Problem Set 1.4

1 For $-u'' = \delta(x-a)$, the solution must be linear on each side of the load. What four conditions determine A, B, C, D if $u(0) = 2$ and $u(1) = 0$?

$$u(x) = Ax + B \quad \text{for} \quad 0 \le x \le a \quad \text{and} \quad u(x) = Cx + D \quad \text{for} \quad a \le x \le 1\,.$$

2 Change Problem 1 to the free-fixed case $u'(0) = 0$ and $u(1) = 4$. Find and solve the four equations for A, B, C, D.

1. $u = -R(x-a) + Ex + F$

$u(0) = F = 2$

$u(1) = a - 1 + E + 2$

$E = -a - 1$

$u = -R(x-a) - (a+1)x + 2$

$0 \le x \le a$:

$u(x) = -(a+1)x + 2$

$A = -(a+1) \quad B = 2$

$C = (a+2) \quad D = a+2$

$a < x \le 1$:

$u(x) = a - x - (a+1)x + 2$

$= -(a+2)x + a + 2$

2 Change Problem 1 to the free-fixed case $u'(0) = 0$ and $u(1) = 4$. Find and solve the four equations for A, B, C, D.

$$u(x) = -R(x-a) + Ex + F$$

$$u'(0) = E = 0 \qquad u(1) = a - 1 + F = 4$$

$$F = 5 - a$$

$$u(x) = -R(x-a) + 5 - a$$

$$0 \le x \le a$$

$$A = 0 \quad B = 5 - a$$

$$a \le x \le 1$$

$$\cancel{a} - x + 5 - \cancel{a} = 5 - x$$

$$C = -1 \quad D = 5$$

4 Solve the equation $-d^2u/dx^2 = \delta(x-a)$ with **fixed-free** boundary conditions $u(0) = 0$ and $u'(1) = 0$. Draw the graphs of $u(x)$ and $u'(x)$.

$$u(x) = -R(x-a) + Cx + D$$

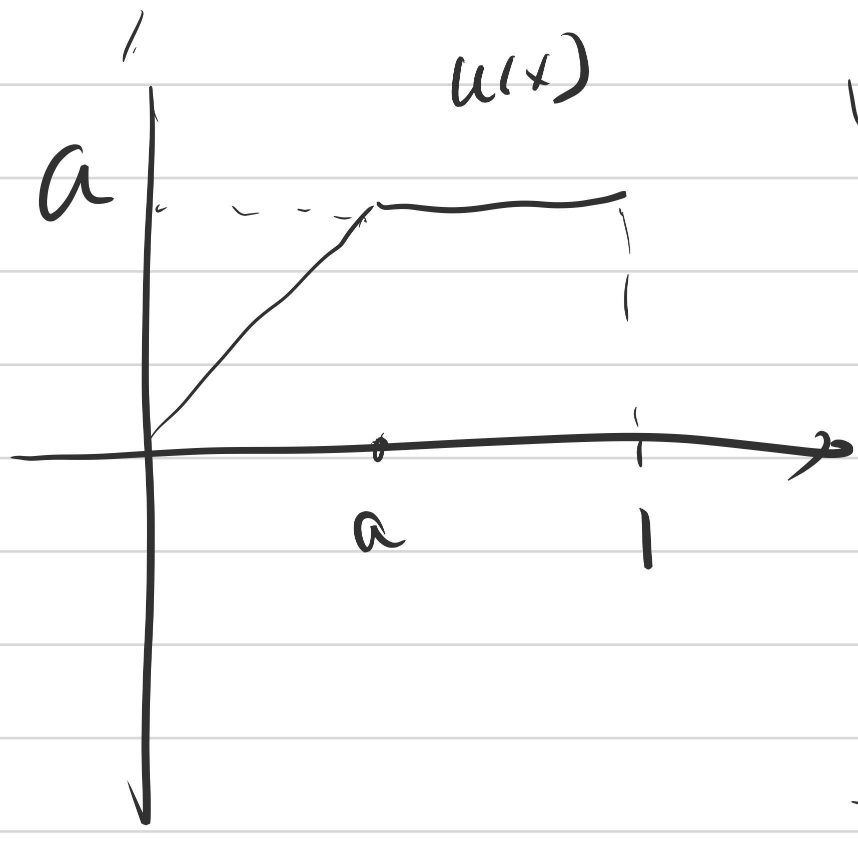
$$u(0) = D = 0$$

$$u'(1) = -S(1-a) + C = C - 1 = 0$$

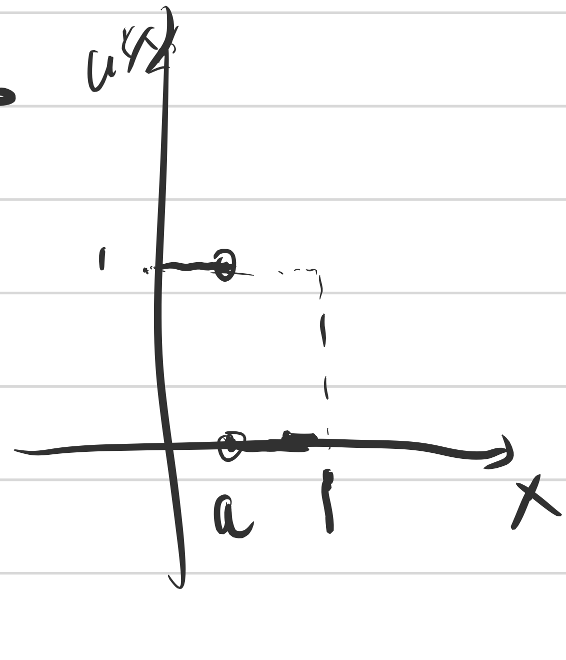
$$C = 1$$

$$u(x) = -R(x-a) + x$$

$$u(x) = \begin{cases} x & 0 \le x \le a \\ a & a \le x \le 1 \end{cases}$$



$$u'(x) = \begin{cases} 1 & 0 \le x < a \\ 0 & 1 \ge x > a \end{cases}$$



10 If you add together the columns of K^{-1} (or T^{-1}), you get a "discrete parabola" that solves the equation $Ku = f$ (or $Tu = f$) with what vector f? Do this addition for K_4^{-1} in Figure 1.9 and T_4^{-1} in Figure 1.10.

$$f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \cdots \quad f_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$K u_1 = f_1$$
$$u_1 = K^{-1} f_1$$

suppose $K^{-1} = [c_1 \cdots c_n]$

$K^{-1} f_1 = c_1 \to$ column one of K^{-1}

$\therefore$ $c_1 + c_2 + \cdots c_n =$

$$K^{-1} f_1 + K^{-1} f_2 + \cdots K^{-1} f_n$$

$$= u_1 + \cdots u_n = K^{-1}(f_1 + \cdots f_n) = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

so the vector f is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(2) for K_4^{-1} we can get

$$\begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

for T_4^{-1} we can get

$$\begin{bmatrix} 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}$$

12 The cubic spline $C(x)$ solves the fourth-order equation $u'''' = \delta(x)$. What is the complete solution $u(x)$ with four arbitrary constants? Choose those constants so that $u(1) = u''(1) = u(-1) = u''(-1) = 0$. This gives the bending of a uniform *simply supported beam* under a point load.

$$u(x) = C(x) + Dx^3 + Ex^2 + Fx + G$$

$$u(1) = \frac{1}{6} + D + E + F + G = 0$$

$$u''(1) = 1 + 6D + 2E = 0$$

$$u(-1) = -D + E - F + G = 0$$

$$u''(-1) = -6D + 2E = 0$$

$$E = -\frac{1}{4} \qquad D = \frac{E}{3} = -\frac{1}{12}$$

$$2E + 2G = -\frac{1}{6} \qquad F = \frac{1}{12} - \frac{1}{4} + \frac{1}{6} = 0$$

$$G = -\frac{1}{12} + \frac{1}{4} = \frac{1}{6}$$

$$u(x) = C(x) - \frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6}$$

$$u(x) = \begin{cases} \frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6} & x \geq 0 \\ -\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6} & x < 0 \end{cases}$$

Problem set 1.5:

2\. **2** When you multiply the eigenvector $y = (\sin \pi h, \sin 2\pi h, \ldots)$ by K, the first row will produce a multiple of $\sin \pi h$. Find that multiplier λ by a double-angle formula for $\sin 2\pi h$:

$$(Ky)_1 = 2 \sin \pi h - 1 \sin 2\pi h = \lambda \sin \pi h \qquad \text{Then } \lambda = \underline{\qquad}.$$

$\sin(2\pi h)$

$= 2 \sin \pi h \cos \pi h$

$(Ky)_1 = (2 - 2\cos \pi h) \sin \pi h$

$\lambda = 2 - 2\cos \pi h$

3 In MATLAB, construct $K = K_5$ and then its eigenvalues by $e = \mathsf{eig}(K)$. That column should be $(2-\sqrt{3}, 2-1, 2-0, 2+1, 2+\sqrt{3})$. Verify that e agrees with $2 * \mathsf{ones}(5,1) - 2 * \mathsf{cos}([1:5] * \mathsf{pi}/6)'$.

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

look the matlabcode to see they are same

23 Find λ's and x's so that $u = e^{\lambda t}x$ solves

$$\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u.$$

What combination $u = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2$ starts from $u(0) = (5, -2)$?

$$\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} - \lambda I = \begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix}$$

$$(4-\lambda)(1-\lambda) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 1$$

$$\lambda_1 = 4$$

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = 1$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u = c_1 e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$u_0 = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\-2\end{bmatrix}$$

$$c_2 = 2$$

$$c_1 = 3$$

$$u(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}3e^{4t}+2e^{t}\\-2e^{t}\end{bmatrix}$$

28 Find the eigenvalues and unit eigenvectors of A and T, and check the trace:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} \qquad \lambda^3 - \lambda - 2 = 0$$

$$\det(A - \lambda I) = 0 \quad (1-\lambda)\lambda^2 + \lambda + \lambda = 0$$

$$= \lambda(\lambda - \lambda^2 + 2) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 0 \quad \lambda_3 = -1$$

Check trace $1 + 0 + 0 = 2 + 0 - 1 = 1$

$$\lambda_1 = 2 \qquad \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \qquad \lambda_2 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\lambda_3 = -1$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$v_3 = \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \quad \det(T - \lambda I)$$

$$= (1-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}$$

$$\lambda_1 + \lambda_2 = 3 = 1+2 = \text{trace}$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2}$$

$$\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$v_1 = \begin{bmatrix} \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} \end{bmatrix} \quad \frac{6+2\sqrt{5}}{4} \quad \frac{10-2\sqrt{5}}{4}$$

$$\lambda_2 = \frac{3-\sqrt{5}}{2}$$

$$\begin{bmatrix} \frac{\sqrt{5}-1}{2} & -1 \\ -1 & -\frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$u_2 = \frac{2}{\sqrt{10-2\sqrt{5}}} \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{10-2\sqrt{5}}} \\ \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} \end{bmatrix}$$