A ternary Diophantine inequality with prime numbers of a special form

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Abstract: Let N be a sufficiently large real number. In this paper, it is proved that, for $1 < c < \frac{973}{856}$ and for any arbitrary large number $E > 0$, the Diophantine inequality

$$\left|p_1^c + p_2^c + p_3^c - N\right| < (\log N)^{-E}$$

is solvable in prime variables p_1, p_2, p_3 such that each of the numbers $p_i + 2\,(i = 1, 2, 3)$ has at most $[\frac{12626}{4865-4280c}]$ prime factors, counted according to multiplicity. This result constitutes an improvement upon the previous result of Zhu [38].

Keywords: Diophantine inequality; exponential sum; prime variable; almost–prime

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1 Introduction and main result

For fixed integer $k \geqslant 1$ and sufficiently large integer N, the well–known Waring–Goldbach problem is devoted to investigating the solvability of the following Diophantine equality

$$N = p_1^k + p_2^k + \cdots + p_s^k \tag{1.1}$$

in prime variables $p_1, p_2, \ldots, p_s$. In this topic, many mathematicians have derived many splendid results. For instance, in 1937, Vinogradov [35] proved that such a representation of the type (1.1) exists for every sufficiently large odd integer with $k =$

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$1, s=3$. Moreover, in 1938, Hua [16] showed that (1.1) is solvable for every sufficiently large integer N satisfying $N \equiv 5 (\bmod 24)$ with $k=2, s=5$.

In 1952, Piatetski–Shapiro [24] studied the following analog of the Waring–Goldbach problem. Suppose that $c>1$ is not an integer and ε is a small positive number. Denote by $H(c)$ the smallest natural number r such that, for every sufficiently large real number N, the Diophantine inequality

$$\left|p_1^c + p_2^c + \cdots + p_s^c - N\right| < \varepsilon \tag{1.2}$$

is solvable in primes $p_1, p_2, \ldots, p_s$. Then it was proved in [24] that

$$\limsup_{c \to +\infty} \frac{H(c)}{c \log c} \leqslant 4.$$

Also, in [24], Piatetski–Shapiro considered the case $r=5$ in (1.2) and proved that $H(c) \leqslant 5$ for $1<c<3/2$. Later, the upper bound $3/2$ for $H(c) \leqslant 5$ was improved successively to

$$\frac{14142}{8923}, \quad \frac{1+\sqrt{5}}{2}, \quad \frac{81}{40}, \quad \frac{108}{53}, \quad 2.041, \quad \frac{52}{25}$$

by Zhai and Cao [36], Garaev [13], Zhai and Cao [37], Shi and Liu [26], Baker and Weingartner [1], Li and Cai [21], respectively.

From these results and the Goldbach–Vinogradov theorem, it is reasonable to conjecture that if c is near to 1, then the Diophantine inequality (1.2) is solvable for $s=3$. This conjecture was first established by Tolev [28] for $1<c<\frac{27}{26}$. Since then, the range of c was enlarged to

$$\frac{15}{14}, \quad \frac{13}{12}, \quad \frac{11}{10}, \quad \frac{237}{214}, \quad \frac{61}{55}, \quad \frac{10}{9}, \quad \frac{43}{36}$$

by Tolev [29], Cai [5], Cai [6] and Kumchev and Nedeva [19] independently, Cao and Zhai [8], Kumchev [20], Baker and Weingartner [2], Cai [7], successively and respectively. The best result up to now belongs to Baker [3] with $1<c<6/5$.

Another central problem in the theory of prime distribution, namely the twin prime conjecture, states that there exist infinitely many primes p such that $p+2$ is also prime. Although this conjecture has resisted all attacks, there have been spectacular partial achievements. Let $\mathcal{P}_r$ denote an almost–prime with at most r prime factors, counted according to multiplicity. One well–known result is due to Chen [9, 10], who proved that there exist infinitely many primes p such that $p+2$ has at most 2 prime factors.

Bearing in mind the result of Chen [9, 10], one may try to study the arithmetical properties of the set of primes p such that $p+2 \in \mathcal{P}_r$ for a fixed $r \geqslant 2$ and, in particular, to establish the solvability of Diophantine equations or inequalities in such primes. For instance, combining the results of Vinogradov [35] and Chen [10], Tolev [30, 31, 32] established such kinds of results, while Matomäki and Shao [22] improved the result of Tolev [32] and proved that every sufficiently large odd integer N can be represented as a sum of three primes p_1, p_2, p_3 such that $p_i + 2 \in \mathcal{P}_2\,(i = 1, 2, 3)$.

Motivated by Tolev [28, 29] and Chen [9, 10], it is reasonable to conjecture that if the constant $c > 1$ is close to one, then inequality (1.2), with a suitable $\varepsilon = \varepsilon(N)$ satisfying $\varepsilon(N) \to 0$ as $N \to \infty$, is solvable in primes p_i such that $p_i + 2$ are almost primes of a certain fixed order for $s = 3$. An attempt to establish a result of this type was first made by Dimitrov [11], he dealt with this problem with $0 < c < 4/21$ and $p_i + 2 = \mathcal{P}_{10}, i = 1, 2, 3$. After that, the next step also belongs to Dimitrov [12] with $1 < c < 121/120$ and $p_i + 2 = \mathcal{P}_{29}, i = 1, 2, 3$.

Later, motivated by Dimitrov [11], Tolev [33] proved that, for $1 < c < \frac{15}{14}$, the Diophantine inequality

$$\left|p_1^c + p_2^c + p_3^c - N\right| < (\log N)^{-E} \tag{1.3}$$

is solvable in primes p_1, p_2, p_3 such that each of the numbers $p_i + 2\,(i = 1, 2, 3)$ has at most $[\frac{369}{180-168c}]$ prime factors, counted according to multiplicity, where $E > 0$ is a sufficiently large constant. Recently, Zhu [38] improved the result of Tolev [33] and showed that for $1 < c < \frac{281}{250}$, (1.3) is solvable in primes p_1, p_2, p_3 such that each of the numbers $p_i + 2\,(i = 1, 2, 3)$ has at most $[\frac{1475}{562-500c}]$ prime factors, counted according to multiplicity.

In this paper, we shall continue to improve the result of Zhu [38], and establish the following theorem.

Theorem 1.1 *Suppose that $1 < c < \frac{973}{856}$ and let N be a sufficiently large real number. Then for any arbitrary large number $E > 0$, the Diophantine inequality*

$$\left|p_1^c + p_2^c + p_3^c - N\right| < (\log N)^{-E} \tag{1.4}$$

is solvable in prime variables p_1, p_2, p_3 such that each of the numbers $p_i + 2\,(i = 1, 2, 3)$ has at most $[\frac{12626}{4865-4280c}]$ prime factors, counted according to multiplicity.

Notation. In this paper, we denote by ε and A an arbitrarily small positive number and an arbitrarily large constant, respectively, which may not be the same in different formula. The letter p, with or without subscript, always denotes a prime number. As usual, we use $d(n), \mu(n), \varphi(n), \Lambda(n)$ to denote Dirichlet's divisor function, Möbius' function, Euler's function and von Mangoldt's function, respectively. Moreover, we shall use (m, n) and $[m, n]$ for the greatest common divisor and the least common multiple of the integers m and n, respectively. We write $e(t)=\exp(2\pi it)$. $f(x)\ll g(x)$ means that $f(x)=O(g(x))$; $f(x)\asymp g(x)$ means that $f(x)\ll g(x)\ll f(x)$. Suppose that $E>0$ is any arbitrary large number. In addition, we define

$$1<c<\frac{973}{856},\qquad X=N^{\frac{1}{c}},\qquad \delta=\frac{973}{856}-c,\qquad \xi=\frac{18c}{25}-\frac{2}{5},\qquad \eta=\frac{20}{59}\delta,$$

$$z=X^{\eta},\qquad D=X^{\delta},\qquad \tau=X^{\xi-c},\qquad P(z)=\prod_{2<p<z}p,\qquad \Xi=(\log X)^{E+3}.$$

2 Preliminary Lemmas

In this section, we shall give some preliminary lemmas, which are necessary in the proof of Theorem 1.1.

Lemma 2.1 *Let a and b be real numbers with $0<b<a/4$, and let r be a positive integer. Then there exists a function $\vartheta(y)$ which is r times continuously differentiable and such that*

$$\begin{cases}\vartheta(y)=1, & \textit{for}\quad |y|\leqslant a-b,\\ 0<\vartheta(y)<1, & \textit{for}\quad a-b<|y|<a+b,\\ \vartheta(y)=0, & \textit{for}\quad |y|\geqslant a+b,\end{cases}$$

and its Fourier transform

$$\Theta(x)=\int_{-\infty}^{+\infty}\vartheta(y)e(-xy)\mathrm{d}y$$

satisfies the inequality

$$|\Theta(x)|\leqslant\min\left(2a,\frac{1}{\pi|x|},\frac{1}{\pi|x|}\left(\frac{r}{2\pi|x|b}\right)^{r}\right).$$

Proof. See Piatetski–Shapiro [24] or Segal [25]. ■

Lemma 2.2 *Suppose that $D>4$ is a real number and let $\lambda^{\pm}(d)$ be the Rosser's functions of level D. Then we have the following properties.*

(1) *For any positive integer d, we have*

$$|\lambda^{\pm}(d)|\leqslant 1,\qquad \lambda^{\pm}(d)=0\quad\textit{if}\quad d>D\quad\textit{or}\quad \mu(d)=0\ .$$

(2) *If n is a positive integer, then*

$$\sum_{d|n}\lambda^{-}(d)\leqslant\sum_{d|n}\mu(d)\leqslant\sum_{d|n}\lambda^{+}(d).$$

(3) *If z is a real number such that $z^2 \leqslant D \leqslant z^3$ and if*

$$P(z)=\prod_{2<p<z} p, \quad \mathfrak{P}=\prod_{2<p<z}\left(1-\frac{1}{p-1}\right), \quad \mathcal{M}^{\pm}=\sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \quad s_0=\frac{\log D}{\log z}, \tag{2.1}$$

then we have

$$\mathfrak{P} \leqslant \mathcal{M}^{+} \leqslant \mathfrak{P}\Big(F(s_0)+O\big((\log D)^{-1/3}\big)\Big),$$
$$\mathfrak{P} \geqslant \mathcal{M}^{-} \geqslant \mathfrak{P}\Big(f(s_0)+O\big((\log D)^{-1/3}\big)\Big),$$

where $F(s)$ and $f(s)$ denote the classical functions in the linear sieve theory defined by

$$F(s)=\frac{2e^{\gamma}}{s} \qquad \text{and} \qquad f(s)=\frac{2e^{\gamma}\log(s-1)}{s}$$

for $2 \leqslant s \leqslant 3$. Here γ stands for the Euler's constant.

Proof. This is a special case of a more general result. For the details, one can see Chapter 4 of Greaves [15]. ∎

Lemma 2.3 *Let*

$$\Lambda_i=\sum_{d|(p_i+2,P(z))} \mu(d), \qquad \Lambda_i^{\pm}=\sum_{d|(p_i+2,P(z))} \lambda^{\pm}(d), \qquad i=1,2,3.$$

Then we have

$$\Lambda_1\Lambda_2\Lambda_3 \geqslant \Lambda_1^{-}\Lambda_2^{+}\Lambda_3^{+}+\Lambda_1^{+}\Lambda_2^{-}\Lambda_3^{+}+\Lambda_1^{+}\Lambda_2^{+}\Lambda_3^{-}-2\Lambda_1^{+}\Lambda_2^{+}\Lambda_3^{+}.$$

Proof. The proof of Lemma 2.3 is exactly the same as that of Lemma 13 in Brüdern and Fouvry [4], so we omit the details herein. ∎

Lemma 2.4 *Let $b-a \geqslant 1$. Let $f(x)$ be a real function on $[a,b]$ such that $|f''(x)| \asymp \Lambda$ uniformly for $x \in [a,b]$ with $\Lambda>0$. Then we have*

$$\sum_{a<n\leqslant b} e(f(n)) \ll (b-a)\Lambda^{\frac{1}{2}}+\Lambda^{-\frac{1}{2}}.$$

Proof. See Corollary 8.13 of Iwaniec and Kowalski [17], or Theorem 5 of Chapter 1 in Karatsuba [18]. ∎

Lemma 2.5 *For any complex numbers z_n, we have*

$$\Bigg|\sum_{a<n\leqslant b} z_n\Bigg|^2 \leqslant \left(1+\frac{b-a}{Q}\right)\sum_{|q|<Q}\left(1-\frac{|q|}{Q}\right)\sum_{a<n,n+q\leqslant b} z_{n+q}\overline{z_n},$$

where Q is any positive integer.

Proof. See Lemma 8.17 of Iwaniec and Kowalski [17]. ∎

Lemma 2.6 *Suppose that* $f(x):[a,b]\to\mathbb{R}$ *has continuous derivatives of arbitrary order on* $[a,b]$*, where* $1\leqslant a<b\leqslant 2a$*. Suppose further that*

$$\big|f^{(j)}(x)\big|\asymp\lambda_1 a^{1-j},\qquad j\geqslant 1,\qquad x\in[a,b].$$

Then for any exponential pair (κ,λ)*, we have*

$$\sum_{a<n\leqslant b}e(f(n))\ll\lambda_1^{\kappa}a^{\lambda}+\lambda_1^{-1}.$$

Proof. See (3.3.4) of Graham and Kolesnik [14]. ∎

Lemma 2.7 *Let*

$$\mathcal{T}(x)=\sum_{d\leqslant D}\sum_{\substack{\mu X<n\leqslant X\\ d|n+2}}e(n^c x).$$

Then for $0<|x|\leqslant 2\Xi$*, we have*

$$\mathcal{T}(x)\ll X^{\frac{c}{6}+\frac{1}{2}+\varepsilon}D^{\frac{1}{2}}+|x|^{-1}X^{1-c}\log X.$$

Proof. Obviously, we have

$$\begin{aligned}\mathcal{T}(x)&=\sum_{d\leqslant D}\sum_{\frac{\mu X+2}{d}<h\leqslant\frac{X+2}{d}}e\big((hd-2)^c x\big)\\&\ll(\log X)\max_{\mathscr{D}\leqslant D}\sum_{\mathscr{D}<D\leqslant 2\mathscr{D}}\Bigg|\sum_{\frac{\mu X+2}{d}<h\leqslant\frac{X+2}{d}}e\big((hd-2)^c x\big)\Bigg|.\end{aligned}\tag{2.2}$$

For the inner sum in (2.2), it follows from Lemma 2.6 with exponential pair $(\kappa,\lambda)=A(\frac{1}{2},\frac{1}{2})=(\frac{1}{6},\frac{2}{3})$ that

$$\begin{aligned}\sum_{\frac{\mu X+2}{d}<h\leqslant\frac{X+2}{d}}e\big((hd-2)^c x\big)&\ll\big(|x|dX^{c-1}\big)^{\frac{1}{6}}\left(\frac{X}{d}\right)^{\frac{2}{3}}+\frac{X^{1-c}}{|x|d}\\&\ll|x|^{\frac{1}{6}}d^{-\frac{1}{2}}X^{\frac{c}{6}+\frac{1}{2}}+\frac{X^{1-c}}{|x|d}.\end{aligned}\tag{2.3}$$

Putting (2.3) into (2.2), we obtain

$$\begin{aligned}\mathcal{T}(x)&\ll(\log X)\max_{\mathscr{D}\leqslant D}\sum_{\mathscr{D}<D\leqslant 2\mathscr{D}}\left(|x|^{\frac{1}{6}}d^{-\frac{1}{2}}X^{\frac{c}{6}+\frac{1}{2}}+\frac{X^{1-c}}{|x|d}\right)\\&\ll X^{\frac{c}{6}+\frac{1}{2}+\varepsilon}D^{\frac{1}{2}}+|x|^{-1}X^{1-c}\log X,\end{aligned}$$

which completes the proof of Lemma 2.7. ■

For fixed constant $\mu \in (0,1)$, we define

$$I(x)=\int_{\mu X}^{X} e(t^c x)\mathrm{d}t. \tag{2.4}$$

Moreover, we define

$$L(x)=\sum_{d\leqslant D}\lambda(d)\sum_{\substack{\mu X<p\leqslant X\\ d|p+2}}(\log p)e(p^c x), \tag{2.5}$$

where $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)|\leqslant 1, \qquad \lambda(d)=0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d)=0\ . \tag{2.6}$$

Lemma 2.8 *Let $I(x)$ and $L(x)$ be defined as above. Suppose that ξ and δ satisfy the following conditions*

$$\xi+7\delta<2 \qquad \textit{and} \qquad 3\xi+6\delta<2. \tag{2.7}$$

Then for $|x|\leqslant\tau$, we have

$$L(x)=\sum_{d\leqslant D}\frac{\lambda(d)}{\varphi(d)}I(x)+O\bigg(\frac{X}{(\log X)^A}\bigg),$$

where $A>0$ is a sufficiently large constant.

Proof. See the process of the proof of Lemma 4.5 of Zhu [38]. Especially, the condition (2.7) comes from (4.21) and (4.22) in Zhu [38]. ■

Lemma 2.9 *Let $I(x)$ and $L(x)$ be defined as above. Then we have*

$$\int_{|x|\leqslant\tau}\big|L(x)\big|^2\mathrm{d}x \ll X^{2-c}(\log X)^6,$$
$$\int_{|x|\leqslant\tau}\big|I(x)\big|^2\mathrm{d}x \ll X^{2-c}(\log X)^4,$$
$$\int_{|x|\leqslant\Xi}\big|L(x)\big|^2\mathrm{d}x \ll X\Xi(\log X)^6.$$

Proof. See Lemma 11 of Tolev [33], or Lemma 4.6 of Zhu [38]. ■

Lemma 2.10 *Let $f(n)$ be a complex–valued function defined for integers $n\in(\mu X, X]$. Then we have*

$$\sum_{\mu X<n\leqslant X}\Lambda(n)f(n)=S_1-S_2-S_3,$$

where

$$S_1 = \sum_{k \leqslant X^{1/3}} \mu(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} (\log \ell) f(k\ell),$$

$$S_2 = \sum_{k \leqslant X^{2/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} f(k\ell),$$

$$S_3 = \sum_{X^{1/3} < k \leqslant X^{2/3}} a(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} \Lambda(\ell) f(k\ell),$$

and where $a(k)$ and $c(k)$ are real numbers satisfying

$$\big|a(k)\big| \leqslant d(k), \qquad \big|c(k)\big| \leqslant \log k.$$

Proof. See the arguments on page 112 of Vaughan [34]. ■

Lemma 2.11 *Suppose that $1 < c < \frac{973}{856}$. Suppose also that the real numbers $\lambda(d)$ satisfy (2.6) and $L(x)$ is defined by (2.5). Then we have*

$$\sup_{\tau \leqslant |x| \leqslant \Xi} \big|L(x)\big| \ll X^{\frac{3-c}{2} - \varepsilon}$$

Proof. Obviously, we have

$$L(x) = \sum_{d \leqslant D} \lambda(d) \sum_{\substack{\mu X < n \leqslant X \\ d | n+2}} \Lambda(n) e(n^c x) + O\big(X^{\frac{1}{2}+\varepsilon}\big) =: L_1(x) + O\big(X^{\frac{1}{2}+\varepsilon}\big).$$

Trivially, we only need to show that

$$\sup_{\tau \leqslant |x| \leqslant \Xi} \big|L_1(x)\big| \ll X^{\frac{3-c}{2} - \varepsilon}$$

for $1 < c < \frac{973}{856}$. We rewrite $L_1(x)$ in the form

$$L_1(x) = \sum_{\mu X < n \leqslant X} \Lambda(n) f(n),$$

where

$$f(n) = \sum_{\substack{d \leqslant D \\ d | n+2}} \lambda(d) e(n^c x).$$

By Lemma 2.10, we can see that

$$L_1(x) = S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{k \leqslant X^{1/3}} \mu(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} (\log \ell) f(k\ell),$$

$$S_2 = \sum_{k \leqslant X^{2/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} f(k\ell),$$

$$S_3 = \sum_{X^{1/3} < k \leqslant X^{2/3}} a(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} \Lambda(\ell) f(k\ell),$$

and $|a(k)| \leqslant d(k), |c(k)| \leqslant \log k$. Moreover, we write $S_2 = S_2^{(1)} + S_2^{(2)}$, where

$$S_2^{(1)} = \sum_{k \leqslant X^{1/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} f(k\ell), \qquad S_2^{(2)} = \sum_{X^{1/3} < k \leqslant X^{2/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k}} f(k\ell).$$

Therefore, we get

$$L_1(x) \ll |S_1| + |S_2^{(1)}| + |S_2^{(2)}| + |S_3|.$$

First, we consider the sum $S_2^{(1)}$. By noting the fact that $\lambda(d) = 0$ for $2|d$, we can represent $S_2^{(1)}$ as

$$S_2^{(1)} = \sum_{\substack{d \leqslant D \\ (d,2)=1}} \lambda(d) \sum_{k \leqslant X^{1/3}} c(k) \sum_{\substack{\frac{\mu X}{k} < \ell \leqslant \frac{X}{k} \\ d|k\ell+2}} e\big((k\ell)^c x\big).$$

Since $(d,2) = 1$ and $d|k\ell + 2$, we have $(k,d) = 1$, and thus $d|k\ell + 2$ is equivalent to $\ell \equiv \ell_0 (\bmod d)$ for some fixed integer ℓ_0. This means that $\ell = \ell_0 + md$ for some integer m. Hence, we obtain

$$S_2^{(1)} = \sum_{\substack{d \leqslant D \\ (d,2)=1}} \lambda(d) \sum_{\substack{k \leqslant X^{1/3} \\ (k,d)=1}} c(k) \sum_{\frac{\mu X}{kd} - \frac{\ell_0}{d} < m \leqslant \frac{X}{kd} - \frac{\ell_0}{d}} e\big(k^c(\ell_0 + md)^c x\big). \tag{2.8}$$

Setting $h(m) = k^c(\ell_0 + md)^c x$, then we have $|h''(m)| \asymp |x| d^2 k^2 X^{c-2}$. By Lemma 2.4, we know that the inner sum over m in (2.8) is

$$\begin{aligned} &\ll \frac{X}{kd}\big(|x|d^2k^2X^{c-2}\big)^{\frac{1}{2}} + \big(|x|d^2k^2X^{c-2}\big)^{-\frac{1}{2}} \\ &\ll |x|^{\frac{1}{2}} X^{\frac{c}{2}} + |x|^{-\frac{1}{2}} k^{-1} d^{-1} X^{1-\frac{c}{2}}. \end{aligned} \tag{2.9}$$

Putting (2.9) into (2.8), we get

$$S_2^{(1)} \ll X^{\varepsilon}\Big(D|x|^{\frac{1}{2}} X^{\frac{c}{2}+\frac{1}{3}} + |x|^{-\frac{1}{2}} X^{1-\frac{c}{2}}\Big). \tag{2.10}$$

For the sum S_1, by partial summation we can get rid of the factor $\log \ell$ and then proceed as the process of $S_2^{(1)}$ to derive that

$$S_1 \ll X^{\varepsilon}\Big(D|x|^{\frac{1}{2}} X^{\frac{c}{2}+\frac{1}{3}} + |x|^{-\frac{1}{2}} X^{1-\frac{c}{2}}\Big). \tag{2.11}$$

Now, we consider the sum S_3. By a splitting argument, we can divide S_3 into $O(\log X)$ sums of the form

$$W(K) := \sum_{K<k\leqslant K_1} a(k) \sum_{\frac{\mu X}{k}<\ell\leqslant\frac{X}{k}} \Lambda(\ell) \sum_{\substack{d\leqslant D\\ d|k\ell+2}} \lambda(d)e\big((k\ell)^c x\big),$$

where

$$K<K_1\leqslant 2K, \qquad X^{\frac{1}{3}}\leqslant K<K_1\leqslant X^{\frac{2}{3}}.$$

Next, we shall consider the case $K\geqslant X^{\frac{1}{2}}$ first. Trivially, we have

$$W(K)\ll X^{\varepsilon}\sum_{K<k\leqslant K_1}\Bigg|\sum_{\frac{\mu X}{k}<\ell\leqslant\frac{X}{k}}\Phi(\ell)\Bigg|,$$

where

$$\Phi(\ell)=\Lambda(\ell)\sum_{\substack{d\leqslant D\\ d|k\ell+2}}\lambda(d)e\big((k\ell)^c x\big).$$

It follows from Cauchy's inequality that

$$|W(K)|^2\ll X^{\varepsilon}K\sum_{K<k\leqslant K_1}\Bigg|\sum_{\frac{\mu X}{k}<\ell\leqslant\frac{X}{k}}\Phi(\ell)\Bigg|^2. \tag{2.12}$$

Suppose that H is an integer which satisfies

$$1\leqslant H\ll\frac{X}{K}.$$

For the innermost sum on the right–hand side of (2.12), we use Lemma 2.5 to derive that

$$\begin{aligned}
\big|W(K)\big|^2 \ll{}& \frac{X^{1+\varepsilon}}{H}\sum_{K<k\leqslant K_1}\sum_{|h|<H}\left(1-\frac{|h|}{H}\right)\sum_{\frac{\mu X}{k}<\ell,\ell+h\leqslant\frac{X}{k}}\Lambda(\ell)\\
&\times\sum_{\substack{d_1\leqslant D\\ d_1|k\ell+2}}\lambda(d_1)e\big(-(k\ell)^c x\big)\Lambda(\ell+h)\sum_{\substack{d_2\leqslant D\\ d_2|k(\ell+h)+2}}\lambda(d_2)e\big((k(\ell+h))^c x\big)\\
\ll{}& \frac{X^{1+\varepsilon}}{H}\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}\lambda(d_1)\lambda(d_2)\sum_{|h|<H}\left(1-\frac{|h|}{H}\right)\\
&\times\sum_{\frac{\mu X}{K_1}<\ell,\ell+h\leqslant\frac{X}{K}}\Lambda(\ell+h)\Lambda(\ell)\sum_{\substack{\widetilde{K}<k\leqslant\widetilde{K_1}\\ d_1|k\ell+2\\ d_2|k(\ell+h)+2}}e\Big(k^c\big((\ell+h)^c-\ell^c\big)x\Big),
\end{aligned}$$

where

$$\widetilde{K}=\max\left(K,\frac{\mu X}{\ell},\frac{\mu X}{\ell+h}\right),\qquad \widetilde{K_1}=\min\left(K_1,\frac{X}{\ell},\frac{X}{\ell+h}\right).$$

By noting that $\lambda(d)=0$ for $2|d$, we can assume that $2\nmid d_1d_2$. Moreover, it follows from $d_1|k\ell+2$ and $d_2|k(\ell+h)+2$ that $(d_1,\ell)=(d_2,\ell+h)=1$. Hence there exists some fixed integer $k_0=k_0(\ell,h,d_1,d_2)$ such that the pair of conditions $d_1|k\ell+2$ and $d_2|k(\ell+h)+2$ is equivalent to $k\equiv k_0(\bmod\,[d_1,d_2])$. This means that $k=k_0+m[d_1,d_2]$ for some integer m. Therefore, we get

$$\begin{aligned}
\mathscr{F} :=& \sum_{\substack{\widetilde{K}<k\leqslant\widetilde{K_1}\\ d_1|k\ell+2\\ d_2|k(\ell+h)+2}} e\Big(k^c\big((\ell+h)^c-\ell^c\big)x\Big)\\
=& \sum_{\frac{\widetilde{K}-k_0}{[d_1,d_2]}<m\leqslant\frac{\widetilde{K_1}-k_0}{[d_1,d_2]}} e\Big(\big(k_0+m[d_1,d_2]\big)^c\big((\ell+h)^c-\ell^c\big)x\Big)\\
=:& \sum_{\frac{\widetilde{K}-k_0}{[d_1,d_2]}<m\leqslant\frac{\widetilde{K_1}-k_0}{[d_1,d_2]}} e(F(m)),
\end{aligned}$$

say. Trivially, for $h=0$, we have $\mathscr{F}\ll K[d_1,d_2]^{-1}$. By the elementary estimate

$$\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}[d_1,d_2]^{-1}\ll(\log D)^3,$$

we can see that the contribution of $\mathscr{F}$ with $h=0$ to $|W(K)|^2$ is $\ll X^{2+\varepsilon}H^{-1}$. For the case $h\neq0$, we have

$$\big|F^{(j)}(m)\big|\asymp|x|h\ell^{c-1}[d_1,d_2]K^{c-1}\cdot\big(K[d_1,d_2]^{-1}\big)^{1-j},\qquad j\geqslant1.$$

It follows from Lemma 2.6 with $(\kappa,\lambda)=BABABA^3BA^2(\frac12,\frac12)=(\frac{75}{278},\frac{161}{278})$ that

$$\begin{aligned}
\mathscr{F}&\ll|x|^{-1}h^{-1}\ell^{1-c}[d_1,d_2]^{-1}K^{1-c}+\big(|x|h\ell^{c-1}[d_1,d_2]K^{c-1}\big)^{\frac{75}{278}}\big(K[d_1,d_2]^{-1}\big)^{\frac{161}{278}}\\
&\ll|x|^{-1}h^{-1}\ell^{1-c}[d_1,d_2]^{-1}K^{1-c}+|x|^{\frac{75}{278}}h^{\frac{75}{278}}\ell^{\frac{75(c-1)}{278}}[d_1,d_2]^{-\frac{43}{139}}K^{\frac{75c}{278}+\frac{43}{139}}.
\end{aligned}$$

From the following estimate

$$\begin{aligned}
\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}[d_1,d_2]^{-\frac{43}{139}}&\ll\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}\left(\frac{(d_1,d_2)}{d_1d_2}\right)^{\frac{43}{139}}=\sum_{1\leqslant r\leqslant D}\sum_{k_1\leqslant\frac{D}{r}}\sum_{k_2\leqslant\frac{D}{r}}\frac{1}{r^{\frac{43}{139}}k_1^{\frac{43}{139}}k_2^{\frac{43}{139}}}\\
&\ll\sum_{1\leqslant r\leqslant D}r^{-\frac{43}{139}}\Bigg(\sum_{k\leqslant\frac{D}{r}}k^{-\frac{43}{139}}\Bigg)^2\ll\sum_{1\leqslant r\leqslant D}r^{-\frac{43}{139}}\left(\frac{D}{r}\right)^{\frac{192}{139}}\ll D^{\frac{96}{139}},
\end{aligned}$$

we can see that the contribution of $\mathscr{F}$ with $h\neq0$ to $|W(K)|^2$ is

$$\ll X^{1+\varepsilon}H^{-1}\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}\sum_{0<|h|<H}\sum_{\frac{\mu X}{K_1}<\ell,\ell+h\leqslant\frac{X}{K}}\Lambda(\ell+h)\Lambda(\ell)$$

$$\times\Big(|x|^{-1}h^{-1}\ell^{1-c}[d_1,d_2]^{-1}K^{1-c}+|x|^{\frac{75}{278}}h^{\frac{75}{278}}\ell^{\frac{75(c-1)}{278}}[d_1,d_2]^{-\frac{43}{139}}K^{\frac{75c}{278}+\frac{43}{139}}\Big)$$

$$\ll X^{1+\varepsilon}H^{-1}|x|^{-1}K^{1-c}\Bigg(\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}[d_1,d_2]^{-1}\Bigg)\Bigg(\sum_{0<|h|<H}h^{-1}\Bigg)\Bigg(\sum_{\frac{\mu X}{K_1}<\ell,\ell+h\leqslant\frac{X}{K}}\ell^{1-c}\Bigg)$$

$$+X^{1+\varepsilon}H^{-1}|x|^{\frac{75}{278}}K^{\frac{75c}{278}+\frac{43}{139}}\Bigg(\sum_{d_1\leqslant D}\sum_{d_2\leqslant D}[d_1,d_2]^{-\frac{43}{139}}\Bigg)$$

$$\times\Bigg(\sum_{0<|h|<H}h^{\frac{75}{278}}\Bigg)\Bigg(\sum_{\frac{\mu X}{K_1}<\ell,\ell+h\leqslant\frac{X}{K}}\ell^{\frac{75(c-1)}{278}}\Bigg)$$

$$\ll X^{3-c+\varepsilon}|x|^{-1}H^{-1}K^{-1}+X^{\frac{75c+481}{278}+\varepsilon}H^{\frac{75}{278}}D^{\frac{96}{139}}|x|^{\frac{75}{278}}K^{-\frac{117}{278}}.$$

Combining the above two cases, we obtain

$$\big|W(K)\big|^2\ll X^{\varepsilon}\Big(X^2H^{-1}+X^{3-c}|x|^{-1}H^{-1}K^{-1}+X^{\frac{75c+481}{278}+\varepsilon}H^{\frac{75}{278}}D^{\frac{96}{139}}|x|^{\frac{75}{278}}K^{-\frac{117}{278}}\Big).\tag{2.13}$$

Taking

$$H_0=X^{\frac{75(1-c)}{353}}K^{\frac{117}{353}}D^{-\frac{192}{353}}|x|^{-\frac{75}{353}},\qquad H=\big[\min(H_0,XK^{-1})\big],$$

it is easy to see that

$$H^{-1}\asymp H_0^{-1}+KX^{-1}.\tag{2.14}$$

By using (2.13) and (2.14), and noting that $K\geqslant X^{\frac{1}{2}}$, we derive that

$$\begin{aligned}\big|W(K)\big|^2&\ll X^{\varepsilon}\Big(X^2(H_0^{-1}+KX^{-1})+X^{3-c}|x|^{-1}K^{-1}(H_0^{-1}+KX^{-1})\\&\qquad+X^{\frac{75c+481}{278}+\varepsilon}H_0^{\frac{75}{278}}D^{\frac{96}{139}}|x|^{\frac{75}{278}}K^{-\frac{117}{278}}\Big)\\&\ll X^{\varepsilon}\Big(X^{\frac{150c+1145}{706}}D^{\frac{192}{353}}|x|^{\frac{75}{353}}+X^{\frac{5}{3}}+X^{\frac{749-278c}{353}}D^{\frac{192}{353}}|x|^{-\frac{278}{353}}+X^{2-c}|x|^{-1}\Big),\end{aligned}$$

which implies that

$$\big|W(K)\big|\ll X^{\varepsilon}\Big(X^{\frac{150c+1145}{1412}}D^{\frac{96}{353}}|x|^{\frac{75}{706}}+X^{\frac{5}{6}}+X^{\frac{749-278c}{706}}D^{\frac{96}{353}}|x|^{-\frac{139}{353}}+X^{1-\frac{c}{2}}|x|^{-\frac{1}{2}}\Big).\tag{2.15}$$

If $K<X^{\frac{1}{2}}$, we write $W(K)$ as

$$W(K)=\sum_{\mu X/K_1<\ell\leqslant X/K}\Lambda(\ell)\sum_{\max(K,\mu X/\ell)<k\leqslant\min(K_1,X/\ell)}a(k)\sum_{\substack{d\leqslant D\\ d|k\ell+2}}\lambda(d)e\big((k\ell)^cx\big).$$

Then we have $X/K\gg X^{\frac{1}{2}}$ and we can proceed as the previous process by changing the roles of the variables k and ℓ reversed. Therefore, we also derive the estimate (2.15) in this case. Consequently, we obtain

$$S_3\ll X^{\varepsilon}\Big(X^{\frac{150c+1145}{1412}}D^{\frac{96}{353}}|x|^{\frac{75}{706}}+X^{\frac{5}{6}}+X^{\frac{749-278c}{706}}D^{\frac{96}{353}}|x|^{-\frac{139}{353}}+X^{1-\frac{c}{2}}|x|^{-\frac{1}{2}}\Big).\tag{2.16}$$

For $S_2^{(2)}$, we can use the same way to give the upper bound estimate by the expression of the right–hand side of (2.16).

Above all, we deduce that

$$\begin{aligned}L_1(x) \ll X^{\varepsilon}\Big(&X^{\frac{c}{2}+\frac{1}{3}}D|x|^{\frac{1}{2}}+X^{\frac{150c+1145}{1412}}D^{\frac{96}{353}}|x|^{\frac{75}{706}}+X^{\frac{5}{6}}\\&+X^{\frac{749-278c}{706}}D^{\frac{96}{353}}|x|^{-\frac{139}{353}}+X^{1-\frac{c}{2}}|x|^{-\frac{1}{2}}\Big)\\ \ll X^{\varepsilon}\Big(&X^{\frac{c}{2}+\frac{1}{3}+\delta}+X^{\frac{150c+1145}{1412}+\frac{96}{353}\delta}+X^{\frac{5}{6}}+X^{\frac{749}{706}+\frac{96}{353}\delta-\frac{139}{353}\xi}+X^{1-\frac{1}{2}\xi}\Big),\end{aligned}$$

and thus, for $1<c<\frac{973}{856}$, there holds

$$\sup_{\tau\leqslant|x|\leqslant\Xi}\big|L_1(x)\big| \ll X^{\frac{3-c}{2}-\varepsilon}.$$

This completes the proof Lemma 2.11. ■

Lemma 2.12 *For $1<c<\frac{973}{856}$, we have*

$$\int_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^3\mathrm{d}x \ll X^{3-c-\varepsilon}.$$

Proof. We have

$$\begin{aligned}\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^3\mathrm{d}x\bigg| &= \bigg|\int_{\tau\leqslant|x|\leqslant\Xi}L(x)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|\\ &= \bigg|\sum_{d\leqslant D}\lambda(d)\sum_{\substack{\mu X<p\leqslant X\\ d|p+2}}(\log p)\int_{\tau\leqslant|x|\leqslant\Xi}e(p^cx)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|\\ &\leqslant (\log X)\sum_{d\leqslant D}\sum_{\substack{\mu X<p\leqslant X\\ d|p+2}}\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}e(p^cx)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|\\ &\leqslant (\log X)\sum_{d\leqslant D}\sum_{\substack{\mu X<n\leqslant X\\ d|n+2}}\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}e(n^cx)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|.\end{aligned}$$

By Cauchy's inequality, we have

$$\begin{aligned}&\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^3\mathrm{d}x\bigg|^2 \ll (\log X)^2\Bigg(\sum_{\mu X<n\leqslant X}\sum_{\substack{d\leqslant D\\ d|n+2}}\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}e(n^cx)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|\Bigg)^2\\ &\ll X(\log X)^2\sum_{\mu X<n\leqslant X}\Bigg(\sum_{\substack{d\leqslant D\\ d|n+2}}\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}e(n^cx)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|\Bigg)^2\\ &\ll X(\log X)^2\sum_{\mu X<n\leqslant X}\Bigg(\sum_{\substack{d\leqslant D\\ d|n+2}}1\Bigg)\sum_{\substack{d\leqslant D\\ d|n+2}}\bigg|\int_{\tau\leqslant|x|\leqslant\Xi}e(n^cx)\overline{L(x)}\big|L(x)\big|\mathrm{d}x\bigg|^2\end{aligned}$$

$$\ll X^{1+\frac{\varepsilon}{4}} \sum_{d\leqslant D} \sum_{\substack{\mu X<n\leqslant X\\ d|n+2}} \left| \int_{\tau\leqslant|x|\leqslant\Xi} e(n^c x)\overline{L(x)}\big|L(x)\big|\mathrm{d}x \right|^2$$

$$= X^{1+\frac{\varepsilon}{4}} \sum_{d\leqslant D} \sum_{\substack{\mu X<n\leqslant X\\ d|n+2}} \left(\int_{\tau\leqslant|x|\leqslant\Xi} e(n^c x)\overline{L(x)}\big|L(x)\big|\mathrm{d}x \right)\left(\int_{\tau\leqslant|y|\leqslant\Xi} \overline{e(n^c y)\overline{L(y)}\big|L(y)\big|}\mathrm{d}y \right)$$

$$\ll X^{1+\frac{\varepsilon}{4}} \int_{\tau\leqslant|y|\leqslant\Xi} \big|L(y)\big|^2\mathrm{d}y \int_{\tau\leqslant|x|\leqslant\Xi} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x. \tag{2.17}$$

For the innermost integral on the right–hand side of (2.17), we have

$$\int_{\tau\leqslant|x|\leqslant\Xi} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x$$
$$\ll \int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ |x-y|\leqslant X^{-c}}} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x + \int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ X^{-c}<|x-y|\leqslant 2\Xi}} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x. \tag{2.18}$$

By the trivial estimate $|\mathcal{T}(x-y)|\ll X\log X$ and Lemma 2.11, we have

$$\int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ |x-y|\leqslant X^{-c}}} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x \ll X(\log X)\times \sup_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^2 \times \int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ |x-y|\leqslant X^{-c}}} \mathrm{d}x$$
$$\ll X^{1-c}(\log X)\times \sup_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^2 \ll X^{4-2c-\varepsilon}. \tag{2.19}$$

It follows from Lemma 2.7 and Lemma 2.9 that

$$\int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ X^{-c}<|x-y|\leqslant 2\Xi}} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x$$
$$\ll \int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ X^{-c}<|x-y|\leqslant 2\Xi}} \big|L(x)\big|^2\left(X^{\frac{c}{6}+\frac{1}{2}+\varepsilon}D^{\frac{1}{2}} + |x-y|^{-1}X^{1-c}\log X\right)\mathrm{d}x$$
$$\ll X^{\frac{c}{6}+\frac{1}{2}+\frac{1}{2}\delta+\varepsilon}\int_{|x|\leqslant\Xi}\big|L(x)\big|^2\mathrm{d}x + X^{1-c+\varepsilon}\times\sup_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^2\times\int_{\substack{\tau\leqslant|x|\leqslant\Xi\\ X^{-c}<|x-y|\leqslant 2\Xi}}\frac{\mathrm{d}x}{|x-y|}$$
$$\ll X^{\frac{c}{6}+\frac{3}{2}+\frac{1}{2}\delta+\varepsilon} + X^{4-2c-\varepsilon} \ll X^{4-2c-\varepsilon}. \tag{2.20}$$

From (2.18), (2.19) and (2.20), we have

$$\int_{\tau\leqslant|x|\leqslant\Xi} \big|L(x)\big|^2\big|\mathcal{T}(x-y)\big|\mathrm{d}x \ll X^{4-2c-\varepsilon}. \tag{2.21}$$

Combining (2.17), (2.21) and Lemma 2.9, we get

$$\left|\int_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^3\mathrm{d}x\right|^2 \ll X^{1+\frac{\varepsilon}{4}}\cdot X^{4-2c-\varepsilon}\int_{\tau\leqslant|y|\leqslant\Xi}\big|L(y)\big|^2\mathrm{d}y \ll X^{6-2c-\frac{\varepsilon}{2}},$$

which implies that

$$\left|\int_{\tau\leqslant|x|\leqslant\Xi}\big|L(x)\big|^3\mathrm{d}x\right| \ll X^{3-c-\varepsilon}.$$

This completes the proof of Lemma 2.12. ∎

3 Proof of Theorem 1.1

Consider the sum

$$\Gamma = \sum_{\substack{\mu X < p_1, p_2, p_3 \leqslant X \\ |p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E} \\ (p_i + 2, P(z)) = 1 \\ i = 1,2,3}} (\log p_1)(\log p_2)(\log p_3). \tag{3.1}$$

In order to prove Theorem 1.1, we only need to show that $\Gamma > 0$. Suppose that $\Theta(x)$ and $\vartheta(x)$ are the functions which are defined in Lemma 2.1 with parameters $a = \frac{7}{8}(\log N)^{-E}, b = \frac{1}{8}(\log N)^{-E}$ and $r = [\log^2 X]$. Therefore, we get

$$\vartheta(y) = 0 \quad \text{if} \quad |y| \geqslant (\log N)^{-E}, \quad 0 < \vartheta(y) < 1 \quad \text{if} \quad |y| < (\log N)^{-E}. \tag{3.2}$$

Obviously, it follows from (3.1) and (3.2) that

$$\Gamma \geqslant \widetilde{\Gamma} := \sum_{\substack{\mu X < p_1, p_2, p_3 \leqslant X \\ (p_i + 2, P(z)) = 1 \\ i = 1,2,3}} (\log p_1)(\log p_2)(\log p_3)\vartheta(p_1^c + p_2^c + p_3^c - N). \tag{3.3}$$

By the definition of Λ_i in Lemma 2.3, we know that

$$\Lambda_i = \sum_{d | (p_i + 2, P(z))} \mu(d) = \begin{cases} 1, & \text{if } (p_i + 2, P(z)) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

which combined with Lemma 2.3 yields that

$$\begin{aligned} \widetilde{\Gamma} = & \sum_{\mu X < p_1, p_2, p_3 \leqslant X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1\Lambda_2\Lambda_3\vartheta(p_1^c + p_2^c + p_3^c - N) \\ \geqslant & \sum_{\mu X < p_1, p_2, p_3 \leqslant X} (\log p_1)(\log p_2)(\log p_3)\vartheta(p_1^c + p_2^c + p_3^c - N) \\ & \times \left(\Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+\right) \\ = & \Gamma_1 + \Gamma_2 + \Gamma_3 - 2\Gamma_4, \end{aligned}$$

say. Trivially, by the symmetric property, we can see that

$$\begin{aligned} \Gamma_1 = \Gamma_2 = \Gamma_3 = & \sum_{\mu X < p_1, p_2, p_3 \leqslant X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1^-\Lambda_2^+\Lambda_3^+\vartheta(p_1^c + p_2^c + p_3^c - N), \\ \Gamma_4 = & \sum_{\mu X < p_1, p_2, p_3 \leqslant X} (\log p_1)(\log p_2)(\log p_3)\Lambda_1^+\Lambda_2^+\Lambda_3^+\vartheta(p_1^c + p_2^c + p_3^c - N), \end{aligned}$$

and thus

$$\widetilde{\Gamma} \geqslant 3\Gamma_1 - 2\Gamma_4. \tag{3.4}$$

Let $\lambda^{\pm}(d)$ be the Rosser's functions of level D, and define

$$\begin{aligned}L^{\pm}(x) :&= \sum_{\mu X<p\leqslant X}(\log p)e(p^c x)\sum_{d|(p+2,P(z))}\lambda^{\pm}(d)\\ &= \sum_{d|P(z)}\lambda^{\pm}(d)\sum_{\substack{\mu X<p\leqslant X\\ d|p+2}}(\log p)e(p^c x).\end{aligned}$$

According to the Fourier's inverse transformation, we have

$$\begin{aligned}\Gamma_1 =& \sum_{\mu X<p_1,p_2,p_3\leqslant X}(\log p_1)(\log p_2)(\log p_3)\Lambda_1^-\Lambda_2^+\Lambda_3^+\\ &\times\int_{-\infty}^{+\infty}\Theta(x)e\big((p_1^c+p_2^c+p_3^c-N)x\big)\mathrm{d}x\\ =& \int_{-\infty}^{+\infty}L^-(x)(L^+(x))^2\Theta(x)e(-Nx)\mathrm{d}x\\ =& \bigg(\int_{|x|\leqslant\tau}+\int_{\tau<|x|<\Xi}+\int_{|x|\geqslant\Xi}\bigg)L^-(x)(L^+(x))^2\Theta(x)e(-Nx)\mathrm{d}x\\ =& \Gamma_1^{(1)}+\Gamma_1^{(2)}+\Gamma_1^{(3)},\end{aligned}\tag{3.5}$$

say. Similarly, for Γ_4, we also have

$$\begin{aligned}\Gamma_4 &= \int_{-\infty}^{+\infty}(L^+(x))^3\Theta(x)e(-Nx)\mathrm{d}x\\ &= \bigg(\int_{|x|\leqslant\tau}+\int_{\tau<|x|<\Xi}+\int_{|x|\geqslant\Xi}\bigg)(L^+(x))^3\Theta(x)e(-Nx)\mathrm{d}x\\ &=: \Gamma_4^{(1)}+\Gamma_4^{(2)}+\Gamma_4^{(3)}.\end{aligned}\tag{3.6}$$

By the trivial estimate $L^{\pm}(x)\ll X^{1+\varepsilon}$ and Lemma 2.1, we get

$$\begin{aligned}\Gamma_1^{(3)},\Gamma_4^{(3)} &\ll X^{3+\varepsilon}\int_{\Xi}^{\infty}\frac{1}{\pi|x|}\bigg(\frac{r}{2\pi|x|b}\bigg)^r\mathrm{d}x\\ &\ll X^{3+\varepsilon}\bigg(\frac{r}{2\pi b}\bigg)^r\int_{\Xi}^{\infty}\frac{\mathrm{d}x}{x^{r+1}}\ll X^{3+\varepsilon}\bigg(\frac{r}{2\pi\Xi b}\bigg)^r\\ &\ll \frac{X^{3+\varepsilon}}{(2\pi\log X)^{\log X}}\ll\frac{X^{3+\varepsilon}}{X^{\log\log X+\log(2\pi)}}\ll 1.\end{aligned}\tag{3.7}$$

It follows from (3.3), (3.4), (3.5), (3.6) and (3.7) that

$$\Gamma\geqslant\Big(3\Gamma_1^{(1)}-2\Gamma_4^{(1)}\Big)+\Big(3\Gamma_1^{(2)}-2\Gamma_4^{(2)}\Big)+O(1).\tag{3.8}$$

By Lemma 2.8, we know that, for $|x|\leqslant\tau$, there holds

$$L^{\pm}(x)=\mathcal{M}^{\pm}I(x)+O\bigg(\frac{X}{(\log X)^A}\bigg),$$

where $\mathcal{M}^{\pm}$ and $I(x)$ are defined by (2.1) and (2.4), respectively. By noting the identity

$$
\begin{aligned}
& L^{-}(x)\big(L^{+}(x)\big)^{2}-\mathcal{M}^{-}\big(\mathcal{M}^{+}\big)^{2} I^{3}(x) \\
=& \big(L^{+}(x)\big)^{2}\big(L^{-}(x)-\mathcal{M}^{-} I(x)\big)+\mathcal{M}^{-} I(x) L^{+}(x)\big(L^{+}(x)-\mathcal{M}^{+} I(x)\big) \\
& \quad+\mathcal{M}^{-} \mathcal{M}^{+} I^{2}(x)\big(L^{+}(x)-\mathcal{M}^{+} I(x)\big)
\end{aligned}
$$

and the elementary estimate

$$
\mathcal{M}^{\pm} \ll \sum_{d \leqslant D} \frac{1}{\varphi(d)} \ll \log X,
$$

we derive that

$$
\begin{aligned}
& \Big|L^{-}(x)\big(L^{+}(x)\big)^{2}-\mathcal{M}^{-}\big(\mathcal{M}^{+}\big)^{2} I^{3}(x)\Big| \\
\ll & \frac{X}{(\log X)^{A}}\Big(\big|L^{+}(x)\big|^{2}+(\log X)\big|L^{+}(x) I(x)\big|+(\log X)^{2}\big|I(x)\big|^{2}\Big) \\
\ll & \frac{X}{(\log X)^{A-2}}\Big(\big|L^{+}(x)\big|^{2}+\big|I(x)\big|^{2}\Big).
\end{aligned}
$$

Define

$$
\mathcal{I}_{0}=\int_{|x| \leqslant \tau} I^{3}(x) \Theta(x) e(-N x) \mathrm{d} x .
$$

Then from Lemma 2.1 and Lemma 2.9, we derive that

$$
\begin{aligned}
\Big|\Gamma_{1}^{(1)}-\mathcal{M}^{-}\big(\mathcal{M}^{+}\big)^{2} \mathcal{I}_{0}\Big| \leqslant & \int_{|x| \leqslant \tau}\Big|L^{-}(x)\big(L^{+}(x)\big)^{2}-\mathcal{M}^{-}\big(\mathcal{M}^{+}\big)^{2} I^{3}(x)\Big|\big|\Theta(x)\big| \mathrm{d} x \\
\ll & \frac{X}{(\log X)^{A+E-2}}\bigg(\int_{|x| \leqslant \tau}\big|L^{+}(x)\big|^{2} \mathrm{d} x+\int_{|x| \leqslant \tau}\big|I(x)\big|^{2} \mathrm{d} x\bigg) \\
\ll & \frac{X^{3-c}}{(\log X)^{A+E-8}} .
\end{aligned} \tag{3.9}
$$

Set

$$
\mathcal{I}=\int_{-\infty}^{+\infty} I^{3}(x) \Theta(x) e(-N x) \mathrm{d} x .
$$

It follows from Lemma 6 of Tolev [29] that

$$
\mathcal{I} \gg \frac{X^{3-c}}{(\log X)^{E}} . \tag{3.10}
$$

Since $\left|\frac{\mathrm{d}}{\mathrm{d}t}(t^{c} x)\right| \gg|x| X^{c-1}$ for $t \in(\mu X, X]$, by Lemma 4.2 of Titchmarsh [27], we have $|I(x)| \ll(|x| X^{c-1})^{-1}$. Therefore, we obtain

$$
\begin{aligned}
\big|\mathcal{I}-\mathcal{I}_{0}\big| & \ll \int_{\tau}^{\infty}\big|I(x)\big|^{3}\big|\Theta(x)\big| \mathrm{d} x \ll(\log X)^{-E} \int_{\tau}^{\infty}\left(\frac{1}{|x| X^{c-1}}\right)^{3} \mathrm{d} x \\
& \ll(\log X)^{-E} X^{3-3 c} \tau^{-2} \ll \frac{X^{3-c-2 \xi}}{(\log X)^{E}} .
\end{aligned} \tag{3.11}
$$

Combining (3.9) and (3.11), we get

$$\begin{aligned}
\Gamma_1^{(1)} &= \mathcal{M}^-(\mathcal{M}^+)^2\mathcal{I}_0 + O\bigg(\frac{X^{3-c}}{(\log X)^{A+E-8}}\bigg) \\
&= \mathcal{M}^-(\mathcal{M}^+)^2\bigg(\mathcal{I} + O\bigg(\frac{X^{3-c-2\xi}}{(\log X)^{E}}\bigg)\bigg) + O\bigg(\frac{X^{3-c}}{(\log X)^{A+E-8}}\bigg) \\
&= \mathcal{M}^-(\mathcal{M}^+)^2\mathcal{I} + O\bigg(\frac{X^{3-c}}{(\log X)^{A+E-8}}\bigg). \qquad (3.12)
\end{aligned}$$

Similarly, we can get

$$\Gamma_4^{(1)} = (\mathcal{M}^+)^3\mathcal{I} + O\bigg(\frac{X^{3-c}}{(\log X)^{A+E-8}}\bigg). \qquad (3.13)$$

From Mertens' prime number theorem (See [23]), we know that

$$\mathfrak{P} \asymp \frac{1}{\log X},$$

which combined with (3.10), (3.12), (3.13) and Lemma 2.2 yields

$$\begin{aligned}
&3\Gamma_1^{(1)} - 2\Gamma_4^{(1)} = (3\mathcal{M}^- - 2\mathcal{M}^+)(\mathcal{M}^+)^2\mathcal{I} + O\bigg(\frac{X^{3-c}}{(\log X)^{A+E-8}}\bigg) \\
&\geqslant \bigg(3f\bigg(\frac{\log D}{\log z}\bigg) - 2F\bigg(\frac{\log D}{\log z}\bigg)\bigg)(1+O(\log^{-1/3}X))\mathfrak{P}^3\mathcal{I} + O\bigg(\frac{X^{3-c}}{(\log X)^{A+E-8}}\bigg) \\
&= \bigg(3f\bigg(\frac{59}{20}\bigg) - 2F\bigg(\frac{59}{20}\bigg)\bigg)\mathfrak{P}^3\mathcal{I} + O\bigg(\frac{X^{3-c}}{(\log X)^{E+10/3}}\bigg) \\
&= \frac{120e^{\gamma}}{59}\bigg(\log\frac{39}{20} - \frac{2}{3}\bigg)\mathfrak{P}^3\mathcal{I} + O\bigg(\frac{X^{3-c}}{(\log X)^{E+10/3}}\bigg). \qquad (3.14)
\end{aligned}$$

For $\Gamma_1^{(2)}$, by Hölder's inequality, Lemma 2.1 and Lemma 2.12, we obtain

$$\begin{aligned}
\big|\Gamma_1^{(2)}\big| &\ll \int_{\tau\leqslant|x|\leqslant\Xi} |L^-(x)||L^+(x)|^2|\Theta(x)|\mathrm{d}x \\
&\ll (\log X)^{-E}\int_{\tau\leqslant|x|\leqslant\Xi} |L^-(x)||L^+(x)|^2\mathrm{d}x \\
&\ll (\log X)^{-E}\bigg(\int_{\tau\leqslant|x|\leqslant\Xi} |L^-(x)|^3\mathrm{d}x\bigg)^{\frac{1}{3}}\bigg(\int_{\tau\leqslant|x|\leqslant\Xi} |L^+(x)|^3\mathrm{d}x\bigg)^{\frac{2}{3}} \\
&\ll (\log X)^{-E}\cdot X^{3-c-\varepsilon} \ll X^{3-c-\varepsilon}. \qquad (3.15)
\end{aligned}$$

Similarly, we have

$$\big|\Gamma_4^{(2)}\big| \ll (\log X)^{-E}\int_{\tau\leqslant|x|\leqslant\Xi} |L^+(x)|^3\mathrm{d}x \ll X^{3-c-\varepsilon}. \qquad (3.16)$$

According to (3.8), (3.14), (3.15) and (3.16), we deduce that

$$\Gamma \geqslant \Big(3\Gamma_1^{(1)} - 2\Gamma_4^{(1)}\Big) + O\Big(\Big|\Gamma_1^{(2)}\Big| + \Big|\Gamma_4^{(2)}\Big| + 1\Big)$$

$$\geqslant \frac{120e^{\gamma}}{59}\left(\log\frac{39}{20}-\frac{2}{3}\right)\mathfrak{P}^3\mathcal{I}+O\left(\frac{X^{3-c}}{(\log X)^{E+10/3}}\right)+O\big(X^{3-c-\varepsilon}\big)$$
$$\gg \frac{X^{3-c}}{(\log X)^{E+3}}.$$

Therefore, $\Gamma > 0$ for sufficiently large real number N. Then inequality (1.4) would have a solution in primes p_1, p_2, p_3 satisfying

$$(p_1+2, P(z)) = (p_2+2, P(z)) = (p_3+2, P(z)) = 1. \tag{3.17}$$

If the number $p_i + 2$ has l prime factors counted with multiplicity, then from (3.17) and from the condition $\mu X < p_i \leqslant X$, it is easy to find that $l \leqslant \eta^{-1}$, which means that $p_i + 2$ would be almost–prime of order $[\eta^{-1}] = [\frac{12626}{4865-4280c}]$.

This completes the proof of Theorem 1.1.

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