Probability and Mathematical Statistics: Homework #9

Due on September 18, 2022 at 11:59am

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Problem 1 (mention the source of question, *e.g.*, BH CH0 #1)

(a) $P(N=n,X=x,Y=y)(0\le x\le n,0\le y\le n,x+y=n,n\ge 0)=P(X=x,Y=y|N=n)P(N=n)=\binom{n}{x}p^x(1-p)^{n-x}\frac{e^{-\lambda}\lambda^n}{n!}$ since given N and X we can infer what Y is they are not independent.

(b) $P(N=n,X=x)(0\le x\le n,n\ge 0)=P(X=x|N=n)P(N=n)$ which is $\binom{n}{x}p^x(1-p)^{n-x}\frac{e^{-\lambda}\lambda^n}{n!}$ since given n=10,we can know it is impossible that x=15,so they are not independent

(c) $P(X=x,Y=y)(0\le x,0\le y)=P(X=x,Y=y|N=x+y)P(N=x+y)=\binom{x+y}{x}p^x(1-p)^y\frac{e^{-\lambda}\lambda^{x+y}}{(x+y)!}=\frac{e^{-\lambda p}(\lambda p)^x\cdot e^{-\lambda(1-p)}(\lambda(1-p))^y}{x!\cdot y!}=h(x)\cdot h(y)$so they are indepnedent

(d) from c we can get that $P(X=x)=\frac{e^{-\lambda}(\lambda p)^x}{x!}$then Corr$(N,X)=\frac{Cov(N,X)}{\sqrt{Var(N)\cdot Var(X)}}=\frac{E(NX)-E(N)E(X)}{\sqrt{Var(N)Var(X)}}$ $E(NX)=\sum_{n=0}^{\infty}\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}\frac{e^{-\lambda}\lambda^n}{n!}nx=\sum\sum np\binom{n-1}{x-1}p^{x-1}(1-p)^{n-1}\frac{e^{-\lambda}(\lambda)^n}{n!}n=(n(n-1)+n)p\frac{e^{-\lambda}(\lambda)^n}{n!}=p(\lambda^2+\lambda)E(N)=Var(N)=\lambda E(X)=Var(X)=\lambda p$ so the answer is $\sqrt{p}$

Problem 2 (BH CH0 #2)

(a) It is Multivariate Normal since $t_1X + t_2Y + t_3(X + y) = (t_1 + t_3)X + (t_2 + t_3)Y$ a linear combination of X and Y,so they are Multivariate norma

(b) No If $t_1 = t_2 = 1, t_3 = 1$then the sum of them is $(1 + S)(X + Y)$ $P((1 + S)(X + Y) = 0)$ It is equal to add the possibility that $S = -1$ and X+Y=0,the total probability is $\frac{1}{2}$ so it isn't Multivariate

(c) First I want to show that if X is $N(0, 1)$ then $SX$is also $N(0, 1)$,$P(SX = k) = P(X = k|S = 1)P(S = 1) + P(X = -k|S = -1)P(S = -1) = P(X = k)$(according to the symmetry)so SX is also $N(0, 1)$ so SX and SY is also $i.i.dN(0, 1)$ the linear combination of two $i.i.dN(0, 1)$ are also a normal distribution,so it is Multivariate normal

Problem 3 (BH CH0 #3)

(a) $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \rho$ so $Cov(X,Y) = \rho\sigma_1\sigma_2$ if X and $Y - cX$ is independent $Cov(X, Y - cX) = Cov(X,Y) - cCov(X,X) = Cov(X,Y) - cVar(X) = 0$,c=$\frac{\rho\sigma_2}{\sigma_1}$

(b) since $T = X + Y, W = X/Y$ $X = \frac{WT}{W+1}, Y = \frac{T}{W+1}$

$$\begin{vmatrix} \frac{\partial X}{\partial T} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial W} \end{vmatrix} = -\frac{T}{(W+1)^2} \tag{1}$$

so the jocabbian is $\frac{T}{(W+1)^2}$ $f_{T,W}(t,w) = (\lambda)^2 e^{-\lambda t}\frac{t}{(w+1)^2}(0 \le t, 0 \le w)$ the marginal pdf $f_T(t) = \int_0^\infty \lambda^2 e^{-\lambda t} t \frac{1}{(w+1)^2} dw = \lambda^2 e^{-\lambda t} t$,$f_W(w) = \frac{1}{(w+1)^2}$

(c) $P(W = w) = P(X = w - t|Y + Z = t)P(Y + Z = t)$ we can know that

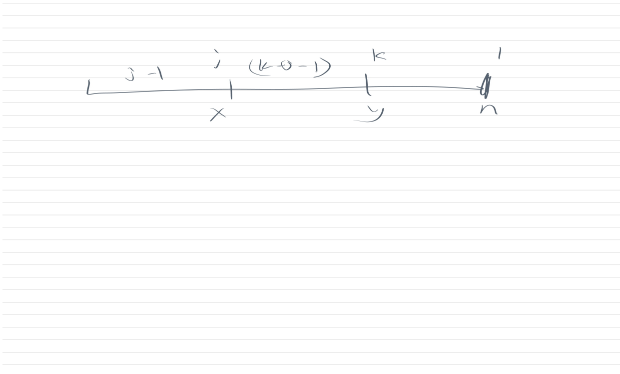
$$P(Y + Z = t) = \begin{cases} t(0 \le t \le 1) \\ 2 - t(1 \le t \le 2)0(other) \end{cases} \tag{2}$$

$(min(w,2) \ge t \ge max(0, w-1))$ then to calculate it

$$\begin{cases} \frac{1}{2}w^2(0 \le w \le 1) \\ 3w - w^2 - \frac{3}{2}(1 \le w \le 2) \\ \frac{w^2-6w+9}{2}(2 \le w \le 3) \\ 0(other) \end{cases} \tag{3}$$

Problem 4 (BH CH0 #4)

(a) $P(U_{(j)} = x, U_{(k)} = y)(0 \le x \le y \le 1)$ we can use the axis



It is equal to $n(n-1)\binom{n-2}{j-1}\binom{n-j-1}{k-j-1}P(X \le x)^{j-1}P(x \le X \le y)^{k-j-1}P(y \le X \le 1)^{n-k} = n(n-1)\binom{n-2}{j-1}\binom{n-j-1}{k-j-1}x^{j-1}(y-x)^{k-j-1}(1-y)^{n-k}$

(b) considering $U_1, U_2, \cdots, U_n$ are all $i.i.d\ Unif(0,1)$ we can see $U_i \le p$ as a success,so the number of success is greater than j is equal to "at least j of the $U_1, U_2 \cdots U_n$ $P(U_(j) \le p)$" $f_{U_{(j)}}(x) = n\binom{n-1}{j-1}x^{j-1}(1-x)^{n-j} = f_B(x)$ then $P(X \ge j) = P(B \le p)$

(c) defining $U_i \le x$ as success,the lefthandside can be interpreted as all possibility of at least j success happen,from the possition 0 to possition x,the righthandside is the sum of possibility of $P(U_(j) \le x)$ since at least j success happen is equal to the j_{th} min is smaller than x,so the both side are equal

(d) from c to take the limit you will see the answer,first the rhs

$$\lim_{n\to\infty}\sum_{i=j}^{\infty}\binom{n}{i}x^k(1-x)^{n-j}$$

let nx= λ since the binom can be approximated as Pois(λ) ,and j to be k+1,so during the process the rhs can be wirtten as $Pois(\lambda)$ process ,$P(X \ge k+1) = 1 - P(X \le k)$ then to calculate the lhs since x=$\frac{\lambda}{n}$ let u to be nt,j=k+1,the left is $\int_0^{\lambda}\frac{n!}{k!(n-k-1)!}(\frac{u}{n})^k(1-\frac{u}{n})^{n-k-1}d\frac{u}{n} = \lim_{n\to\infty}\int_0^{\lambda}\frac{\Gamma(k+1)n!u^k(1-\frac{u}{n})^{n-k-1}}{(n-k-1)!n^{k+1}}du = \int_0^{\lambda}\frac{u^ke^{-t}}{\Gamma(k+1)}du = 1 - P(Z \ge \lambda)$ so lhs is equal to rhs
using the story we can know that the lhs can be represented as a total length of λ let $X_1, X_2, X_3 \cdots X_n$ be the iid $Expo(1)$ then think the length of λ $P(X \le k)$ is equal to in the length of λ,there are at most k arrive,while Gamma$(k+1,1)$,is the total arrival time of $k+1$ iid random variables $P(Z > \lambda)$ means when k+1 arrive the time is larger than λ which is the same as at most k arrive happen in the length of λ

Problem 5 (BH CH0 #5)

(a) $E(p^2(1-p)^2) = \int\limits_0^1 \frac{p^{a+1}(1-p)^{b+1}dp}{\beta(a,b)} = \frac{\beta(a+2,b+2)}{\beta(a,b)} = \frac{\Gamma(a+2)\Gamma(b+2)\Gamma(a+b)}{\Gamma(a+b+4)\Gamma(a)\Gamma(b)} = \frac{(a+1)a(b+1)b}{(a+b+3)(a+b+2)(a+b+1)(a+b)}$

(b) my posterior distribution for p doen't depend on the specific order of outcomes(due to the consistence of bayes'rule) if we use the success to update the posterior distribution $\beta(a+1,b)$,if fail update$\beta(a,b+1)$,so that we can the final distribution unrelated to order.

(c) from the analysis before we can get p as $\beta(7,5)$since the initial posterior p is $Unif(0,1)$ satisfy $\beta(1,1)$

(d) conditional on p ,the first trial and second trial is conditionally independent as the p is fixed,it is two independent bernouli trials,they are uncorrelated, while conditional on historical data ,the indicator of first success is positively related,since if first success,the p will be larger,which means the second success will more likely to happen.

(e) it is same as the possibility of tie happen in the 4_{th} they are tie,$\binom{4}{2}p^2(1-p)^2$ the p is updated by c $\beta(7,5)$ then from a we can know $\binom{4}{2}\frac{8\cdot7\cdot6\cdot5}{12\cdot13\cdot14\cdot15} = \frac{4}{13}$

Problem 6 (BH CH0 #6)