

# A ternary Diophantine inequality with prime numbers of a special form

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**Abstract:** Let  $N$  be a sufficiently large real number. In this paper, it is proved that, for  $1 < c < \frac{973}{856}$  and for any arbitrary large number  $E > 0$ , the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E}$$

is solvable in prime variables  $p_1, p_2, p_3$  such that each of the numbers  $p_i + 2$  ( $i = 1, 2, 3$ ) has at most  $[\frac{12626}{4865-4280c}]$  prime factors, counted according to multiplicity. This result constitutes an improvement upon the previous result of Zhu [38].

**Keywords:** Diophantine inequality; exponential sum; prime variable; almost-prime

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## 1 Introduction and main result

For fixed integer  $k \geq 1$  and sufficiently large integer  $N$ , the well-known Waring–Goldbach problem is devoted to investigating the solvability of the following Diophantine equality

$$N = p_1^k + p_2^k + \cdots + p_s^k \tag{1.1}$$

in prime variables  $p_1, p_2, \dots, p_s$ . In this topic, many mathematicians have derived many splendid results. For instance, in 1937, Vinogradov [35] proved that such a representation of the type (1.1) exists for every sufficiently large odd integer with  $k =$

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1,  $s = 3$ . Moreover, in 1938, Hua [16] showed that (1.1) is solvable for every sufficiently large integer  $N$  satisfying  $N \equiv 5 \pmod{24}$  with  $k = 2, s = 5$ .

In 1952, Piatetski–Shapiro [24] studied the following analog of the Waring–Goldbach problem. Suppose that  $c > 1$  is not an integer and  $\varepsilon$  is a small positive number. Denote by  $H(c)$  the smallest natural number  $r$  such that, for every sufficiently large real number  $N$ , the Diophantine inequality

$$|p_1^c + p_2^c + \cdots + p_s^c - N| < \varepsilon \quad (1.2)$$

is solvable in primes  $p_1, p_2, \dots, p_s$ . Then it was proved in [24] that

$$\limsup_{c \rightarrow +\infty} \frac{H(c)}{c \log c} \leq 4.$$

Also, in [24], Piatetski–Shapiro considered the case  $r = 5$  in (1.2) and proved that  $H(c) \leq 5$  for  $1 < c < 3/2$ . Later, the upper bound  $3/2$  for  $H(c) \leq 5$  was improved successively to

$$\frac{14142}{8923}, \quad \frac{1 + \sqrt{5}}{2}, \quad \frac{81}{40}, \quad \frac{108}{53}, \quad 2.041, \quad \frac{52}{25}$$

by Zhai and Cao [36], Garaev [13], Zhai and Cao [37], Shi and Liu [26], Baker and Weingartner [1], Li and Cai [21], respectively.

From these results and the Goldbach–Vinogradov theorem, it is reasonable to conjecture that if  $c$  is near to 1, then the Diophantine inequality (1.2) is solvable for  $s = 3$ . This conjecture was first established by Tolev [28] for  $1 < c < \frac{27}{26}$ . Since then, the range of  $c$  was enlarged to

$$\frac{15}{14}, \quad \frac{13}{12}, \quad \frac{11}{10}, \quad \frac{237}{214}, \quad \frac{61}{55}, \quad \frac{10}{9}, \quad \frac{43}{36}$$

by Tolev [29], Cai [5], Cai [6] and Kumchev and Nedeva [19] independently, Cao and Zhai [8], Kumchev [20], Baker and Weingartner [2], Cai [7], successively and respectively. The best result up to now belongs to Baker [3] with  $1 < c < 6/5$ .

Another central problem in the theory of prime distribution, namely the twin prime conjecture, states that there exist infinitely many primes  $p$  such that  $p + 2$  is also prime. Although this conjecture has resisted all attacks, there have been spectacular partial achievements. Let  $\mathcal{P}_r$  denote an almost-prime with at most  $r$  prime factors, counted according to multiplicity. One well-known result is due to Chen [9, 10], who proved that there exist infinitely many primes  $p$  such that  $p + 2$  has at most 2 prime factors.

Bearing in mind the result of Chen [9, 10], one may try to study the arithmetical properties of the set of primes  $p$  such that  $p + 2 \in \mathcal{P}_r$  for a fixed  $r \geq 2$  and, in particular, to establish the solvability of Diophantine equations or inequalities in such primes. For

instance, combining the results of Vinogradov [35] and Chen [10], Tolev [30, 31, 32] established such kinds of results, while Matomäki and Shao [22] improved the result of Tolev [32] and proved that every sufficiently large odd integer  $N$  can be represented as a sum of three primes  $p_1, p_2, p_3$  such that  $p_i + 2 \in \mathcal{P}_2$  ( $i = 1, 2, 3$ ).

Motivated by Tolev [28, 29] and Chen [9, 10], it is reasonable to conjecture that if the constant  $c > 1$  is close to one, then inequality (1.2), with a suitable  $\varepsilon = \varepsilon(N)$  satisfying  $\varepsilon(N) \rightarrow 0$  as  $N \rightarrow \infty$ , is solvable in primes  $p_i$  such that  $p_i + 2$  are almost primes of a certain fixed order for  $s = 3$ . An attempt to establish a result of this type was first made by Dimitrov [11], he dealt with this problem with  $0 < c < 4/21$  and  $p_i + 2 = \mathcal{P}_{10}, i = 1, 2, 3$ . After that, the next step also belongs to Dimitrov [12] with  $1 < c < 121/120$  and  $p_i + 2 = \mathcal{P}_{29}, i = 1, 2, 3$ .

Later, motivated by Dimitrov [11], Tolev [33] proved that, for  $1 < c < \frac{15}{14}$ , the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E} \quad (1.3)$$

is solvable in primes  $p_1, p_2, p_3$  such that each of the numbers  $p_i + 2$  ( $i = 1, 2, 3$ ) has at most  $\lfloor \frac{369}{180-168c} \rfloor$  prime factors, counted according to multiplicity, where  $E > 0$  is a sufficiently large constant. Recently, Zhu [38] improved the result of Tolev [33] and showed that for  $1 < c < \frac{281}{250}$ , (1.3) is solvable in primes  $p_1, p_2, p_3$  such that each of the numbers  $p_i + 2$  ( $i = 1, 2, 3$ ) has at most  $\lfloor \frac{1475}{562-500c} \rfloor$  prime factors, counted according to multiplicity.

In this paper, we shall continue to improve the result of Zhu [38], and establish the following theorem.

**Theorem 1.1** *Suppose that  $1 < c < \frac{973}{856}$  and let  $N$  be a sufficiently large real number. Then for any arbitrary large number  $E > 0$ , the Diophantine inequality*

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E} \quad (1.4)$$

*is solvable in prime variables  $p_1, p_2, p_3$  such that each of the numbers  $p_i + 2$  ( $i = 1, 2, 3$ ) has at most  $\lfloor \frac{12626}{4865-4280c} \rfloor$  prime factors, counted according to multiplicity.*

**Notation.** In this paper, we denote by  $\varepsilon$  and  $A$  an arbitrarily small positive number and an arbitrarily large constant, respectively, which may not be the same in different formula. The letter  $p$ , with or without subscript, always denotes a prime number. As usual, we use  $d(n), \mu(n), \varphi(n), \Lambda(n)$  to denote Dirichlet's divisor function, Möbius' function, Euler's function and von Mangoldt's function, respectively. Moreover, we shall use  $(m, n)$  and  $[m, n]$  for the greatest common divisor and the least common

multiple of the integers  $m$  and  $n$ , respectively. We write  $e(t) = \exp(2\pi it)$ .  $f(x) \ll g(x)$  means that  $f(x) = O(g(x))$ ;  $f(x) \asymp g(x)$  means that  $f(x) \ll g(x) \ll f(x)$ . Suppose that  $E > 0$  is any arbitrary large number. In addition, we define

$$1 < c < \frac{973}{856}, \quad X = N^{\frac{1}{c}}, \quad \delta = \frac{973}{856} - c, \quad \xi = \frac{18c}{25} - \frac{2}{5}, \quad \eta = \frac{20}{59}\delta,$$

$$z = X^\eta, \quad D = X^\delta, \quad \tau = X^{\xi-c}, \quad P(z) = \prod_{2 < p < z} p, \quad \Xi = (\log X)^{E+3}.$$

## 2 Preliminary Lemmas

In this section, we shall give some preliminary lemmas, which are necessary in the proof of Theorem 1.1.

**Lemma 2.1** *Let  $a$  and  $b$  be real numbers with  $0 < b < a/4$ , and let  $r$  be a positive integer. Then there exists a function  $\vartheta(y)$  which is  $r$  times continuously differentiable and such that*

$$\begin{cases} \vartheta(y) = 1, & \text{for } |y| \leq a - b, \\ 0 < \vartheta(y) < 1, & \text{for } a - b < |y| < a + b, \\ \vartheta(y) = 0, & \text{for } |y| \geq a + b, \end{cases}$$

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{+\infty} \vartheta(y) e(-xy) dy$$

satisfies the inequality

$$|\Theta(x)| \leq \min \left( 2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{r}{2\pi|x|b} \right)^r \right).$$

**Proof.** See Piatetski-Shapiro [24] or Segal [25]. ■

**Lemma 2.2** *Suppose that  $D > 4$  is a real number and let  $\lambda^\pm(d)$  be the Rosser's functions of level  $D$ . Then we have the following properties.*

(1) *For any positive integer  $d$ , we have*

$$|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0.$$

(2) *If  $n$  is a positive integer, then*

$$\sum_{d|n} \lambda^-(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^+(d).$$

(3) If  $z$  is a real number such that  $z^2 \leq D \leq z^3$  and if

$$P(z) = \prod_{2 < p < z} p, \quad \mathfrak{P} = \prod_{2 < p < z} \left(1 - \frac{1}{p-1}\right), \quad \mathcal{M}^\pm = \sum_{d|P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}, \quad s_0 = \frac{\log D}{\log z}, \quad (2.1)$$

then we have

$$\begin{aligned} \mathfrak{P} &\leq \mathcal{M}^+ \leq \mathfrak{P} \left( F(s_0) + O((\log D)^{-1/3}) \right), \\ \mathfrak{P} &\geq \mathcal{M}^- \geq \mathfrak{P} \left( f(s_0) + O((\log D)^{-1/3}) \right), \end{aligned}$$

where  $F(s)$  and  $f(s)$  denote the classical functions in the linear sieve theory defined by

$$F(s) = \frac{2e^\gamma}{s} \quad \text{and} \quad f(s) = \frac{2e^\gamma \log(s-1)}{s}$$

for  $2 \leq s \leq 3$ . Here  $\gamma$  stands for the Euler's constant.

**Proof.** This is a special case of a more general result. For the details, one can see Chapter 4 of Greaves [15]. ■

**Lemma 2.3** *Let*

$$\Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d), \quad \Lambda_i^\pm = \sum_{d|(p_i+2, P(z))} \lambda^\pm(d), \quad i = 1, 2, 3.$$

*Then we have*

$$\Lambda_1 \Lambda_2 \Lambda_3 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+.$$

**Proof.** The proof of Lemma 2.3 is exactly the same as that of Lemma 13 in Brüdern and Fouvry [4], so we omit the details herein. ■

**Lemma 2.4** *Let  $b - a \geq 1$ . Let  $f(x)$  be a real function on  $[a, b]$  such that  $|f''(x)| \asymp \Lambda$  uniformly for  $x \in [a, b]$  with  $\Lambda > 0$ . Then we have*

$$\sum_{a < n \leq b} e(f(n)) \ll (b-a)\Lambda^{\frac{1}{2}} + \Lambda^{-\frac{1}{2}}.$$

**Proof.** See Corollary 8.13 of Iwaniec and Kowalski [17], or Theorem 5 of Chapter 1 in Karatsuba [18]. ■

**Lemma 2.5** *For any complex numbers  $z_n$ , we have*

$$\left| \sum_{a < n \leq b} z_n \right|^2 \leq \left(1 + \frac{b-a}{Q}\right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \sum_{a < n, n+q \leq b} z_{n+q} \overline{z_n},$$

where  $Q$  is any positive integer.

**Proof.** See Lemma 8.17 of Iwaniec and Kowalski [17]. ■

**Lemma 2.6** *Suppose that  $f(x) : [a, b] \rightarrow \mathbb{R}$  has continuous derivatives of arbitrary order on  $[a, b]$ , where  $1 \leq a < b \leq 2a$ . Suppose further that*

$$|f^{(j)}(x)| \asymp \lambda_1 a^{1-j}, \quad j \geq 1, \quad x \in [a, b].$$

*Then for any exponential pair  $(\kappa, \lambda)$ , we have*

$$\sum_{a < n \leq b} e(f(n)) \ll \lambda_1^\kappa a^\lambda + \lambda_1^{-1}.$$

**Proof.** See (3.3.4) of Graham and Kolesnik [14]. ■

**Lemma 2.7** *Let*

$$\mathcal{T}(x) = \sum_{d \leq D} \sum_{\substack{\mu X < n \leq X \\ d|n+2}} e(n^c x).$$

*Then for  $0 < |x| \leq 2\Xi$ , we have*

$$\mathcal{T}(x) \ll X^{\frac{c}{6} + \frac{1}{2} + \varepsilon} D^{\frac{1}{2}} + |x|^{-1} X^{1-c} \log X.$$

**Proof.** Obviously, we have

$$\begin{aligned} \mathcal{T}(x) &= \sum_{d \leq D} \sum_{\substack{\frac{\mu X+2}{d} < h \leq \frac{X+2}{d}}} e((hd-2)^c x) \\ &\ll (\log X) \max_{\mathscr{D} \leq D} \sum_{\mathscr{D} < D \leq 2\mathscr{D}} \left| \sum_{\substack{\frac{\mu X+2}{d} < h \leq \frac{X+2}{d}}} e((hd-2)^c x) \right|. \end{aligned} \quad (2.2)$$

For the inner sum in (2.2), it follows from Lemma 2.6 with exponential pair  $(\kappa, \lambda) = A(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{6}, \frac{2}{3})$  that

$$\begin{aligned} \sum_{\substack{\frac{\mu X+2}{d} < h \leq \frac{X+2}{d}}} e((hd-2)^c x) &\ll (|x|dX^{c-1})^{\frac{1}{6}} \left(\frac{X}{d}\right)^{\frac{2}{3}} + \frac{X^{1-c}}{|x|d} \\ &\ll |x|^{\frac{1}{6}} d^{-\frac{1}{2}} X^{\frac{c}{6} + \frac{1}{2}} + \frac{X^{1-c}}{|x|d}. \end{aligned} \quad (2.3)$$

Putting (2.3) into (2.2), we obtain

$$\begin{aligned} \mathcal{T}(x) &\ll (\log X) \max_{\mathscr{D} \leq D} \sum_{\mathscr{D} < D \leq 2\mathscr{D}} \left( |x|^{\frac{1}{6}} d^{-\frac{1}{2}} X^{\frac{c}{6} + \frac{1}{2}} + \frac{X^{1-c}}{|x|d} \right) \\ &\ll X^{\frac{c}{6} + \frac{1}{2} + \varepsilon} D^{\frac{1}{2}} + |x|^{-1} X^{1-c} \log X, \end{aligned}$$

which completes the proof of Lemma 2.7. ■

For fixed constant  $\mu \in (0, 1)$ , we define

$$I(x) = \int_{\mu X}^X e(t^c x) dt. \quad (2.4)$$

Moreover, we define

$$L(x) = \sum_{d \leq D} \lambda(d) \sum_{\substack{\mu X < p \leq X \\ d|p+2}} (\log p) e(p^c x), \quad (2.5)$$

where  $\lambda(d)$  are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d) = 0. \quad (2.6)$$

**Lemma 2.8** *Let  $I(x)$  and  $L(x)$  be defined as above. Suppose that  $\xi$  and  $\delta$  satisfy the following conditions*

$$\xi + 7\delta < 2 \quad \text{and} \quad 3\xi + 6\delta < 2. \quad (2.7)$$

*Then for  $|x| \leq \tau$ , we have*

$$L(x) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(x) + O\left(\frac{X}{(\log X)^A}\right),$$

where  $A > 0$  is a sufficiently large constant.

**Proof.** See the process of the proof of Lemma 4.5 of Zhu [38]. Especially, the condition (2.7) comes from (4.21) and (4.22) in Zhu [38]. ■

**Lemma 2.9** *Let  $I(x)$  and  $L(x)$  be defined as above. Then we have*

$$\begin{aligned} \int_{|x| \leq \tau} |L(x)|^2 dx &\ll X^{2-c} (\log X)^6, \\ \int_{|x| \leq \tau} |I(x)|^2 dx &\ll X^{2-c} (\log X)^4, \\ \int_{|x| \leq \Xi} |L(x)|^2 dx &\ll X \Xi (\log X)^6. \end{aligned}$$

**Proof.** See Lemma 11 of Tolev [33], or Lemma 4.6 of Zhu [38]. ■

**Lemma 2.10** *Let  $f(n)$  be a complex-valued function defined for integers  $n \in (\mu X, X]$ . Then we have*

$$\sum_{\mu X < n \leq X} \Lambda(n) f(n) = S_1 - S_2 - S_3,$$

where

$$\begin{aligned} S_1 &= \sum_{k \leq X^{1/3}} \mu(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} (\log \ell) f(k\ell), \\ S_2 &= \sum_{k \leq X^{2/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} f(k\ell), \\ S_3 &= \sum_{X^{1/3} < k \leq X^{2/3}} a(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} \Lambda(\ell) f(k\ell), \end{aligned}$$

and where  $a(k)$  and  $c(k)$  are real numbers satisfying

$$|a(k)| \leq d(k), \quad |c(k)| \leq \log k.$$

**Proof.** See the arguments on page 112 of Vaughan [34]. ■

**Lemma 2.11** Suppose that  $1 < c < \frac{973}{856}$ . Suppose also that the real numbers  $\lambda(d)$  satisfy (2.6) and  $L(x)$  is defined by (2.5). Then we have

$$\sup_{\tau \leq |x| \leq \Xi} |L(x)| \ll X^{\frac{3-c}{2}-\varepsilon}$$

**Proof.** Obviously, we have

$$L(x) = \sum_{d \leq D} \lambda(d) \sum_{\substack{\mu X < n \leq X \\ d|n+2}} \Lambda(n) e(n^c x) + O(X^{\frac{1}{2}+\varepsilon}) =: L_1(x) + O(X^{\frac{1}{2}+\varepsilon}).$$

Trivially, we only need to show that

$$\sup_{\tau \leq |x| \leq \Xi} |L_1(x)| \ll X^{\frac{3-c}{2}-\varepsilon}$$

for  $1 < c < \frac{973}{856}$ . We rewrite  $L_1(x)$  in the form

$$L_1(x) = \sum_{\mu X < n \leq X} \Lambda(n) f(n),$$

where

$$f(n) = \sum_{\substack{d \leq D \\ d|n+2}} \lambda(d) e(n^c x).$$

By Lemma 2.10, we can see that

$$L_1(x) = S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{k \leq X^{1/3}} \mu(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} (\log \ell) f(k\ell),$$



$$S_2 = \sum_{k \leq X^{2/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} f(k\ell),$$

$$S_3 = \sum_{X^{1/3} < k \leq X^{2/3}} a(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} \Lambda(\ell) f(k\ell),$$

and  $|a(k)| \leq d(k), |c(k)| \leq \log k$ . Moreover, we write  $S_2 = S_2^{(1)} + S_2^{(2)}$ , where

$$S_2^{(1)} = \sum_{k \leq X^{1/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} f(k\ell), \quad S_2^{(2)} = \sum_{X^{1/3} < k \leq X^{2/3}} c(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} f(k\ell).$$

Therefore, we get

$$L_1(x) \ll |S_1| + |S_2^{(1)}| + |S_2^{(2)}| + |S_3|.$$

First, we consider the sum  $S_2^{(1)}$ . By noting the fact that  $\lambda(d) = 0$  for  $2|d$ , we can represent  $S_2^{(1)}$  as

$$S_2^{(1)} = \sum_{\substack{d \leq D \\ (d,2)=1}} \lambda(d) \sum_{k \leq X^{1/3}} c(k) \sum_{\substack{\frac{\mu X}{k} < \ell \leq \frac{X}{k} \\ d|k\ell+2}} e((k\ell)^c x).$$

Since  $(d,2) = 1$  and  $d|k\ell + 2$ , we have  $(k,d) = 1$ , and thus  $d|k\ell + 2$  is equivalent to  $\ell \equiv \ell_0 \pmod{d}$  for some fixed integer  $\ell_0$ . This means that  $\ell = \ell_0 + md$  for some integer  $m$ . Hence, we obtain

$$S_2^{(1)} = \sum_{\substack{d \leq D \\ (d,2)=1}} \lambda(d) \sum_{\substack{k \leq X^{1/3} \\ (k,d)=1}} c(k) \sum_{\frac{\mu X}{kd} - \frac{\ell_0}{d} < m \leq \frac{X}{kd} - \frac{\ell_0}{d}} e(k^c(\ell_0 + md)^c x). \quad (2.8)$$

Setting  $h(m) = k^c(\ell_0 + md)^c x$ , then we have  $|h''(m)| \asymp |x|d^2 k^2 X^{c-2}$ . By Lemma 2.4, we know that the inner sum over  $m$  in (2.8) is

$$\begin{aligned} &\ll \frac{X}{kd} (|x|d^2 k^2 X^{c-2})^{\frac{1}{2}} + (|x|d^2 k^2 X^{c-2})^{-\frac{1}{2}} \\ &\ll |x|^{\frac{1}{2}} X^{\frac{c}{2}} + |x|^{-\frac{1}{2}} k^{-1} d^{-1} X^{1-\frac{c}{2}}. \end{aligned} \quad (2.9)$$

Putting (2.9) into (2.8), we get

$$S_2^{(1)} \ll X^\varepsilon \left( D|x|^{\frac{1}{2}} X^{\frac{c}{2}+\frac{1}{3}} + |x|^{-\frac{1}{2}} X^{1-\frac{c}{2}} \right). \quad (2.10)$$

For the sum  $S_1$ , by partial summation we can get rid of the factor  $\log \ell$  and then proceed as the process of  $S_2^{(1)}$  to derive that

$$S_1 \ll X^\varepsilon \left( D|x|^{\frac{1}{2}} X^{\frac{c}{2}+\frac{1}{3}} + |x|^{-\frac{1}{2}} X^{1-\frac{c}{2}} \right). \quad (2.11)$$

Now, we consider the sum  $S_3$ . By a splitting argument, we can divide  $S_3$  into  $O(\log X)$  sums of the form

$$W(K) := \sum_{K < k \leq K_1} a(k) \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} \Lambda(\ell) \sum_{\substack{d \leq D \\ d|k\ell+2}} \lambda(d) e((k\ell)^c x),$$

where

$$K < K_1 \leq 2K, \quad X^{\frac{1}{3}} \leq K < K_1 \leq X^{\frac{2}{3}}.$$

Next, we shall consider the case  $K \geq X^{\frac{1}{2}}$  first. Trivially, we have

$$W(K) \ll X^\varepsilon \sum_{K < k \leq K_1} \left| \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} \Phi(\ell) \right|,$$

where

$$\Phi(\ell) = \Lambda(\ell) \sum_{\substack{d \leq D \\ d|k\ell+2}} \lambda(d) e((k\ell)^c x).$$

It follows from Cauchy's inequality that

$$|W(K)|^2 \ll X^\varepsilon K \sum_{K < k \leq K_1} \left| \sum_{\frac{\mu X}{k} < \ell \leq \frac{X}{k}} \Phi(\ell) \right|^2. \quad (2.12)$$

Suppose that  $H$  is an integer which satisfies

$$1 \leq H \ll \frac{X}{K}.$$

For the innermost sum on the right-hand side of (2.12), we use Lemma 2.5 to derive that

$$\begin{aligned} |W(K)|^2 &\ll \frac{X^{1+\varepsilon}}{H} \sum_{K < k \leq K_1} \sum_{|h| < H} \left(1 - \frac{|h|}{H}\right) \sum_{\substack{\frac{\mu X}{k} < \ell, \ell+h \leq \frac{X}{k}}} \Lambda(\ell) \\ &\quad \times \sum_{\substack{d_1 \leq D \\ d_1|k\ell+2}} \lambda(d_1) e(-(k\ell)^c x) \Lambda(\ell+h) \sum_{\substack{d_2 \leq D \\ d_2|k(\ell+h)+2}} \lambda(d_2) e((k(\ell+h))^c x) \\ &\ll \frac{X^{1+\varepsilon}}{H} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \lambda(d_1) \lambda(d_2) \sum_{|h| < H} \left(1 - \frac{|h|}{H}\right) \\ &\quad \times \sum_{\substack{\frac{\mu X}{K_1} < \ell, \ell+h \leq \frac{X}{K}}} \Lambda(\ell+h) \Lambda(\ell) \sum_{\substack{\tilde{K} < k \leq \tilde{K}_1 \\ d_1|k\ell+2 \\ d_2|k(\ell+h)+2}} e(k^c((\ell+h)^c - \ell^c)x), \end{aligned}$$

where

$$\tilde{K} = \max\left(K, \frac{\mu X}{\ell}, \frac{\mu X}{\ell+h}\right), \quad \tilde{K}_1 = \min\left(K_1, \frac{X}{\ell}, \frac{X}{\ell+h}\right).$$

By noting that  $\lambda(d) = 0$  for  $2|d$ , we can assume that  $2 \nmid d_1 d_2$ . Moreover, it follows from  $d_1|k\ell + 2$  and  $d_2|k(\ell + h) + 2$  that  $(d_1, \ell) = (d_2, \ell + h) = 1$ . Hence there exists some fixed integer  $k_0 = k_0(\ell, h, d_1, d_2)$  such that the pair of conditions  $d_1|k\ell + 2$  and  $d_2|k(\ell + h) + 2$  is equivalent to  $k \equiv k_0 \pmod{[d_1, d_2]}$ . This means that  $k = k_0 + m[d_1, d_2]$  for some integer  $m$ . Therefore, we get

$$\begin{aligned} \mathcal{F} &:= \sum_{\substack{\widetilde{K} < k \leq \widetilde{K}_1 \\ d_1|k\ell+2 \\ d_2|k(\ell+h)+2}} e\left(k^c((\ell+h)^c - \ell^c)x\right) \\ &= \sum_{\substack{\frac{\widetilde{K}-k_0}{[d_1, d_2]} < m \leq \frac{\widetilde{K}_1-k_0}{[d_1, d_2]}} e\left((k_0 + m[d_1, d_2])^c((\ell+h)^c - \ell^c)x\right) \\ &=: \sum_{\substack{\frac{\widetilde{K}-k_0}{[d_1, d_2]} < m \leq \frac{\widetilde{K}_1-k_0}{[d_1, d_2]}} e(F(m)), \end{aligned}$$

say. Trivially, for  $h = 0$ , we have  $\mathcal{F} \ll K[d_1, d_2]^{-1}$ . By the elementary estimate

$$\sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-1} \ll (\log D)^3,$$

we can see that the contribution of  $\mathcal{F}$  with  $h = 0$  to  $|W(K)|^2$  is  $\ll X^{2+\varepsilon} H^{-1}$ . For the case  $h \neq 0$ , we have

$$|F^{(j)}(m)| \asymp |x| h \ell^{c-1} [d_1, d_2] K^{c-1} \cdot (K[d_1, d_2]^{-1})^{1-j}, \quad j \geq 1.$$

It follows from Lemma 2.6 with  $(\kappa, \lambda) = BABABA^3 BA^2(\frac{1}{2}, \frac{1}{2}) = (\frac{75}{278}, \frac{161}{278})$  that

$$\begin{aligned} \mathcal{F} &\ll |x|^{-1} h^{-1} \ell^{1-c} [d_1, d_2]^{-1} K^{1-c} + (|x| h \ell^{c-1} [d_1, d_2] K^{c-1})^{\frac{75}{278}} (K[d_1, d_2]^{-1})^{\frac{161}{278}} \\ &\ll |x|^{-1} h^{-1} \ell^{1-c} [d_1, d_2]^{-1} K^{1-c} + |x|^{\frac{75}{278}} h^{\frac{75}{278}} \ell^{\frac{75(c-1)}{278}} [d_1, d_2]^{-\frac{43}{139}} K^{\frac{75c}{278} + \frac{43}{139}}. \end{aligned}$$

From the following estimate

$$\begin{aligned} \sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-\frac{43}{139}} &\ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left( \frac{(d_1, d_2)}{d_1 d_2} \right)^{\frac{43}{139}} = \sum_{1 \leq r \leq D} \sum_{k_1 \leq \frac{D}{r}} \sum_{k_2 \leq \frac{D}{r}} \frac{1}{r^{\frac{43}{139}} k_1^{\frac{43}{139}} k_2^{\frac{43}{139}}} \\ &\ll \sum_{1 \leq r \leq D} r^{-\frac{43}{139}} \left( \sum_{k \leq \frac{D}{r}} k^{-\frac{43}{139}} \right)^2 \ll \sum_{1 \leq r \leq D} r^{-\frac{43}{139}} \left( \frac{D}{r} \right)^{\frac{192}{139}} \ll D^{\frac{96}{139}}, \end{aligned}$$

we can see that the contribution of  $\mathcal{F}$  with  $h \neq 0$  to  $|W(K)|^2$  is

$$\ll X^{1+\varepsilon} H^{-1} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \sum_{0 < |h| < H} \sum_{\substack{\frac{\mu X}{K_1} < \ell, \ell+h \leq \frac{X}{K}}} \Lambda(\ell+h) \Lambda(\ell)$$

$$\begin{aligned}
& \times \left( |x|^{-1} h^{-1} \ell^{1-c} [d_1, d_2]^{-1} K^{1-c} + |x|^{\frac{75}{278}} h^{\frac{75}{278}} \ell^{\frac{75(c-1)}{278}} [d_1, d_2]^{-\frac{43}{139}} K^{\frac{75c}{278} + \frac{43}{139}} \right) \\
& \ll X^{1+\varepsilon} H^{-1} |x|^{-1} K^{1-c} \left( \sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-1} \right) \left( \sum_{0 < |h| < H} h^{-1} \right) \left( \sum_{\frac{\mu X}{K_1} < \ell, \ell+h \leq \frac{X}{K}} \ell^{1-c} \right) \\
& + X^{1+\varepsilon} H^{-1} |x|^{\frac{75}{278}} K^{\frac{75c}{278} + \frac{43}{139}} \left( \sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-\frac{43}{139}} \right) \\
& \times \left( \sum_{0 < |h| < H} h^{\frac{75}{278}} \right) \left( \sum_{\frac{\mu X}{K_1} < \ell, \ell+h \leq \frac{X}{K}} \ell^{\frac{75(c-1)}{278}} \right) \\
& \ll X^{3-c+\varepsilon} |x|^{-1} H^{-1} K^{-1} + X^{\frac{75c+481}{278}+\varepsilon} H^{\frac{75}{278}} D^{\frac{96}{139}} |x|^{\frac{75}{278}} K^{-\frac{117}{278}}.
\end{aligned}$$

Combining the above two cases, we obtain

$$|W(K)|^2 \ll X^\varepsilon \left( X^2 H^{-1} + X^{3-c} |x|^{-1} H^{-1} K^{-1} + X^{\frac{75c+481}{278}+\varepsilon} H^{\frac{75}{278}} D^{\frac{96}{139}} |x|^{\frac{75}{278}} K^{-\frac{117}{278}} \right). \quad (2.13)$$

Taking

$$H_0 = X^{\frac{75(1-c)}{353}} K^{\frac{117}{353}} D^{-\frac{192}{353}} |x|^{-\frac{75}{353}}, \quad H = [\min(H_0, XK^{-1})],$$

it is easy to see that

$$H^{-1} \asymp H_0^{-1} + KX^{-1}. \quad (2.14)$$

By using (2.13) and (2.14), and noting that  $K \geq X^{\frac{1}{2}}$ , we derive that

$$\begin{aligned}
|W(K)|^2 & \ll X^\varepsilon \left( X^2 (H_0^{-1} + KX^{-1}) + X^{3-c} |x|^{-1} K^{-1} (H_0^{-1} + KX^{-1}) \right. \\
& \quad \left. + X^{\frac{75c+481}{278}+\varepsilon} H_0^{\frac{75}{278}} D^{\frac{96}{139}} |x|^{\frac{75}{278}} K^{-\frac{117}{278}} \right) \\
& \ll X^\varepsilon \left( X^{\frac{150c+1145}{706}} D^{\frac{192}{353}} |x|^{\frac{75}{353}} + X^{\frac{5}{3}} + X^{\frac{749-278c}{353}} D^{\frac{192}{353}} |x|^{-\frac{278}{353}} + X^{2-c} |x|^{-1} \right),
\end{aligned}$$

which implies that

$$|W(K)| \ll X^\varepsilon \left( X^{\frac{150c+1145}{1412}} D^{\frac{96}{353}} |x|^{\frac{75}{706}} + X^{\frac{5}{6}} + X^{\frac{749-278c}{706}} D^{\frac{96}{353}} |x|^{-\frac{139}{353}} + X^{1-\frac{c}{2}} |x|^{-\frac{1}{2}} \right). \quad (2.15)$$

If  $K < X^{\frac{1}{2}}$ , we write  $W(K)$  as

$$W(K) = \sum_{\mu X/K_1 < \ell \leq X/K} \Lambda(\ell) \sum_{\max(K, \mu X/\ell) < k \leq \min(K_1, X/\ell)} a(k) \sum_{\substack{d \leq D \\ d|k\ell+2}} \lambda(d) e((k\ell)^c x).$$

Then we have  $X/K \gg X^{\frac{1}{2}}$  and we can proceed as the previous process by changing the roles of the variables  $k$  and  $\ell$  reversed. Therefore, we also derive the estimate (2.15) in this case. Consequently, we obtain

$$S_3 \ll X^\varepsilon \left( X^{\frac{150c+1145}{1412}} D^{\frac{96}{353}} |x|^{\frac{75}{706}} + X^{\frac{5}{6}} + X^{\frac{749-278c}{706}} D^{\frac{96}{353}} |x|^{-\frac{139}{353}} + X^{1-\frac{c}{2}} |x|^{-\frac{1}{2}} \right). \quad (2.16)$$

For  $S_2^{(2)}$ , we can use the same way to give the upper bound estimate by the expression of the right-hand side of (2.16).

Above all, we deduce that

$$\begin{aligned} L_1(x) &\ll X^\varepsilon \left( X^{\frac{c}{2} + \frac{1}{3}} D |x|^{\frac{1}{2}} + X^{\frac{150c+1145}{1412}} D^{\frac{96}{353}} |x|^{\frac{75}{706}} + X^{\frac{5}{6}} \right. \\ &\quad \left. + X^{\frac{749-278c}{706}} D^{\frac{96}{353}} |x|^{-\frac{139}{353}} + X^{1-\frac{c}{2}} |x|^{-\frac{1}{2}} \right) \\ &\ll X^\varepsilon \left( X^{\frac{c}{2} + \frac{1}{3} + \delta} + X^{\frac{150c+1145}{1412} + \frac{96}{353}\delta} + X^{\frac{5}{6}} + X^{\frac{749}{706} + \frac{96}{353}\delta - \frac{139}{353}\xi} + X^{1-\frac{1}{2}\xi} \right), \end{aligned}$$

and thus, for  $1 < c < \frac{973}{856}$ , there holds

$$\sup_{\tau \leq |x| \leq \Xi} |L_1(x)| \ll X^{\frac{3-c}{2} - \varepsilon}.$$

This completes the proof Lemma 2.11. ■

**Lemma 2.12** For  $1 < c < \frac{973}{856}$ , we have

$$\int_{\tau \leq |x| \leq \Xi} |L(x)|^3 dx \ll X^{3-c-\varepsilon}.$$

**Proof.** We have

$$\begin{aligned} &\left| \int_{\tau \leq |x| \leq \Xi} |L(x)|^3 dx \right| = \left| \int_{\tau \leq |x| \leq \Xi} L(x) \overline{L(x)} |L(x)| dx \right| \\ &= \left| \sum_{d \leq D} \lambda(d) \sum_{\substack{\mu X < p \leq X \\ d|p+2}} (\log p) \int_{\tau \leq |x| \leq \Xi} e(p^c x) \overline{L(x)} |L(x)| dx \right| \\ &\leq (\log X) \sum_{d \leq D} \sum_{\substack{\mu X < p \leq X \\ d|p+2}} \left| \int_{\tau \leq |x| \leq \Xi} e(p^c x) \overline{L(x)} |L(x)| dx \right| \\ &\leq (\log X) \sum_{d \leq D} \sum_{\substack{\mu X < n \leq X \\ d|n+2}} \left| \int_{\tau \leq |x| \leq \Xi} e(n^c x) \overline{L(x)} |L(x)| dx \right|. \end{aligned}$$

By Cauchy's inequality, we have

$$\begin{aligned} &\left| \int_{\tau \leq |x| \leq \Xi} |L(x)|^3 dx \right|^2 \ll (\log X)^2 \left( \sum_{\mu X < n \leq X} \sum_{\substack{d \leq D \\ d|n+2}} \left| \int_{\tau \leq |x| \leq \Xi} e(n^c x) \overline{L(x)} |L(x)| dx \right| \right)^2 \\ &\ll X (\log X)^2 \sum_{\mu X < n \leq X} \left( \sum_{\substack{d \leq D \\ d|n+2}} \left| \int_{\tau \leq |x| \leq \Xi} e(n^c x) \overline{L(x)} |L(x)| dx \right| \right)^2 \\ &\ll X (\log X)^2 \sum_{\mu X < n \leq X} \left( \sum_{\substack{d \leq D \\ d|n+2}} 1 \right) \sum_{\substack{d \leq D \\ d|n+2}} \left| \int_{\tau \leq |x| \leq \Xi} e(n^c x) \overline{L(x)} |L(x)| dx \right|^2 \end{aligned}$$

$$\begin{aligned}
&\ll X^{1+\frac{\varepsilon}{4}} \sum_{d \leq D} \sum_{\substack{\mu X < n \leq X \\ d|n+2}} \left| \int_{\tau \leq |x| \leq \Xi} e(n^c x) \overline{L(x)} |L(x)| dx \right|^2 \\
&= X^{1+\frac{\varepsilon}{4}} \sum_{d \leq D} \sum_{\substack{\mu X < n \leq X \\ d|n+2}} \left( \int_{\tau \leq |x| \leq \Xi} e(n^c x) \overline{L(x)} |L(x)| dx \right) \left( \int_{\tau \leq |y| \leq \Xi} \overline{e(n^c y) \overline{L(y)}} |L(y)| dy \right) \\
&\ll X^{1+\frac{\varepsilon}{4}} \int_{\tau \leq |y| \leq \Xi} |L(y)|^2 dy \int_{\tau \leq |x| \leq \Xi} |L(x)|^2 |\mathcal{T}(x-y)| dx. \tag{2.17}
\end{aligned}$$

For the innermost integral on the right-hand side of (2.17), we have

$$\begin{aligned}
&\int_{\tau \leq |x| \leq \Xi} |L(x)|^2 |\mathcal{T}(x-y)| dx \\
&\ll \int_{\substack{\tau \leq |x| \leq \Xi \\ |x-y| \leq X^{-c}}} |L(x)|^2 |\mathcal{T}(x-y)| dx + \int_{\substack{\tau \leq |x| \leq \Xi \\ X^{-c} < |x-y| \leq 2\Xi}} |L(x)|^2 |\mathcal{T}(x-y)| dx. \tag{2.18}
\end{aligned}$$

By the trivial estimate  $|\mathcal{T}(x-y)| \ll X \log X$  and Lemma 2.11, we have

$$\begin{aligned}
\int_{\substack{\tau \leq |x| \leq \Xi \\ |x-y| \leq X^{-c}}} |L(x)|^2 |\mathcal{T}(x-y)| dx &\ll X(\log X) \times \sup_{\tau \leq |x| \leq \Xi} |L(x)|^2 \times \int_{\substack{\tau \leq |x| \leq \Xi \\ |x-y| \leq X^{-c}}} dx \\
&\ll X^{1-c}(\log X) \times \sup_{\tau \leq |x| \leq \Xi} |L(x)|^2 \ll X^{4-2c-\varepsilon}. \tag{2.19}
\end{aligned}$$

It follows from Lemma 2.7 and Lemma 2.9 that

$$\begin{aligned}
&\int_{\substack{\tau \leq |x| \leq \Xi \\ X^{-c} < |x-y| \leq 2\Xi}} |L(x)|^2 |\mathcal{T}(x-y)| dx \\
&\ll \int_{\substack{\tau \leq |x| \leq \Xi \\ X^{-c} < |x-y| \leq 2\Xi}} |L(x)|^2 \left( X^{\frac{c}{6} + \frac{1}{2} + \varepsilon} D^{\frac{1}{2}} + |x-y|^{-1} X^{1-c} \log X \right) dx \\
&\ll X^{\frac{c}{6} + \frac{1}{2} + \frac{1}{2}\delta + \varepsilon} \int_{|x| \leq \Xi} |L(x)|^2 dx + X^{1-c+\varepsilon} \times \sup_{\tau \leq |x| \leq \Xi} |L(x)|^2 \times \int_{\substack{\tau \leq |x| \leq \Xi \\ X^{-c} < |x-y| \leq 2\Xi}} \frac{dx}{|x-y|} \\
&\ll X^{\frac{c}{6} + \frac{3}{2} + \frac{1}{2}\delta + \varepsilon} + X^{4-2c-\varepsilon} \ll X^{4-2c-\varepsilon}. \tag{2.20}
\end{aligned}$$

From (2.18), (2.19) and (2.20), we have

$$\int_{\tau \leq |x| \leq \Xi} |L(x)|^2 |\mathcal{T}(x-y)| dx \ll X^{4-2c-\varepsilon}. \tag{2.21}$$

Combining (2.17), (2.21) and Lemma 2.9, we get

$$\left| \int_{\tau \leq |x| \leq \Xi} |L(x)|^3 dx \right|^2 \ll X^{1+\frac{\varepsilon}{4}} \cdot X^{4-2c-\varepsilon} \int_{\tau \leq |y| \leq \Xi} |L(y)|^2 dy \ll X^{6-2c-\frac{\varepsilon}{2}},$$

which implies that

$$\left| \int_{\tau \leq |x| \leq \Xi} |L(x)|^3 dx \right| \ll X^{3-c-\varepsilon}.$$

This completes the proof of Lemma 2.12. ■

### 3 Proof of Theorem 1.1

Consider the sum

$$\Gamma = \sum_{\substack{\mu X < p_1, p_2, p_3 \leq X \\ |p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E} \\ (p_i + 2, P(z)) = 1 \\ i=1,2,3}} (\log p_1)(\log p_2)(\log p_3). \quad (3.1)$$

In order to prove Theorem 1.1, we only need to show that  $\Gamma > 0$ . Suppose that  $\Theta(x)$  and  $\vartheta(x)$  are the functions which are defined in Lemma 2.1 with parameters  $a = \frac{7}{8}(\log N)^{-E}$ ,  $b = \frac{1}{8}(\log N)^{-E}$  and  $r = \lfloor \log^2 X \rfloor$ . Therefore, we get

$$\vartheta(y) = 0 \quad \text{if } |y| \geq (\log N)^{-E}, \quad 0 < \vartheta(y) < 1 \quad \text{if } |y| < (\log N)^{-E}. \quad (3.2)$$

Obviously, it follows from (3.1) and (3.2) that

$$\Gamma \geq \tilde{\Gamma} := \sum_{\substack{\mu X < p_1, p_2, p_3 \leq X \\ (p_i + 2, P(z)) = 1 \\ i=1,2,3}} (\log p_1)(\log p_2)(\log p_3) \vartheta(p_1^c + p_2^c + p_3^c - N). \quad (3.3)$$

By the definition of  $\Lambda_i$  in Lemma 2.3, we know that

$$\Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d) = \begin{cases} 1, & \text{if } (p_i + 2, P(z)) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

which combined with Lemma 2.3 yields that

$$\begin{aligned} \tilde{\Gamma} &= \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1 \Lambda_2 \Lambda_3 \vartheta(p_1^c + p_2^c + p_3^c - N) \\ &\geq \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \vartheta(p_1^c + p_2^c + p_3^c - N) \\ &\quad \times (\Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+) \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3 - 2\Gamma_4, \end{aligned}$$

say. Trivially, by the symmetric property, we can see that

$$\begin{aligned} \Gamma_1 = \Gamma_2 = \Gamma_3 &= \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \vartheta(p_1^c + p_2^c + p_3^c - N), \\ \Gamma_4 &= \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \vartheta(p_1^c + p_2^c + p_3^c - N), \end{aligned}$$

and thus

$$\tilde{\Gamma} \geq 3\Gamma_1 - 2\Gamma_4. \quad (3.4)$$

Let  $\lambda^\pm(d)$  be the Rosser's functions of level  $D$ , and define

$$\begin{aligned} L^\pm(x) &:= \sum_{\mu X < p \leq X} (\log p) e(p^c x) \sum_{d|(p+2, P(z))} \lambda^\pm(d) \\ &= \sum_{d|P(z)} \lambda^\pm(d) \sum_{\substack{\mu X < p \leq X \\ d|p+2}} (\log p) e(p^c x). \end{aligned}$$

According to the Fourier's inverse transformation, we have

$$\begin{aligned} \Gamma_1 &= \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \\ &\quad \times \int_{-\infty}^{+\infty} \Theta(x) e((p_1^c + p_2^c + p_3^c - N)x) dx \\ &= \int_{-\infty}^{+\infty} L^-(x) (L^+(x))^2 \Theta(x) e(-Nx) dx \\ &= \left( \int_{|x| \leq \tau} + \int_{\tau < |x| < \Xi} + \int_{|x| \geq \Xi} \right) L^-(x) (L^+(x))^2 \Theta(x) e(-Nx) dx \\ &= \Gamma_1^{(1)} + \Gamma_1^{(2)} + \Gamma_1^{(3)}, \end{aligned} \tag{3.5}$$

say. Similarly, for  $\Gamma_4$ , we also have

$$\begin{aligned} \Gamma_4 &= \int_{-\infty}^{+\infty} (L^+(x))^3 \Theta(x) e(-Nx) dx \\ &= \left( \int_{|x| \leq \tau} + \int_{\tau < |x| < \Xi} + \int_{|x| \geq \Xi} \right) (L^+(x))^3 \Theta(x) e(-Nx) dx \\ &=: \Gamma_4^{(1)} + \Gamma_4^{(2)} + \Gamma_4^{(3)}. \end{aligned} \tag{3.6}$$

By the trivial estimate  $L^\pm(x) \ll X^{1+\varepsilon}$  and Lemma 2.1, we get

$$\begin{aligned} \Gamma_1^{(3)}, \Gamma_4^{(3)} &\ll X^{3+\varepsilon} \int_{\Xi}^{\infty} \frac{1}{\pi|x|} \left( \frac{r}{2\pi|x|b} \right)^r dx \\ &\ll X^{3+\varepsilon} \left( \frac{r}{2\pi b} \right)^r \int_{\Xi}^{\infty} \frac{dx}{x^{r+1}} \ll X^{3+\varepsilon} \left( \frac{r}{2\pi \Xi b} \right)^r \\ &\ll \frac{X^{3+\varepsilon}}{(2\pi \log X)^{\log X}} \ll \frac{X^{3+\varepsilon}}{X^{\log \log X + \log(2\pi)}} \ll 1. \end{aligned} \tag{3.7}$$

It follows from (3.3), (3.4), (3.5), (3.6) and (3.7) that

$$\Gamma \geq \left( 3\Gamma_1^{(1)} - 2\Gamma_4^{(1)} \right) + \left( 3\Gamma_1^{(2)} - 2\Gamma_4^{(2)} \right) + O(1). \tag{3.8}$$

By Lemma 2.8, we know that, for  $|x| \leq \tau$ , there holds

$$L^\pm(x) = \mathcal{M}^\pm I(x) + O\left( \frac{X}{(\log X)^A} \right),$$



where  $\mathcal{M}^\pm$  and  $I(x)$  are defined by (2.1) and (2.4), respectively. By noting the identity

$$\begin{aligned} & L^-(x)(L^+(x))^2 - \mathcal{M}^-(\mathcal{M}^+)^2 I^3(x) \\ &= (L^+(x))^2 (L^-(x) - \mathcal{M}^- I(x)) + \mathcal{M}^- I(x) L^+(x) (L^+(x) - \mathcal{M}^+ I(x)) \\ & \quad + \mathcal{M}^- \mathcal{M}^+ I^2(x) (L^+(x) - \mathcal{M}^+ I(x)) \end{aligned}$$

and the elementary estimate

$$\mathcal{M}^\pm \ll \sum_{d \leq D} \frac{1}{\varphi(d)} \ll \log X,$$

we derive that

$$\begin{aligned} & \left| L^-(x)(L^+(x))^2 - \mathcal{M}^-(\mathcal{M}^+)^2 I^3(x) \right| \\ & \ll \frac{X}{(\log X)^A} \left( |L^+(x)|^2 + (\log X) |L^+(x)I(x)| + (\log X)^2 |I(x)|^2 \right) \\ & \ll \frac{X}{(\log X)^{A-2}} \left( |L^+(x)|^2 + |I(x)|^2 \right). \end{aligned}$$

Define

$$\mathcal{I}_0 = \int_{|x| \leq \tau} I^3(x) \Theta(x) e(-Nx) dx.$$

Then from Lemma 2.1 and Lemma 2.9, we derive that

$$\begin{aligned} \left| \Gamma_1^{(1)} - \mathcal{M}^-(\mathcal{M}^+)^2 \mathcal{I}_0 \right| & \leq \int_{|x| \leq \tau} \left| L^-(x)(L^+(x))^2 - \mathcal{M}^-(\mathcal{M}^+)^2 I^3(x) \right| |\Theta(x)| dx \\ & \ll \frac{X}{(\log X)^{A+E-2}} \left( \int_{|x| \leq \tau} |L^+(x)|^2 dx + \int_{|x| \leq \tau} |I(x)|^2 dx \right) \\ & \ll \frac{X^{3-c}}{(\log X)^{A+E-8}}. \end{aligned} \tag{3.9}$$

Set

$$\mathcal{I} = \int_{-\infty}^{+\infty} I^3(x) \Theta(x) e(-Nx) dx.$$

It follows from Lemma 6 of Tolev [29] that

$$\mathcal{I} \gg \frac{X^{3-c}}{(\log X)^E}. \tag{3.10}$$

Since  $\left| \frac{d}{dt}(t^c x) \right| \gg |x| X^{c-1}$  for  $t \in (\mu X, X]$ , by Lemma 4.2 of Titchmarsh [27], we have  $|I(x)| \ll (|x| X^{c-1})^{-1}$ . Therefore, we obtain

$$\begin{aligned} |\mathcal{I} - \mathcal{I}_0| & \ll \int_{\tau}^{\infty} |I(x)|^3 |\Theta(x)| dx \ll (\log X)^{-E} \int_{\tau}^{\infty} \left( \frac{1}{|x| X^{c-1}} \right)^3 dx \\ & \ll (\log X)^{-E} X^{3-3c} \tau^{-2} \ll \frac{X^{3-c-2\xi}}{(\log X)^E}. \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11), we get

$$\begin{aligned}
\Gamma_1^{(1)} &= \mathcal{M}^-(\mathcal{M}^+)^2 \mathcal{I}_0 + O\left(\frac{X^{3-c}}{(\log X)^{A+E-8}}\right) \\
&= \mathcal{M}^-(\mathcal{M}^+)^2 \left(\mathcal{I} + O\left(\frac{X^{3-c-2\xi}}{(\log X)^E}\right)\right) + O\left(\frac{X^{3-c}}{(\log X)^{A+E-8}}\right) \\
&= \mathcal{M}^-(\mathcal{M}^+)^2 \mathcal{I} + O\left(\frac{X^{3-c}}{(\log X)^{A+E-8}}\right).
\end{aligned} \tag{3.12}$$

Similarly, we can get

$$\Gamma_4^{(1)} = (\mathcal{M}^+)^3 \mathcal{I} + O\left(\frac{X^{3-c}}{(\log X)^{A+E-8}}\right). \tag{3.13}$$

From Mertens' prime number theorem (See [23]), we know that

$$\mathfrak{P} \asymp \frac{1}{\log X},$$

which combined with (3.10), (3.12), (3.13) and Lemma 2.2 yields

$$\begin{aligned}
3\Gamma_1^{(1)} - 2\Gamma_4^{(1)} &= (3\mathcal{M}^- - 2\mathcal{M}^+)(\mathcal{M}^+)^2 \mathcal{I} + O\left(\frac{X^{3-c}}{(\log X)^{A+E-8}}\right) \\
&\geq \left(3f\left(\frac{\log D}{\log z}\right) - 2F\left(\frac{\log D}{\log z}\right)\right) (1 + O(\log^{-1/3} X)) \mathfrak{P}^3 \mathcal{I} + O\left(\frac{X^{3-c}}{(\log X)^{A+E-8}}\right) \\
&= \left(3f\left(\frac{59}{20}\right) - 2F\left(\frac{59}{20}\right)\right) \mathfrak{P}^3 \mathcal{I} + O\left(\frac{X^{3-c}}{(\log X)^{E+10/3}}\right) \\
&= \frac{120e^\gamma}{59} \left(\log \frac{39}{20} - \frac{2}{3}\right) \mathfrak{P}^3 \mathcal{I} + O\left(\frac{X^{3-c}}{(\log X)^{E+10/3}}\right).
\end{aligned} \tag{3.14}$$

For  $\Gamma_1^{(2)}$ , by Hölder's inequality, Lemma 2.1 and Lemma 2.12, we obtain

$$\begin{aligned}
|\Gamma_1^{(2)}| &\ll \int_{\tau \leq |x| \leq \Xi} |L^-(x)| |L^+(x)|^2 |\Theta(x)| dx \\
&\ll (\log X)^{-E} \int_{\tau \leq |x| \leq \Xi} |L^-(x)| |L^+(x)|^2 dx \\
&\ll (\log X)^{-E} \left(\int_{\tau \leq |x| \leq \Xi} |L^-(x)|^3 dx\right)^{\frac{1}{3}} \left(\int_{\tau \leq |x| \leq \Xi} |L^+(x)|^3 dx\right)^{\frac{2}{3}} \\
&\ll (\log X)^{-E} \cdot X^{3-c-\varepsilon} \ll X^{3-c-\varepsilon}.
\end{aligned} \tag{3.15}$$

Similarly, we have

$$|\Gamma_4^{(2)}| \ll (\log X)^{-E} \int_{\tau \leq |x| \leq \Xi} |L^+(x)|^3 dx \ll X^{3-c-\varepsilon}. \tag{3.16}$$

According to (3.8), (3.14), (3.15) and (3.16), we deduce that

$$\Gamma \geq \left(3\Gamma_1^{(1)} - 2\Gamma_4^{(1)}\right) + O\left(|\Gamma_1^{(2)}| + |\Gamma_4^{(2)}| + 1\right)$$

$$\begin{aligned} &\geq \frac{120e^\gamma}{59} \left( \log \frac{39}{20} - \frac{2}{3} \right) \mathfrak{P}^3 \mathcal{I} + O\left( \frac{X^{3-c}}{(\log X)^{E+10/3}} \right) + O(X^{3-c-\varepsilon}) \\ &\gg \frac{X^{3-c}}{(\log X)^{E+3}}. \end{aligned}$$

Therefore,  $\Gamma > 0$  for sufficiently large real number  $N$ . Then inequality (1.4) would have a solution in primes  $p_1, p_2, p_3$  satisfying

$$(p_1 + 2, P(z)) = (p_2 + 2, P(z)) = (p_3 + 2, P(z)) = 1. \quad (3.17)$$

If the number  $p_i + 2$  has  $l$  prime factors counted with multiplicity, then from (3.17) and from the condition  $\mu X < p_i \leq X$ , it is easy to find that  $l \leq \eta^{-1}$ , which means that  $p_i + 2$  would be almost-prime of order  $[\eta^{-1}] = \left\lfloor \frac{12626}{4865-4280c} \right\rfloor$ .

This completes the proof of Theorem 1.1.

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