Probability and Mathematical Statistics: Homework #10

Due on September 18, 2022 at 11:59am

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Problem 1 (mention the source of question, *e.g.*, BH CH0 #1)

(a) $E(X|X \leq 1) = \sum k(P(X=k|X \geq 1)) = \sum \frac{kP(X=k, X\geq 1)}{P(X\geq 1)} = \sum \frac{ke^{-\lambda}(\lambda)^k}{k!(1-e^{-\lambda}} = \frac{\lambda}{1-e^{-\lambda}}$

(b)

$$Var(X|X \geq 1) = E(X^2|X \geq 1) - (E(X|X \geq 1))^2$$
$$= \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda}\lambda^k}{k!(1-e^{-\lambda})} - \sum \frac{ke^{-\lambda}\lambda^k}{k!(1-e^{-\lambda})^2}$$
$$= \frac{\lambda^2+\lambda}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}$$

Problem 2 (BH CH0 #2)

(a) let $Y = X^2$ and $X \sim unif(0,1)$ then $E(\frac{X}{X+Y}) = E(\frac{1}{X+1}) = \int_0^1 \frac{x}{x+1}dx = 1 - \ln 2$, $\frac{E(X)}{E(X+Y)} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}} = \frac{3}{5}$

(b) it is true,since they are iid we can know $E(\frac{X}{X+Y}) = E(\frac{Y}{X+Y})$(by symmetry), since add them up you can get E($\frac{X+Y}{X+Y}$)=1,so $E(\frac{X}{X+Y}) = \frac{1}{2}$,then the rhs $\frac{E(X)}{E(X+Y)} = \frac{E(X)}{E(X)+E(Y)}$ since X and Y are iid,so it is equal to $\frac{1}{2}$ lhs equal to rhs so it is true.

(c) let T be (X+Y),let W be $\frac{X}{X+Y}$ X and Y are independent ,and $\sim Gamma(a,\lambda)$ and $\sim Gamma(b,\lambda)$,T and W are independent,so does T^c and W^c,so we have $E(T^cW^c) = E(T^c)E(W^c)$,so $E(X^c) = E((X+Y)^c)E((\frac{X}{X+Y})^c)$,so $\frac{E(X^c)}{E(X+Y)^c} = E((\frac{X}{X+Y})^c)$

Problem 3 (BH CH0 #3)

(a) let X be $\frac{T_1}{T_2}$ Y be $T_1 + T_2$ the jocabbian of $|\frac{\partial(t1,t2)}{\partial(x,y)}| = \frac{y}{(x+1)^2}$

$f_{X,Y}(x,y) = (\lambda_1)^2 e^{-\lambda_1 y} y \frac{1}{(x+1)^2}$ it can be divided into $f(x) \cdot g(y)$ so it is independent

(b) using lotp,this is to compute $\int_0^\infty P(T_1 < T_2|T_2 = t_2) f_{T_2}(t_2) dt_2 = \int_0^\infty (1 - e^{-\lambda_1 t_2}) \lambda_2 e^{-\lambda_2 t_2} dt_2 = \int_0^\infty \lambda_2 e^{-\lambda_2 t_2} - \lambda_2 e^{-(\lambda_1+\lambda_2) t_2} dt_2 = \frac{\lambda_1}{\lambda_1+\lambda_2}$,especially when $\lambda_1 = \lambda_2$ it is equal to $\frac{1}{2}$

(c) it is equal to find the minum expectation of wait and serve,to calculate the wait time $E(min(T_1, T_2))$ min (T_1, T_2) satisfy $\sim Expo(\lambda_1 + \lambda_2)$ so the expectation is $\frac{1}{\lambda_1+\lambda_2}$ then the wait time,suppose the random variable of waiting time is W $f_W(w) = f_W(w|T_1 < T_2)P(T_1 < T_2) + f_W(w|T_1 \geq T_2)P(T_1 \geq T_2)$ conditional on$T_1 < T_2$ it is $Expo(\lambda_1)$ conditional on $T_1 \geq T_2$ the W is $Expo(\lambda_2)$ so the expectation of serving time is $\frac{2}{\lambda_1+\lambda_2}$,add them up ,they are totally $\frac{3}{\lambda_1+\lambda_2}$

Problem 4 (BH CH0 #4)

(a) let I_j is the position of j_{th} can form a sequence of "CATCAT",the total sequence $E(\sum I_j) = E(I_1 + I_2 + I_3 \cdots + I_n) = 110(p_1p_2p_3)^2$,$E(I_j) = (p_1p_2p_3)^2$

(b) considering when the first A appears before C ,it is similiar as FS distribution each trial one by one,if it isn't A and C then continue which has probability $1 - p_1 - p_2$ when first A happen it finish so the total probability is $\sum_{t=0}^{\infty}(1 - p_1 - p_2)^t p_1 = \frac{p_1}{p_1+p_2}$

(c) the prior distribution of p is $Beta(1, 1)$ after observing the data ,in the observed data the number of success is updated by 1,and the number of failure is updated by 2,using the Beta-Binomial conjugacy we can know the posterior is $Beta(2, 3)$ let X be the number of C in the sequence we've observed,to calculate it first we get $f(p|X = 1) = \frac{p(1-p)^2}{\beta(2,3)}$,then $\int_0^1 \frac{p^2(1-p)^2}{\beta(2,3)} dp = 0.4$

Problem 5 (BH CH0 #5)

(a) let the event of HT happen be divided into two independent event W_1 be the number of tosses until first H happens which satify $FS(p)$,and W_2 be the number of tosses after the first H happen first T happen which satift $FS(1-p)$,so $E(W_1+W_2)=E(W_1)+E(W_2)=\frac{1}{(1-p)}+\frac{1}{p}$

(b) using the conditional expectation let W_{HH} be the number of toss when first HT happens $E(W_{HH})=E(W_{HH}|\text{first toss H})P(\text{first toss H})$+E($W_{HH}$—first toss T)P(first toss T),
$E(W_{HH}|\text{first toss H})=E(W_{HH}|)$first toss H,second toss H)P(second H)+E(W_{HH}|first toss H,second toss T) P(second toss T)=2p+(E(W_{HH})+2)(1-p) so the total E is $\frac{1}{p}+\frac{1}{p^2}$

(c) for (a) the total expectation is $\int_0^1 E(W_{HH}|p)f(p)dp=\int_0^1(\frac{1}{p}+\frac{1}{1-p})(\frac{p^{a-1}(1-p)^{b-1}}{\beta(a,b)})dp=\frac{\beta(a-1,b)+\beta(a,b-1)}{\beta(a,b)}=\frac{\Gamma(a-1)\Gamma(a+b)}{\Gamma(a+b-1)\Gamma(a)}+\frac{\Gamma(b-1)\Gamma(a+b)}{\Gamma(a+b-1)\Gamma(a)}=\frac{a+b-1}{a-1}+\frac{a+b-1}{b-1}$ for(b) the total expectation is $\int_0^1(\frac{1}{p}+\frac{1}{p^2})\frac{p^{a-1}(1-p)^{b-1}}{\beta(a,b)}dp=\frac{\beta(a-1,b)}{\beta(a,b)}+\frac{\beta(a-2,b)}{\beta(a,b)}=\frac{a+b-1}{a-1}+\frac{(a+b-1)(a+b-2)}{(a-1)(a-2)}$