\theta$. Then



From the figure,

$$(\cos(-\theta), \sin(-\theta)) = (a, -b) = (\cos\theta, -\sin\theta)$$

$$\left(\cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right)\right) = (b, a) = (\sin\theta, \cos\theta)$$

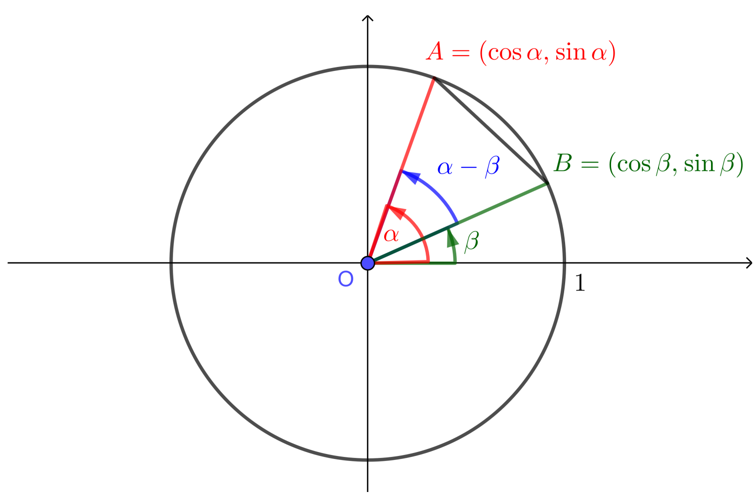
$$(\cos(\theta + \pi), \sin(\theta + \pi)) = (-a, -b) = (-\cos\theta, -\sin\theta)$$

$$\tan(\theta + \pi) = \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

Sum and Difference of Formulas

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}
\end{aligned}$$

Proof. We first prove $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. Consider the points $A = (\cos\alpha, \sin\alpha)$ and $B = (\cos\beta, \sin\beta)$ on the unit circle.



By applying the cosine rule to the triangle $\triangle OAB$,

$$OA^2 + OB^2 - 2OA \cdot OB \cdot \cos(\alpha - \beta) = AB^2$$

$$
\begin{aligned}
1 + 1 - 2\cos(\alpha - \beta) &= (\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 \\
&= \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\alpha \\
&= (\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}
$$

The other 5 formulae can be deduced as follows.

$$
\begin{aligned}
\cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) \\
&= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha + \beta) &= \cos\left(\frac{\pi}{2} - \alpha - \beta\right) \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\
&= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta
\end{aligned}
$$

$$
\begin{aligned}
\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
&= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}} \\
&= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}
\end{aligned}
$$

$$
\begin{aligned}
\tan(\alpha - \beta) &= \tan(\alpha + (-\beta)) \\
&= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)} \\
&= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}
$$

$\square$

By taking $\alpha = \beta = \theta$ in the sum and difference formulae above, we obtain the following important double angle formulas.

Double Angle Formulas

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

By changing subject in the double angle formulas of $\cos 2\theta$ and replacing θ by $\frac{\theta}{2}$, we obtain these half angle formulas.

Half Angle Formulas

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \quad \text{or} \quad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \quad \text{or} \quad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Next, we convert between product and sum/difference of sine and cosine.

Product to Sum Formulas

$$\begin{aligned}\sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]\\ \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]\\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]\\ \cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]\end{aligned}$$

Proof. For product to sum formulas, it is easy to prove from the right hand side. For example,

$$\begin{aligned}\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] &= \frac{1}{2}(\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\ &= \sin\alpha\cos\beta\end{aligned}$$

□

Sum to Product Formulas

$$\begin{aligned}\sin\alpha+\sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\ \sin\alpha-\sin\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\ \cos\alpha+\cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\ \cos\alpha-\cos\beta &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\end{aligned}$$

Proof. The sum to product formulas can be proved using the product sum formulas. For example,

$$\begin{aligned}2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= 2\left(\frac{1}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]\\ &= \cos\alpha+\cos\beta\end{aligned}$$

□

Example 5. Find the exact value of the followings:

(a) $\tan 75°$; (b) $\cos\frac{\pi}{12}$ (c) $\sin\frac{\pi}{24}\sin\frac{7\pi}{24}$.

Solution. 1.

$$\begin{aligned}\tan 75° &= \tan(30°+45°)\\ &= \frac{\tan 30°+\tan 45°}{1-\tan 30°\tan 45°}\\ &= \frac{\frac{1}{\sqrt{3}}+1}{1-(\frac{1}{\sqrt{3}})(1)}\\ &= \frac{1+\sqrt{3}}{\sqrt{3}-1}\end{aligned}$$

2.

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

3.

$$\begin{aligned}\sin\frac{\pi}{24}\sin\frac{7\pi}{24} &= \frac{1}{2}\left[\cos\left(\frac{\pi}{24}-\frac{7\pi}{24}\right)-\cos\left(\frac{\pi}{24}+\frac{7\pi}{24}\right)\right]\\ &= \frac{1}{2}\left[\cos\left(-\frac{\pi}{4}\right)-\cos\frac{\pi}{3}\right]\\ &= \frac{1}{2}\left[\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right]\\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}-1}{4}\end{aligned}$$

■

Example 6. Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Solution.

$$\begin{aligned}\text{L.H.S.} &= \sin 3\theta\\ &= \sin(\theta+2\theta)\\ &= \sin\theta\cos 2\theta+\cos\theta\sin 2\theta\\ &= \sin\theta(1-2\sin^2\theta)+\cos\theta(2\sin\theta\cos\theta)\\ &= \sin\theta-2\sin^3\theta+2\sin\theta(1-\sin^2\theta)\\ &= 3\sin\theta-4\sin^3\theta\\ &= \text{ R.H.S.}\end{aligned}$$

■

Example 7. Prove that $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution.

$$\begin{aligned}\text{R.H.S.} &= \frac{1-\tan^2\theta}{1+\tan^2\theta} \\ &= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} \\ &= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} \\ &= \cos 2\theta \\ &= \text{L.H.S.}\end{aligned}$$

■

Example 8. Let $f(x) = \cos x$. Simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

Solution.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\cos(x+h)-\cos x}{h} \\ &= \frac{-2\sin(\frac{x+h+x}{2})\sin(\frac{x+h-x}{2})}{h} \\ &= \frac{-2\sin(x+\frac{h}{2})\sin\frac{h}{2}}{h}\end{aligned}$$

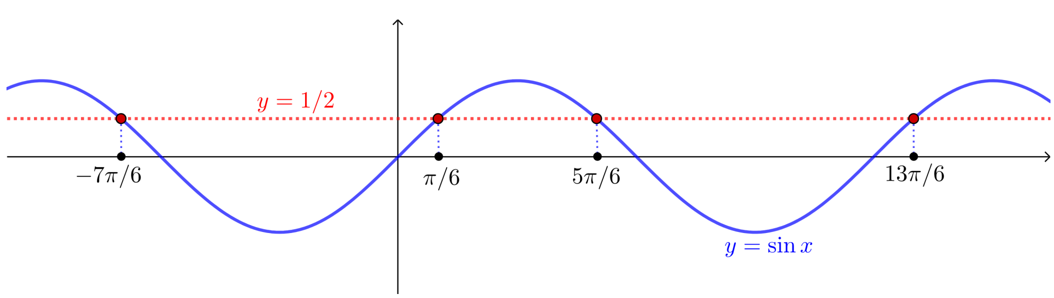
■

Remarks: By taking limit $h \to 0$, we obtain that

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{-2\sin(x+\frac{h}{2})\sin\frac{h}{2}}{h} = \lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right)\cdot\frac{sin\frac{h}{2}}{\frac{h}{2}} = -\sin x.$$

3.5 Inverse Functions

We would like to define inverse functions for trigonometric functions, for example, the inverse sine function $\sin^{-1} x$. To do so, we consider the graph of $\sin x$.

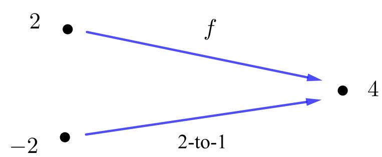


A problem arises here: What should be $\sin^{-1}\left(\frac{1}{2}\right)$? $\frac{\pi}{6}$? $\frac{5\pi}{6}$? $\frac{13\pi}{6}$ or $-\frac{7\pi}{6}$? Which one should be the correct value? While finding the inverse of $\sin x$, we observe that $\sin x$ is not one-to-one. To get around this, we should restrict the function $\sin x$ to a smaller domain.

One-to-one Function

Definition 3. A function $f(x)$ is called one-to-one if $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$.

Example 9. Let $f(x) = x^2$. Note that $f(2) = 4 = f(-2)$. In other words, f sends both 2 and -2 to 4.



Therefore, f is not one-to-one.

The definition of one-to-one functions above can be rephrased as follows:

$f(x)$ is one-to-one if $f(x_1) = f(x_2) \implies x_1 = x_2$.

Example 10. Let $g(x) = 2x + 3$. If $g(x_1) = g(x_2)$, then

$$\begin{aligned} & 2x_1 + 3 = 2x_2 + 3 \\ \implies \quad & 2x_1 = 2x_2 \\ \implies \quad & x_1 = x_2 \end{aligned}$$

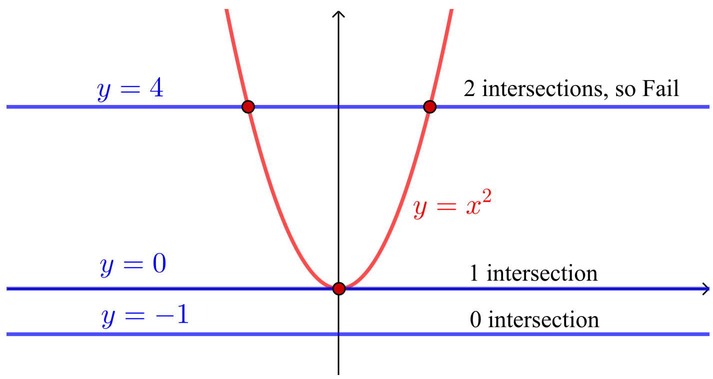
Therefore, g is one-to-one.

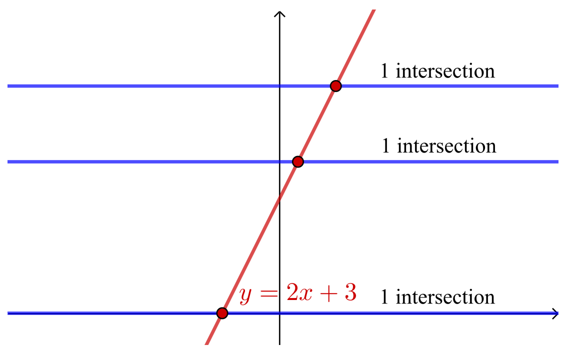
Horizontal Line Test

It is also possible to determine whether a function is one-to-one from its graph.

Horizontal Line Test If every horizontal line has at most one intersection with the graph of $f(x)$, then f is one-to-one.

The graph $y = x^2$ has two intersections with the horizontal line $y = 4$. It fails the horizontal line test and so x^2 is not one-to-one.



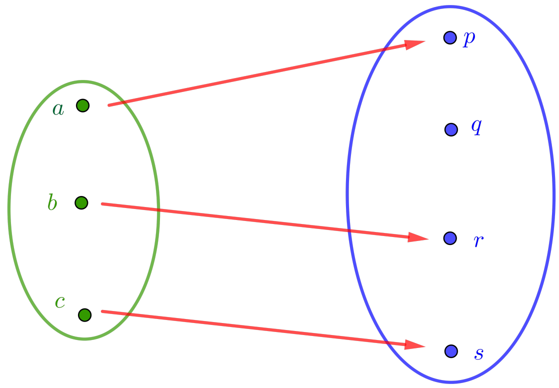


The graph $y = 2x + 3$ has one intersection with every horizontal line. Hence, it passes the horizontal line test and so $2x + 3$ is one-to-one.

An important property for one-to-one functions is the existence of inverses.

If f is one-to-one, then its inverse f^{-1} can be defined.

For example, if a function f is defined as below,

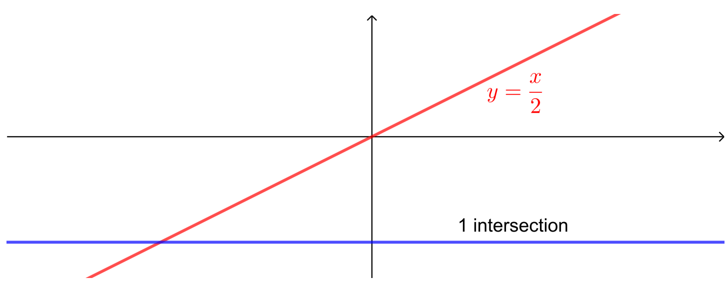


then f is one-to-one, f^{-1} can be defined by $f^{-1}(p) = a$, $f^{-1}(r) = b$ and $f^{-1}(s) = c$. Note that $f^{-1}(q)$ is not defined because q is not in the range of f. In general,

For an one-to-one function f,

$$\begin{array}{ccc} D_{f^{-1}} = R_f & \text{and} & R_{f^{-1}} = D_f \\ (f^{-1} \circ f)(x) = x & \text{for} & x \in D_f \\ (f \circ f^{-1})(x) = x & \text{for} & x \in D_{f^{-1}} \end{array}$$

Example 11. Let $f(x) = \dfrac{x}{2}$.

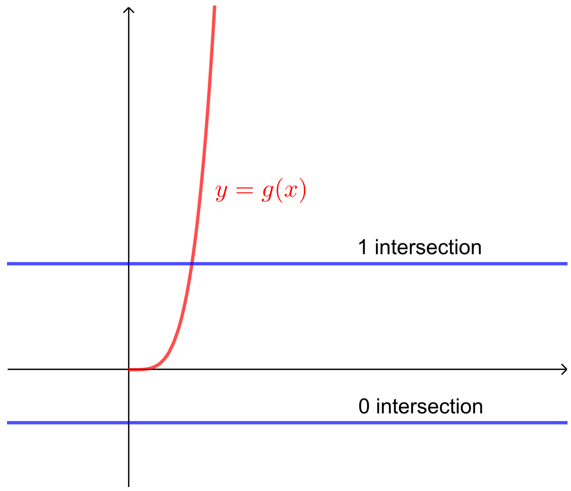


Its graph passes the horizontal line test. It implies that f is one-to-one and so f^{-1} can be defined. Note that f divides a number by 2. Its inverse f^{-1} does the reverse process of multiplying a number by 2 and so $f^{-1}(x) = 2x$.

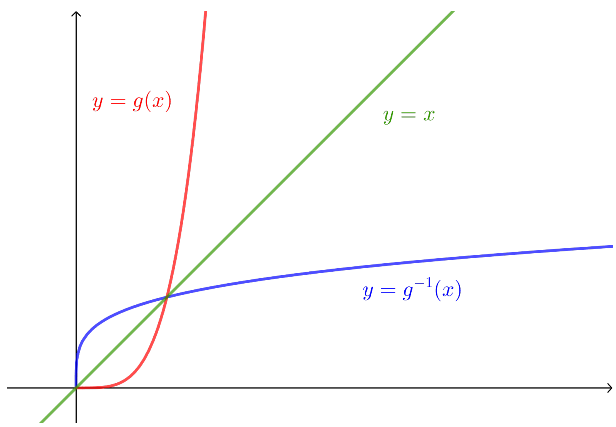
$$x \underset{f^{-1} \quad (\times 2)}{\overset{f \quad (\times \frac{1}{2})}{\rightleftharpoons}} \frac{x}{2}$$

Also, note that $D_{f^{-1}} = R_f = (-\infty, \infty)$ and $R_{f^{-1}} = D_f = (-\infty, \infty)$.

Example 12. Since $(-1)^4 = 1^4$, the function x^4 is not one-to-one on $\mathbb{R}$. To define its inverse, we restrict it to a smaller domain so that it becomes one-to-one. Consider $g : [0, \infty) \longrightarrow \mathbb{R}$ defined by $g(x) = x^4$.



The graph of g passes the horizontal line test. Hence, g is one-to-one and has an inverse. Its inverse is $g^{-1}(x) = \sqrt[4]{x}$ with domain $D_{g^{-1}} = R_g = [0, \infty)$ and range $R_{g^{-1}} = D_g = [0, \infty)$. Below are the graphs of $g(x)$ and $g^{-1}(x)$.



Note that the graph of g and g^{-1} are reflections of each other across the line $y = x$. In general,

> Let $f(x)$ be a one-to-one function. The graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other across the line $y = x$.

It is true because for a point (a, b) on the xy-plane,

$$\begin{aligned} (a,b) \text{ is on the graph } y = f(x) \quad &\Leftrightarrow \quad b = f(a) \\ &\Leftrightarrow \quad a = f^{-1}(b) \\ &\Leftrightarrow \quad (b,a) \text{ is on the graph } y = f^{-1}(x). \end{aligned}$$

Note that (a, b) and (b, a) are reflections of each other across the line $y = x$.

Finding the Inverse of a One-to-one Function

Here are a few steps to find the inverse of a function:

1. Let $y = f(x)$.
2. Express x in terms of y.
3. If $x = g(y)$, then $f^{-1}(x) = g(x)$.

Example 13. Let $f(x) = \dfrac{3x+1}{x+2}$. Find $f^{-1}(x)$, its domain and range.

Solution. First, let $y = f(x) = \dfrac{3x+1}{x+2}$, then

$$\begin{aligned} y(x+2) &= 3x+1 \\ xy - 3x &= 1 - 2y \\ x &= \frac{1-2y}{y-3} = g(y) \end{aligned}$$

Therefore, $f^{-1}(x) = g(x) = \dfrac{1-2x}{x-3}$. Its domain and range are

$$D_{f^{-1}} = \mathbb{R} \setminus \{3\} \quad \text{and} \quad R_{f^{-1}} = D_f = \mathbb{R} \setminus \{-2\}.$$

Remark.

1. We can deduce from $f^{-1}(x)$ that the range of $f(x)$ is $R_f = D_{f^{-1}} = \mathbb{R} \setminus \{3\}$.
2. We can check our formula of $f^{-1}(x)$ by computing the composition:

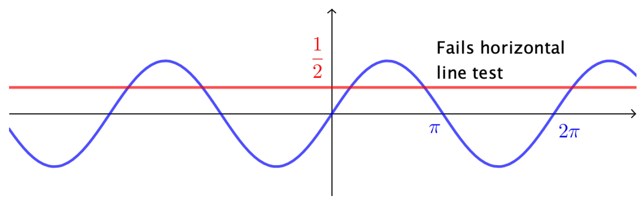
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{3x+1}{x+2}\right) \\ &= \frac{1 - 2\left(\frac{3x+1}{x+2}\right)}{\frac{3x+1}{x+2} - 3} \\ &= \frac{x+2-2(3x+1)}{3x+1-3(x+2)} \\ &= \frac{-5x}{-5} \\ &= x. \end{aligned}$$

   Hence, our computation is correct.

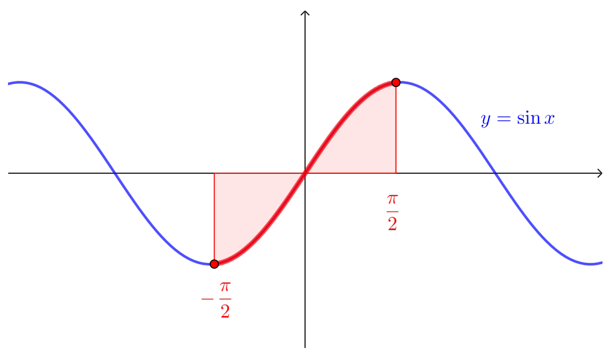
∎

3.6 Inverses of Trigonometric Functions

Back to the previous example about defining $\sin^{-1}$.



The problem is that $\sin x$ is not one-to-one on $\mathbb{R}$, so to find its inverse, we need to restrict the function to a smaller domain.



As seen from the figure above, if we restrict the domain of $\sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the restricted function

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}$$

is one-to-one. Also, its range is $[-1, 1]$, same as the original function $\sin : \mathbb{R} \longrightarrow \mathbb{R}$. We define arcsine to be the inverse of the function $\sin$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

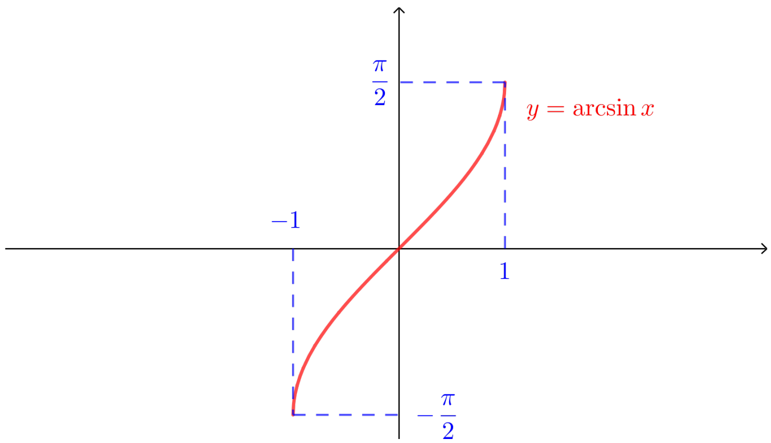
$$\boxed{\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

In other words, if $x \in [-1, 1]$, then $\arcsin x$ is equal to the unique $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = x$. For example,

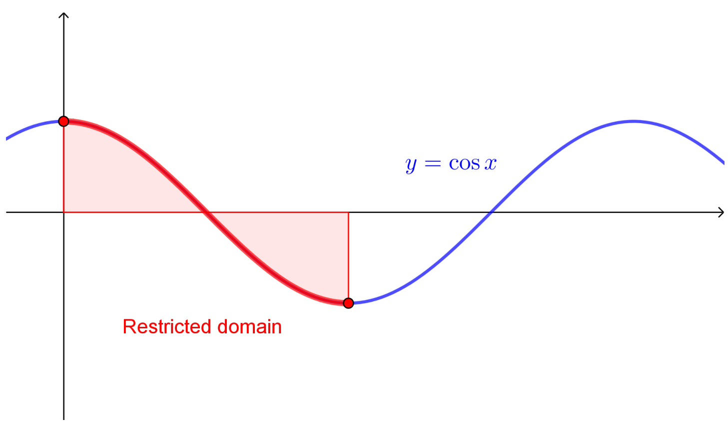
$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \text{ and } -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

As an inverse function, the graph $y = \arcsin x$ is obtained by reflecting the graph $y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ across the line $y = x$.



Next, we consider the cosine function.



Similar to sine, the cosine function is not one-to-one on $\mathbb{R}$, but if we restrict the domain to $[0, \pi]$, then the restricted function

$$\cos : [0, \pi] \longrightarrow \mathbb{R}$$

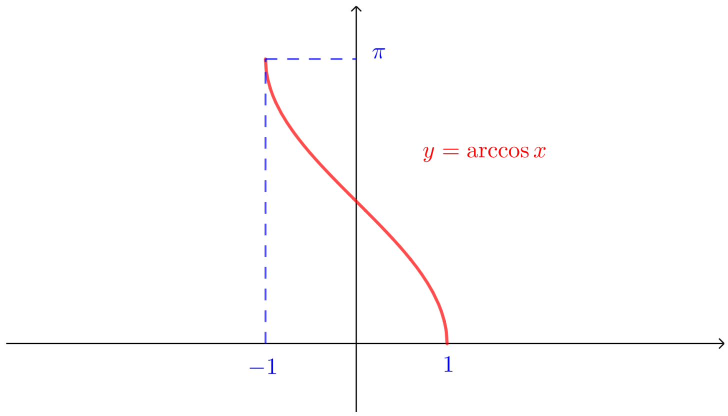
is one-to-one with range $[-1, 1]$. We define arccosine to be the inverse of this restriction.

$$\boxed{\arccos : [-1, 1] \longrightarrow [0, \pi]}$$

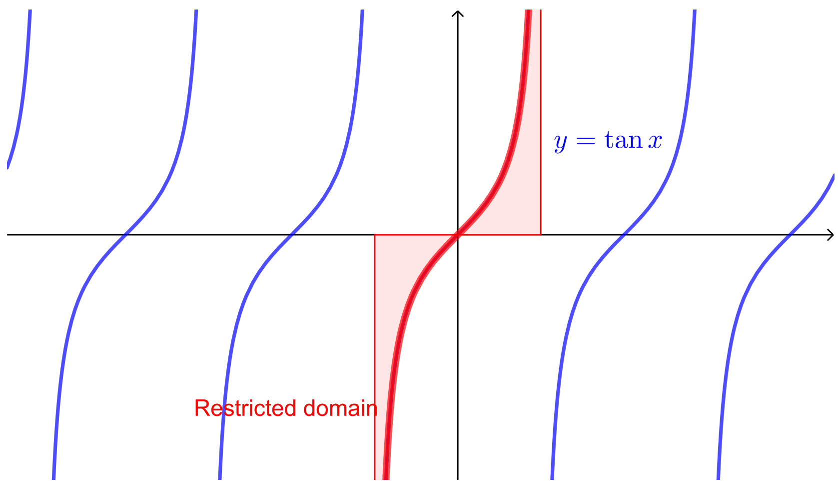
For example,

$$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \text{ and } \frac{5\pi}{6} \in [0, \pi] \implies \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

The graph $y = \arccos x$ is obtained by reflecting the graph $y = \cos x$ on $[0, \pi]$ across the line $y = x$.



Finally, we consider the tangent function.



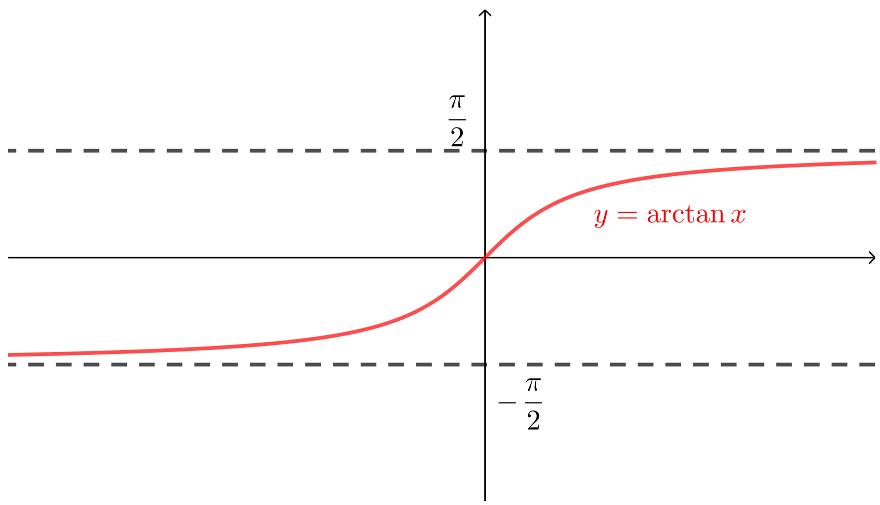
If we restrict the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

is one-to-one with range $\mathbb{R}$. We define arctangent to be its inverse.

$$\boxed{\arctan : (-\infty, \infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

The graph $y = \arctan x$ is obtained by reflecting the graph $y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ across the line $y = x$.



Summary

The domains and ranges of the three inverse trigonometric functions discussed are summarized below.

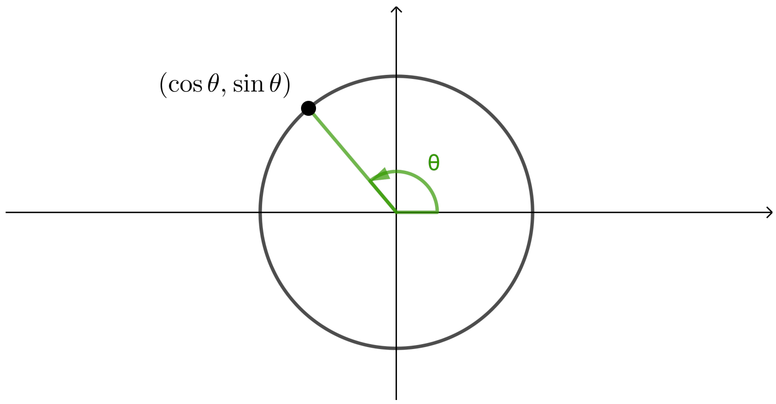
$$\begin{aligned}
&\arcsin : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
&\arccos : [-1,1] \longrightarrow [0,\pi] \\
&\arctan : (-\infty,\infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
\end{aligned}$$

For each of these functions, the range is specifically chosen such that the sine, cosine or tangent function is one-to-one. This range is called the principal values of the inverse trigonometric function.

Remark. We denote the three inverse trigonometric functions above as $\arcsin, \arccos$ and $\arctan$. It is also common to use the following standard notation of inverse functions for them.

$$\arcsin = \sin^{-1} \qquad \arccos = \cos^{-1} \qquad \arctan = \tan^{-1}$$

We will do some examples. Recall that $x = \cos\theta$ and $y = \sin\theta$ are the coordinates of the point representing θ on the unit circle.



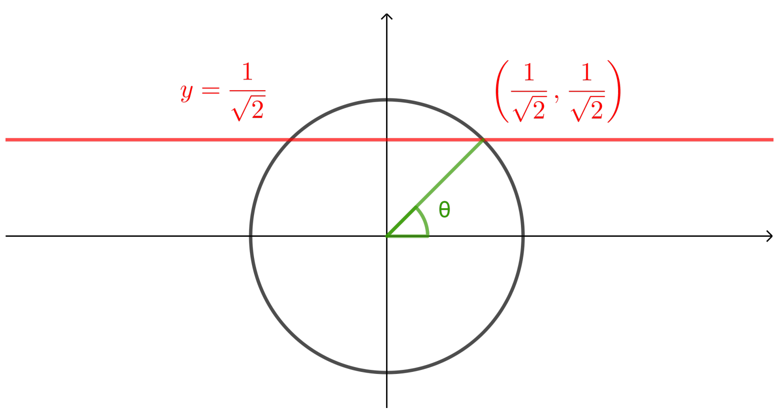
Example 14. Find the value of $\theta = \arcsin\left(\frac{1}{\sqrt{2}}\right)$.

Solution. $\theta = \arcsin\left(\frac{1}{\sqrt{2}}\right)$ and the range of arcsin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore,

$$\sin\theta = \frac{1}{\sqrt{2}} \text{ and } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The unique θ satisfying these is $\frac{\pi}{4}$. Hence, $\theta = \frac{\pi}{4}$. ■

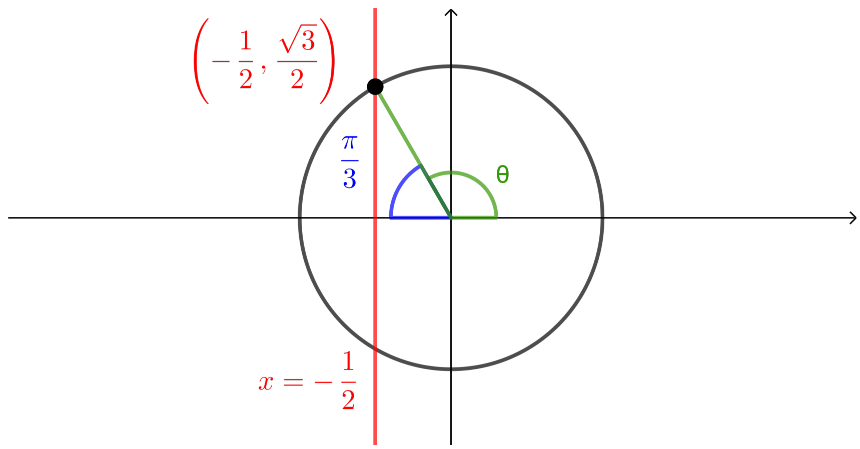
Remark. It is also possible to find θ above graphically. If $\sin\theta = \frac{1}{\sqrt{2}}$, then the point representing θ on the unit circle has y-coordinate $\frac{1}{\sqrt{2}}$.



The line $y = \frac{1}{\sqrt{2}}$ has two intersections with the unit circle. Since $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it can be concluded that $\theta = \frac{\pi}{4}$.

Example 15. Find the value of $\theta = \arccos\left(-\frac{1}{2}\right)$.

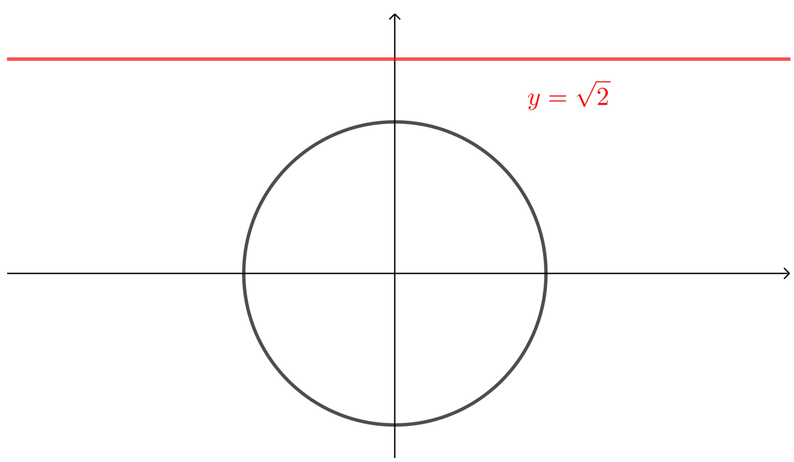
Solution. Since $\cos\theta = -\frac{1}{2}$, the point representing θ on the unit circle has x-coordinate $-\frac{1}{2}$. Also, $\theta \in R_{\arccos} = [0, \pi]$.



From the figure above, $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. ■

Example 16. Find the value of $\theta = \arcsin\sqrt{2}$.

Solution. Since $\sqrt{2} \notin [-1, 1] = D_{\arcsin}$, $\arcsin\sqrt{2}$ is undefined. Graphically, it is because the unit circle and the line $y = 2$ have no intersection. $\sin\theta \neq \sqrt{2}$ for any θ.



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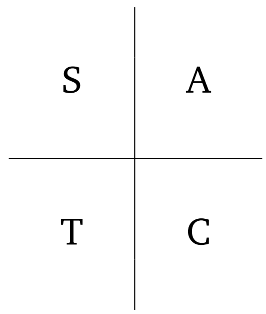
Example 17. Find the value of $\sin\left(2\arccos\left(-\frac{2}{3}\right)\right)$.

Solution. Let $\theta = \arccos\left(-\frac{2}{3}\right)$. Then

$$\sin\left(2\arccos\left(-\frac{2}{3}\right)\right) = \sin 2\theta = 2\sin\theta\cos\theta.$$

Clearly, $\cos\theta = -\frac{2}{3}$. Also, $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(-\frac{2}{3}\right)^2 = \frac{5}{9}$. Since $\theta \in R_{\text{arccos}} = [0, \pi]$, $\sin\theta \geq 0$ as seen from the CAST diagram.



Hence, $\sin\theta = \frac{\sqrt{5}}{3}$ and

$$\sin\left(2\arccos\left(-\frac{2}{3}\right)\right) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}.$$

■

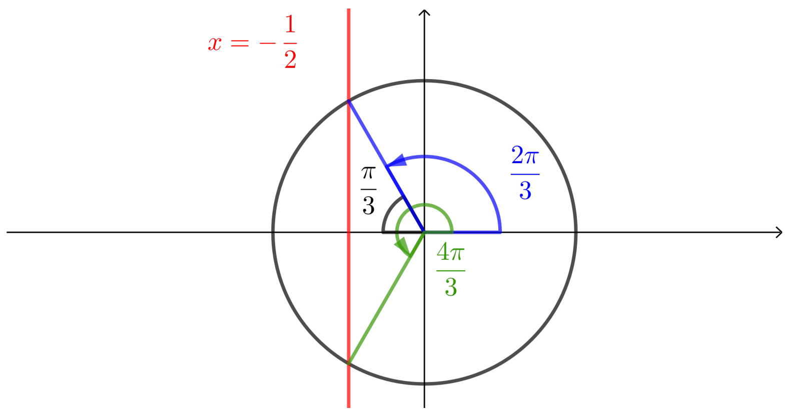
3.7 Trigonometric Equations

In this section, we solve equations involving trigonometric functions.

Example 18. Solve $\cos\theta = -\frac{1}{2}$ for

1. $0 \leq \theta \leq 2\pi$;
2. $\theta \in \mathbb{R}$.

Solution. The unit circle centered at origin intersects the line $x = -\frac{1}{2}$ at two points, representing $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.



Hence, for $0 \le \theta \le 2\pi$, the solution of $\cos\theta = -\frac{1}{2}$ is $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Since $\cos\theta$ has period 2π, the general solutions of $\cos\theta = -\frac{1}{2}$ for $\theta \in \mathbb{R}$ are

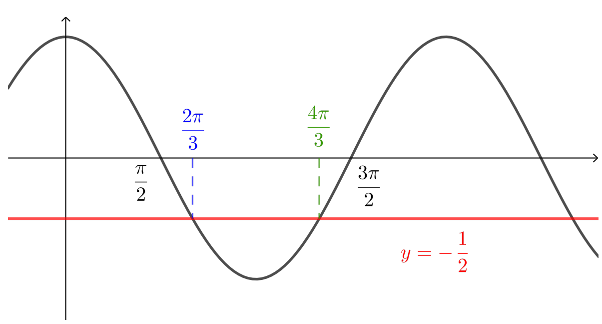
$$\theta = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad \frac{4\pi}{3} + 2k\pi, \quad \text{where } k \in \mathbb{Z}.$$

■

Remark. 1. Simply applying arccos to the given equation gives only one of the solutions

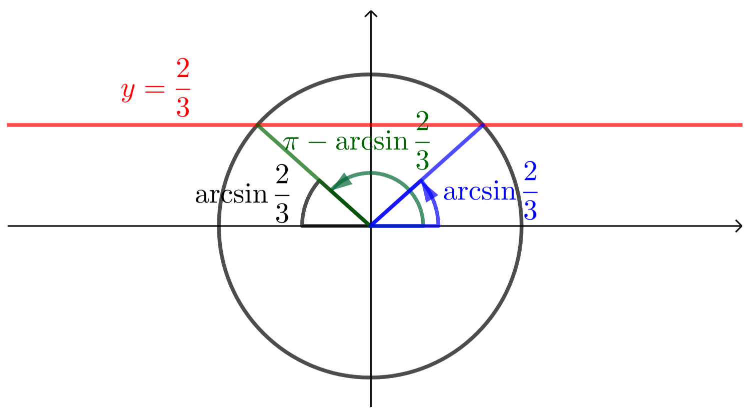
$$\theta = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

2. Below are the graphs $y = \cos x$ and $y = -\frac{1}{2}$. The x-coordinates of the intersections are the solutions of $\cos\theta = -\frac{1}{2}$.



Example 19. Solve $\sin\theta = \frac{2}{3}$ for

1. $0 \le \theta \le 2\pi$;
2. $\theta \in \mathbb{R}$.



Solution. From the figure, for $0 \leq \theta \leq 2\pi$, the solutions are $\theta = \arcsin\frac{2}{3}$ or $\pi - \arcsin\frac{2}{3}$. Since $\sin\theta$ has period 2π, the general solution of $\sin\theta = \frac{2}{3}$ for $\theta \in \mathbb{R}$ are

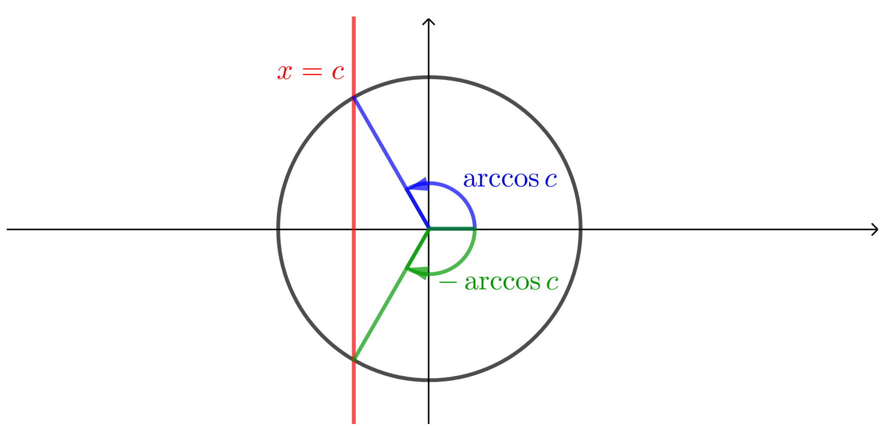
$$\theta = 2k\pi + \arcsin\frac{2}{3} \quad \text{or} \quad (2k+1)\pi - \arcsin\frac{2}{3}, \quad \text{where } k \in \mathbb{Z}.$$

∎

In general, for $-1 \leq c \leq 1$ and $\theta \in \mathbb{R}$, we have the following general solutions.

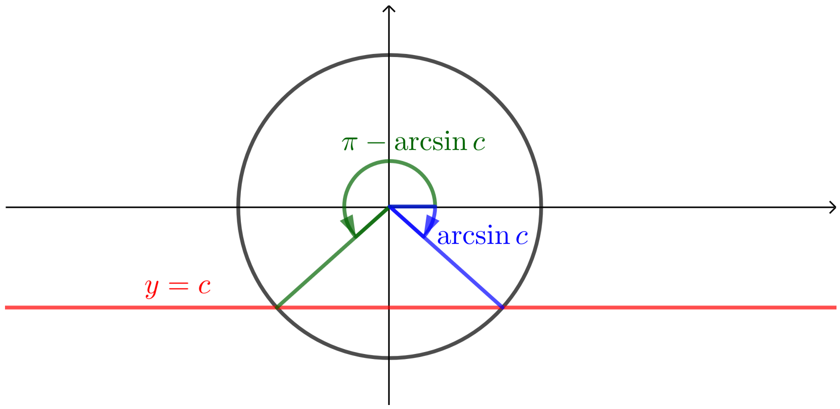
For $\cos\theta = c$, the general solutions are

$$\theta = 2k\pi \pm \arccos c, \text{ where } k \in \mathbb{Z}.$$



For $\sin\theta = c$, the general solutions are

$$\theta = 2k\pi + \arcsin c \text{ or } (2k+1)\pi - \arcsin c, \text{ where } k \in \mathbb{Z}.$$

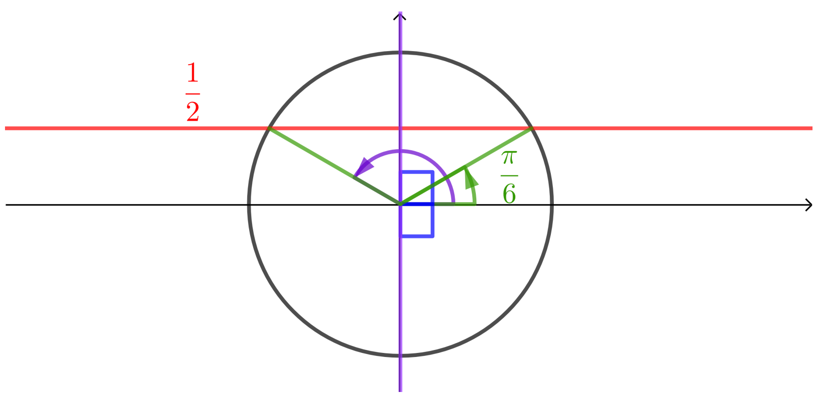


Example 20. Solve the following equations.

1. $\sin 2x = \cos x$ for $x \in \mathbb{R}$.

   Solution.

$$\begin{aligned} 2\sin x \cos x &= \cos x \\ \cos x(2\sin x - 1) &= 0 \\ \cos x = 0 \text{ or } \sin x &= \frac{1}{2} \\ x = 2k\pi \pm \frac{\pi}{2} &\text{ or } 2k\pi + \frac{\pi}{6} \text{ or } 2k\pi + \frac{5\pi}{6} \end{aligned}$$



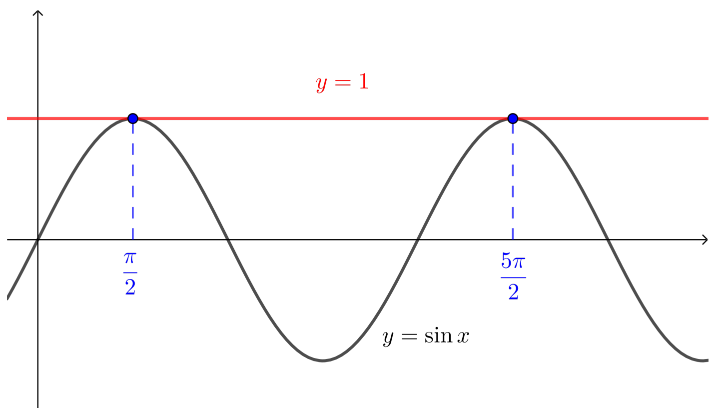
■

2. $\cos^2 2\theta + 4\sin 2\theta = 4$ for $0 \le \theta \le 2\pi$.

Solution.

$$\begin{aligned}(1-\sin^2 2\theta)+4\sin 2\theta &= 4\\ -\sin^2 2\theta+4\sin 2\theta-3 &= 0\\ \sin^2 2\theta-4\sin 2\theta+3 &= 0\\ (\sin 2\theta-1)(\sin 2\theta-3) &= 0\\ \sin 2\theta &= 1 \text{ or } 3 \text{ (no solution)}\end{aligned}$$

If $0 \le \theta \le 2\pi$, then $0 \le 2\theta \le 4\pi$.



Therefore,

$$\begin{aligned}2\theta &= \frac{\pi}{2} \text{ or } \frac{5\pi}{2}\\ \theta &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}\end{aligned}$$

■

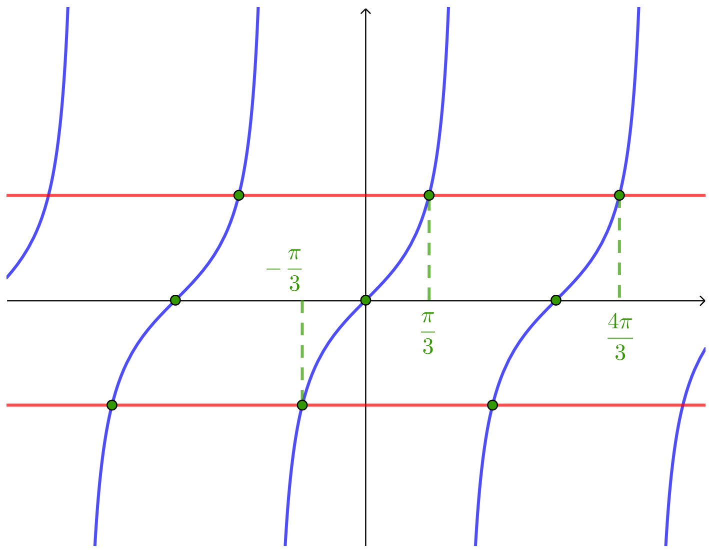
3. $\tan x + \tan 2x = 0$ for $x \in \mathbb{R}$.

Solution.

$$\begin{aligned}\tan x + \frac{2\tan x}{1-\tan^2 x} &= 0\\ \tan x(1-\tan^2 x)+2\tan x &= 0\\ \tan x(3-\tan^2 x) &= 0\\ \tan x = 0 \quad &\text{or} \quad \tan^2 x = 3\\ \tan x = 0 \quad &\text{or} \quad \pm\sqrt{3}\end{aligned}$$

Since $\tan x$ has a period of π, we consider the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of length π. The solutions of $\tan x = 0$ or $\pm\sqrt{3}$ over this interval are $x = 0$ or $\pm\frac{\pi}{3}$. Therefore, the general solution is

$$x = k\pi \quad \text{or} \quad k\pi \pm \frac{\pi}{3}, \quad \text{where } k \in \mathbb{Z}.$$



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Remark. In general, if $c \in \mathbb{R}$, then

> For $\tan\theta = c$, the general solutions are
>
> $$\theta = k\pi + \arctan c, \text{ where } k \in \mathbb{Z}.$$

4. $\sin x \sin 2x = \cos 3x \cos 4x$ for $0 \leq x \leq \dfrac{\pi}{2}$.

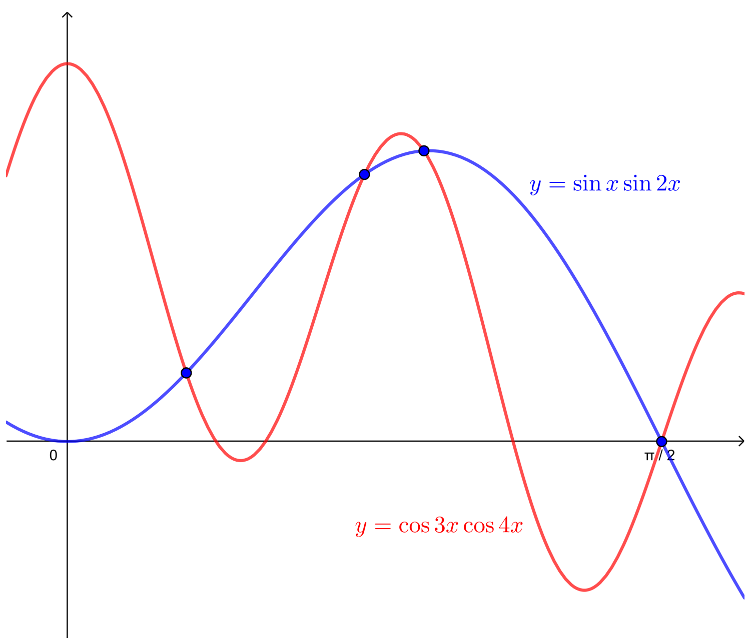
   Solution. By product-to-sum and sum-to-product formulas,

$$\begin{aligned}
\frac{1}{2}[\cos(x-2x)-\cos(x+2x)] &= \frac{1}{2}[\cos(3x-4x)+\cos(3x+4x)] \\
-\cos 3x &= \cos 7x \\
\cos 7x + \cos 3x &= 0 \\
2\cos\frac{7x+3x}{2}\cos\frac{7x-3x}{2} &= 0 \\
\cos 5x \cos 2x &= 0 \\
\cos 5x = 0 \quad \text{or} \quad \cos 2x &= 0
\end{aligned}$$

   Since $0 \leq 5x \leq \dfrac{5\pi}{2}$ and $0 \leq 2x \leq \pi$,

$$\begin{aligned}
5x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \quad \text{or} \quad 2x = \frac{\pi}{2} \\
x &= \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2} \quad \text{or} \quad x = \frac{\pi}{4}
\end{aligned}$$

   Therefore, the solutions are $x = \dfrac{\pi}{10}, \dfrac{\pi}{4}, \dfrac{3\pi}{10}$ or $\dfrac{\pi}{2}$.



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3.8 More Techniques in Trigonometry

Technique of subsidiary angle

Example 21. Let $f(x) = \sin x + \sqrt{3} \cos x$.

1. Express $f(x)$ in the form $A \sin(x + c)$ with $A \geq 0$;
2. Find the maximum/minimum value(s) of $f(x)$ and the corresponding x;
3. Solve the equation $f(x) = 1$.

Solution. 1. Note that

$$\begin{aligned} A \sin(x + c) &= A(\sin x \cos c + \cos x \sin c) \\ &= A \cos c \sin x + A \sin c \cos x \\ f(x) &= 1 \sin x + \sqrt{3} \cos x \end{aligned}$$

By comparing coefficients, we want to find A and c such that

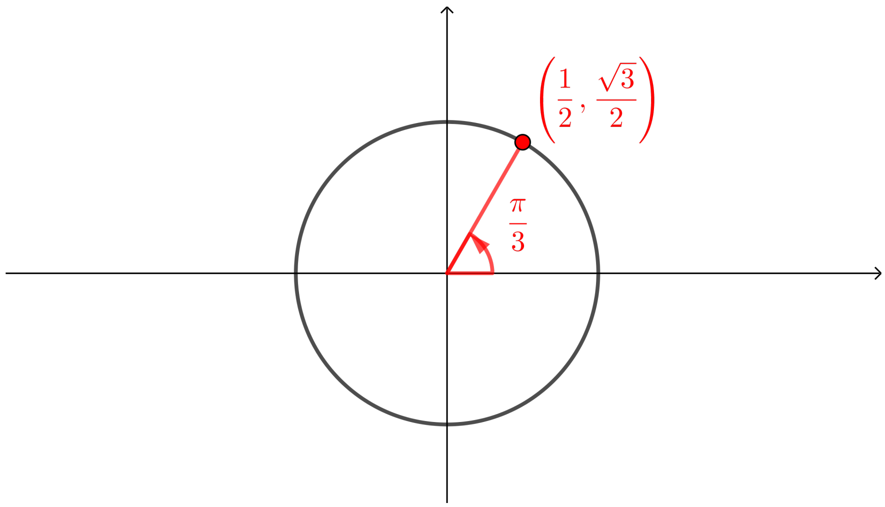
$$\begin{cases} A \cos c = 1 \\ A \sin c = \sqrt{3} \end{cases} \tag{*}$$

$$\begin{aligned} (A \cos c)^2 + (A \sin c)^2 &= 1^2 + (\sqrt{3})^2. \\ A^2 &= 4 \\ A &= 2 \end{aligned}$$

Put $A = 2$ back to the equations $(*)$, we have

$$\cos c = \frac{1}{2} \text{ and } \sin c = \frac{\sqrt{3}}{2}$$

We can take $c = \frac{\pi}{3}$. Therefore, $f(x) = 2 \sin\left(x + \frac{\pi}{3}\right)$.



2. It can be deduced from $f(x) = 2\sin\left(x + \frac{\pi}{3}\right)$ that the maximum value of $f(x)$ is 2, and it occurs when

$$\begin{aligned} \sin\left(x + \frac{\pi}{3}\right) &= 1 \\ x + \frac{\pi}{3} &= 2k\pi + \frac{\pi}{2}, \quad \text{where } k \in \mathbb{Z} \\ x &= 2k\pi + \frac{\pi}{6} \end{aligned}$$

Similarly, the minimum value of $f(x)$ is -2 when

$$\begin{aligned} \sin\left(x + \frac{\pi}{3}\right) &= -1 \\ x + \frac{\pi}{3} &= 2k\pi - \frac{\pi}{2}, \quad \text{where } k \in \mathbb{Z} \\ x &= 2k\pi - \frac{5\pi}{6} \end{aligned}$$

3.

$$\begin{aligned} f(x) &= 1 \\ 2\sin\left(x + \frac{\pi}{3}\right) &= 1 \\ \sin\left(x + \frac{\pi}{3}\right) &= \frac{1}{2} \\ x + \frac{\pi}{3} &= 2k\pi + \frac{\pi}{6} \text{ or } 2k\pi + \frac{5\pi}{6} \\ x &= 2k\pi - \frac{\pi}{6} \text{ or } 2k\pi + \frac{\pi}{2}, \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

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Example 22. Similarly, the functions $g(x) = \cos x - \sin x$ and $h(x) = 3\cos x + 4\sin x$ can be expressed in the form $A\cos(x + c)$, with $A > 0$, below.

$$g(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) \quad \text{and} \quad h(x) = 5\cos\left(x - \arctan\frac{4}{3}\right)$$

The details are left as exercises.

t-substitution

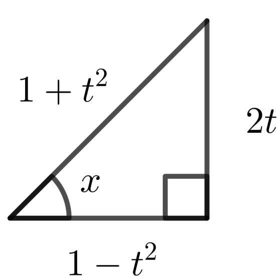
Let $t = \tan\frac{x}{2}$. Then

$$\boxed{\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}}$$

The proofs of these formulas are straightforward. For example, by the double angle formula

$$\tan x = \tan\left(2 \cdot \frac{x}{2}\right) = \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$$

The following figure may help you to remember the formulas.



These formulas can be used to convert rational functions in $\sin x$ and $\cos x$ to rational functions in t. It is useful for integration in calculus.

Example 23. Express $\dfrac{\sin x + \cos x}{1 + \sin x}$ as a rational function of t.

Solution.

$$\begin{aligned}
\frac{\sin x + \cos x}{1 + \sin x} &= \frac{\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2}}{1 + \dfrac{2t}{1+t^2}} \\
&= \frac{2t + 1 - t^2}{1 + t^2 + 2t} \\
&= \frac{-t^2 + 2t + 1}{(1+t)^2}
\end{aligned}$$

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Trigonometric Substitution for Integration

Example 24. Assume $0 \leq \theta \leq \dfrac{\pi}{2}$. Express the following in terms of θ with the given substitution and simplify.

1. $x\sqrt{25 - x^2}$, where $x = 5\sin\theta$

   Solution.

$$\begin{aligned}
x\sqrt{25 - x^2} &= 5\sin\theta\sqrt{25 - (5\sin\theta)^2} \\
&= 5\sin\theta\sqrt{25(1 - \sin^2\theta)} \\
&= 5\sin\theta \cdot \sqrt{25\cos^2\theta} \\
&= 5\sin\theta \cdot 5\cos\theta && \text{(Since } 0 \leq \theta \leq \frac{\pi}{2},\ \cos\theta \geq 0\text{)} \\
&= 25\sin\theta\cos\theta \text{ or } \frac{25}{2}\sin 2\theta
\end{aligned}$$

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2. $\dfrac{x}{\sqrt{4x^2 + 8x + 5}}$, where $x + 1 = \dfrac{\tan\theta}{2}$.

Solution.

$$\begin{aligned}
\frac{x}{\sqrt{4x^2+8x+5}} &= \frac{x}{\sqrt{4(x^2+2x+1)+1}} \\
&= \frac{x}{\sqrt{4(x+1)^2+1}} \\
&= \frac{\dfrac{\tan\theta}{2}-1}{\sqrt{\tan^2\theta+1}} \\
&= \frac{\tan\theta-2}{2\sqrt{\sec^2\theta}} \\
&= \frac{\tan\theta-2}{2\sec\theta} && \left(\text{Since } 0\le\theta\le\frac{\pi}{2},\ \sec\theta\ge 0\right) \\
&= \frac{1}{2}\cos\theta(\tan\theta-2) \\
&= \frac{1}{2}\sin\theta-\cos\theta
\end{aligned}$$

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