PDE 作业

1 第一章

习题 1 利用 Gauss-Green 公式证明

1. 若 $u, v \in C^1(\Omega) \cap C(\bar{\Omega})$, 则

$$\int_\Omega u_{x_i} v \mathrm{d}x = -\int_\Omega u v_{x_i} \mathrm{d}x + \int_{\partial\Omega} u v n_i \mathrm{d}s.$$

2. 若 $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, 则

$$\int_\Omega \Delta u \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n} \mathrm{d}s.$$

3. 若 $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$, 则

$$\int_\Omega u\Delta v - v\Delta u \mathrm{d}x = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \mathrm{d}s.$$

证明. 1. 令 $F = (0, 0, \cdots, uv, \cdots, 0)^T$, 根据 Gauss-Green 公式,

$$\int_\Omega u_{x_i} v \mathrm{d}x = \int_\Omega \nabla \cdot F \mathrm{d}x = \int_{\partial\Omega} F \cdot n \mathrm{d}s = \int_{\partial\Omega} u v n_i \mathrm{d}s.$$

2. 注意到 $\Delta u = \nabla \cdot \nabla u$, 那么

$$\int_\Omega \Delta u \mathrm{d}x = \int_\Omega \nabla \cdot \nabla u \mathrm{d}x = \int_{\partial\Omega} \nabla u \cdot n \mathrm{d}s = \int_{\partial\Omega} \frac{\partial u}{\partial n} \mathrm{d}s.$$

3. 根据第 1 问的结论,

$$\begin{aligned}
\int_\Omega u\Delta v - v\Delta u \mathrm{d}x &= \int_\Omega \sum_{i=1}^n \left(u v_{x_i x_i} - v u_{x_i x_i}\right) \mathrm{d}x \\
&= -\int_\Omega \sum_{i=1}^n (u_{x_i} v_{x_i} - u_{x_i} v_{x_i}) \mathrm{d}x + \int_{\partial\Omega} \sum_{i=1}^n (u v_{x_i} n_i - v u_{x_i} n_i) \mathrm{d}s \\
&= \int_{\partial\Omega} (u \nabla v \cdot n - v \nabla u \cdot n) \mathrm{d}s = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \mathrm{d}s.
\end{aligned}$$

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习题 2 将下列方程化为标准型.

1. $\sum_{i=1}^{n} u_{x_ix_i} + \sum_{1\leqslant j<j\leqslant n} u_{x_ix_j} = 0.$

2. $u_{xx} + 2u_{xy} + 2u_{yy} = 0.$

解. (1) 令矩阵

$$A = \begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{bmatrix}, \quad P = \begin{bmatrix} -1/\sqrt{2} & \cdots & -1/\sqrt{2} & 1/\sqrt{n} \\ 1/\sqrt{2} & \cdots & 0 & 1/\sqrt{n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1/\sqrt{2} & 1/\sqrt{n} \end{bmatrix},$$

那么 $P^TAP = \operatorname{diag}\{1/2, \cdots, 1/2, (n+1)/2\}$, 令

$$\begin{aligned} y_1 &= -\frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2 - \cdots - \frac{1}{\sqrt{2}}x_{n-1} + \frac{1}{\sqrt{n}}x_n, \\ y_2 &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{n}}x_n, \\ &\cdots\cdots \\ y_n &= \frac{1}{\sqrt{2}}x_{n-1} + \frac{1}{\sqrt{n}}x_n, \end{aligned}$$

那么标准型为

$$\frac{1}{2}u_{y_1y_1} + \cdots + \frac{1}{2}u_{y_{n-1}y_{n-1}} + \frac{n+1}{2}u_{y_ny_n} = 0.$$

(2) 令

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \end{bmatrix},$$

那么 $P^TAP = \operatorname{diag}\left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\}$, 令

$$\begin{aligned} s &= \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}x + \frac{-\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}y, \\ t &= \frac{2}{\sqrt{10-2\sqrt{5}}}x + \frac{2}{\sqrt{10+2\sqrt{5}}}y. \end{aligned}$$

得到标准型为

$$\frac{3+\sqrt{5}}{2}u_{ss} + \frac{3-\sqrt{5}}{2}u_{tt} = 0.$$

习题 3 设

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} \alpha(x)v^2\mathrm{d}s - \int_\Omega fv\mathrm{d}x - \int_{\partial\Omega} gv\mathrm{d}s,$$

其中 $\alpha(x) \geqslant 0$. 考虑以下三个问题:

问题 I(变分问题): 求 $u \in M = C^1(\bar{\Omega})$, 使得

$$J(u) = \min_{v \in M} J(v).$$

问题 II: 求 $u \in M = C^1(\bar{\Omega})$, 使得它对于任意的 $v \in M$, 都满足

$$\int_\Omega (\nabla u \cdot \nabla v + u \cdot v - fv)\mathrm{d}x + \int_{\partial\Omega} (\alpha(x)uv - gv)\mathrm{d}s = 0.$$

问题 III(第三边值问题): 求 $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, 满足以下边值问题

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ \frac{\partial u}{\partial n} + \alpha(x)u = g, & x \in \partial\Omega. \end{cases}$$

1. 证明问题 I 与问题 II 等价.

2. 当 $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ 时, 证明问题 I、II、III 等价.

证明. (1) "$\Rightarrow$". 令 $j(\varepsilon) = J(u + \varepsilon v)$, 则 $j(\varepsilon) \geqslant j(0)$, 那么 $j'(0) = 0$. 又

$$\begin{aligned} j(\varepsilon) &= \frac{1}{2}\int_\Omega (|\nabla(u+\varepsilon)v|^2 + (u+\varepsilon v)^2)\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} \alpha(x)(u+\varepsilon v)^2 \mathrm{d}s \\ &\quad - \int_\Omega f(x)(u+\varepsilon v)\mathrm{d}x - \int_{\partial\Omega} g(x)(u+\varepsilon v)\mathrm{d}s \\ &= \frac{1}{2}\int_\Omega \left[\left(\frac{\partial u}{\partial x} + \varepsilon\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \varepsilon\frac{\partial v}{\partial y}\right)^2 + (u+\varepsilon v)^2\right]\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\partial\Omega} \alpha(x)(u+\varepsilon v)^2\mathrm{d}s - \int_\Omega f(x)(u+\varepsilon v)\mathrm{d}x - \int_{\partial\Omega} g(x)(u+\varepsilon v)\mathrm{d}s. \end{aligned}$$

那么

$$\begin{aligned} j'(\varepsilon) = \int_\Omega &\left[\left(\frac{\partial u}{\partial x} + \varepsilon\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} + \varepsilon\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y} + (u+\varepsilon v)v\right]\mathrm{d}x \\ &+ \int_{\partial\Omega} \alpha(x)(u+\varepsilon v)v\mathrm{d}s - \int_\Omega f(x)v\mathrm{d}x - \int_{\partial\Omega} g(v)v\mathrm{d}s. \end{aligned}$$

则

$$\begin{aligned} j'(0) &= \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + uv\right)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)uv\mathrm{d}s - \int_\Omega fv\mathrm{d}x - \int_{\partial\Omega} gv\mathrm{d}s \\ &= 0. \end{aligned}$$

“$\Leftarrow$”.

$$\begin{aligned}J(v)-J(u)&=\frac{1}{2}\int_{\Omega}(|\nabla v|^2+v^2-|\nabla u|^2-u^2)\mathrm{d}x+\frac{1}{2}\int_{\partial\Omega}\alpha(x)(v^2-u^2)\mathrm{d}s\\&\quad-\int_{\Omega}f(v-u)\mathrm{d}x-\int_{\partial\Omega}g(v-u)\mathrm{d}s\\&=\frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+v^2-\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial x}\right)^2-u^2\right]\mathrm{d}x\\&\quad+\frac{1}{2}\int_{\partial\Omega}\alpha(x)(v^2-u^2)\mathrm{d}s-\int_{\Omega}f(v-u)\mathrm{d}x-\int_{\partial\Omega}g(v-u)\mathrm{d}s.\end{aligned}$$

又已知

$$\begin{aligned}\int_{\Omega}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+u^2-fu\right]\mathrm{d}x+\int_{\partial\Omega}(\alpha(x)u^2-gu)\mathrm{d}s=0,\\\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+uv-fv\right)\mathrm{d}x+\int_{\partial\Omega}(\alpha(x)uv-gv)\mathrm{d}s=0,\end{aligned}$$

联立以上两式, 可得

$$J(v)-J(u)=\frac{1}{2}\int_{\partial\Omega}\alpha(x)(u-v)^2\mathrm{d}s+\frac{1}{2}\int_{\Omega}(|\nabla u-\nabla v|^2+(u-v)^2)\mathrm{d}x\geqslant 0.$$

(2) II$\Rightarrow$III. 根据习题 1第一问,

$$\int_{\Omega}(u\cdot v-fv-v\Delta u)\mathrm{d}x\mathrm{d}y+\int_{\partial\Omega}(\alpha(x)uv+\frac{\partial u}{\partial n}v-gv)\mathrm{d}s=0.$$

选取 $v\in C_0^{\infty}(\Omega)$, 那么

$$\int_{\Omega}(u\cdot v-fv-v\Delta u)\mathrm{d}x\mathrm{d}y=0.$$

又因为 $u-f-\Delta u$ 连续, 那么

$$u-\Delta u=f.$$

将上式代入, 得到,

$$\int_{\partial\Omega}(\alpha(x)uv+\frac{\partial u}{\partial n}v-gv)\mathrm{d}s=0,\quad v\in M.$$

那么

$$\alpha(x)u+\frac{\partial u}{\partial n}-g=0.$$

III$\Rightarrow$II. 将两式代入即证. ∎

习题 4 (1) 证明在自变量代换

$$\begin{cases}\xi = x - at,\\ \eta = x + at\end{cases}$$

下, 波动方程 $u_{tt} - a^2u_{xx} = 0$ 具有形式

$$u_{\xi\tau} = 0.$$

(2) 证明在自变量代换

$$\begin{cases}\xi = x - \alpha t,\\ \tau = t\end{cases}$$

下, 方程 $u_t + \alpha u_x = a^2u_{xx}$ 具有形式

$$u_\tau = a^2u_{\xi\xi}.$$

解. (1)

$$u_t = u + \xi(-a) + u_\eta a,$$
$$u_x = u_\xi + u_\eta.$$

那么

$$u_{tt} = a^2(u_{\xi\xi} + u_{\eta\eta} - 2u_{\eta\xi}),$$
$$u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}.$$

代入原方程可得 $u_{\eta\xi} = 0$, 方程的通解是

$$u(x,t) = f(x - at) + g(x + at).$$

(2)

$$u_t = u_\xi(-\alpha) + u_\tau,$$
$$u_x = u_\xi,$$

那么

$$u_{xx} = u_{\xi\xi}.$$

代入原方程可得 $u_\tau = a^2u_{\xi\xi}$.

习题 5 若 u 是 Laplace 方程 $\Delta u = 0$ 的解, 如果 $u(x)$ 只是向径 $r = |x|$ 的函数, 即 $u(x) = \tilde{u}(r)$, 试写出 $\tilde{u}(r)$ 适合的常微分方程.

证明. 因为 $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$, 那么,

$$
\begin{aligned}
u_{x_1} &= u_r \frac{x_1}{r}, \\
u_{x_2} &= u_r \frac{x_2}{r}, \\
&\cdots\cdots \\
u_{x_n} &= u_r \frac{x_n}{r}.
\end{aligned}
$$

则

$$
u_{x_i x_i} = \frac{u_{rr} r x_i^2 - u_r x_i^2 + u_r r^2}{r^3}.
$$

代入, 可得

$$
\tilde{u}_{rr} r + (n-1)\tilde{u}_r = 0.
$$