Pole-Placement Design – A Polynomial Approach

Overview

- A Simple Design Problem
- The Diophantine Equation
- More Realistic Assumptions

A Simple Design Problem

- System:

$$\mathsf{A}(q)y[k] = \mathsf{B}(q)u[k] \tag{1}$$

- Assumptions:
  - $\mathsf{A} = q^{n_a} + a_1 q^{n_a - 1} + \cdots + a_{n_a - 1} q + a_{n_a}$ is monic
  - $\deg \mathsf{A} > \deg \mathsf{B}$.
  - Disturbances are widely spread impulses

- Specifications are given by the closed-loop characteristic polynomial, and the controller may have certain properties, for example, integral action

- General linear controller (u_c: command signal, y: measured output, u: control signal):

$$\mathsf{R}(q)u[k] = \mathsf{T}(q)u_c[k] - \mathsf{S}(q)y[k] \tag{2}$$

- $\mathsf{R}(q)$ is chosen to be monic.
- Feedforward part:

$$H_{ff}(z) = \frac{\mathsf{T}(z)}{\mathsf{R}(z)} \tag{3}$$

- Feedback part:

$$H_{fb}(z) = \frac{\mathsf{S}(z)}{\mathsf{R}(z)} \tag{4}$$

- Goal: Causal controller with no time-delay $\Rightarrow$

$$\deg \mathsf{R} = \deg \mathsf{S} \tag{5}$$

Solving the design process

- Eliminating $u[k]$ between the process model (1) and the controller (2) gives

$$(\mathsf{A}(q)\mathsf{R}(q) + \mathsf{B}(q)\mathsf{S}(q))\, y[k] = \mathsf{B}(q)\mathsf{T}(q)u_c[k] \tag{6}$$

- Closed-loop characteristic polynomial:

$$\mathsf{A}_{cl}(z) = \mathsf{A}(z)\mathsf{R}(z) + \mathsf{B}(z)\mathsf{S}(z) \tag{7}$$

- The pole-placement design is to find polynomials S and R that satisfy Eq. (7) for given A, B, and A_{cl}.

- Eq. (7) is called *Diophantine Equation*.

- Factorising the A_{cl} polynomial:

$$\mathsf{A}_{cl}(z) = \mathsf{A}_c(z)\mathsf{A}_o(z) \tag{8}$$

- We call $\mathsf{A}_c(z)$ the *controller polynomial* and $\mathsf{A}_o(z)$ the *observer polynomial*.
- In order to determine the polynomial T, we calculate the pulse-transfer function from the command signal to the output:

$$Y(z) = \frac{\mathsf{B}(z)\mathsf{T}(z)}{A_{cl}(z)} U_c(z) = \frac{\mathsf{B}(z)\mathsf{T}(z)}{\mathsf{A}_c(z)\mathsf{A}_o(z)} U_c(z) \tag{9}$$

- Zeros of the open-loop system are also zeros of the closed-loop system (unless $\mathsf{B}(z)$ and $\mathsf{A}_{cl}(z)$ have common factors).
- Let's choose the polynomial T so that is cancels the observer polynomial A_o:

$$\mathsf{T}(z) = t_0 \mathsf{A}_o(z) \tag{10}$$

- The response to command signals is then given by

$$\mathsf{Y}(z) = \frac{t_0 \mathsf{B}(z)}{\mathsf{A}_c(z)} U_c(z) \tag{11}$$

where t_0 is chosen to obtain a desired static gain for the system (e.g., for unit gain: $t_0 = \mathsf{A}_c(1)/\mathsf{B}(1)$).

The Diophantine Equation - Minimal-Degree Solution:

- The equation

$$\mathsf{A}_{cl}(z) = \mathsf{A}(z)\mathsf{R}(z) + \mathsf{B}(z)\mathsf{S}(z) \tag{12}$$

has a solution only if the greatest common divisor of A and B divides A_{cl}.

- Number of controller parameters when $\deg \mathsf{R} = \deg \mathsf{S}$:

$$n_p = 2(\deg \mathsf{R} + 1) = 2 \deg \mathsf{R} + 2 \tag{13}$$

- Degree of A_{cl} (remind that $\deg(\mathsf{AR}) > \deg(\mathsf{BS})$):

$$\deg(\mathsf{A}_{cl}) = \max(\deg(\mathsf{AR}), \deg(\mathsf{BS})) = \deg(\mathsf{AR}) = \deg(\mathsf{A}) + \deg(\mathsf{R}) \tag{14}$$

- Number of equations (coefficients of A_{cl}):

$$n_e = \deg(\mathsf{A}_{cl}) + 1 = \deg(\mathsf{A}) + \deg(\mathsf{R}) + 1 \tag{15}$$

- Unique minimal solution:

$$
\begin{aligned}
n_e &= n_p & (16)\\
\deg \mathsf{A} + \deg \mathsf{R} + 1 &= 2 \deg \mathsf{R} + 2 & (17)
\end{aligned}
$$

$\Rightarrow$

$$\deg \mathsf{R} = \deg \mathsf{A} - 1 \tag{18}$$

- Degree of the closed-loop polynomial for the minimum-degree solution:

$$\deg \mathsf{A}_{cl} = 2 \deg(\mathsf{A}) - 1 \tag{19}$$

- The Diophantine equation can be solved using matrix calculations ($n = n_a > n_b$):

$$\begin{pmatrix} \overline{a}_0 & 0 & 0 & \cdots & 0 & \overline{b}_0 & 0 & 0 & \cdots & 0 \\ \overline{a}_1 & \overline{a}_0 & 0 & \cdots & 0 & \overline{b}_1 & \overline{b}_0 & 0 & \cdots & 0 \\ \overline{a}_2 & \overline{a}_1 & \overline{a}_0 & \cdots & 0 & \overline{b}_2 & \overline{b}_1 & \overline{b}_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a}_n & \overline{a}_{n-1} & \overline{a}_{n-2} & \cdots & \overline{a}_0 & \overline{b}_n & \overline{b}_{n-1} & b_{n-2} & \cdots & \overline{b}_0 \\ 0 & \overline{a}_n & \overline{a}_{n-1} & \cdots & \overline{a}_1 & 0 & \overline{b}_n & \overline{b}_{n-1} & \cdots & \overline{b}_1 \\ 0 & 0 & \overline{a}_n & \cdots & \overline{a}_2 & 0 & 0 & \overline{b}_n & \cdots & \overline{b}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{a}_n & 0 & 0 & 0 & \cdots & \overline{b}_n \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n_a-1} \\ s_0 \\ \vdots \\ s_{n_a-1} \end{pmatrix} = \begin{pmatrix} a_{cl,0} \\ a_{cl,1} \\ a_{cl,2} \\ \vdots \\ a_{cl,n} \\ a_{cl,n+1} \\ a_{cl,n+2} \\ \vdots \\ a_{cl,2n-1} \end{pmatrix}$$

$$\overline{\mathsf{B}} = \overline{b}_0 q^n + \overline{b}_1 q^{n-1} + \cdots + \overline{b}_{n-1} q + \overline{b}_n = b_0 q^{n_b} + b_1 q^{n_b-1} + \cdots + b_{n_b-1} q + b_{n_b} \quad (21)$$

$$\overline{\mathsf{A}} = \overline{a}_0 q^n + \overline{a}_1 q^{n-1} + \cdots + \overline{a}_{n-1} q + \overline{a}_n = 1 q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a-1} q + a_{n_a} \quad (22)$$

More Realistic Assumptions

Cancellations of Poles and Zeros

- Factorisation of A and B:

$$\mathsf{A} = \mathsf{A}^{+}\mathsf{A}^{-} \tag{23}$$

$$\mathsf{B} = \mathsf{B}^{+}\mathsf{B}^{-} \tag{24}$$

- A^{+} and B^{+} are stable (well damped) factors that can be cancelled (both should be chosen monic).

- Poles that shall be cancelled must be controller zeros and zeros that must be cancelled must be controller poles:

$$\mathsf{R} = \mathsf{B}^{+}\mathsf{R}_{d}\overline{\mathsf{R}} \tag{25}$$

$$\mathsf{S} = \mathsf{A}^{+}\mathsf{S}_{d}\overline{\mathsf{S}} \tag{26}$$

where R_d and S_d are fixed pre-determined parts of the controller (see next subsection).

- Closed-loop polynomial:

$$\mathsf{A}_{cl} = \mathsf{AR} + \mathsf{BS} = \mathsf{A}^+\mathsf{B}^+(\mathsf{R}_d\overline{\mathsf{R}}\mathsf{A}^- + \mathsf{S}_d\overline{\mathsf{S}}\mathsf{B}^-) = \mathsf{A}^+\mathsf{B}^+\overline{\mathsf{A}}_{cl} \tag{27}$$

- Cancelled zeros and poles are part of the closed-loop polynomial and must therefore be well damped!

- Cancelling the common factors we find that the polynomials $\overline{\mathsf{R}}$ and $\overline{\mathsf{S}}$ must satisfy:

$$\overline{\mathsf{R}}\mathsf{R}_d\mathsf{A}^- + \overline{\mathsf{S}}\mathsf{S}_d\mathsf{B}^- = \overline{\mathsf{A}}_{cl} \tag{28}$$

- Minimal-degree solution:

$$\begin{aligned} n_e &= n_p && (29)\\ \deg \overline{\mathsf{A}}_{cl} + 1 &= \deg \overline{\mathsf{R}} + \deg \overline{\mathsf{S}} + 2 && (30)\\ \max(\deg \overline{\mathsf{R}}\mathsf{R}_d\mathsf{A}^-, \deg \overline{\mathsf{S}}\mathsf{S}_d\mathsf{B}^-) + 1 &= \deg \overline{\mathsf{R}} + \deg \overline{\mathsf{S}} + 2 && (31) \end{aligned}$$

- For $\deg \overline{\mathsf{R}}\mathsf{R}_d\mathsf{A}^- > \deg \overline{\mathsf{S}}\mathsf{S}_d\mathsf{B}^-$ we obtain:

$$\begin{aligned} \deg \overline{\mathsf{S}} &= \deg \mathsf{A}^- + \deg \mathsf{R}_d - 1 && (32)\\ \deg(\overline{\mathsf{R}}\mathsf{R}_d\mathsf{A}^-) &= \max(\deg \overline{\mathsf{A}}_{cl}, \deg \overline{\mathsf{S}}\mathsf{S}_d\mathsf{B}^-) = \deg \overline{\mathsf{A}}_{cl} && (33) \end{aligned}$$

$$
\begin{aligned}
\deg \mathsf{S} &= \deg \mathsf{R} && (34) \\
\underbrace{\deg \mathsf{A}^- + \deg \mathsf{R}_d - 1}_{\deg \overline{\mathsf{S}}} + \deg \mathsf{S}_d + \deg \mathsf{A}^+ &= \\
\underbrace{\deg \overline{\mathsf{A}}_{cl} - \deg \mathsf{A}^- - \deg \mathsf{R}_d}_{\deg \overline{\mathsf{R}}} + \deg \mathsf{R}_d + \deg \mathsf{B}^+ & && (35)
\end{aligned}
$$

Solving this equation for $\deg \overline{\mathsf{A}}_{cl}$ yields:

$$
\begin{aligned}
\deg \overline{\mathsf{A}}_{cl} &= 2 \deg \mathsf{A} + \deg \mathsf{R}_d + \deg \mathsf{S}_d - \deg \mathsf{B}^+ - \deg \mathsf{A}^+ - 1 && (36) \\
\deg \mathsf{A}_{cl} &= 2 \deg \mathsf{A} + \deg \mathsf{R}_d + \deg \mathsf{S}_d - 1 && (37)
\end{aligned}
$$

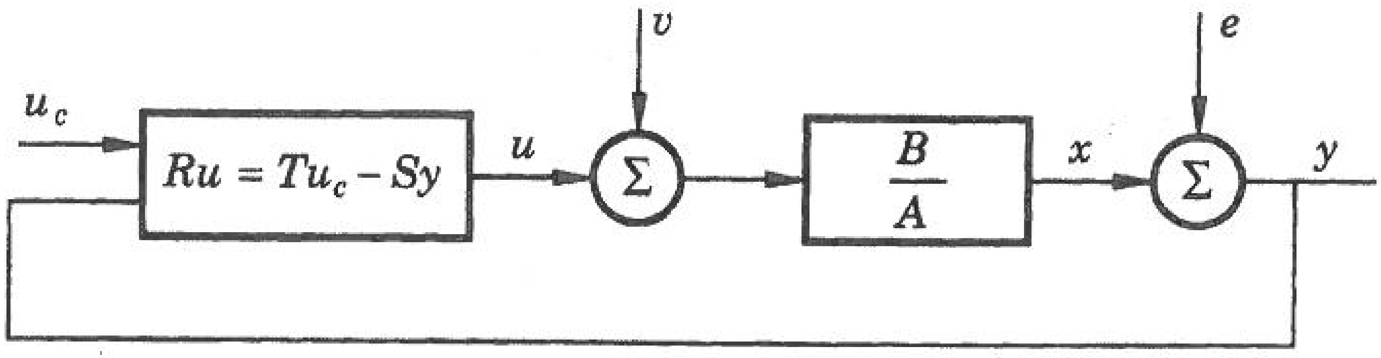
- In order to solve Eq. (28) we create polynomials $\overline{\mathsf{A}}$ and $\overline{\mathsf{B}}$ with $\deg \overline{\mathsf{A}} = \deg \overline{\mathsf{B}} = n = \max(\deg(\mathsf{A}^{-}\mathsf{R}_d), \deg(\mathsf{B}^{-}\mathsf{S}_d))$:

$$\overline{\mathsf{A}} = \mathsf{A}^{-}\mathsf{R}_d \tag{38}$$

$$\overline{\mathsf{B}} = \mathsf{B}^{-}\mathsf{S}_d \tag{39}$$

$$\deg \overline{\mathsf{A}}_{cl} + 1 \left\{ \underbrace{\begin{pmatrix} \ddots & \vdots & \vdots \\ \cdots & \overline{a}_{\deg \overline{\mathsf{A}}} & \overline{a}_{\deg \overline{\mathsf{A}}-1} \\ \cdots & 0 & \overline{a}_{\deg \overline{\mathsf{A}}} \end{pmatrix}}_{\deg \overline{\mathsf{R}}+1} \underbrace{\begin{matrix} \ddots & \vdots & \vdots \\ \cdots & \overline{b}_{\deg \overline{\mathsf{B}}} & \overline{a}_{\deg \overline{\mathsf{B}}-1} \\ \cdots & 0 & \overline{b}_{\deg \overline{\mathsf{B}}} \end{matrix}}_{\deg \overline{\mathsf{S}}+1} \right. \begin{pmatrix} \overline{r}_0 \\ \vdots \\ \overline{s}_0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \overline{a}_{cl_0} \\ \overline{a}_{cl_1} \\ \vdots \end{pmatrix} \tag{40}$$

Handling disturbances



$$
\begin{aligned}
x &= \frac{\mathrm{BT}}{\mathrm{AR}+\mathrm{SB}}u_c + \frac{\mathrm{BR}}{\mathrm{AR}+\mathrm{SB}}v - \frac{\mathrm{BS}}{\mathrm{AR}+\mathrm{SB}}e \\
y &= \frac{\mathrm{BT}}{\mathrm{AR}+\mathrm{SB}}u_c + \frac{\mathrm{BR}}{\mathrm{AR}+\mathrm{SB}}v + \frac{\mathrm{AR}}{\mathrm{AR}+\mathrm{SB}}e \\
u &= \frac{\mathrm{AT}}{\mathrm{AR}+\mathrm{SB}}u_c - \frac{\mathrm{BS}}{\mathrm{AR}+\mathrm{SB}}v - \frac{\mathrm{AS}}{\mathrm{AR}+\mathrm{SB}}e
\end{aligned}
$$

- To avoid steady-state errors due to constant load disturbances the static gain from the disturbance v to y must be zero:

$$\mathsf{B}(1)\mathsf{R}(1) = 0$$

If $\mathsf{B}(1) \neq 0$ then we must require that $\mathsf{R}(1) = 0$. This means that $\mathsf{R}_d = z - 1$ is a factor of $\mathsf{R}(z)$ or that the controller is required to have integral action.

- Elimination of periodic load disturbances (with period $n \cdot \Delta$) by using $\mathsf{R}_d = z^n - 1$:

$$v((k+n)\Delta) - v(k\Delta) = (q^n - 1)v(k\Delta) = 0$$

- Elimination of sinusoidal load disturbances with frequency ω_0:

$$\mathsf{R}_d = z^{-2} + z\cos(\omega_0 \Delta) + 1$$

- Eliminating the effect of measurement noise at Nyquist frequency:

$$\mathsf{S}_d = z + 1$$

Pre-filter revisited

- Let us factorise the polynomial $\mathsf{A}_{cl} = \underbrace{\mathsf{A}^{+}\overline{\mathsf{A}}_{o}}_{\mathsf{A}_o}\underbrace{\mathsf{B}^{+}\overline{\mathsf{A}}_{c}}_{\mathsf{A}_c}$
- Let's choose the polynomial T so that is cancels the observer polynomial A_o:

$$\mathsf{T}(z) = t_0 \mathsf{A}_o(z) \tag{41}$$

- The response to command signals is then given by

$$\mathsf{Y}(z) = \frac{t_0 \mathsf{B}(z)}{\mathsf{A}_c(z)} U_c(z) = H_m(z) U_c(z) \tag{42}$$

  where t_0 is chosen to obtain a desired static gain for the system (e.g., for unit gain: $t_0 = \mathsf{A}_c(1)/\mathsf{B}(1)$).