Probability and Mathematical Statistics: Homework #7

Due on September 18, 2022 at 11:59am

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Problem 1 (mention the source of question, *e.g.*, BH CH0 #1)

(a) To show it

(i) When X and Y are both discrete by definition,$P(Y=y|X=x)=\frac{P(Y=y,X=x)}{P(X=x)}$ since $P(X=x|Y=y)=\frac{P(X=x,Y=y)}{P(Y=y)}$,so $P(Y=y|X=x)=\frac{P(X=x|Y=y)P(Y=y)}{P(X=x)}$

(ii) When Y is continuous,X is discrete,

$$\begin{aligned}
&f_Y(y|X=x)=\lim_{\epsilon\to 0}\frac{P(y-\epsilon\le Y\le y+\epsilon|X=x)}{2\epsilon}\\
&=\lim_{\epsilon\to 0}\frac{P(y-\epsilon\le Y\le y+\epsilon,X=x)}{2\epsilon\cdot P(X=x)}\\
&=\lim_{\epsilon\to 0}\frac{P(X=x|y-\epsilon\le Y\le y+\epsilon)\cdot P(y-\epsilon\le Y\le y+\epsilon)}{P(X=x)\cdot 2\epsilon}\\
&=\frac{P(X=x|Y=y)f_Y(y)}{P(X=x)}
\end{aligned}\tag{1}$$

(iii) when X continuous ,Y discrete

$$\begin{aligned}
&P(Y=y|X=x)=\lim_{\epsilon\to 0}P(Y=y|x-\epsilon\le X\le x+\epsilon)\\
&=\lim_{\epsilon\to 0}\frac{P(Y=y,x-\epsilon\le X\le x+\epsilon)}{P(x-\epsilon\le X\le x+\epsilon)}\\
&=\lim_{\epsilon\to 0}\frac{\frac{P(x-\epsilon\le X\le x+\epsilon|Y=y}{2\epsilon)}P(Y=y)}{\frac{P(x-\epsilon\le X\le X+\epsilon)}{2\epsilon}}\\
&=\frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}
\end{aligned}\tag{2}$$

(iv) when X and Y are continous ,by definition $f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}$,since $f_{X,Y}(x,y)=f_{X|Y}(x|y)f_Y(y)$,so $f_{Y|X}(y|x)=\frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$

Problem 1 continued on next page. . .

(b) (i) since conditional PMF is also valid PMF so,$\sum_y P(Y=y|X=x)=1=\sum_y \frac{P(Y=y,X=x)}{P(X=x)}=1$ thus $\sum_y P(Y=y,X=x)=P(X=x)$,then

$$\sum_y P(X=x|Y=y)P(Y=y)=\sum_y P(X=x,Y=y)=P(X=x)$$

(ii) When X is discrete,Y is continous

$$\begin{aligned} &P(X=x|Y=y)=\frac{f_Y(y|X=x)}{f_Y(y)}\cdot P(X=x) \\ &\int_{-\infty}^{+\infty} P(X=x|Y=y)f_Y(y)dy=\int_{-\infty}^{+\infty} P(X=x)f_Y(y|X=x)dy \\ &=P(X=x) \end{aligned} \tag{3}$$

(iii) when X is continous ,and Y is discrete, $f_X(x)=\lim_{\epsilon\to 0}\frac{P(x-\epsilon\leq X\leq x+\epsilon)}{2\epsilon}$ $=\sum_y \frac{P(x-\epsilon\leq X\leq x+\epsilon|Y=y)P(Y=y)}{2\epsilon}=\sum_y f_X(x|Y=y)P(Y=y)$

(iv) when X and Y are both continous $f_X(x)=\int_{-\infty}^{+\infty} f_{X,Y}(X=x,Y=y)dy$ using bayes' rule $=\int_{-\infty}^{+\infty} f_{X|Y}(x|y)f_Y(y)dy$

Problem 2 (BH CH0 #2)

Solution

(a) To find CDF we can integral PDF $F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}dt = \frac{tan^{-1}(x)}{\pi} + \frac{1}{2}$

(b) by using integral F(x)=0(when $x < 1$),F(x)=$\int_1^x at^{-a-1}dt$=1$-x^{-a}$(when x$\geq$ 1) ,to check the validation,$f(x) \geq 0 forallx$,$\lim_{x\to\infty} F(x) = 1 - 0 = 1$

(c) let Y=max(Z-c,0) then we know when Z$\geq cY = Z - c$ else Y=0 then $E(Y) = \int_c^{\infty}(z-c)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}} - c(1 - \Phi(c))$ =$\varphi(c) - c + c\Phi(c)$

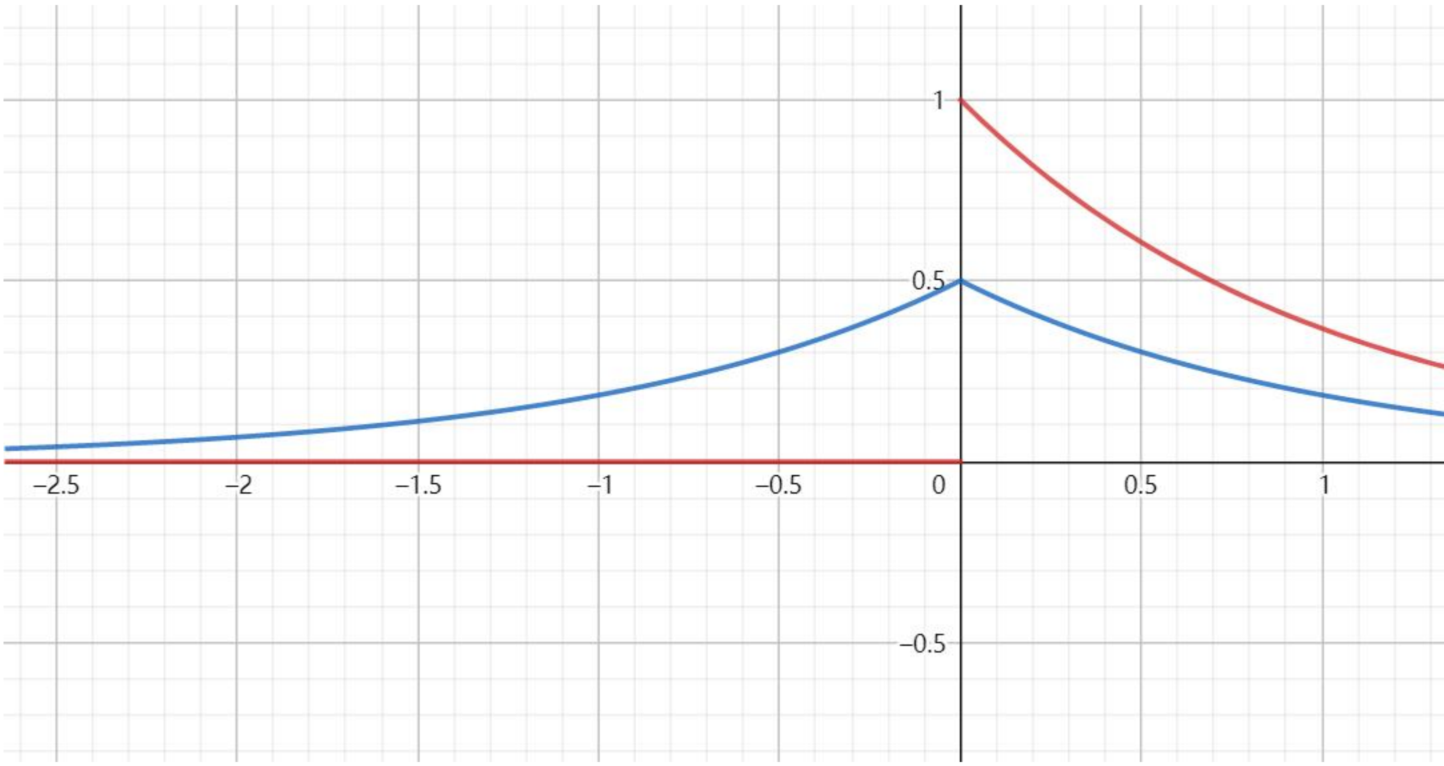
Problem 3 (BH CH0 #3)

(a) from drawing the line ,I can easily get that ($G\delta t = T$(the G_{th} failure,the first trial begin at when t=0)

(b) P($T > t$)=P(G$\Delta t > t$),considering when t=k$\Delta t, k \in Z$,then $G \geq k+1$,when $(k-1)\Delta t < t < k\Delta t$,we calculate $G \geq k$,so we just choose $G \geq \lfloor \frac{t}{\Delta t} \rfloor + 1$,since G satify Geometry distrbution so $P(G \geq \lfloor \frac{t}{\Delta t} \rfloor + 1)$ =$\sum_{k=\lfloor \frac{t}{\Delta t} \rfloor + 1}^{\infty}(1-\lambda\Delta t)^k \lambda\Delta t = (1-\lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor + 1}$,then CDF is $1-(1-\lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor + 1}$

(c) considering the limit of $\lim_{\Delta t \to 0}(1-\lambda\Delta t)^{\frac{t}{\Delta t}} \leq \lim_{\Delta t \to 0}(1-\lambda\Delta t)^{\frac{t}{\Delta t}+1} = \lim((1-\lambda\Delta t)^{\frac{1}{\lambda\Delta t}})^{\lambda t}(1-\lambda\Delta t) = e^{-\lambda t}$,so the CDF is $1-e^{-\lambda t}$satisfy $Expo(\lambda)$

Problem 4 (BH CH0 #4)



(a) The amplitude is half of the expo distribution,so the right of laplace distribution is actually exponential distribution but with an area smaller.

(b) using lotp P(SX> t)=P(X>t—S=1)P(S=1)+P(X<-t—S=-1)P(S=-1)=$\frac{1}{2}(P(X > t) + P(X < -t))$ when t>0,P(SX >t)=$\frac{e^{-t}}{2}$ when t $\leq$0 then P(SX>t)=$\frac{2-e^{t}}{2}$,so we can get the PDF equal to the laplace function

Problem 5 (BH CH0 #5)

(a) $r = \frac{1}{n}\sum(x_i - \overline{x})(y_i - \overline{y}) = \frac{1}{n}\sum(x_i y_i - \overline{x}y_i - \overline{y}x_i + \overline{xy}) = \frac{1}{n}\sum x_i y_i - \frac{\overline{x}\sum y_i}{n} - \frac{\overline{y}\sum x_i}{n} + \overline{x}\cdot\overline{y} = \overline{xy} - \overline{x}\cdot\overline{y} = E(XY) - E(X)E(Y) = COV(X,Y)$

(b) to consider the average signed area of expectation formed by $(X,Y),(\hat{X},\hat{Y})$ which indicates $E((X - \hat{X})(Y - \hat{Y}))$ since two trial is independent there are $\frac{1}{n^2}$possibility to choose $(x_i, y_i),(x_j, y_j)$ so the total expectation is $\frac{\sum_j \sum_i (x_i - x_j)(y_i - y_j)}{n^2} = \frac{\sum_j \sum_i x_i y_i - x_j y_i - y_i x_i + x_j y_i}{n^2} = 2\overline{xy} - 2\overline{x}\cdot\overline{y} = 2Cov(X,Y) = 2r$,so the sample variance is a constant time of the total signed area of the rectangle

(c) (i) since it is the constant time of the average signed area of two indepent trials,changing the x and y axis doesn't change the area,so $Cov(W1,W2) = Cov(W2,W1)$

(ii) the linearscaling of the random variable will also lead to the area to be extended to $a_1 a_2$ times,so $Cov(a1W1, a2W2) = a1a2Cov(W1,W2)$

(iii) the translation of axis doesn't change the area ,so $Cov(W1 + a1, W2 + a2) = Cov(W1,W2)$

(iv) the area with x=x1,y=y1+y2, is the same area as two area sum up x=x1,y=y1,and x=x1,y=y2,so Cov(W1,W2+W3)=Cov(W1,W2)+Cov(W1,W3)