泛函分析作业

习题 1. 当 $\mathscr{X}$ 是复 Hilbert 空间, $T \in \mathscr{L}(\mathscr{X})$, $T^* = T \Leftrightarrow (Tx, x) \in \mathbb{R}, \forall x \in \mathscr{X}$.

证明. "$\Rightarrow$". 若 $T^* = T$, 则 $(Tx, y) = (x, Ty)$, 令 $x = y$, $(Tx, x) = (x, Tx) = \overline{(Tx, x)}$, 即 $(Tx, x) \in \mathbb{R}$.

"$\Leftarrow$". 对 $\forall x, (Tx, x) = (x, Tx)$.

$$
\begin{aligned}
(Tx + Ty, x + y) &= (Tx, x) + (Ty, y) + (Tx, y) + (Ty, x), \\
(x + y, Tx + Ty) &= (x, Tx) + (y, Ty) + (x, Ty) + (y, Tx),
\end{aligned}
$$

则

$$
\begin{aligned}
\mathrm{Im}(Tx, y) + \mathrm{Im}(Ty, x) = 0, \\
\mathrm{Im}(x, Ty) + \mathrm{Im}(y, Tx) = 0,
\end{aligned}
$$

即

$$
\mathrm{Im}(y, Tx) = \mathrm{Im}(Ty, x),
$$

同理, 对 $(Tx + T(iy), x + iy)$ 和 $(x + iy, Tx + T(iy))$ 做类似的分解, 可得

$$
\mathrm{Re}(Tx, y) = \mathrm{Re}(x, Ty),
$$

即 $(Tx, y) = (x, Ty)$, $T^* = T$. ■

习题 2. 设 $\mathscr{X}$ 是 Hilbert 空间, $T_1, T_2 \in \mathscr{L}(\mathscr{X})$, $T_1^* = T_1, T_2^* = T_2$, $T_1T_2 = T_2T_1 \Leftrightarrow (T_1T_2)^* = T_1T_2$.

证明. "$\Rightarrow$". $(T_1T_2)^* = T_2^*T_1^* = T_2T_1 = T_1T_2$.

"$\Leftarrow$". 因为 T_1T_2 是自伴的, $(T_1T_2)^{**} = (T_1T_2)^*$, 即

$$
T_1T_2 = T_1^{**}T_2^{**} = T_2^*T_1^* = T_2T_1. \quad ■
$$

习题 3. 设 $\mathscr{X}$ 是 Hilbert 空间, $T \in \mathscr{L}(\mathscr{X})$, 证明: $\mathrm{Ker}(T^*) = R(T)^\perp = \left(\overline{R(T)}\right)^\perp$.

证明. 对 $\forall x \in \mathrm{Ker}(T^*)$, 对 $\forall y, (Ty, x) = (y, T^*x) = 0$, 则 $x \in R(T)^\perp$.

对 $\forall x \in R(T)^\perp$, 对 $\forall y, (y, T^*x) = (Ty, x) = 0$, 则 $T^*x = \theta$, $x \in \mathrm{Ker}(T^*)$.

综上, $\mathrm{Ker}(T^*) = R(T)^\perp$ 且 $R(T)^\perp = \left(\overline{R(T)}\right)^\perp$. ■

习题 4. 证明: $({}^\perp M)^\perp = \overline{M}$, 且 ${}^\perp(N^\perp) \supset \overline{N}$.

证明. 对 $\forall x \in \overline{M}, f \in {}^{\perp}M$, $f(x) = 0$, 则 $\overline{M} \subset ({}^{\perp}M)^{\perp}$. 对 $\forall x \in ({}^{\perp}M)^{\perp}$, 若 $x \notin \overline{M}$, 则存在 $\delta > 0$, 使得 $\rho(x, \overline{M}) = \delta > 0$, 根据 Hahn-Banach 定理, 存在 $g \in X^*$, $g|_{\overline{M}} = 0$, $g(x) = \delta$, 则 $g \in {}^{\perp}M$ 但 $g(x) \neq 0$, 则 $x \notin ({}^{\perp}M)^{\perp}$, 矛盾. 综上 $({}^{\perp}M)^{\perp} = \overline{M}$.

对 $\forall f \in \overline{N}$, 存在 $f_n \in N$, $f_n \to f$. 对 $\forall x \in N^{\perp}$, $f_n(x) = 0$, 则 $|f_n(x) - f(x)| \leqslant \|f_n - f\|\|x\| \to 0$, 则 $f(x) = 0$, 即 $\overline{N} \subset {}^{\perp}(N^{\perp})$. ■

习题 5. 设 $\mathscr{X}, \mathscr{Y}$ 是 B^* 空间, $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, 则 $\mathrm{Ker}(T^*) = {}^{\perp}R(T), \mathrm{Ker}(T) = R(T^*)^{\perp}$.

证明. 因为 $f \in \mathrm{Ker}(T^*) \Leftrightarrow T^*f = f \circ T = 0 \Leftrightarrow f \in {}^{\perp}R(T)$, $\mathrm{Ker}(T^*) = {}^{\perp}R(T)$.

同样, $x \in R(T^*)^{\perp} \Leftrightarrow$ 对 $\forall f \in Y^*, T^*f(x) = 0 = f(Tx) \Leftrightarrow Tx = 0$, 其中最后一个等价条件是根据 Hahn-Banach 定理. 则 $\mathrm{Ker}(T) = R(T^*)^{\perp}$. ■

习题 6. 设 $\mathscr{X} = \left\{\xi = (x_1, x_2, \cdots) \in l^2 : \sum_{n=1}^{\infty} |nx_n|^2 < \infty\right\}, \|\xi\|_{\mathscr{X}} = \left(\sum_{n=1}^{\infty} |nx_n|^2\right)^{\frac{1}{2}}$, $T : \mathscr{X} \to l^2, Tx = x$, 证明:$\overline{R(T)} = l^2$.

证明. 因为 $\overline{R(T)} = ({}^{\perp}R(T))^{\perp} = (\mathrm{Ker}\, T^*)^{\perp}$, 对 $\forall f \in \mathrm{Ker}\, T^*$, $T^*f = f \circ T = f \equiv 0$, 则 $\mathrm{Ker}\, T^* = \{0\}$, 那么 $\overline{R(T)} = \{0\}^{\perp} = l^2$. ■

习题 7. 若 $\mathscr{X}$ 是自反空间, 则 $\mathscr{X}^*$ 是自反空间.

证明. 令 $J : \mathscr{X} \longrightarrow \mathscr{X}^{**} \in \mathscr{L}(\mathscr{X}^{***}, \mathscr{X}^*)$ 是自然嵌入映射, $\mathscr{X}$ 自反, 则 J 是双射, $(J^{-1})^* \in \mathscr{L}(\mathscr{X}^*, \mathscr{X}^{***})$, 下面证明对 $\forall \psi \in \mathscr{X}^{***}$, s.t. 对 $\forall x^{**} \in \mathscr{X}^{**}$, $\langle \psi, x^{**} \rangle = \langle x^{**}, J^*\psi \rangle$.

因为对 $\forall x^{**} \in \mathscr{X}^{**}$, 存在 $x_0 \in \mathscr{X}$, $\langle \psi, x^{**} \rangle = \langle \psi, J(x_0) \rangle = \langle \psi \circ J, x_0 \rangle = \langle J^*\psi, x_0 \rangle = \langle J_{x_0}, J^*\psi \rangle = \langle x^{**}, J^*\psi \rangle$, 则 $\mathscr{X}^*$ 是自反的. ■

习题 8. 设 $\mathscr{X}$ 是 B 空间, 若 $\mathscr{X}^*$ 是自反空间, 则 $\mathscr{X}$ 是自反空间.

证明.