Probability and Mathematical Statistics: Homework #8

Due on September 18, 2022 at 11:59am

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Problem 1 (mention the source of question, *e.g.*, BH CH0 #1)

(a) the MGF of X is $E(e^{tX}) = E(e^{t(I(A_1)+I(A_2)+I(A_3)+\cdots)})$ since all the events are independent this is equal to $\prod_{i=0}^{n} E(e^{tI(A_i)})$ considering A_i since it is a bernoulli trial , $E(e^{tI(A_i)}) = p_i e^t + q_i$,so the total is $\prod_{i=0}^{n}(p_i e^t + 1 - p_i)$

(b) the MGF is equal to $\prod_{i=1}^{n}(p_i(e^t-1)+1)$ using the approximation ,we can get this equal to $\prod_{i=0}^{n}(e^{p_i(e^t-1)})$,since the total sum of $p_i = \lambda$,so this is equal to $e^{\lambda(e^t-1)}$ since the event can be an approximation of possion process, considering the true possion process $X \sim \text{Pois}(\lambda)$and its MGF $E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}\frac{e^{-\lambda}(\lambda)^k}{k!} = \sum_{k=0}^{\infty} e^{tk}\frac{e^{-\lambda}(e^t\lambda)^k}{k!} = e^{\lambda(e^t-1)}$,since it is approximation about pois,so in the limit context the approximation is equal to pois.

Problem 2 (BH CH0 #2)

(a) to calculate P(L=l,M=m),when $m < l$ the joint PMF is equal to 0,when $m = l$, if and only if $X = Y = l$ the PMF=$q^{2l}p^2$,when $m > l$,the PMF is the sum of $P(X = l, Y = M) + P(X = m, Y = l) = 2q^{l+m}p^2$ since given L=10 $P(M = 5|L = 10) = 0, P(M = 5) \neq 0$,so they are not independent.

(b) $P(L = l) = \sum_{k=l}^{\infty} P(L = l, M = k) = q^{2l}p^2 + \sum_{m=l+1}^{\infty} 2q^{l+m}p^2 = q^{2l}(p^2 + 2pq)$,using the story ,considering L=l,it means in the l trial before,both X and Y is failure q^{2l},considering the l+1 trial there must be at least one success the both success p^2,only one success $2pq$ add them all and multiply the previous chance,we can get that$q^{2l}(p^2 + 2pq)$,on another way ,we can see at least one success as one bernoulli success,so the PMF of L it has $Bern(1 - q^2)$ since $p^2 + 2pq + q^2 = 1, p^2 + 2pq = 1 - q^2$

(c) since $L + M = X + Y$ $E(L + M) = E(X + Y)E(L) + E(M) = 2E(X) = 2\frac{q}{p}$,$E(L) = \sum_{l=0}^{n} lq^{2l}(p^2 + 2pq) = \frac{q^2}{1-q^2}$,so $E(M) = \frac{q(1+2q)}{p(1+q)}$

(d) P(L=l,M-L=k) when k=0 the joint PMF is $q^{2l}p^2$,when k$>$ 0,then PMF is $P(X = l, Y = l + k) + P(X = l + k, Y = l) = 2q^{2l+k}p^2$,it can be divided into $f(l)g(k)$ where P$(L = l) = q^{2l}(p^2 + 2pq)$ g(k)=$\frac{2p^2q^k}{p^2+2pq}(k > 0), \frac{p^2}{p^2+2pq}(k = 0)$ since it can be divided into two function $f(l)g(k)$,it is independent

Problem 3 (BH CH0 #3)

(a) P(T$\leq t|X=x$)=$P(Y \leq t-x)$ since Y$\sim$ $Expo(\lambda)$,the CDF is $1-e^{-\lambda(t-x)}(t>x)$,CDF is $0t \leq x$

(b) we can easily get this by differntiating $f_{T|X}(t|X=x) = \lambda e^{-\lambda(t-x)}$ (when t$\geq$ x),else(0),to check the validation,the PDF is obviously larger than 0,then $\int_x \infty \lambda e^{-\lambda(t-x)} dt = 1$

(c) let $t > 0$ $f_{X|T}(x|t) = \frac{f_{T|X}(t|x) f_X(x)}{f_T(t)} \propto f_{T|X}(t|x) f_X(x) = \lambda^2 e^{-\lambda t}$ it is a constant so the event $X|T \sim Unif(0,T)$ so the conditional PDF of X is $\frac{1}{t}$

(d) $f_T(t) = \frac{f_{T|X}(t|x) f_X(x)}{fX|T(x|t)} = \frac{\lambda^2 e^{-\lambda t}}{\frac{1}{t}} = \lambda^2 t e^{-\lambda t}$

Problem 4 (BH CH0 #4)

(a) the marginal CDF is $P(M \le m) = P(U_1 \le m, U_2 \le m, U_3 \le m)$since the random variable is independent $=m^3$ so PDF is $3m^2$,$P(M \le m) = P(L \le l, M \le m) + P(L \ge l, M \ge m)$ $P(L \ge l, M \le m) = P(l \le U_1 \le m, l \le U_2 \le m, l \le U_3 \le m)$ when $l > m$,this is equal to 0,so the joint CDF is $m^3 - (m-l)^3$,then differntiating it we can get the joint PDF is $6(m-l)$

(b) The marginal PDF of $f_L(l) = 3(1-l)^2$ it is the sum of $(P(U_i = l, U_j \ge l, U_k \ge l))$,$f_{M|L}(m|l) = \frac{f(m,l)}{f_L(l)} = \frac{2(m-l)}{(1-l)^2}$

Problem 5 (BH CH0#5)

(a) let q=1-p,let X be the trial vector$\sim Mult_k(n,(p^2,2pq,q^2))$ $P(X_1=x_1,X_2=x_2,X_3=x_3)=\frac{n!}{x_1!x_2!x_3!}p^{2x_1}(2pq)^{x_2}q^{2x_3}$

(b) We can see it as the Bernoulli trial $\sim(1-q^2)$ when choosing a person,if he is aa then failure,otherwise success,so the distribution of number in the sample who have an A$\sim Bin(n,1-q^2)$

(c) it can be seen as 2n Bernoulli trials when assign A success,assign a failure,so for the total genes ,let Y be the genes contain A$Y\sim Bin(2n,p)$

(d) x_1,x_2,x_3 be the observed data,according to (a) the possibility is $\propto p^{2x_1}(pq)^{x_2}q^{2x_3}$,let $L(p)=p^{2x_1}(pq)^{x_2}q^{2x_3}$to maximum the possibility,we can take the log for the both side and take the differntiating ,let it to be 0,$logL(p)=(2x_1+x_2)logp+(x_2+2x_3)log(1-p)$ let its derivative to be zero,I can get that $p=\frac{2x_1+x_2}{2n}$

(e) We can see it as many times of Bernoulli trials ,the observed data Y indicate how many aa are found$\sim Bin(n,q^2)$ so to find $L(q)$ we want to maximum$q^{2y}(1-q^2)^{n-y}$ using the same technique,we can get $log(L(q))=2yLog(q)+(n-y)log(1-q^2)$ then to calculate the derivative,let it equal to 0,we can get $q=\sqrt{\frac{y}{n}}$ so p=1 $-\sqrt{\frac{y}{n}}$