Priority Queue

- **API**
- **elementary implementations**
- **binary heaps**
- **heapsort**
- **event-driven simulation**

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.

Queue. Remove the item least recently added.

Randomized queue. Remove a random item.

Priority queue. Remove the largest (or smallest) item.

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Priority queue API

Requirement. Generic items are `Comparable`.

`public class MaxPQ<Key extends Comparable<Key>>`		
	`MaxPQ()`	*create an empty priority queue*
	`MaxPQ(Key[] a)`	*create a priority queue with given keys*
`void`	`insert(Key v)`	*insert a key into the priority queue*
`Key`	`delMax()`	*return and remove the largest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`max()`	*return the largest key*
`int`	`size()`	*number of entries in the priority queue*

Priority queue applications

- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Computational number theory. [sum of powers]
- Artificial intelligence. [A* search]
- Statistics. [maintain largest M values in a sequence]
- Operating systems. [load balancing, interrupt handling]
- Discrete optimization. [bin packing, scheduling]
- Spam filtering. [Bayesian spam filter]

Generalizes: stack, queue, randomized queue.

Priority queue client example

Challenge. Find the largest M items in a stream of N items (N huge, M large).

- Fraud detection: isolate $$ transactions.
- File maintenance: find biggest files or directories.

Constraint. Not enough memory to store N items.

```
% more tinyBatch.txt
Turing      6/17/1990   644.08
vonNeumann  3/26/2002  4121.85
Dijkstra    8/22/2007  2678.40
vonNeumann  1/11/1999  4409.74
Dijkstra   11/18/1995   837.42
Hoare       5/10/1993  3229.27
vonNeumann  2/12/1994  4732.35
Hoare       8/18/1992  4381.21
Turing      1/11/2002    66.10
Thompson    2/27/2000  4747.08
Turing      2/11/1991  2156.86
Hoare       8/12/2003  1025.70
vonNeumann 10/13/1993  2520.97
Dijkstra    9/10/2000   708.95
Turing     10/12/1993  3532.36
Hoare       2/10/2005  4050.20
```

```
% java TopM 5 < tinyBatch.txt
Thompson    2/27/2000  4747.08
vonNeumann  2/12/1994  4732.35
vonNeumann  1/11/1999  4409.74
Hoare       8/18/1992  4381.21
vonNeumann  3/26/2002  4121.85
```

sort key

Priority queue client example

Challenge. Find the largest M items in a stream of N items (N huge, M large).

```
MinPQ<Transaction> pq = new MinPQ<Transaction>();

while (StdIn.hasNextLine())
{
   String line = StdIn.readLine();
   Transaction item = new Transaction(line);
   pq.insert(item);
   if (pq.size() > M)
      pq.delMin();
}
```

use a min-oriented pq

Transaction data type is Comparable

pq contains largest M items

order of growth of finding the largest M in a stream of N items

implementation	time	space
sort	N log N	N
elementary PQ	M N	M
binary heap	N log M	M
best in theory	N	M

Priority queue: unordered and ordered array implementation

operation	*argument*	*return value*	*size*	*contents (unordered)*							*contents (ordered)*						
insert	P		1	P							P						
insert	Q		2	P	Q						P	Q					
insert	E		3	P	Q	E					E	P	Q				
remove max		Q	2	P	E						E	P					
insert	X		3	P	E	X					E	P	X				
insert	A		4	P	E	X	A				A	E	P	X			
insert	M		5	P	E	X	A	M			A	E	M	P	X		
remove max		X	4	P	E	M	A				A	E	M	P			
insert	P		5	P	E	M	A	P			A	E	M	P	P		
insert	L		6	P	E	M	A	P	L		A	E	L	M	P	P	
insert	E		7	P	E	M	A	P	L	E	A	E	E	L	M	P	P
remove max		P	6	E	M	A	P	L	E		A	E	E	L	M	P	

A sequence of operations on a priority queue

Priority queue elementary implementations

Challenge. Implement all operations efficiently.

Maintain order: Eager

Unordered: Lazy approach

order-of-growth of running time for priority queue with N items

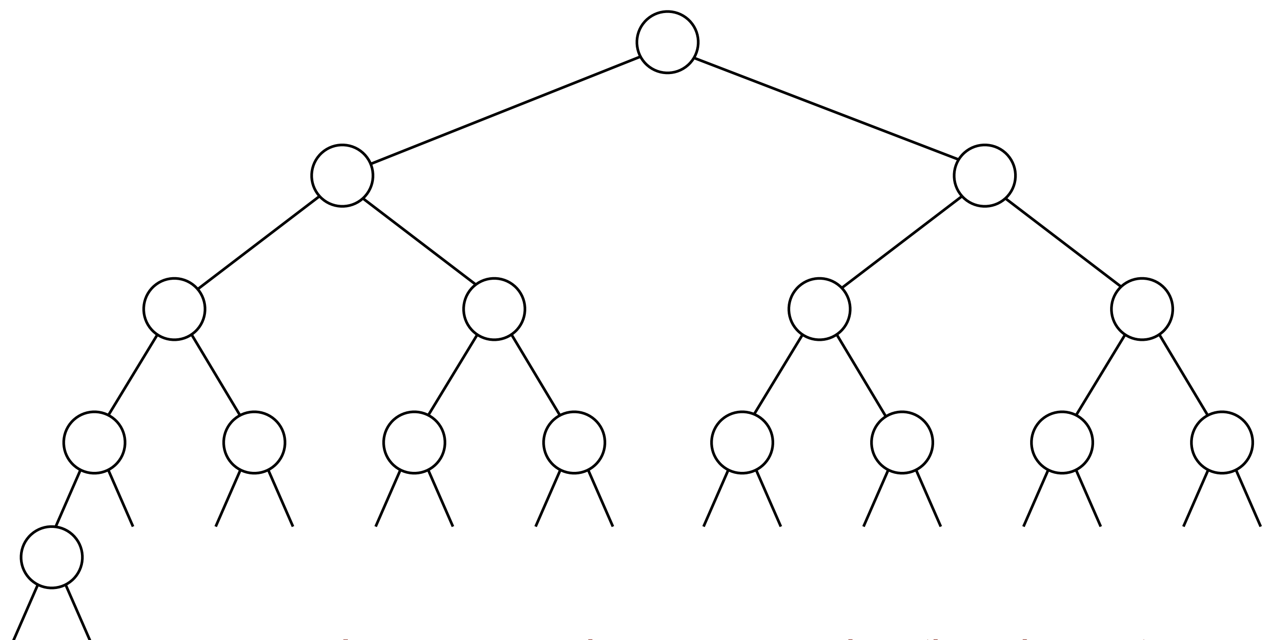
implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	log N	log N	log N

- API
- elementary implementations
- **binary heaps**
- heapsort
- event-driven simulation

Binary tree

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete tree with N = 16 nodes (height = 4)

Property. Height of complete tree with N nodes is $\lfloor \lg N \rfloor$.

Pf. Height only increases when N is a power of 2.

A complete binary tree in nature

Binary heap representations

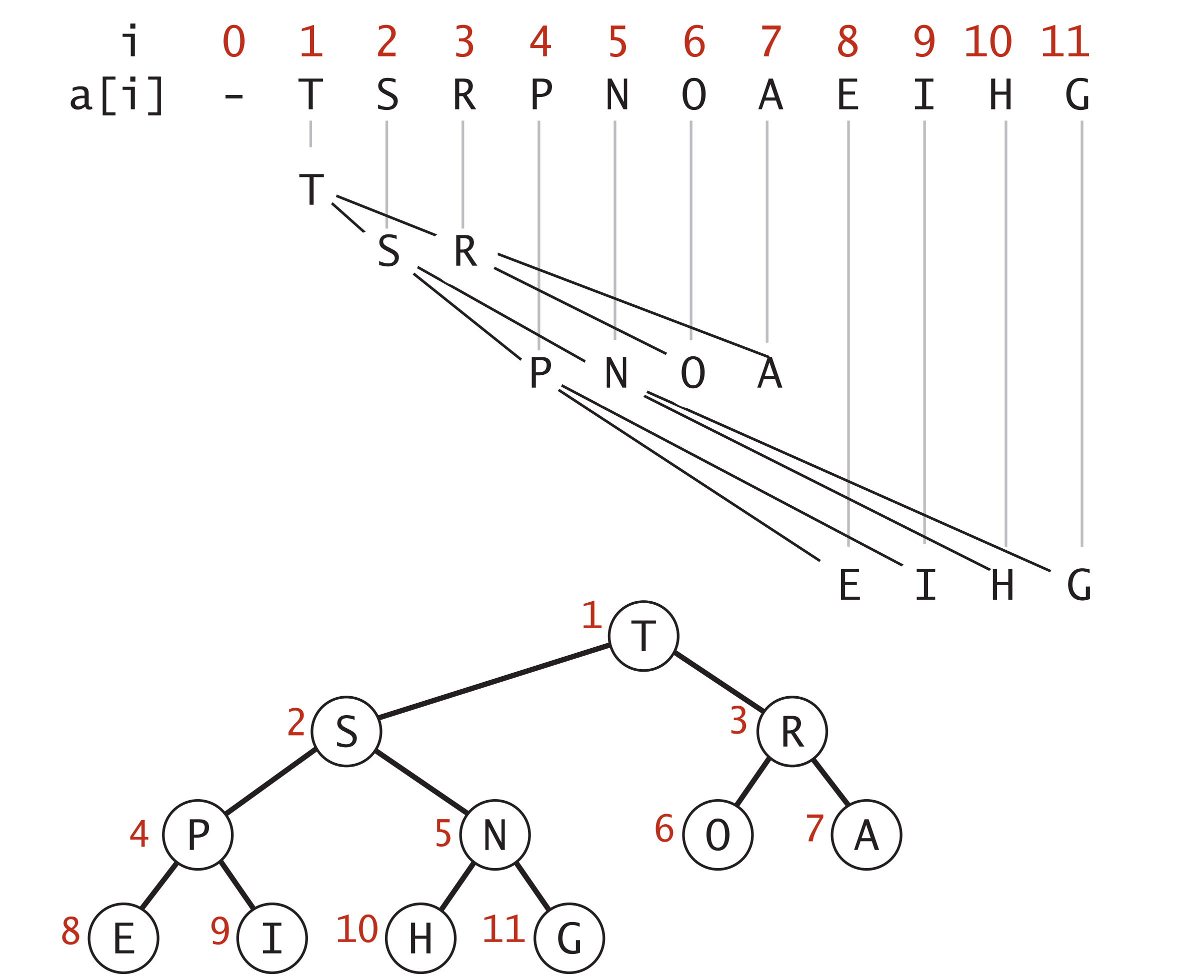
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- No smaller than children's keys.

Array representation.

- Take nodes in level order.
- No explicit links needed!



Heap representations

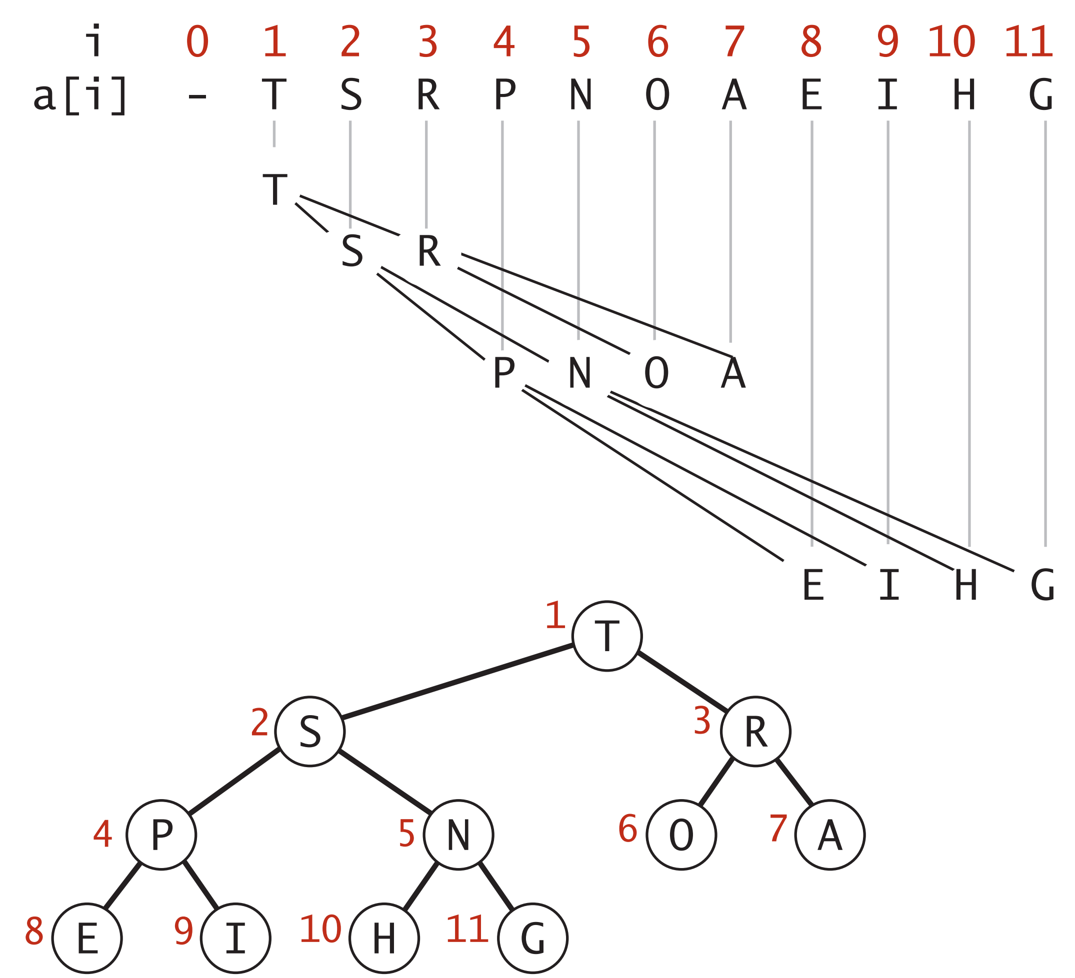
Binary heap properties

Proposition. Largest key is `a[1]`, which is root of binary tree.

indices start at 1

Proposition. Can use array indices to move through tree.

- Parent of node at `k` is at `k/2`.
- Children of node at `k` are at `2k` and `2k+1`.



Heap representations

Promotion in a heap

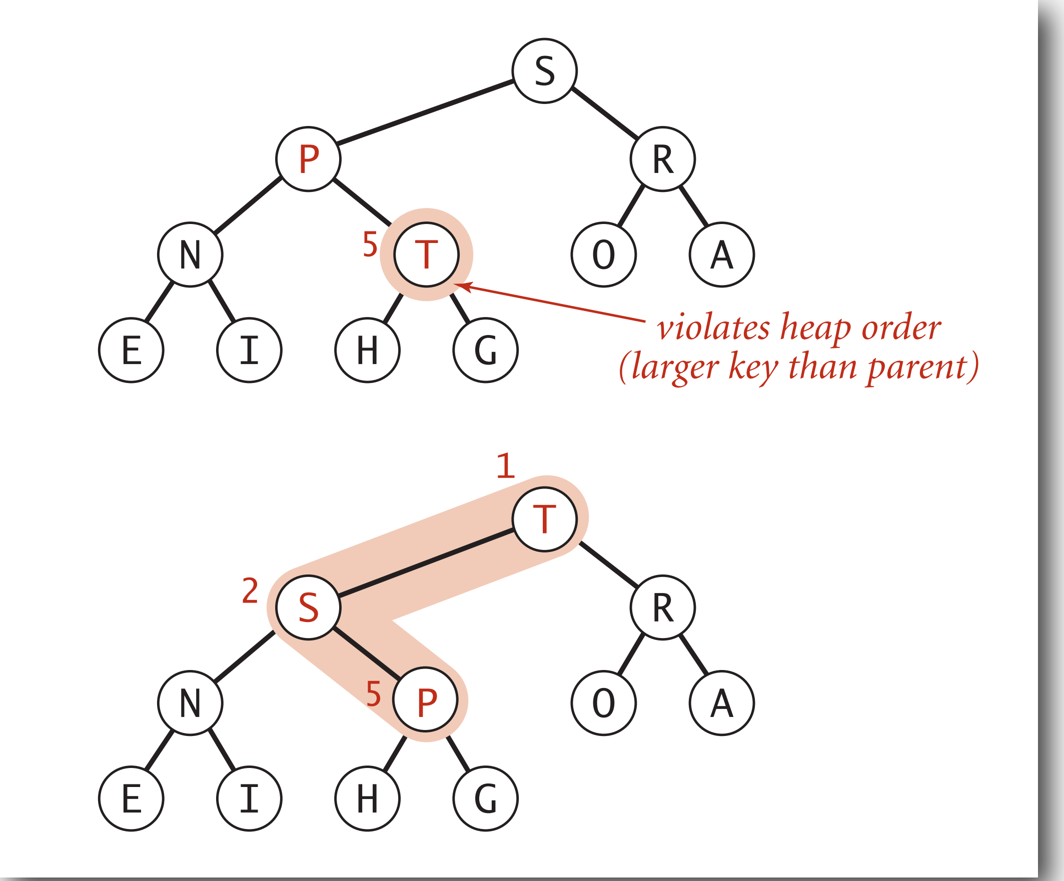
Scenario. Node's key becomes larger key than its parent's key.

To eliminate the violation:

- Exchange key in node with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2



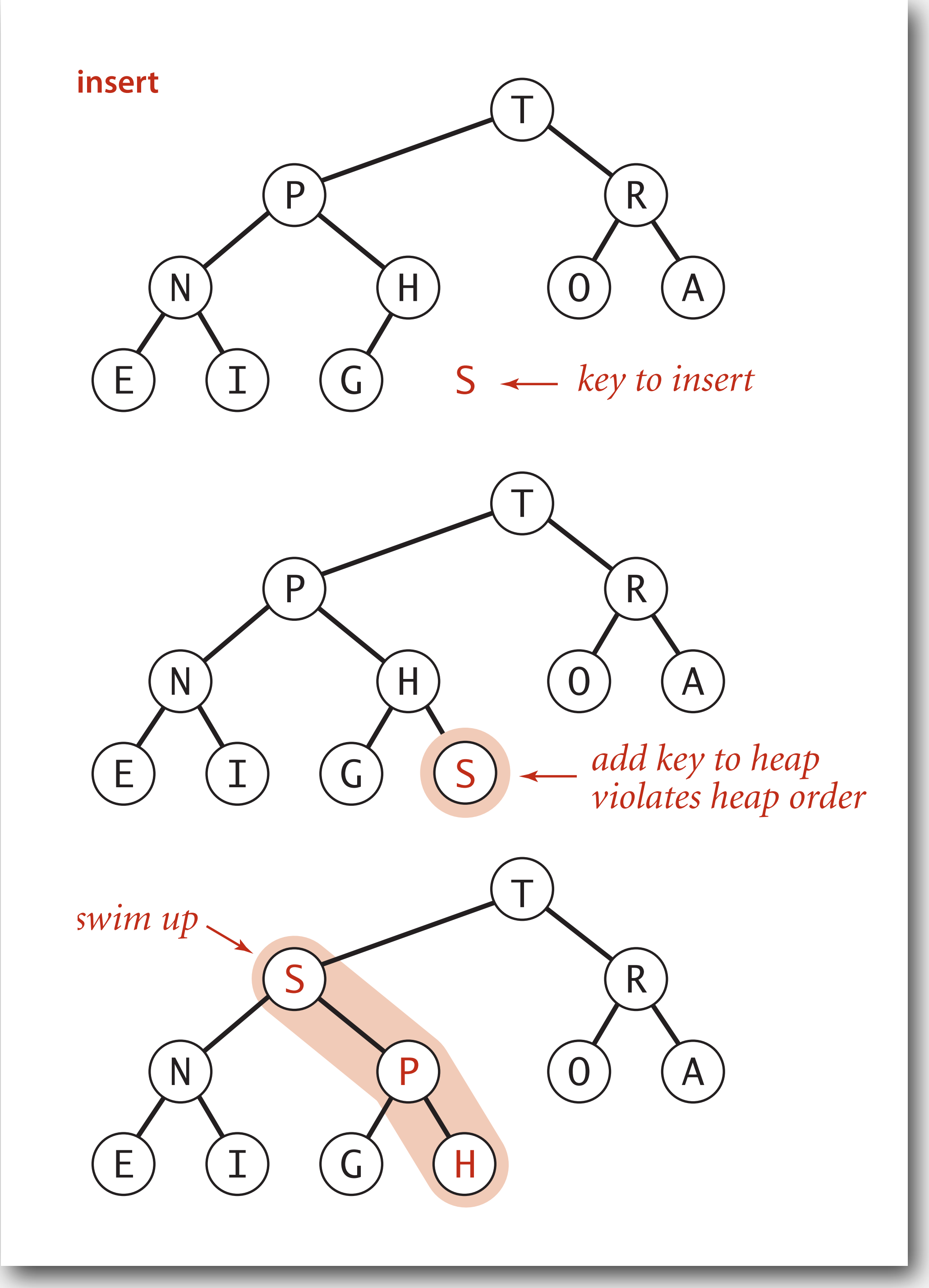
Peter principle. Node promoted to level of incompetence.

Insertion in a heap

Insert. Add node at end, then swim it up.

Cost. At most $1 + \lg N$ compares.

```
public void insert(Key x)
{
   pq[++N] = x;
   swim(N);
}
```



Demotion in a heap

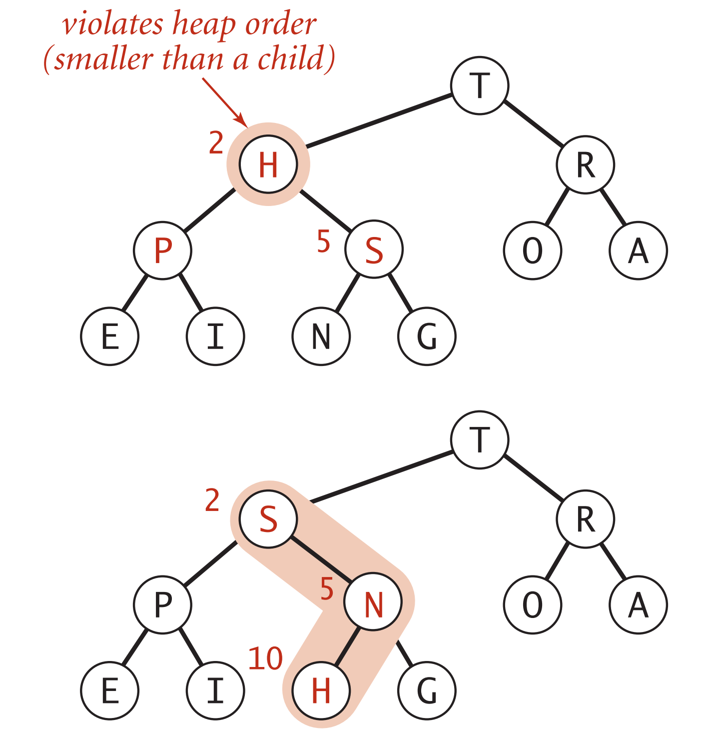
Scenario. Node's key becomes smaller than one (or both) of its children's keys.

To eliminate the violation:

- Exchange key in node with key in larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2k and 2k+1



Top-down reheapify (sink)

Power struggle. Better subordinate promoted.

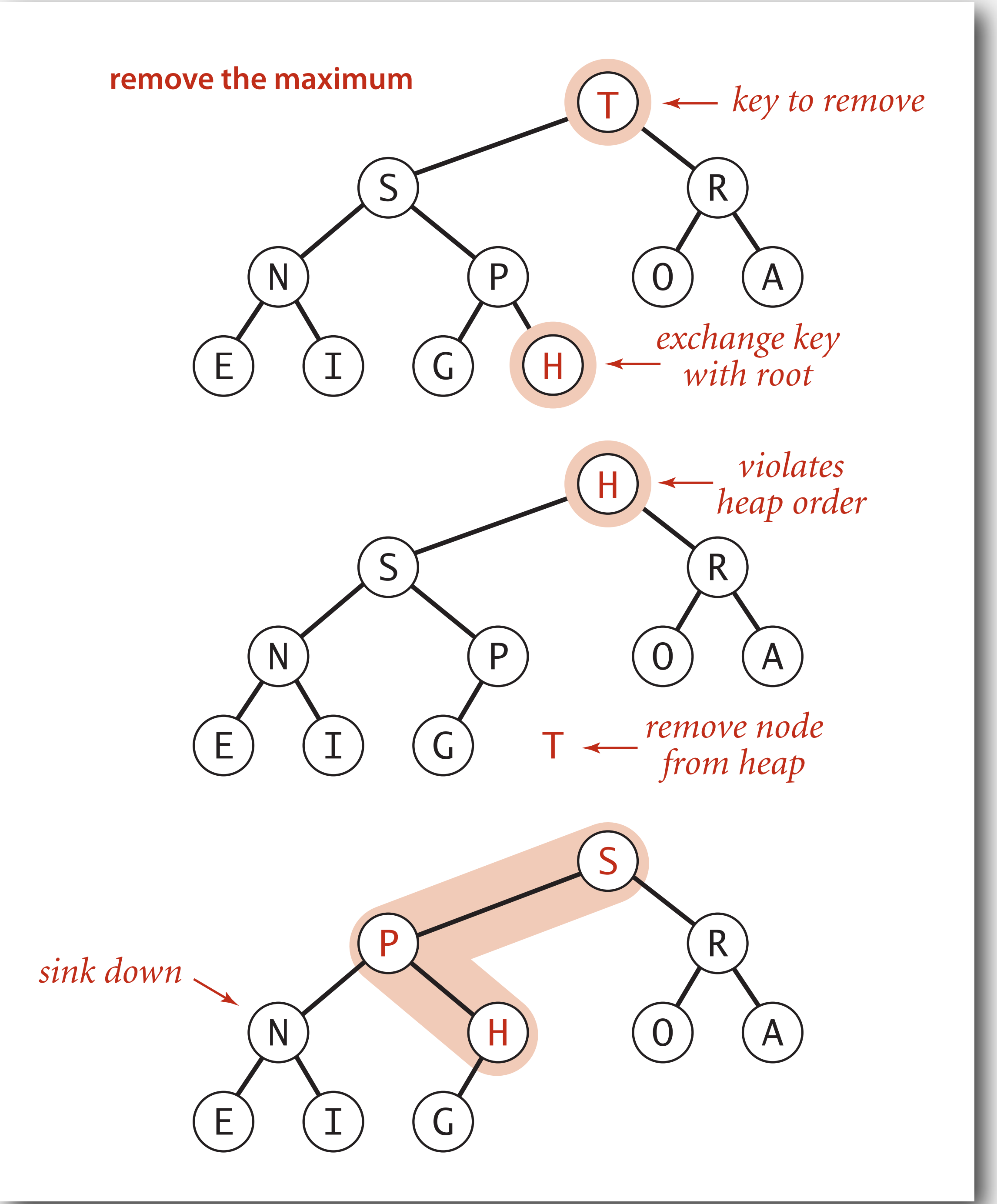
Delete the maximum in a heap

Delete max. Exchange root with node at end, then sink it down.

Cost. At most $2 \lg N$ compares.

```
public Key delMax()
{
   Key max = pq[1];
   exch(1, N--);
   sink(1);
   pq[N+1] = null;
   return max;
}
```

prevent loitering



Binary heap demo

Priority queues implementation cost summary

order-of-growth of running time for priority queue with N items

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	log N	log N	1
d-ary heap	log	d log	1
Fibonacci	1	log N	1
impossible	1	1	1

← why impossible?

† amortized

Binary heap considerations

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

leads to log N amortized time per op

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

easy to implement with **`sink()`** and **`swim()`** [stay tuned]

- API
- elementary implementations
- binary heaps
- **heapsort**
- event-driven simulation

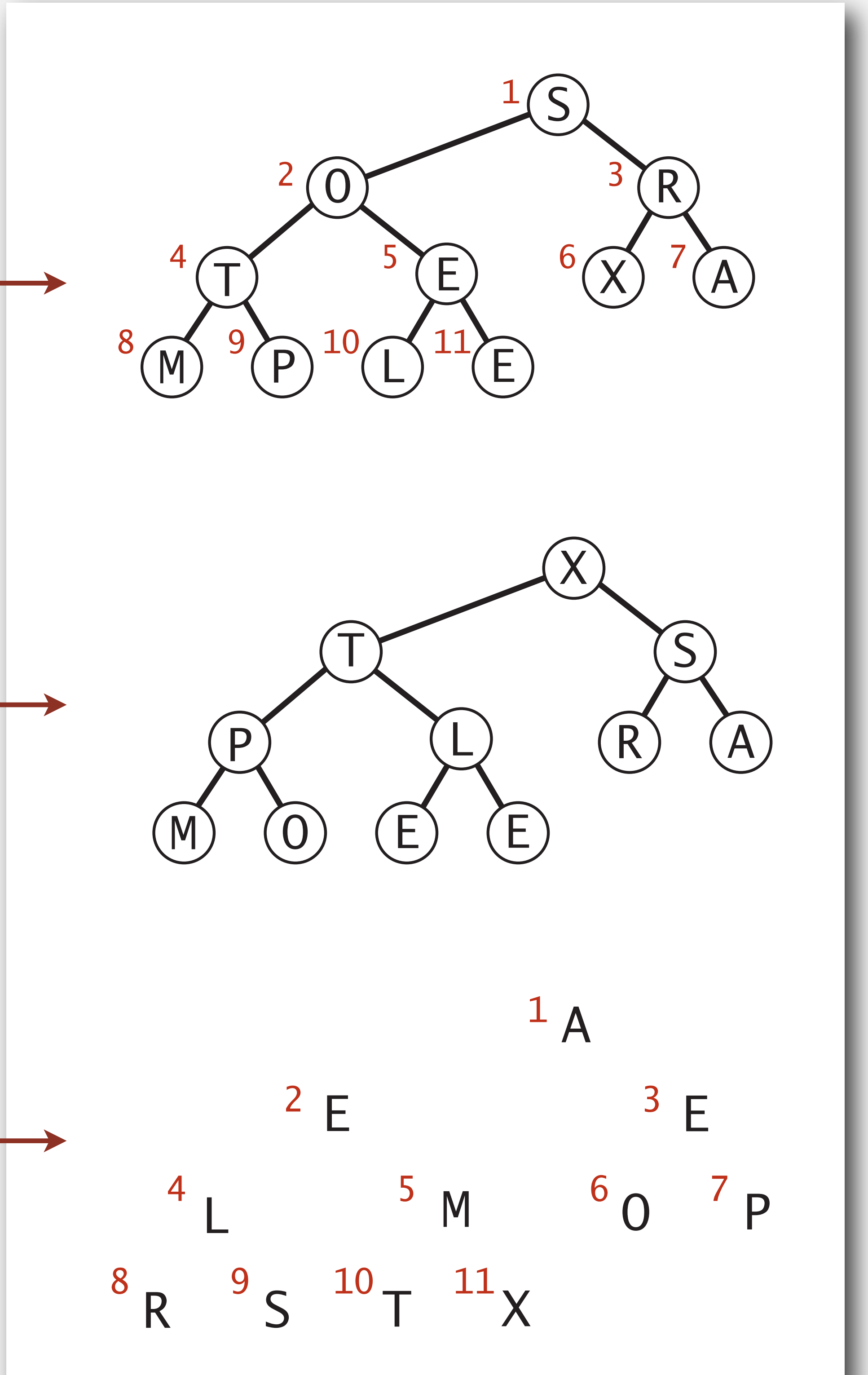
Basic plan for in-place sort.

- Create max-heap with all N keys.
- Repeatedly remove the maximum key.

start with array of keys in arbitrary order

build a max-heap (in place)

sorted result (in place)

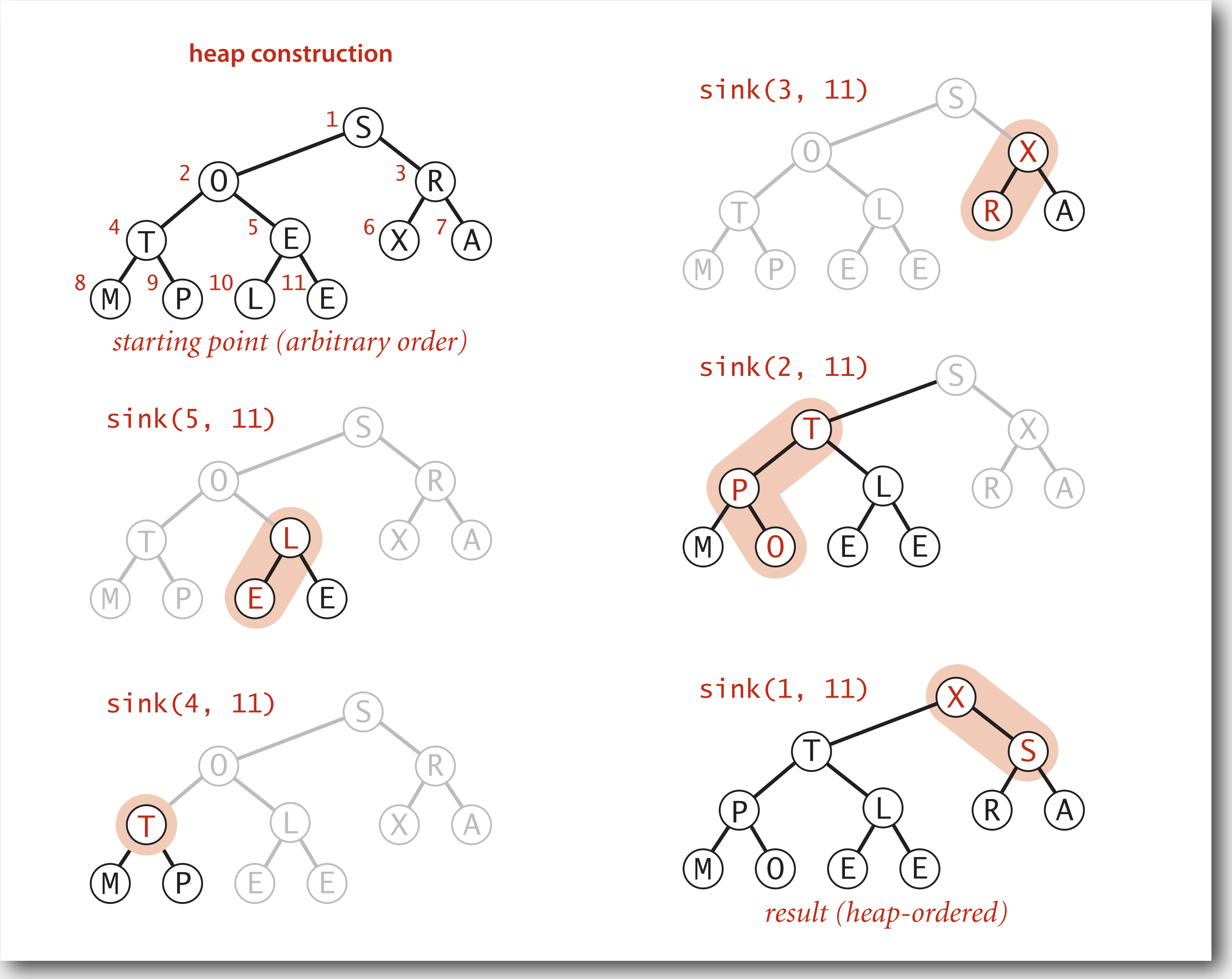


Heapsort demo

Heapsort: heap construction

First pass. Build heap using bottom-up method.

```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```

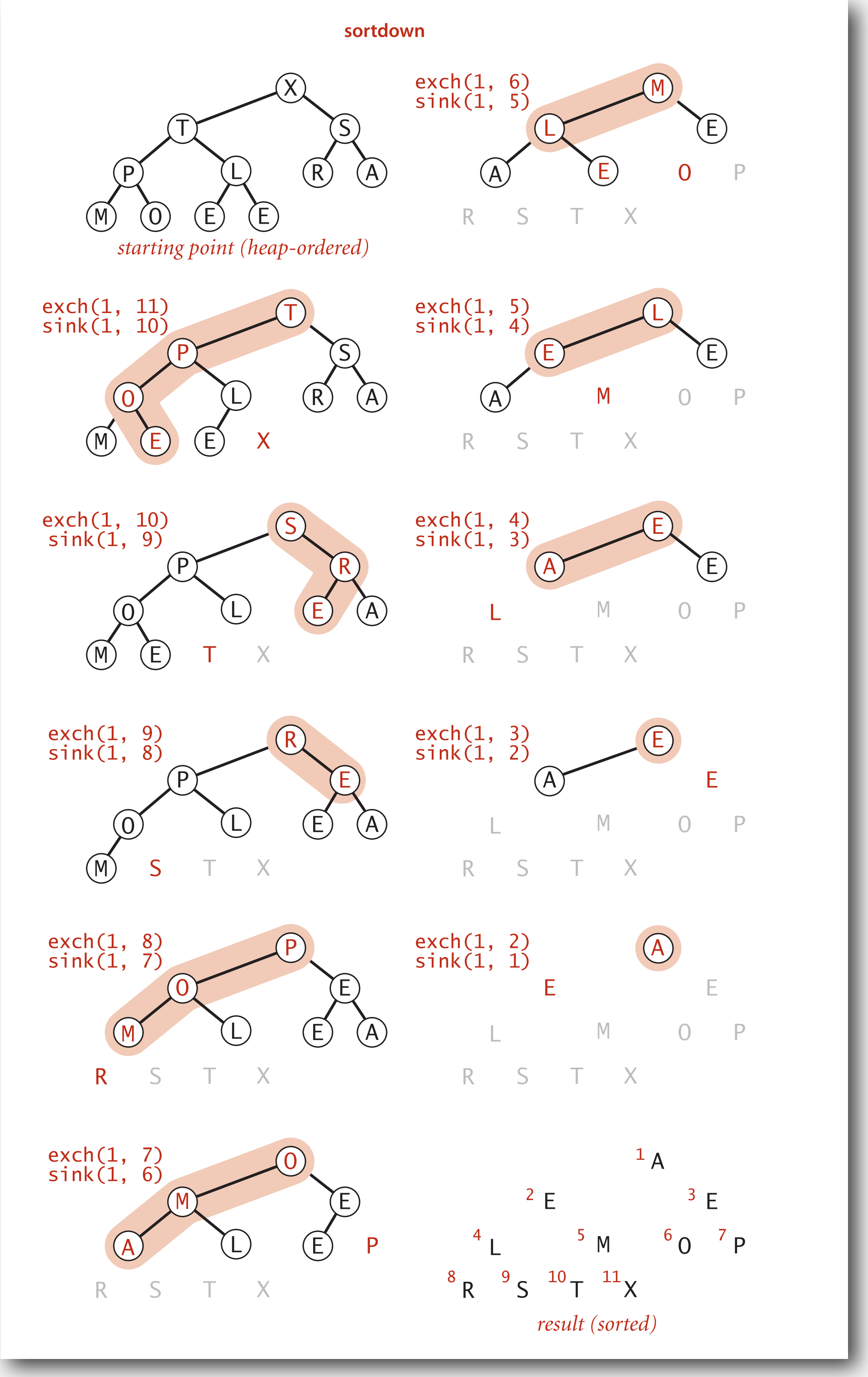


Heapsort: sortdown

Second pass.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



Heapsort: trace

		a[i]											
N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

Heapsort trace (array contents just after each sink)

Heapsort: mathematical analysis

Proposition. Heap construction uses fewer than $2N$ compares and exchanges.

Proposition. Heapsort uses at most $2 N \lg N$ compares and exchanges.

Significance. In-place sorting algorithm with $N \log N$ worst-case.

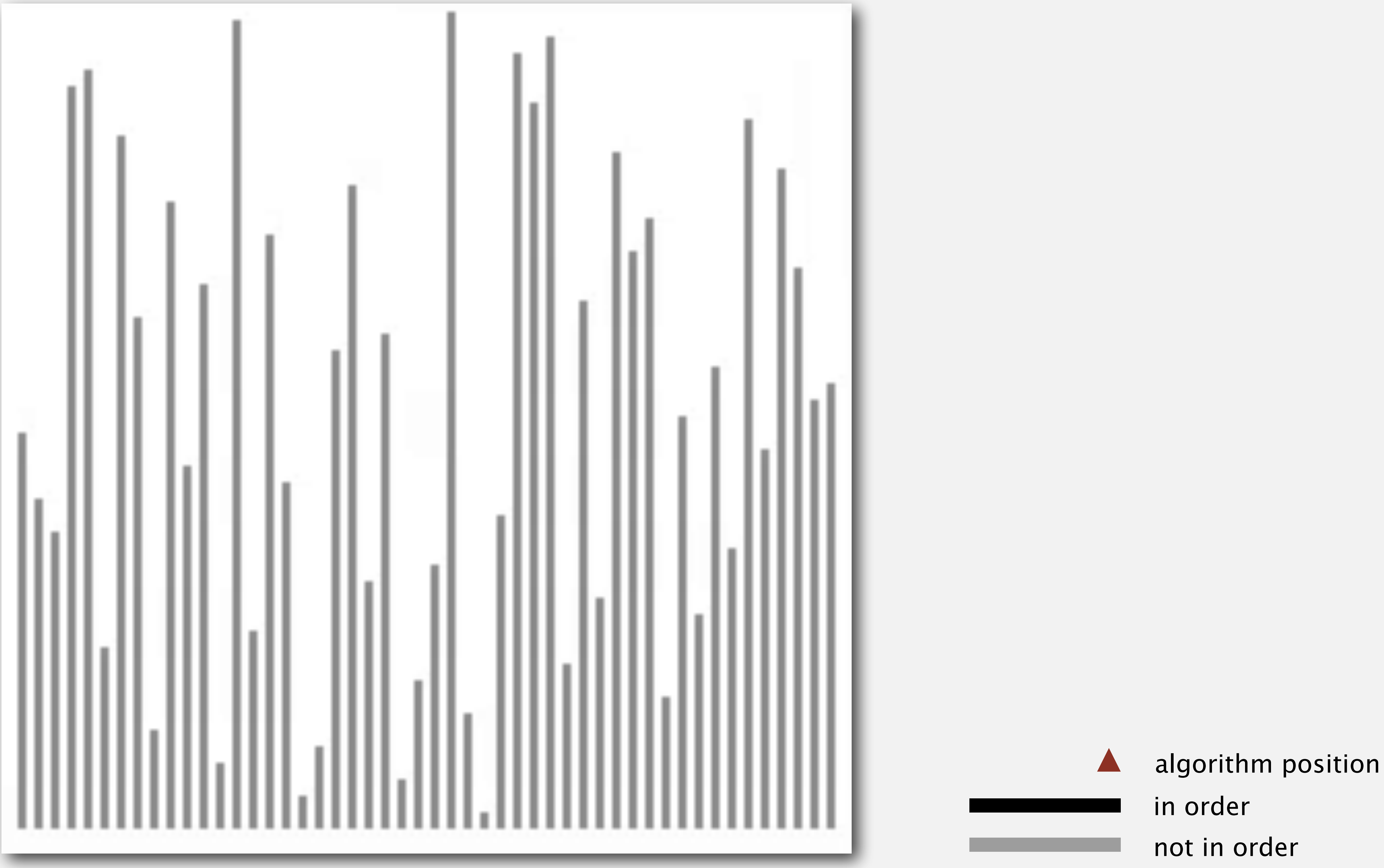
- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case. ← N log N worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache memory.
- Not stable.

Heapsort animation

50 random items



http://www.sorting-algorithms.com/heap-sort

- selection
- duplicate keys
- comparators
- Perspective of sorts...

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Supply chain management.
- Load balancing on a parallel computer.

. . .

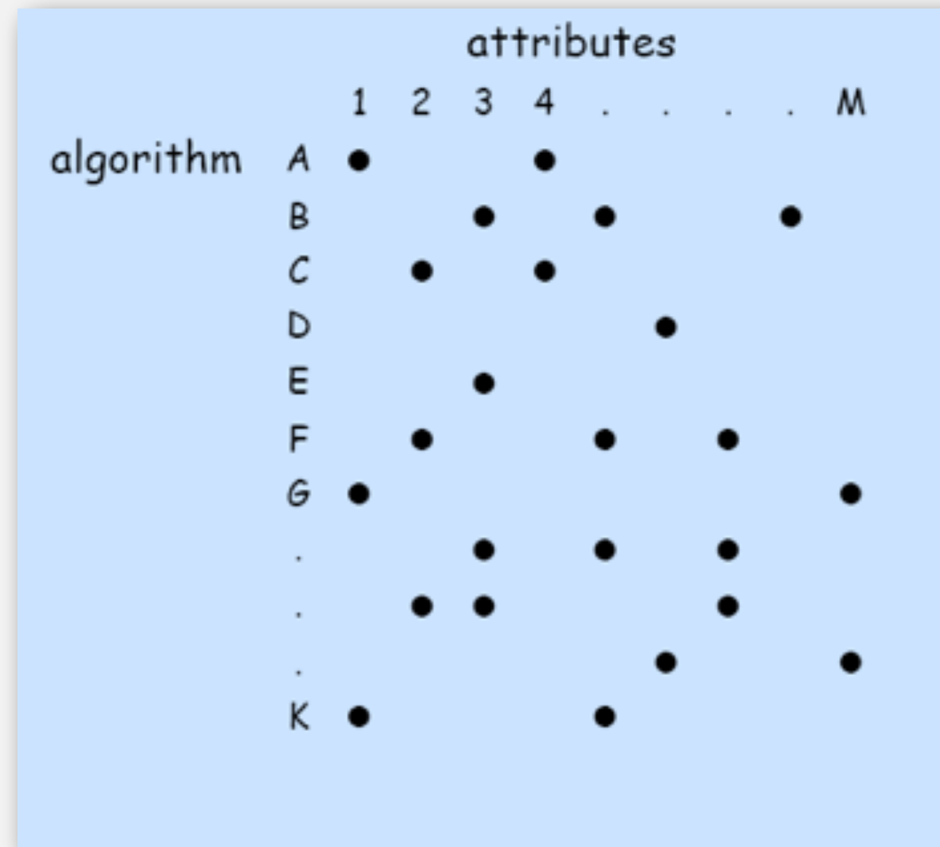
non-obvious applications

Every system needs (and has) a system sort!

System sort: Which algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- Deterministic?
- Keys all distinct?
- Multiple key types?
- Linked list or arrays?
- Large or small items?
- Is your array randomly ordered?
- Need guaranteed performance?



many more combinations of attributes than algorithms

Elementary sort may be method of choice for some combination.

Cannot cover all combinations of attributes.

Q. Is the system sort good enough?

A. Usually.

Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	x		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	x	x	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	x		?	?	N	tight code, subquadratic
merge		x	N lg N	N lg N	N lg N	N log N guarantee, stable
quick	x		$N^2 / 2$	2 N ln N	N lg N	N log N probabilistic guarantee fastest in practice
3-way quick	x		$N^2 / 2$	2 N ln N	N	improves quicksort in presence of duplicate keys
???	x	x	N lg N	N lg N	N lg N	holy sorting grail

Sorting algorithms: summary

	inplace?	stable?	worst	average	best	remarks
selection	x		N	N	N	N exchanges
insertion	x	x	N	N	N	use for small N or partially ordered
shell	x		?	?	N	tight code, subquadratic
quick	x		N	2 N ln N	N lg N	N log N probabilistic guarantee fastest in practice
3-way quick	x		N	2 N ln N	N	improves quicksort in presence of duplicate keys
merge		x	N lg N	N lg N	N lg N	N log N guarantee, stable
heap	x		2 N lg N	2 N lg N	N lg N	N log N guarantee, in-place
???	x	x	N lg N	N lg N	N lg N	holy sorting grail

Which sorting algorithm?

lifo	find	data	data	data	data	hash	data
fifo	fifo	fifo	fifo	exch	fifo	fifo	exch
data	data	find	find	fifo	lifo	data	fifo
type	exch	hash	hash	find	type	link	find
hash	hash	heap	heap	hash	hash	leaf	hash
heap	heap	lifo	lifo	heap	heap	heap	heap
sort	less	link	link	leaf	link	exch	leaf
link	left	list	list	left	sort	node	left
list	leaf	push	push	less	find	lifo	less
push	lifo	root	root	lifo	list	left	lifo
find	push	sort	sort	link	push	find	link
root	root	type	type	list	root	path	list
leaf	list	leaf	leaf	sort	leaf	list	next
tree	tree	left	tree	tree	null	next	node
null	null	node	null	null	path	less	null
path	path	null	path	path	tree	root	path
node	node	path	node	node	exch	sink	push
left	link	tree	left	type	left	swim	root
less	sort	exch	less	root	less	null	sink
exch	type	less	exch	push	node	sort	sort
sink	sink	next	sink	sink	next	type	swap
swim	swim	sink	swim	swim	sink	tree	swim
next	next	swap	next	next	swap	push	tree
swap	swap	swim	swap	swap	swim	swap	type
original	?	?	?	?	?	?	sorted

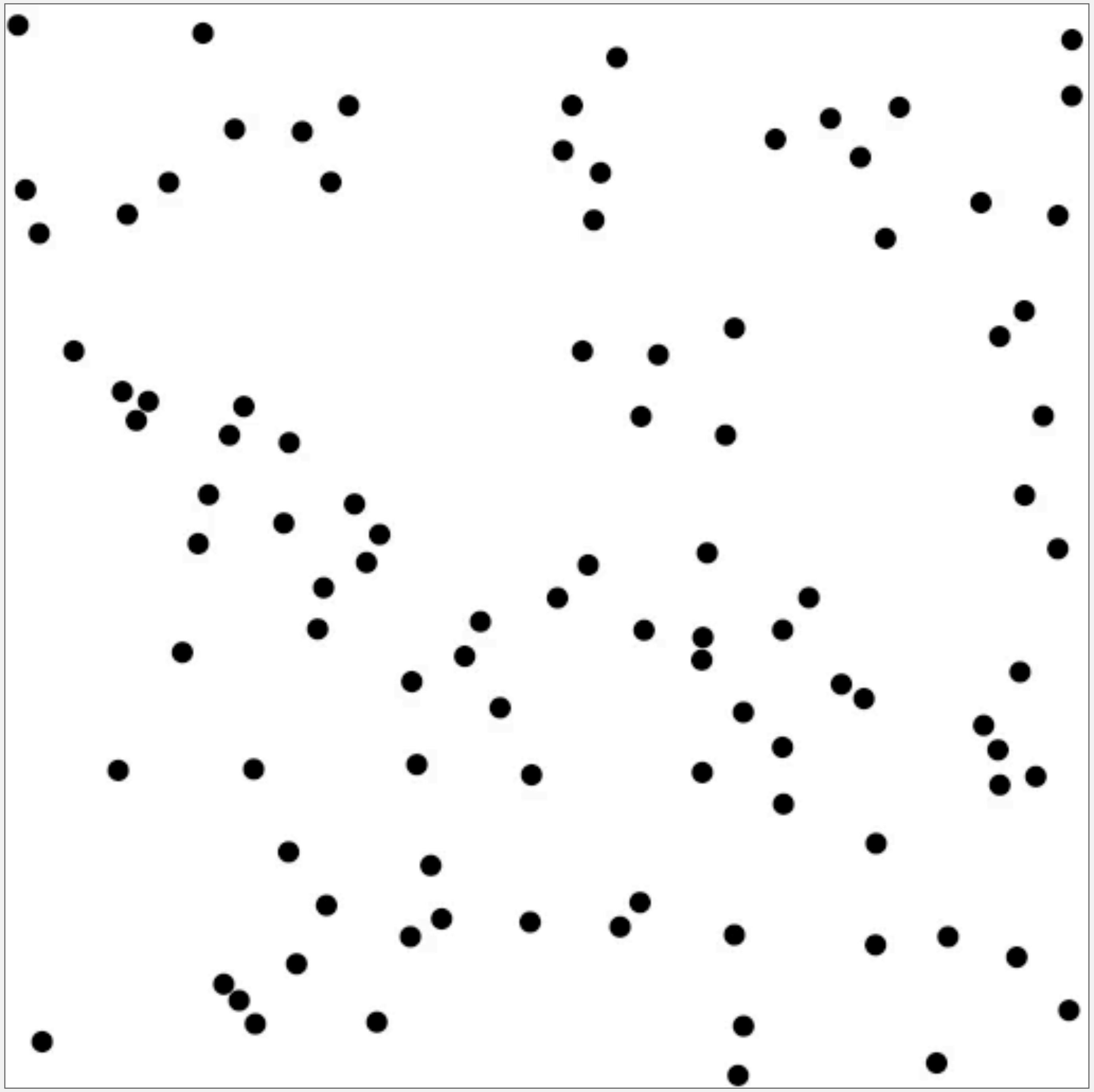
Which sorting algorithm?

original	quicksort	mergesort	insertion	selection	merge BU	shellsort	sorted
lifo	find	data	data	data	data	hash	data
fifo	fifo	fifo	fifo	exch	fifo	fifo	exch
data	data	find	find	fifo	lifo	data	fifo
type	exch	hash	hash	find	type	link	find
hash	hash	heap	heap	hash	hash	leaf	hash
heap	heap	lifo	lifo	heap	heap	heap	heap
sort	less	link	link	leaf	link	exch	leaf
link	left	list	list	left	sort	node	left
list	leaf	push	push	less	find	lifo	less
push	lifo	root	root	lifo	list	left	lifo
find	push	sort	sort	link	push	find	link
root	root	type	type	list	root	path	list
leaf	list	leaf	leaf	sort	leaf	list	next
tree	tree	left	tree	tree	null	next	node
null	null	node	null	null	path	less	null
path	path	null	path	path	tree	root	path
node	node	path	node	node	exch	sink	push
left	link	tree	left	type	left	swim	root
less	sort	exch	less	root	less	null	sink
exch	type	less	exch	push	node	sort	sort
sink	sink	next	sink	sink	next	type	swap
swim	swim	sink	swim	swim	sink	tree	swim
next	next	swap	next	next	swap	push	tree
swap	swap	swim	swap	swap	swim	swap	type

- API
- elementary implementations
- binary heaps
- heapsort
- **event-driven simulation**

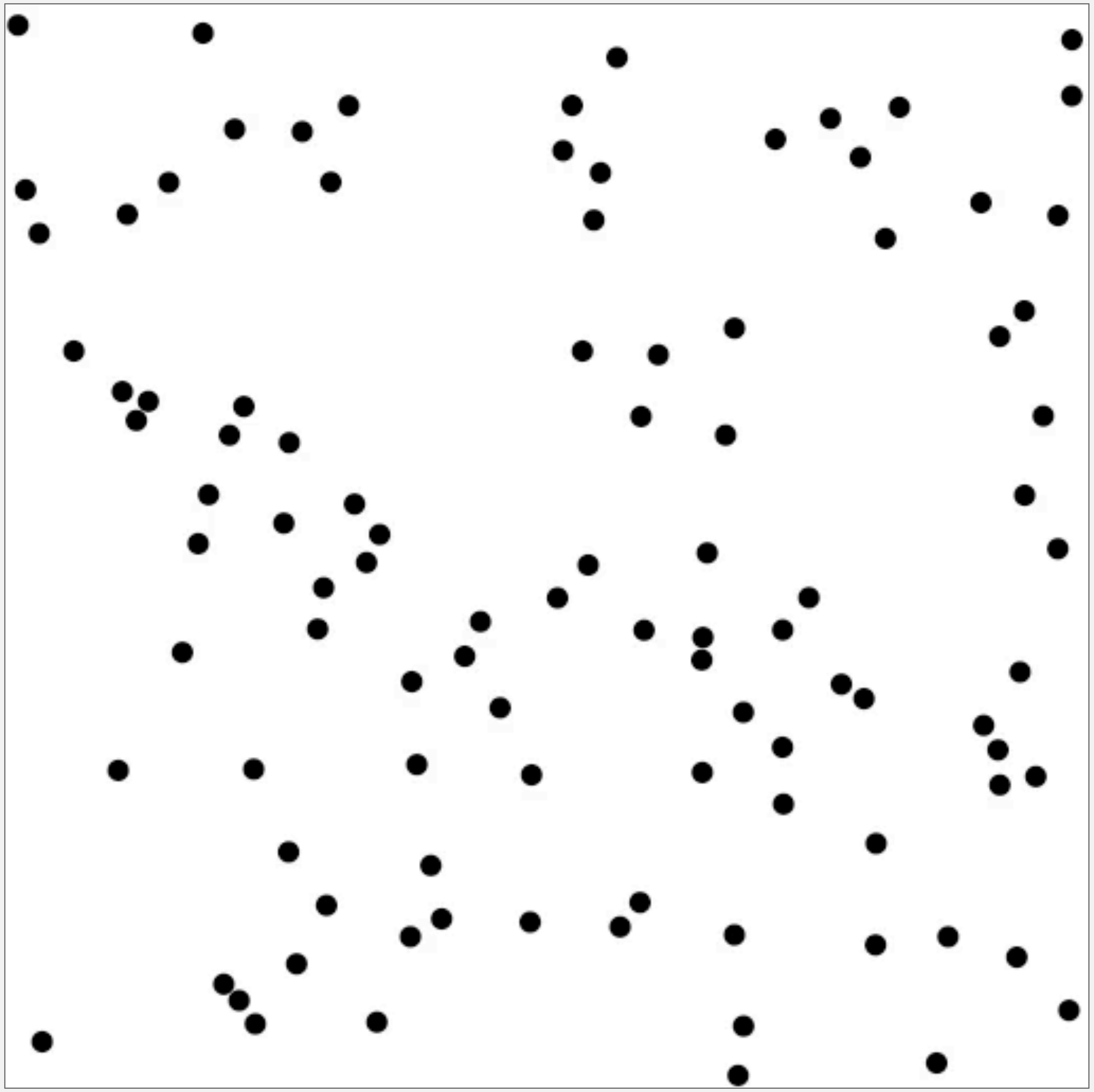
Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.



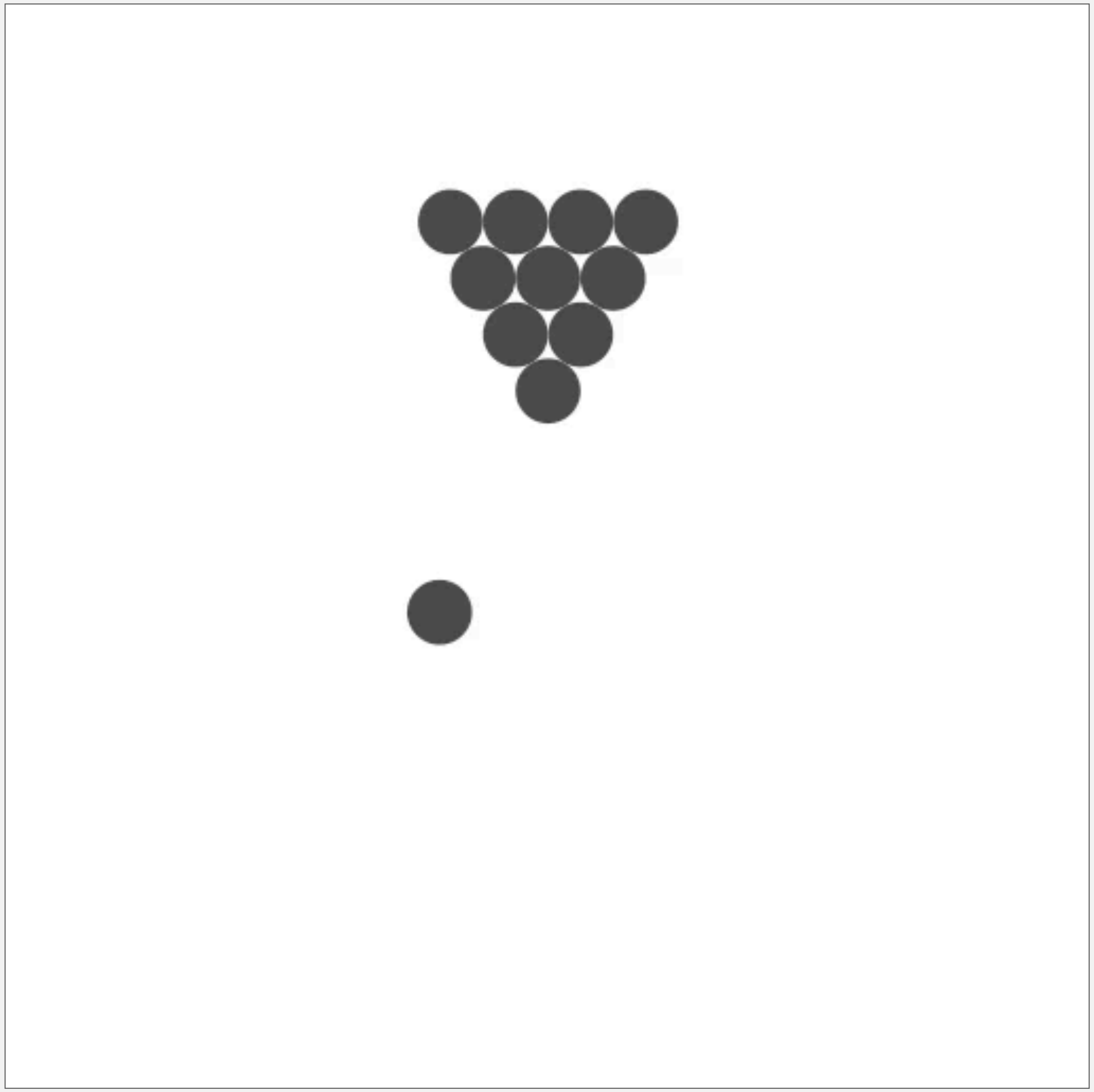
Simulation example 1

```
% java CollisionSystem 100
```



Simulation example 2

```
% java CollisionSystem < billiards.txt
```



Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.

Hard disc model.

- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces.

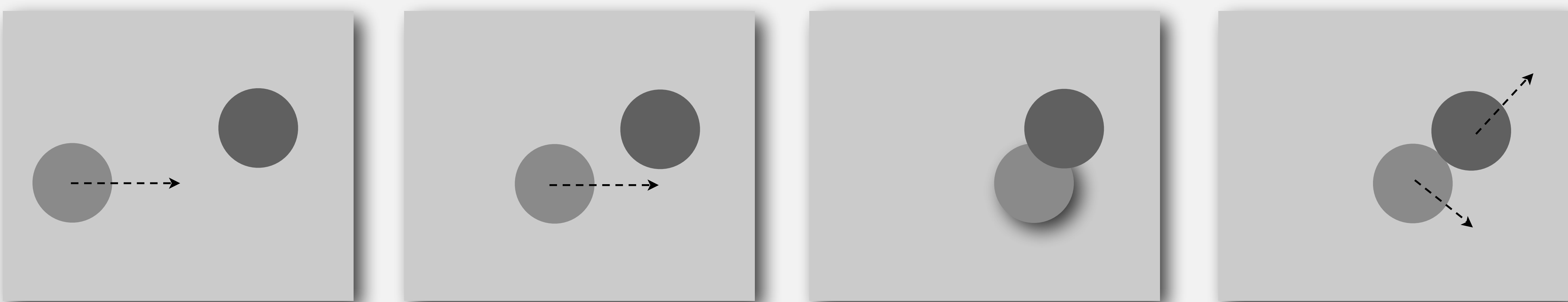
temperature, pressure, diffusion constant

motion of individual atoms and molecules

Significance. Relates macroscopic observables to microscopic dynamics.

- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

- Discretize time in quanta of size dt.
- Update the position of each particle after every dt units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.



t

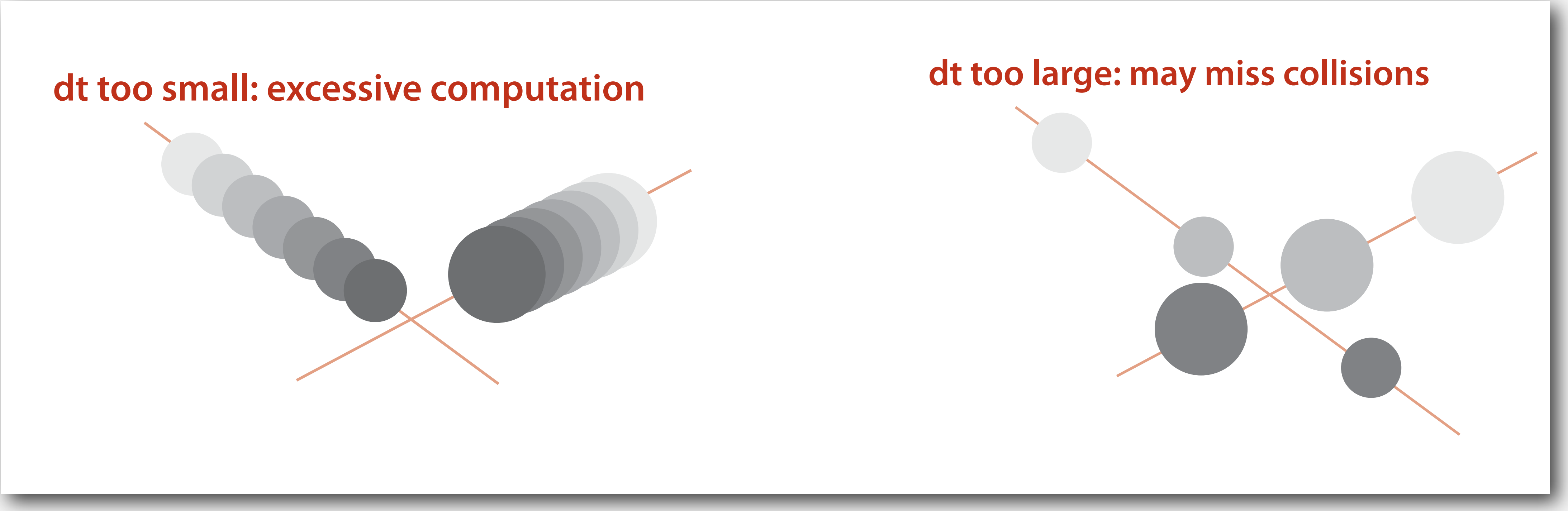
t + dt

t + 2 dt
(collision detected)

t + Δt
(roll back clock)

Main drawbacks.

- ~ $N^2 / 2$ overlap checks per time quantum.
- Simulation is too slow if dt is very small.
- May miss collisions if dt is too large.
  (if colliding particles fail to overlap when we are looking)



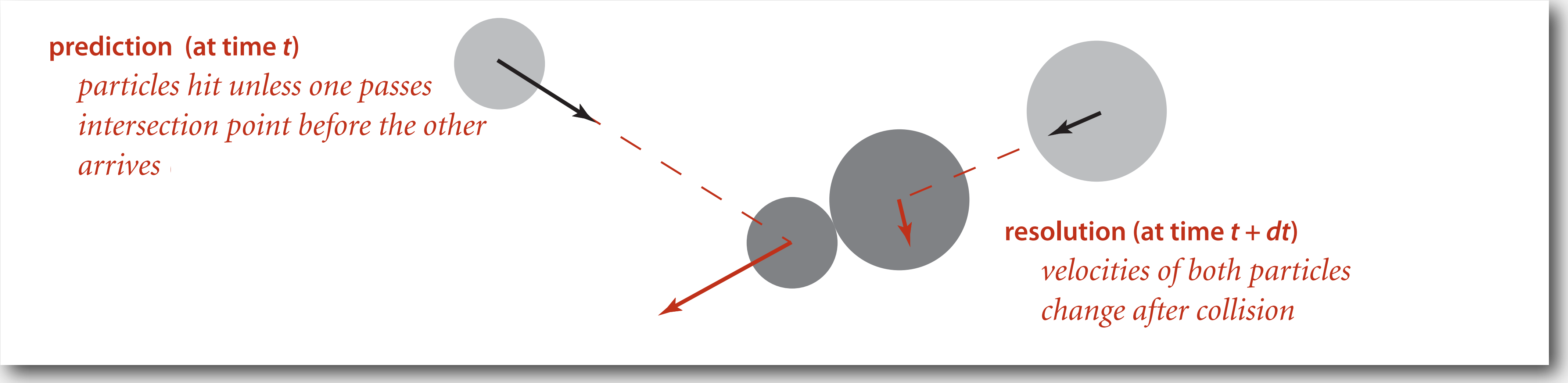
Event-driven simulation

Change state only when something happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain PQ of collision events, prioritized by time.
- Remove the min = get next collision.

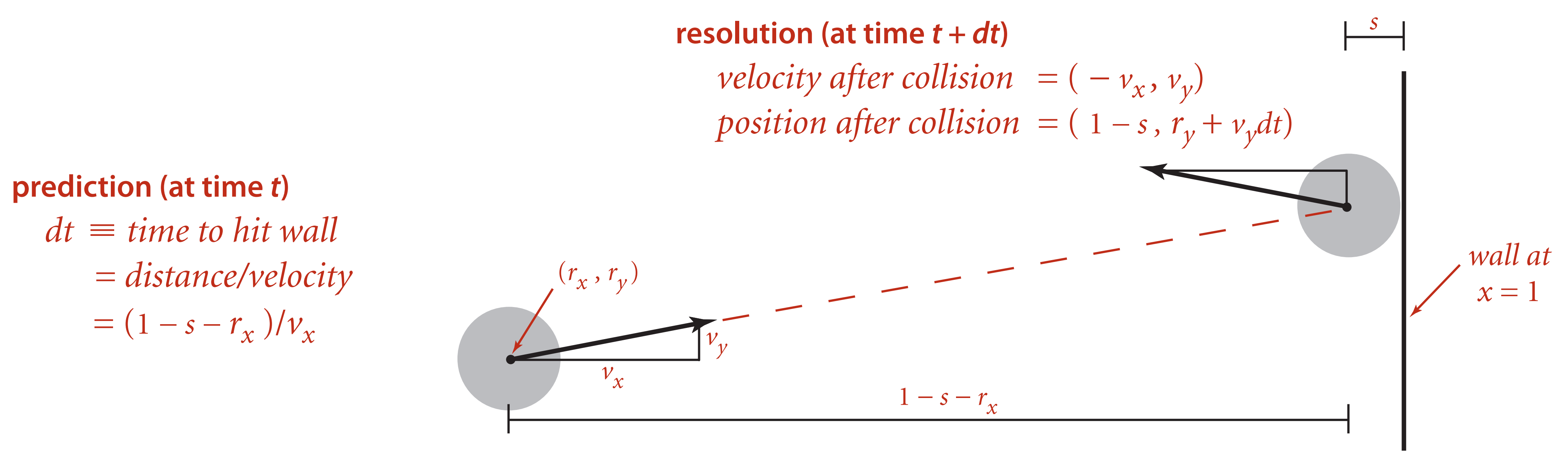
Collision prediction. Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

Collision resolution. If collision occurs, update colliding particle(s) according to laws of elastic collisions.



Collision prediction and resolution.

- Particle of radius s at position (rx, ry).
- Particle moving in unit box with velocity (vx, vy).
- Will it collide with a vertical wall? If so, when?

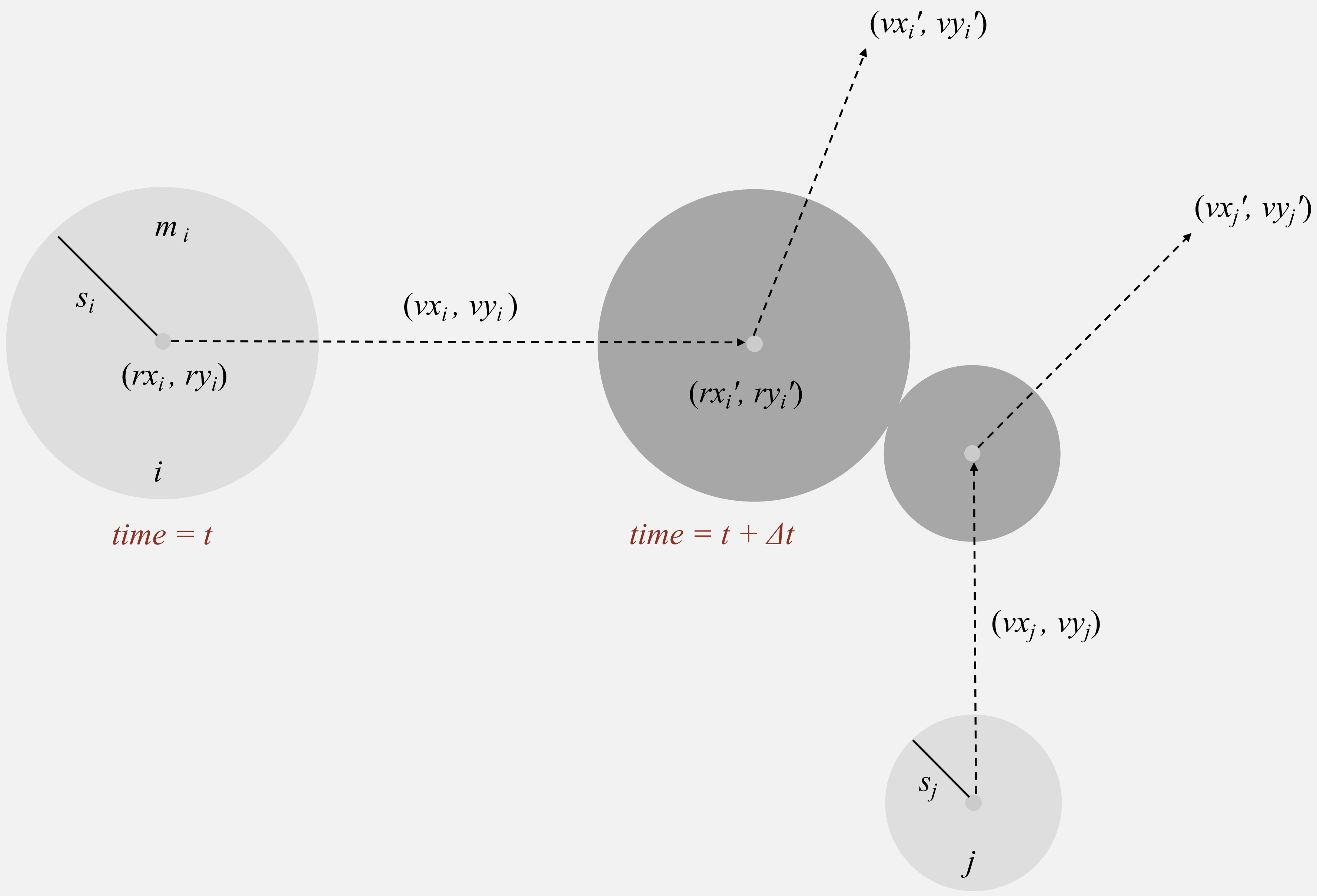


Predicting and resolving a particle-wall collision

Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \geq 0 \\ \infty & \text{if } d < 0 \\ -\dfrac{\Delta v \cdot \Delta r + \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d = (\Delta v \cdot \Delta r)^2 - (\Delta v \cdot \Delta v)\ (\Delta r \cdot \Delta r - \sigma^2) \qquad \sigma = \sigma_i + \sigma_j$$

$$\Delta v = (\Delta vx,\ \Delta vy) = (vx_i - vx_j,\ vy_i - vy_j)$$
$$\Delta r = (\Delta rx,\ \Delta ry) = (rx_i - rx_j,\ ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$
$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$
$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

Important note: This is high-school physics, so we won't be testing you on it!

Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$
\begin{aligned}
vx_i' &= vx_i + Jx \,/\, m_i \\
vy_i' &= vy_i + Jy \,/\, m_i \\
vx_j' &= vx_j - Jx \,/\, m_j \\
vy_j' &= vy_j - Jy \,/\, m_j
\end{aligned}
$$

Newton's second law
(momentum form)

$$
Jx = \frac{J\,\Delta rx}{\sigma}, \quad Jy = \frac{J\,\Delta ry}{\sigma}, \quad J = \frac{2\,m_i\,m_j\,(\Delta v \cdot \Delta r)}{\sigma(m_i + m_j)}
$$

impulse due to normal force
(conservation of energy, conservation of momentum)

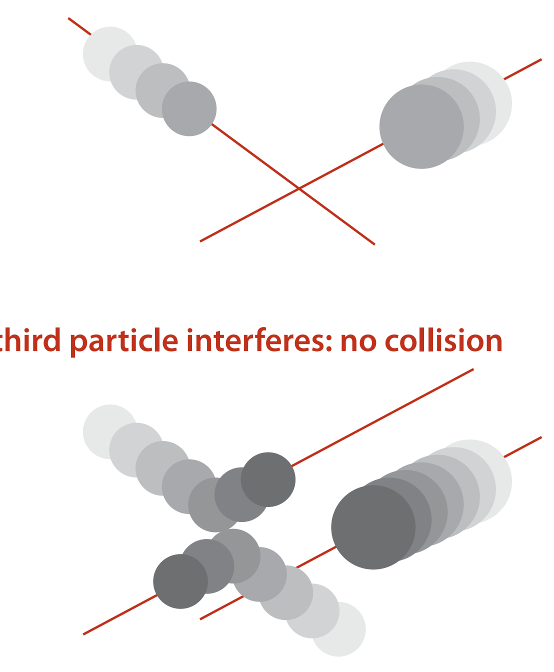
Important note: This is high-school physics, so we won't be testing you on it!

Collision system: event-driven simulation main loop

Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

"potential" since collision may not happen if some other collision intervenes



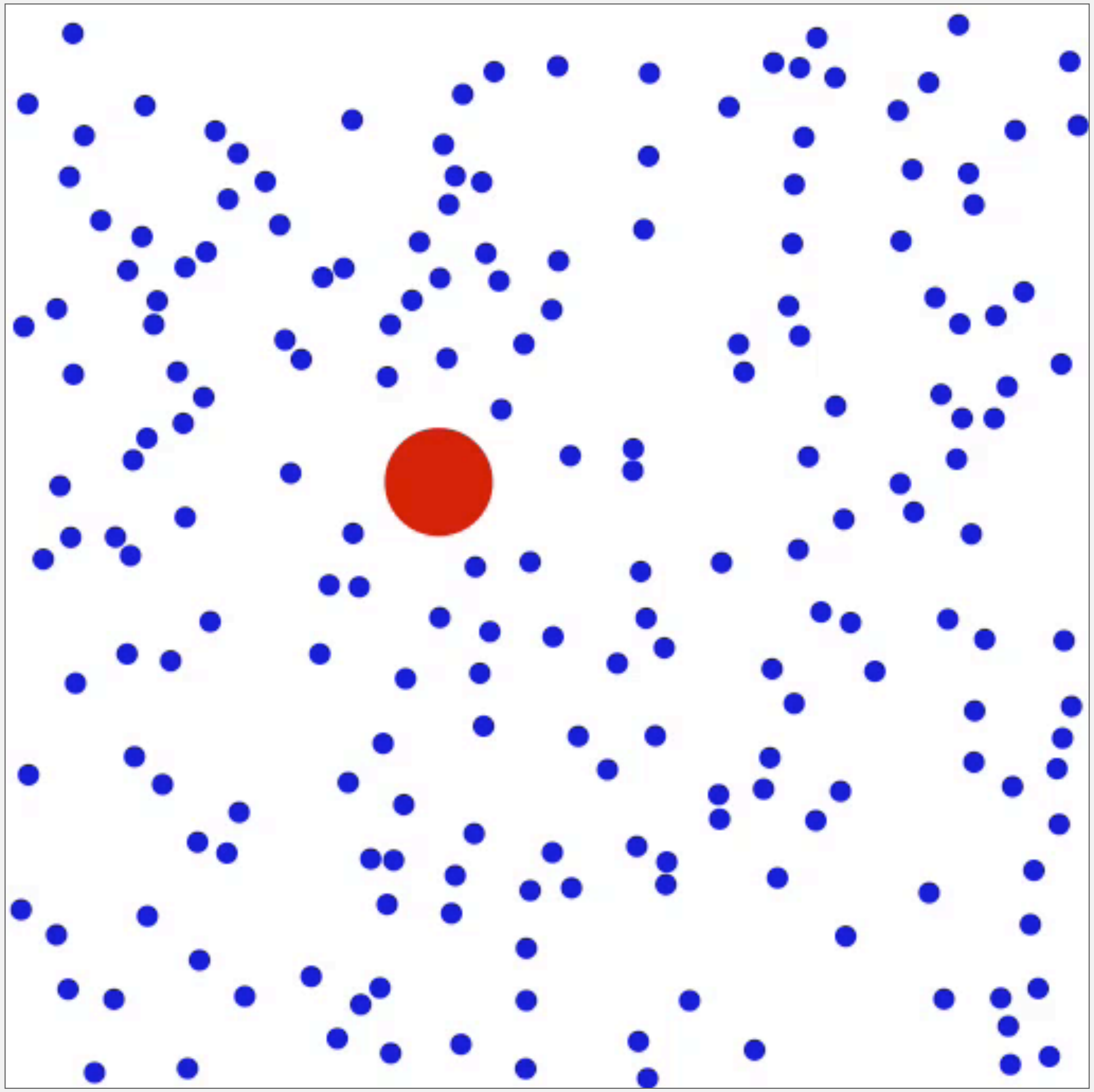
An invalidated event

Main loop.

- Delete the impending event from PQ (min priority = t).
- If the event has been invalidated, ignore it.
- Advance all particles to time t, on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

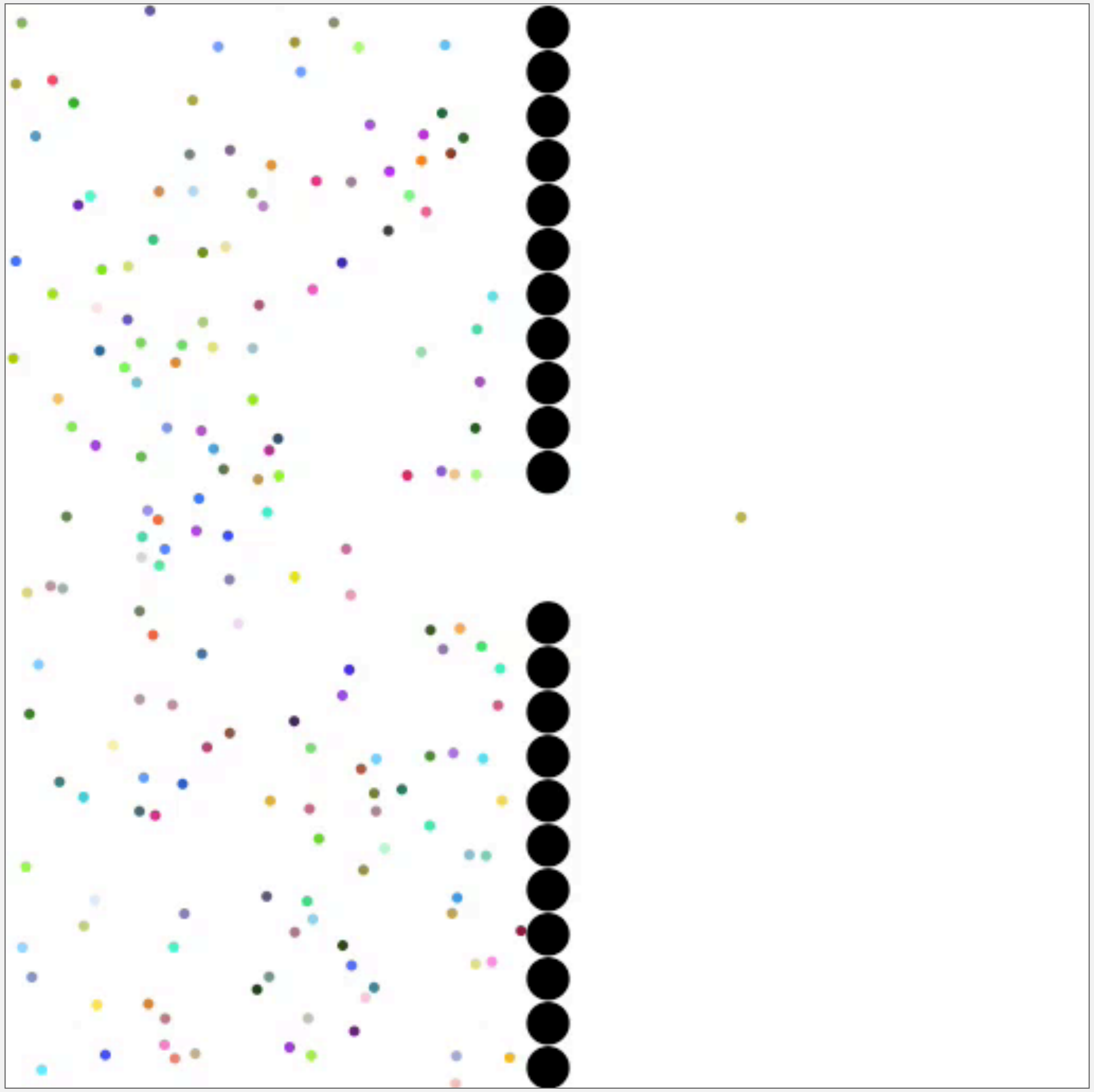
Simulation example 3

```
% java CollisionSystem < brownian.txt
```



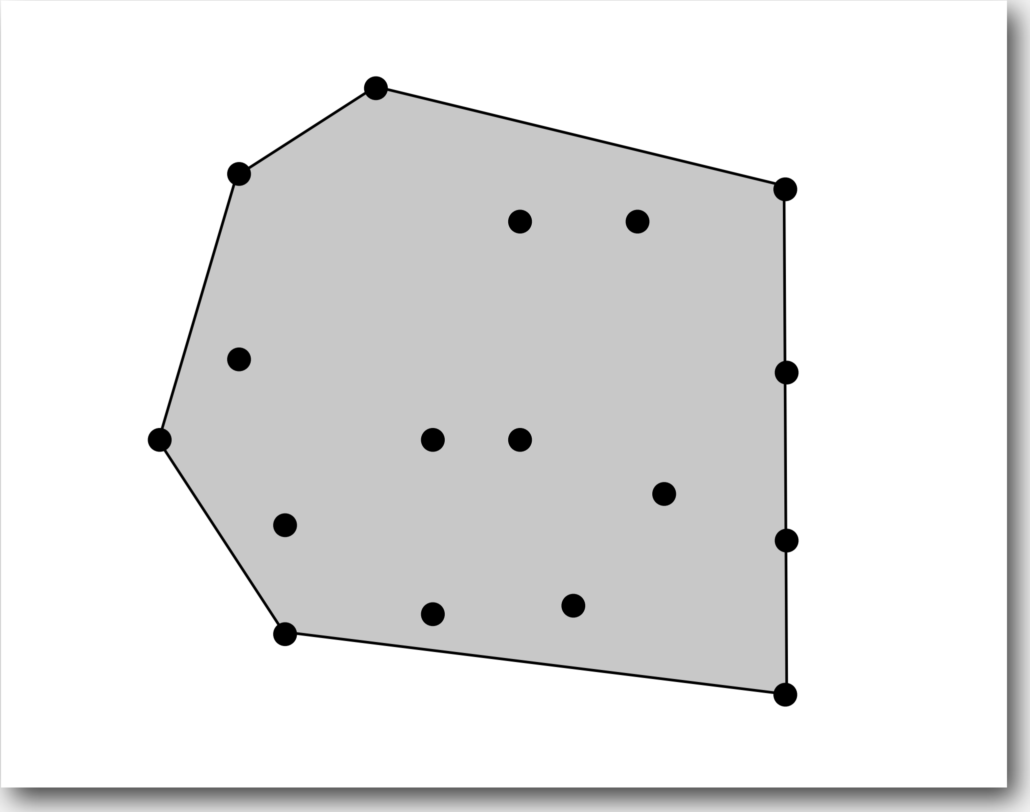
Simulation example 4

```
% java CollisionSystem < diffusion.txt
```



Convex hull

The convex hull of a set of N points is the smallest perimeter fence enclosing the points.

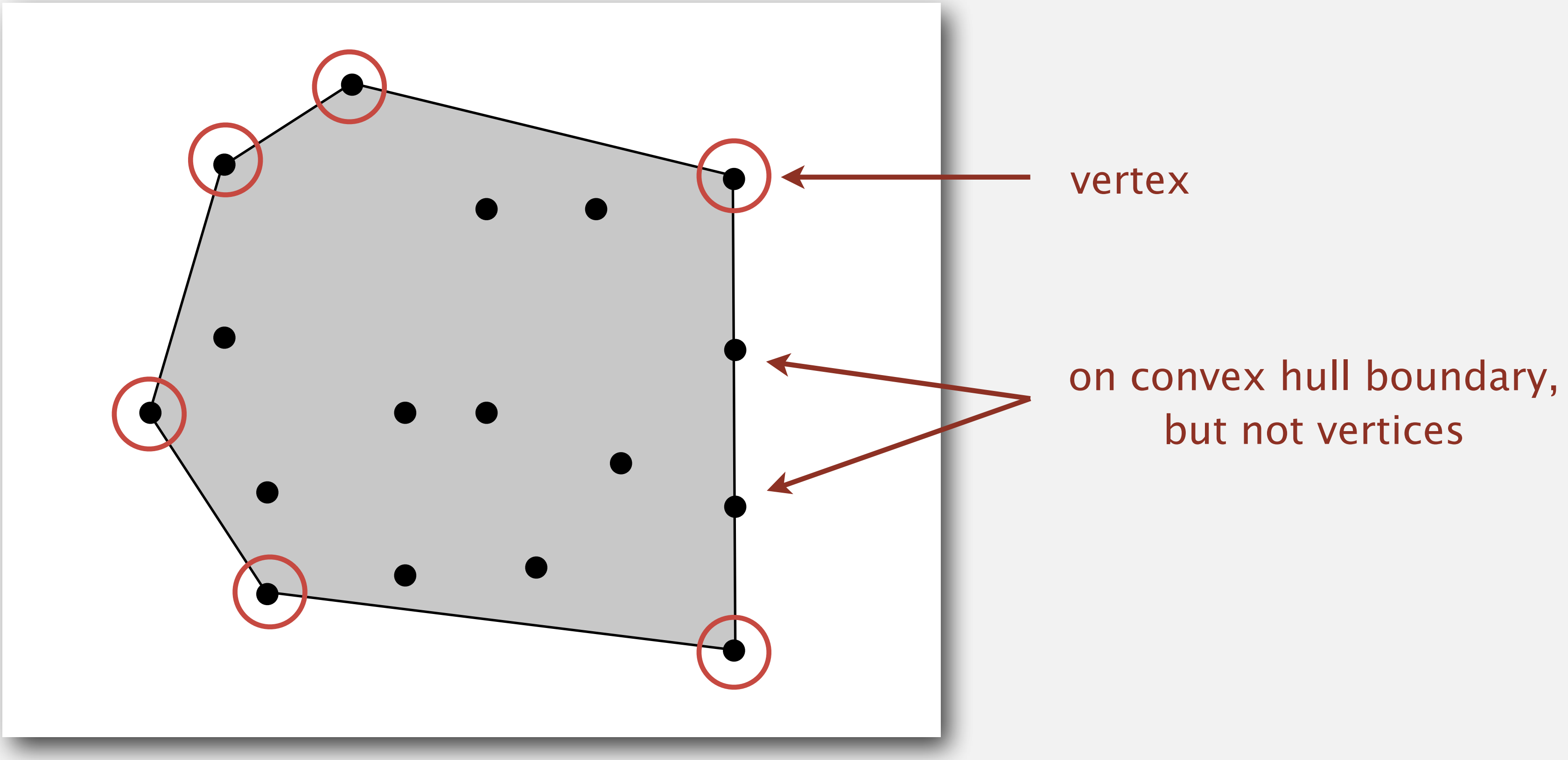


Equivalent definitions.

- Smallest convex set containing all the points.
- Smallest area convex polygon enclosing the points.
- Convex polygon enclosing the points, whose vertices are points in the set.

Convex hull

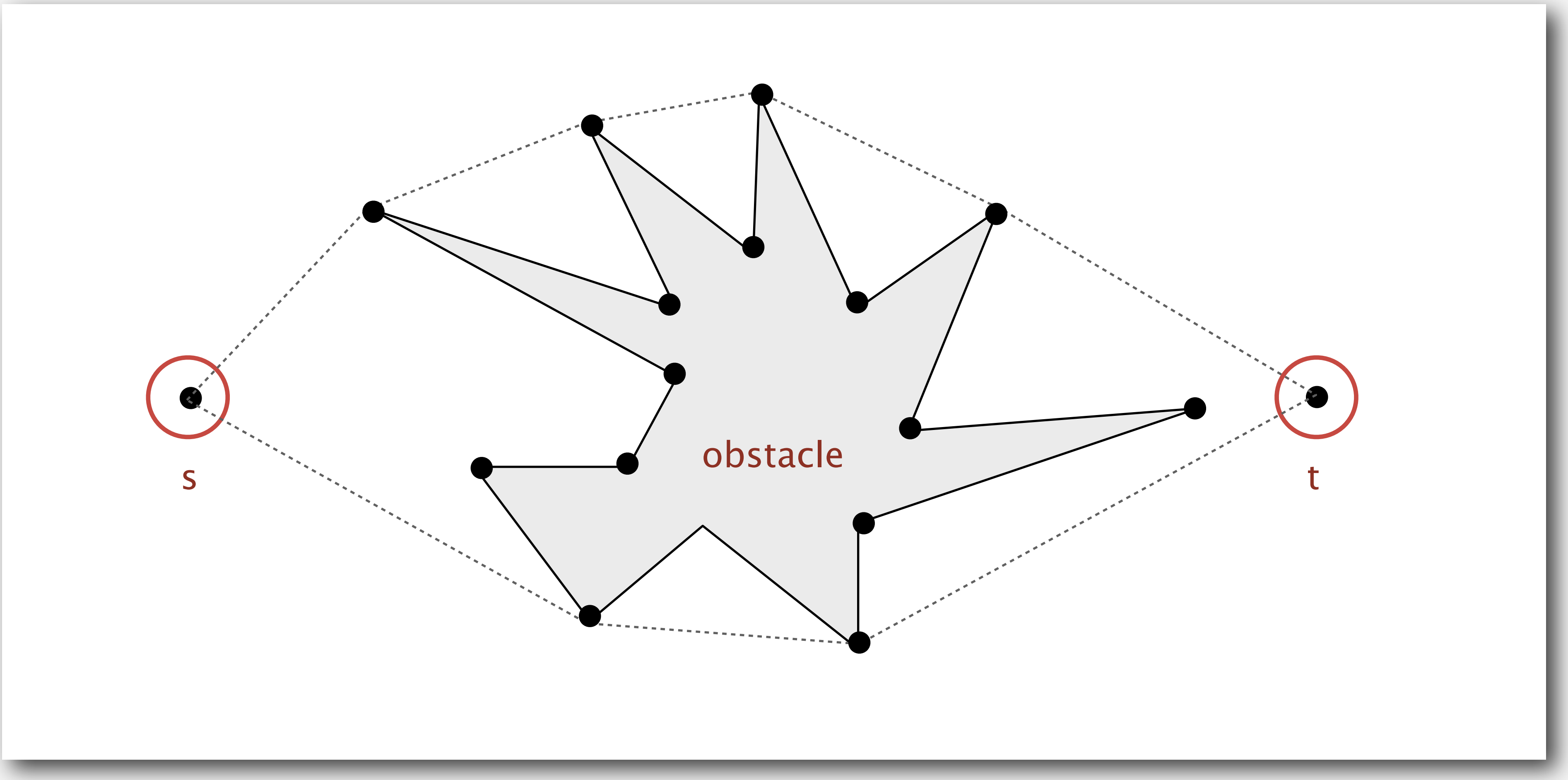
The convex hull of a set of N points is the smallest perimeter fence enclosing the points.



Convex hull output. Sequence of vertices in counterclockwise order.

Convex hull application: motion planning

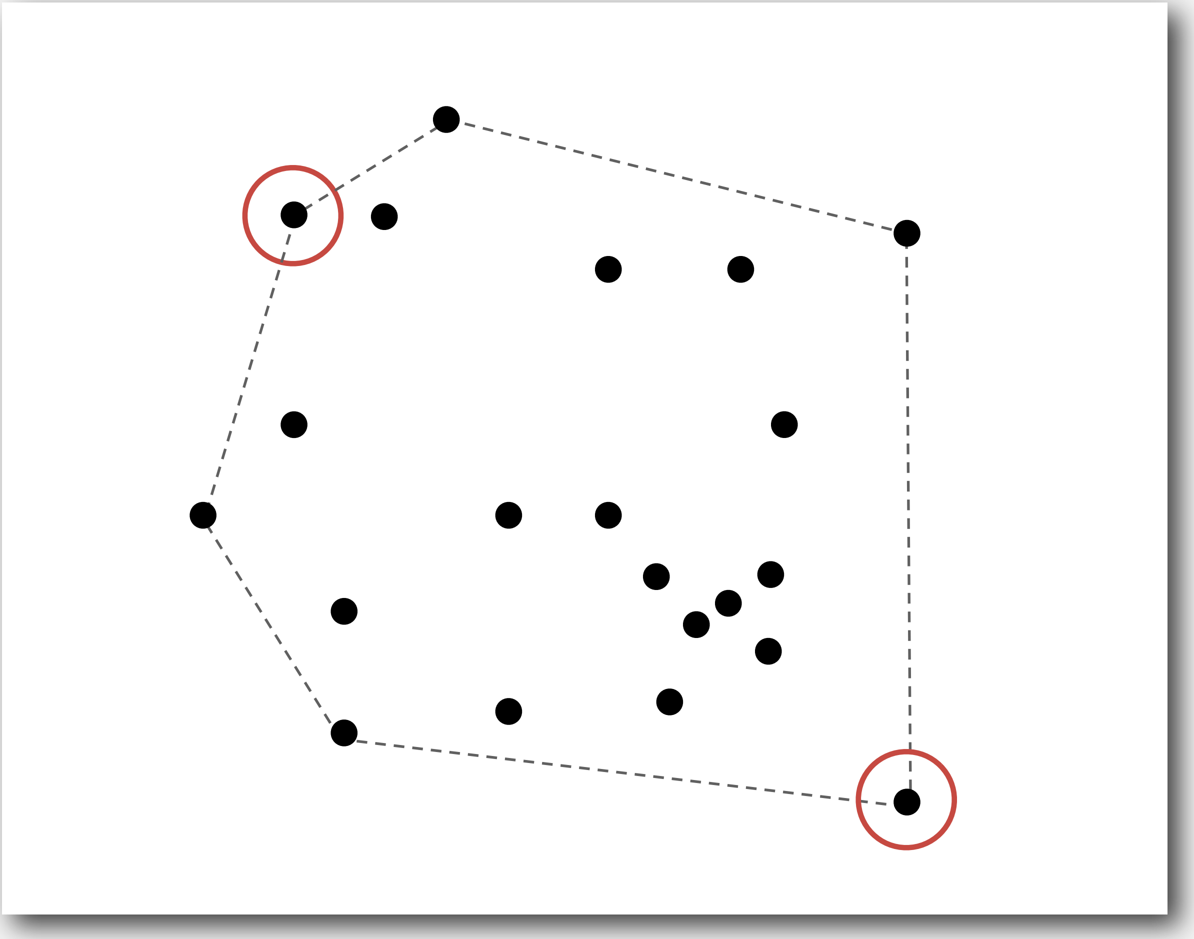
Robot motion planning. Find shortest path in the plane from s to t that avoids a polygonal obstacle.



Fact. Shortest path is either straight line from s to t or it is one of two polygonal chains of convex hull.

Convex hull application: farthest pair

Farthest pair problem. Given N points in the plane, find a pair of points with the largest Euclidean distance between them.

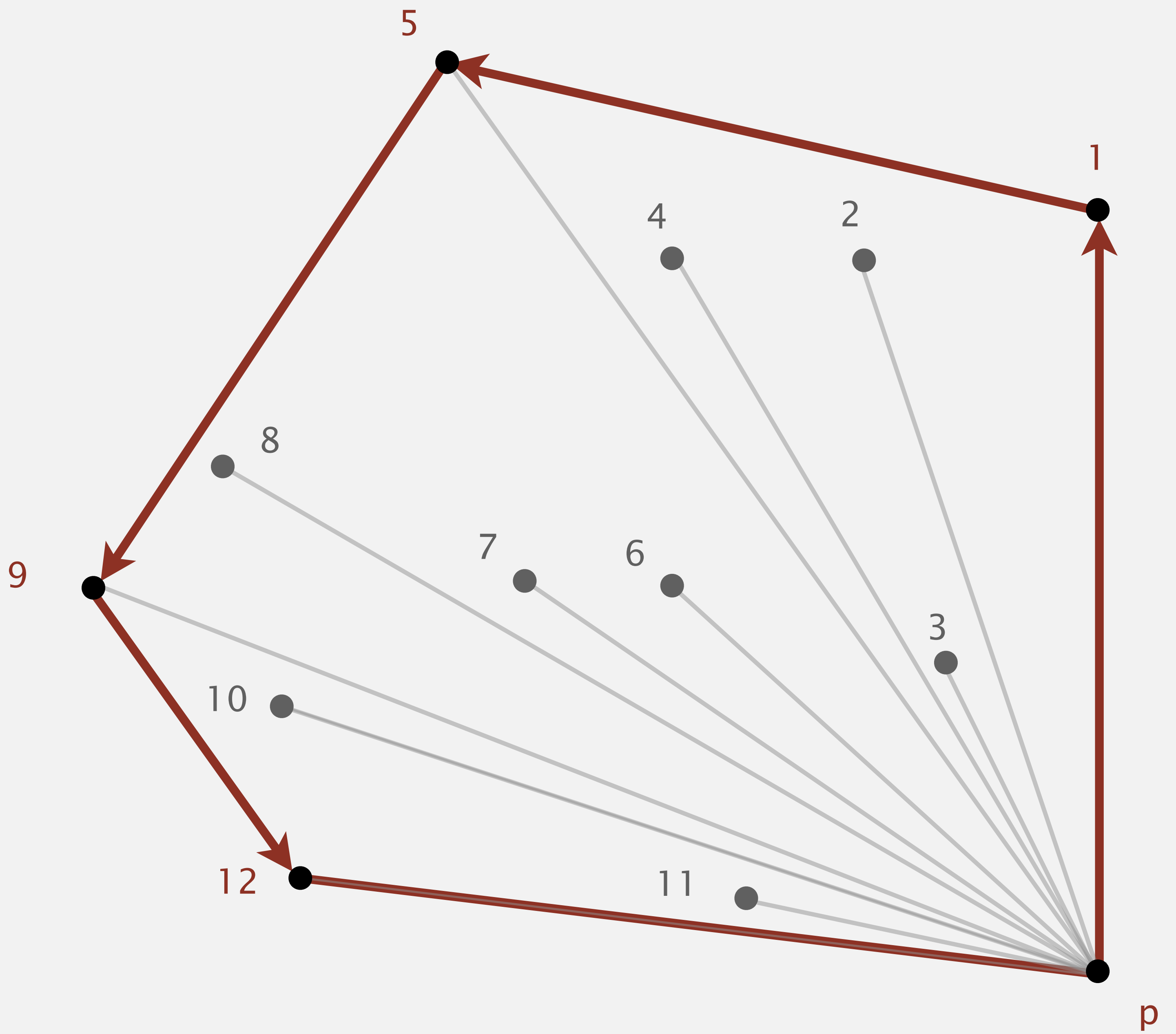


Fact. Farthest pair of points are extreme points on convex hull.

Convex hull: geometric properties

Fact. Can traverse the convex hull by making only counterclockwise turns.

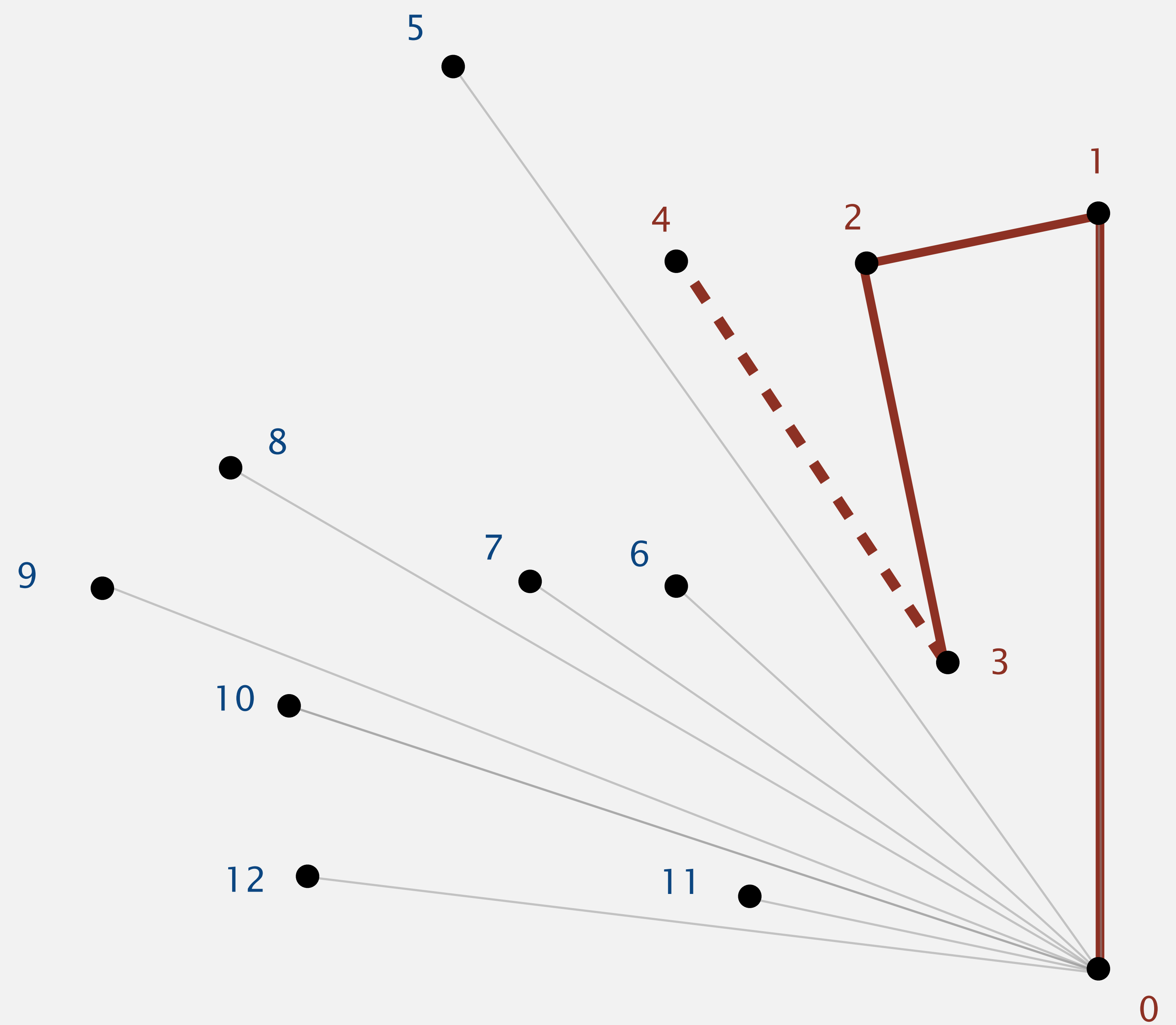
Fact. The vertices of convex hull appear in increasing order of polar angle with respect to point p with lowest y-coordinate.



Graham scan demo

Convex hull: Graham scan

- Choose point p with smallest y-coordinate.
- Sort points by polar angle with p.
- Consider points in order, and discard unless that would create a ccw turn.



Graham scan: implementation challenges

Q. How to find point p with smallest y-coordinate?

A. Define a total order, comparing y-coordinate.

Q. How to sort points by polar angle with respect to p ?

A. Define a total order for each point p.

Q. How to determine whether $p_1 \rightarrow p_2 \rightarrow p_3$ is a counterclockwise turn?

A. Computational geometry.

Q. How to sort efficiently?

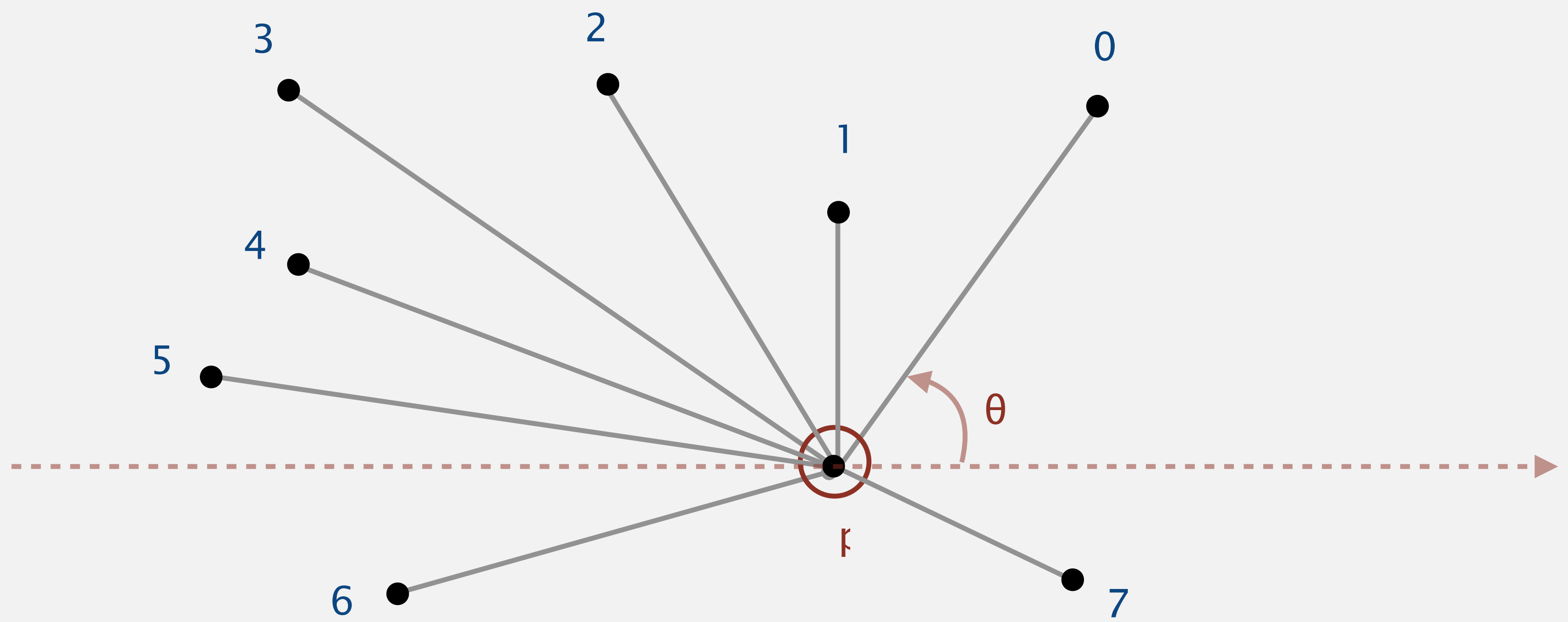
A. Mergesort sorts in $N \log N$ time.

Q. How to handle degeneracies (three or more points on a line)?

A. Requires some care, but not hard.

Polar order

Polar order. Given a point p, order points by the polar angle they make with p.




```
Arrays.sort(points, p.POLAR_ORDER);
```

Application. Graham scan algorithm for convex hull.

High-school trig solution. Compute polar angle θ w.r.t. p using atan2().

Drawback. Evaluating a trigonometric function is expensive.