PDE 作业

第二章第二次

习题 1. 求解三维波动方程的 Cauchy 问题:

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u\big|_{t=0} = 0, \quad t \geqslant 0, \\ u_t\big|_{t=0} = x^3 + y^2 z. \end{cases}$$

解. 根据 Kirchoff 公式, 解为

$$u(x,t) = \frac{1}{4\pi a^2 t} \int_{S_{at}(0)} \psi(x+y)\mathrm{d}y,$$

进一步计算, 可得

$$u(x,t) = \frac{at^2}{3}\left[x_1^3 + x_2^3 x_3 + (3x_1 + x_3)\frac{a^2t^2}{3}\right].$$

习题 2. 用降维法导出一维波动方程 Cauchy 问题的求解公式.

解. 考虑一维波动方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), & \mathbb{R} \times (0,\infty), \\ u\big|_{t=0} = \varphi(x), & x \in \mathbb{R}, \\ u_t\big|_{t=0} = \psi(x), & x \in \mathbb{R}. \end{cases}$$

令 $x_1 = x, \tilde{u}(x_1,x_2,t) = u(x_1,t), \tilde{\varphi}(x_1,x_2) = \varphi(x_1), \tilde{\psi}(x_1,x_2) = \psi(x_1)$, 那么原方程 Cauchy 问题化为二维的波动方程的 Cauchy 问题, 得到解为

$$\begin{aligned} \tilde{u}(x_1,x_2,t) = &\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint_{\Sigma_{at}(x)} \frac{\varphi(y_1)}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\mathrm{d}y\right] \\ &+ \frac{1}{2\pi a}\iint_{\Sigma_{at}(x)} \frac{\psi(y_1)}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\mathrm{d}y \\ &+ \frac{1}{2\pi a}\int_0^t \iint_{\Sigma_{a(t-\tau)}(x)} \frac{f(y_1,\tau)}{\sqrt{a^2(t-\tau)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\mathrm{d}y\mathrm{d}\tau. \end{aligned}$$

又因为

$$
\begin{aligned}
&\iint_{\Sigma_{at}(x)} \frac{\varphi(y_1)}{\sqrt{a^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\mathrm{d}y \\
=&\int_{x_1-at}^{x_1+at}\mathrm{d}y_1\int_{-\sqrt{a^2t^2-(y_1-x_1)^2}}^{\sqrt{a^2t^2-(y_1-x_1)^2}}\frac{\varphi(y_1)}{\sqrt{a^2t^2-(y_1-x_1)^2-y_2^2}}\mathrm{d}y_2 \\
=&\int_{x_1-at}^{x_1+at}\mathrm{d}y_1\int_{-1}^{1}\frac{\varphi(y_1)}{\sqrt{1-y_2^2}}\mathrm{d}y_2 \\
=&\pi\int_{x_1-at}^{x_1+at}\varphi(y_1)\mathrm{d}y_1,
\end{aligned}
$$

同理对剩下的积分也有类似的推导,

$$
\begin{aligned}
u(x_1,t)=\frac{1}{2a}\frac{\partial}{\partial t}\left(\int_{x_1-at}^{x_1+at}\varphi(y_1)\mathrm{d}y_1\right)+\frac{1}{2a}\int_{x_1-at}^{x_1+at}\psi(y_1)\mathrm{d}y_1 \\
+\frac{1}{2a}\int_0^t\left(\int_{x_1-a(t-\tau)}^{x_1+a(t-\tau)}f(y_1,\tau)\mathrm{d}y_1\right)\mathrm{d}\tau.
\end{aligned}
$$

习题 3. 求解二维波动方程的 Cauchy 问题:

$$
\begin{cases}
u_{tt}=a^2(u_{xx}+u_{yy}), \\
u\big|_{t=0}=x^2(x+y). \\
u_t\big|_{t=0}=0.
\end{cases}
$$

解. 根据 Poisson 公式, 可以得到

$$
u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\left(\iint_{\Sigma_{at}(x,y)}\frac{x_1^2(x_1+y_1)}{\sqrt{a^2t^2-(x_1-x)^2-(y_1-y)^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2\right),
$$

进一步计算,

$$
u(x,y,t)=a^2t^2(x+y)+x^2(x+y)+\frac{3\pi a^2t^2x}{4}.
$$

习题 4. 求一下特征值问题的特征函数:

$$
(4)\quad
\begin{cases}
X''(x)+\lambda X(x)=0, & 0<x<l, \\
X(0)=X'(l)+hX(l)=0\ (h>0\text{是常数}).
\end{cases}
$$

解. 因为特征值问题的特征值均大于 0, 那么

$$
X(x)=c_1\sin\left(\sqrt{\lambda}x\right)+c_2\cos\left(\sqrt{\lambda}x\right),
$$

代入条件得 $c_2=0, \sqrt{\lambda}\cos\left(\sqrt{\lambda}l\right)+h\sin\left(\sqrt{\lambda}l\right)=0$, 那么 $\sqrt{\lambda}l\neq\frac{\pi}{2}+k\pi$, 即 λ 是 $\sqrt{\lambda}=-h\tan\left(\sqrt{\lambda}l\right)$ 的正数解, 且方程有正数解 λ_n, 那么特征方程为

$$X_n(x)=\sin\left(\sqrt{\lambda_n}x\right).$$

习题 5. 用分离变量法求解下列定解问题:

$$(2)\begin{cases}Lu=0, & (x,t)\in Q,\\ u\big|_{x=0}=u_x\big|_{x=l}=0, & t\geqslant 0,\\ u\big|_{t=0}=x(x-2l), u_t\big|_{t=0}=0, & 0\leqslant x\leqslant l.\end{cases}$$

解. 令 $v=u-x(x-2l)$,

$$\begin{cases}Lv=2a^2,\\ v\big|_{x=0}=v_x\big|_{x=0}=0,\\ v\big|_{t=0}=v_t\big|_{t=0}=0.\end{cases}$$

令 $v(x,t)=X(x)T(t)$, 那么

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda,$$

即

$$X''(x)+\lambda X(x)=0,$$
$$T''(t)+\lambda a^2T(t)=0.$$

且 $X(0)=X'(l)=T(0)=T'(0)=0$.

类似习题 4可得 $X_k(x)=\sin\frac{(2k+1)\pi}{2l}x$, 令 $v(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin\frac{(2k+1)\pi}{2l}x$ 和 $2a^2=\sum_{k=1}^{\infty}f_k\sin\frac{(2k+1)\pi}{2l}x=\sum_{k=1}^{\infty}\frac{8a^2}{(2k+1)\pi}\sin\frac{(2k+1)\pi}{2l}x$, 则

$$\begin{cases}T_k''(t)+a^2T_k(t)\left(\frac{(2k+1)\pi}{2l}\right)^2=\frac{8a^2}{(2k+1)\pi},\\ T_k(0)=0, T_k'(0)=0.\end{cases}$$

解, 得 $T_k(t)=\frac{32l^2}{((2k+1)\pi)^3}\left(1-\cos a\left(\frac{(2k+1)\pi}{2l}\right)t\right)$, 即

$$v(x,t)=\sum_{k=1}^{\infty}\frac{32l^2}{((2k+1)\pi)^3}\left(1-\cos a\left(\frac{(2k+1)\pi}{2l}\right)t\right)\sin\frac{(2k+1)\pi}{2l}x,$$

那么

$$u(x,t)=x(x-2l)+\sum_{k=1}^{\infty}\frac{32l^2}{((2k+1)\pi)^3}\left(1-\cos a\left(\frac{(2k+1)\pi}{2l}\right)t\right)\sin\frac{(2k+1)\pi}{2l}x.$$

习题 6. 设 $u(x,t)$ 适合定解问题:

$$\begin{cases} Lu = f(x,t), & (x,t) \in Q, \\ -\frac{\partial u}{\partial x} + \alpha\, u\big|_{x=0} = \mu_1(t), & t \geqslant 0, \\ \frac{\partial u}{\partial x} + \beta\, u\big|_{x=l} = \mu(t), & t \geqslant 0, \\ u\big|_{t=0} = \varphi(x),\, u_t\big|_{t=0} = \psi(x), & 0 \leqslant x \leqslant l, \end{cases}$$

试引进辅助函数, 把边值条件齐次化, 设

(a) $a > 0, \beta > 0$; (b) $\alpha = \beta = 0$.

解. (a) 令

$$v(x,t) = \frac{l-x}{l}(-u_x(x,t) + \alpha u_1(x,t) - \mu_1(t)) + \frac{x}{l}(u_x(x,t) + \beta u(x,t) - \mu(t)), \tag{1}$$

则 $v\big|_{x=0} = -u_x(0,t) + \alpha u(0,t) - \mu_1(t) = 0$, $v\big|_{x=l} = u_x(l,t) + \beta u(x,t) - \mu(t) = 0$.

(b) 令式 (1) 中的 $\alpha = \beta = 0$,$v(x,t) = \frac{l-x}{l}(-u_x(x,t) - \mu_1(t)) + \frac{x}{l}(u_x(x,t) - \mu(t))$.

习题 7. 用分离变量法求解下列定解问题:

$$(2)\quad \begin{cases} Lu = -2b\frac{\partial u}{\partial t} + g, & (x,t) \in Q, \\ u\big|_{x=0} = u\big|_{x=t} = 0, & t \geqslant 0, \\ u\big|_{t=0} = u_t\big|_{t=0} = 0, & 0 \leqslant x \leqslant l. \end{cases}$$

解. 首先先令 $Lu = 0$, 然后令 $U(x,t) = T(t)X(x)$, 类似习题 4, 得到 $X_k(x) = \sin\frac{k\pi}{l}x$, $\lambda_n = \frac{n\pi}{l}$.

对于原方程, 令 $u(x) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi}{l}x, g = \sum_{n=1}^{\infty} g_n \sin\frac{k\pi}{l}x$, 则

$$T_n''(t) + a^2\left(\frac{k\pi}{l}\right)^2 T_n(t) + 2bT_n'(t) = g_n,$$

且 $T_n(0) = 0, T_n'(0) = 0$, 又因为 $g_n = \frac{1-(-1)^n}{n\pi g}$, 求解上面的常微分方程, 得到:

1. 当 $\Delta = 4b^2 - 4a^2\left(\frac{n\pi}{l}\right)^2 \geqslant 0$ 时,

$$\begin{aligned} T_n(t) = \frac{l^2 g_n}{a^2 n^2 \pi^2} &+ \frac{(b - \sqrt{b^2 - a^2\lambda_n^2})g_n}{2\lambda_n^2 a^2\sqrt{b^2 - a^2\lambda_n^2}} \exp\left(\left(-b - \sqrt{b^2 - a^2\lambda_n^2}\right)t\right) \\ &+ \frac{(b + \sqrt{b^2 - a^2\lambda_n^2})g_n}{\lambda_n^2 a^2} \exp\left((-b + \sqrt{b^2 - a^2\lambda_n^2})t\right). \end{aligned}$$

2. 当 $\Delta < 0$ 时,

$$
\begin{aligned}
T_n(t) = &\frac{l^2 g_n}{a^2 n^2 \pi^2} - \frac{g_n}{\lambda_n^2 a^2} e^{-bt} \cos\left(\sqrt{a^2\lambda_n^2 - b^2} t\right) \\
&+ \frac{b g_n}{\lambda_n^2 a^2 \sqrt{a^2\lambda_n^2 - b^2}} e^{-bt} \sin\left(\sqrt{a^2\lambda_n^2 - b^2} t\right).
\end{aligned}
$$

那么

$$
\begin{aligned}
u(x,t) = &\sum_{n \leqslant \frac{bl}{a\pi}} \left[\frac{l^2 g_n}{a^2 n^2 \pi^2} + \frac{(b - \sqrt{b^2 - a^2\lambda_n^2}) g_n}{2\lambda_n^2 a^2 \sqrt{b^2 - a^2\lambda_n^2}} \exp\left(\left(-b - \sqrt{b^2 - a^2\lambda_n^2}\right) t\right) \right. \\
&\left. + \frac{(b + \sqrt{b^2 - a^2\lambda_n^2}) g_n}{\lambda_n^2 a^2} \exp\left((-b + \sqrt{b^2 - a^2\lambda_n^2}) t\right) \right] \sin \frac{n\pi}{l} x \\
&+ \sum_{n > \frac{bl}{a\pi}} \left[\frac{l^2 g_n}{a^2 n^2 \pi^2} - \frac{g_n}{\lambda_n^2 a^2} e^{-bt} \cos\left(\sqrt{a^2\lambda_n^2 - b^2} t\right) \right. \\
&\left. + \frac{b g_n}{\lambda_n^2 a^2 \sqrt{a^2\lambda_n^2 - b^2}} e^{-bt} \sin\left(\sqrt{a^2\lambda_n^2 - b^2} t\right) \right] \sin \frac{n\pi}{l} x.
\end{aligned}
$$

习题 8. 考虑定解问题:

$$
\begin{cases}
u_{tt} - u_{xx} = f(x,t), & (x,t) \in Q, \\
u\big|_{x=0} = u\big|_{x=l} = 0, & 0 \leqslant t \leqslant T, \\
u\big|_{t=0} = \varphi(x),\, u_t\big|_{t=0} \psi(x), & 0 \leqslant x \leqslant l.
\end{cases}
$$

试问对 φ, ψ, f 加什么条件才能保证由 Fourier 方法所得的解是古典解? 试证明之!

证明. 根据条件,φ 至少二阶连续可微,ψ 至少连续可微, 因为 $u \in C^2$, 那么 $\lim_{x\to 0} u(x,0) = \lim_{t\to 0} u(0,t)$, 则 $\varphi(0) = 0$, 同样可得 $\varphi(l) = 0$. 又因为 $u_t(0,t) = u_t(l,t) = 0$, 那么类似 $\psi(0) = \psi(l) = 0$. 又因为 $u_{xx}(x,0) = \varphi''(x), u_{tt}(0,t) = u_{tt}(l,t) = 0$, 则 $\varphi''(0) = -f(0,0), \varphi''(l) = -f(l,0)$.

可将原问题分为三部分.

$$
(1) \begin{cases}
Lu = 0 \\
u\big|_{t=0} = \varphi(x),\, u_t\big|_{t=0} = 0 \\
u\big|_{x=0} = 0,\, u\big|_{x=l} = 0
\end{cases} \text{, 解是 } u_1.
$$

$$
(2) \begin{cases}
Lu = 0 \\
u\big|_{t=0} = 0,\, u_t\big|_{t=0} = \psi(x) \\
u\big|_{x=0} = 0,\, u\big|_{x=l} = 0
\end{cases} \text{, 解是 } u_2.
$$

(3) $\begin{cases} Lu = f(x,t) \\ u\big|_{t=0} = 0,\, u_t\big|_{t=0} = 0 \\ u\big|_{x=0} = 0,\, u\big|_{x=l} = 0 \end{cases}$, 解是 u_3.

原方程解 $u = u_1 + u_2 + u_3$. 且 u_1, u_2, u_3 满足 $u_2 = M_\varphi(x,t), u_1 = \frac{\partial}{\partial t} M_\varphi(x,t), u_3 = \int_0^t M_{f_\tau}(t,x;\tau)\,\mathrm{d}\tau$. 当 $\varphi \in C^3, \psi \in C^2$ 时,$u_1, u_2 \in C^2$, 要让 $u_3 \in C^2$, 只需使 $M_{f_\tau}(t,x;\tau) \in C^2$, 它是

$$\begin{cases} Lu = 0 \\ u\big|_{t=\tau} = \varphi(x),\, u_t\big|_{t=\tau} = f(x,\tau) \\ u\big|_{x=0} = 0,\, u\big|_{x=l} = 0 \end{cases}$$

的解, 又因为 $0 \leqslant \tau \leqslant t$, 则 $f \in C^2, f(0,t) = f(l,t) = 0$. ■

习题 9. 用能量不等式证明一维波动方程带有第三边值条件的初边值问题解的唯一性.

证明. 考虑方程

$$\begin{cases} Lu = f(x,t) \\ u\big|_{t=0} = \varphi(x) \\ u_t\big|_{t=0} = \psi(x) \\ -\frac{\partial u}{\partial x}\big|_{x=0} + \alpha_1 u(0,t) = g_1(t) \\ \frac{\partial u}{\partial x}\big|_{x=l} + \alpha_2 u(l,t) = g_2(t) \end{cases}$$

因为这个方程的解集是方程

$$\begin{cases} Lu = f(x,t) \\ u\big|_{t=0} = \varphi(x) \\ u_t\big|_{t=0} = \psi(x) \end{cases} \tag{2}$$

的子集, 且对这个方程

$$\iint_{k_\tau} u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant M_1 \left[\int_{\Omega_0} (\varphi^2 + \psi^2 + a^2\varphi_x^2)\,\mathrm{d}x + \iint_{K_\tau} f^2\,\mathrm{d}x\,\mathrm{d}t \right],$$

则方程 (2) 的解具有唯一性, 即原方程解有唯一性. ■