Lecture 14—Linear Quadratic (LQ) Optimal Control

- Continuous-time LQ Regulation (LQR) control and Riccati Equation
- Infinite Horizon Solution (ARE)
- Discrete-time LQ control
  - Principle of Optimality
  - Discrete-time RE and DARE
- Properties of LQR
  - Robustness
  - Pole locations (root locus analysis)
  - Asymptotic behavior (cheap control)
- LQI (integration)—extension to non-zero set-point
- FSLQ (Frequency Shaping)

Introduction

- LQR in many ways is the beginning of systematic state-space designs for linear MIMO systems (LQG or H2, Hinf)
- Solution for a convex, least-square optimization problem with attractive properties
  - Stable closed-loop
  - Guaranteed level of stability robustness
  - Tuning (re-design) is intuitive
  - Fast and easy computation

Continuous-Time Linear Quadratic Optimal Control

The basic *linear quadratic* (LQ) problem is an optimal control problem for which the system under control is *linear* and the performance index is *quadratic* with non-zero initial conditions and no external disturbance inputs (regulation problems, i.e. LQR).

linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

quadratic cost function

$$J = \mathbf{x}^T(t_f)\cdot\mathbf{S}\cdot\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\cdot\mathbf{Q}(t)\cdot\mathbf{x}(t) + \mathbf{u}^T(t)\cdot\mathbf{R}(t)\cdot\mathbf{u}(t)\right]dt$$

$$J = \mathbf{y}^T(t_f)\cdot\mathbf{S}_y\cdot\mathbf{y}(t_f) + \int_{t_0}^{t_f}\left[\mathbf{y}^T(t)\cdot\mathbf{Q}_{\mathbf{y}}(t)\cdot\mathbf{y}(t) + \mathbf{u}^T(t)\cdot\mathbf{R}(t)\cdot\mathbf{u}(t)\right]dt$$

$$S = \mathbf{C}^T\cdot S_y\cdot\mathbf{C} \qquad \mathbf{Q}(t) = \mathbf{C}^T\cdot\mathbf{Q}_{\mathbf{y}}(t)\cdot\mathbf{C}$$

J is to be minimized by the selected control u(t).

-- tradeoff between regulation error and control effort.

Assumptions for (finite horizon) LQR problem

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t)$$

$$J = \mathbf{x}^T(t_f) \cdot \mathbf{S} \cdot \mathbf{x}(t_f) + \int_{t_0}^{t_f} \left[\mathbf{x}^T(t) \cdot \mathbf{Q}(t) \cdot \mathbf{x}(t) + \mathbf{u}^T(t) \cdot \mathbf{R}(t) \cdot \mathbf{u}(t) \right] dt$$

1. **S** and **Q**(*t*) are symmetric positive semi-definite.
2. **R** is symmetric positive definite.

The matrices **S**, **Q**(*t*), and **R**(*t*) are the design parameters to be selected, and represent the “cost” of error and control signals.
The only constraint on the selection of **S**, **Q**(*t*), and **R**(*t*) is that they have to be symmetric and satisfies their respective positive-definiteness constraints.

$$\mathbf{Q}(t) \uparrow \quad \mathbf{x}(t) \downarrow \qquad\qquad \mathbf{R}(t) \uparrow \quad \mathbf{u}(t) \downarrow$$

Ex9.1 Weighting Matrices Selection

$$G_1(s) = \frac{X_1(s)}{U_1(s)} = \frac{b_1}{s(s+a_1)} \quad \text{and} \quad G_2(s) = \frac{X_2(s)}{U_2(s)} = \frac{b_2}{s(s+a_2)}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \ddot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_2(t) \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & -a_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a_2 \end{array}\right] \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \\ x_2(t) \\ \dot{x}_2(t) \end{bmatrix} + \left[\begin{array}{c|c} 0 & 0 \\ b_1 & 0 \\ \hline 0 & 0 \\ 0 & b_2 \end{array}\right] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Control objective: to regulate both *x*1 and *x*2 to zero (under disturbance and non-zero initial conditions), as well as to minimize the difference between the two positions during the transients.

Ex9.1 (cont.)

$$J = \int_{t_0}^{t_f} \left[q_1 \cdot x_1^2 + q_2 \cdot x_2^2 + \rho \left(x_1 - x_2 \right)^2 + r_1 \cdot u_1^2 + r_2 \cdot u_2^2 \right] dt$$

$$= \int_{t_0}^{t_f} \left\{ \begin{bmatrix} x_1 & x_2 & (x_1 - x_2) \end{bmatrix} \underbrace{\begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & \rho \end{bmatrix}}_{\mathbf{Q_y}} \begin{bmatrix} x_1 \\ x_2 \\ (x_1 - x_2) \end{bmatrix} + \begin{bmatrix} u_1 & u_2 \end{bmatrix} \underbrace{\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dt$$

$$= \int_{t_0}^{t_f} \left\{ \mathbf{x}^T \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{C}^T} \underbrace{\begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & \rho \end{bmatrix}}_{\mathbf{Q_y}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \begin{bmatrix} u_1 & u_2 \end{bmatrix} \underbrace{\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dt$$

Derivation of Continuous-time LQR Solution

Let **H**(t) be an $n \times n$ symmetric matrix

$$\int_{t_0}^{t_f} \frac{d}{dt}[\mathbf{x}^T \cdot \mathbf{H}(t) \cdot \mathbf{x}]\,dt = \mathbf{x}^T(t_f)\mathbf{H}(t_f)\mathbf{x}(t_f) - \mathbf{x}^T(t_0)\mathbf{H}(t_0)\mathbf{x}(t_0)$$

$$= \int_{t_0}^{t_f} \left(\dot{\mathbf{x}}^T \mathbf{H}(t)\mathbf{x} + \mathbf{x}^T \dot{\mathbf{H}}(t)\mathbf{x} + \mathbf{x}^T \mathbf{H}(t)\dot{\mathbf{x}}\right) dt$$

$$= \int_{t_0}^{t_f} \left[(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})^T \mathbf{H}(t)\mathbf{x} + \mathbf{x}^T \dot{\mathbf{H}}(t)\mathbf{x} + \mathbf{x}^T \mathbf{H}(t)(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})\right] dt$$

$$= \int_{t_0}^{t_f} \left[\mathbf{x}^T \left(\mathbf{A}^T \mathbf{H}(t) + \mathbf{H}(t)\mathbf{A} + \dot{\mathbf{H}}(t)\right)\mathbf{x} + \mathbf{u}^T \mathbf{B}^T \mathbf{H}(t)\mathbf{x} + \mathbf{x}^T \mathbf{H}(t)\mathbf{B}\mathbf{u}\right] dt$$

true for all H!

Select **H**(t) so that it satisfies

$$\boxed{\dot{\mathbf{H}}(t) + \mathbf{A}^T \mathbf{H}(t) + \mathbf{H}(t)\mathbf{A} = \mathbf{H}(t)\mathbf{B}\mathbf{R}^{-1}(t)\mathbf{B}^T \mathbf{H}(t) - \mathbf{Q}(t), \quad \text{where} \quad \mathbf{H}(t_f) = \mathbf{S}}$$

$$\Rightarrow \quad -\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \mathbf{x}^T(t_o)\mathbf{H}(t_o)\mathbf{x}(t_o)$$

$$+ \int_{t_o}^{t_f} \left[\mathbf{x}^T(\mathbf{H}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H} - \mathbf{Q})\mathbf{x} + \mathbf{u}^T\mathbf{B}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{H}\mathbf{B}\mathbf{u}\right] dt = 0$$

Derivation (cont.)

Original cost function

$$J = \mathbf{x}^T(t_f)\cdot\mathbf{S}\cdot\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\cdot\mathbf{Q}(t)\cdot\mathbf{x}(t) + \mathbf{u}^T(t)\cdot\mathbf{R}(t)\cdot\mathbf{u}(t)\right]dt$$

New equation

$$-\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \mathbf{x}^T(t_o)\mathbf{H}(t_o)\mathbf{x}(t_o)$$

$$+\int_{t_o}^{t_f}\left[\mathbf{x}^T(\mathbf{H}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}-\mathbf{Q})\mathbf{x} + \mathbf{u}^T\mathbf{B}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{H}\mathbf{B}\mathbf{u}\right]dt = 0$$

Add together

$$J = \mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]dt - \mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \mathbf{x}^T(t_0)\mathbf{H}(t_0)\mathbf{x}(t_0)$$

$$+\int_{t_o}^{t_f}\left[\mathbf{x}^T(\mathbf{H}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}-\mathbf{Q})x + \mathbf{u}^T\mathbf{B}^T\mathbf{H}x + x^T\mathbf{H}\mathbf{B}\mathbf{u}\right]dt$$

$$= \mathbf{x}^T(t_0)\mathbf{H}(t_0)\mathbf{x}(t_0) + \int_{t_o}^{t_f}\left[(\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}\mathbf{x}+\mathbf{u})^T\mathbf{R}(\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}\mathbf{x}+\mathbf{u})\right]dt$$

J is minimized if

$$\boxed{\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}(t)\cdot\mathbf{x}(t)} \qquad \boxed{J^* = \mathbf{x}^T(t_0)\mathbf{H}(t_0)\mathbf{x}(t_0)}$$

Summarizing the continuous LQR results

Linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t) \qquad \mathbf{x}(t_o) = \mathbf{x}_o \qquad \mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t)$$

Quadratic performance index

$$J = \mathbf{x}^T(t_f) \cdot \mathbf{S} \cdot \mathbf{x}(t_f) + \int_{t_0}^{t_f} \left[\mathbf{x}^T(t) \cdot \mathbf{Q}(t) \cdot \mathbf{x}(t) + \mathbf{u}^T(t) \cdot \mathbf{R}(t) \cdot \mathbf{u}(t) \right] dt$$

Optimal control law (linear state feedback form)

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}(t) \cdot \mathbf{x}(t) = -\mathbf{K}_{LQ}(t) \cdot \mathbf{x}(t)$$

where H is solved from the (continuous-time) Riccati equation (backwards, off-line)

$$\boxed{\dot{\mathbf{H}}(t) + \mathbf{A}^T\mathbf{H}(t) + \mathbf{H}(t)\mathbf{A} = \mathbf{H}(t)\mathbf{B}\mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{H}(t) - \mathbf{Q}(t), \quad \text{where} \quad \mathbf{H}(t_f) = \mathbf{S}}$$

Optimal cost

$$\boxed{J^* = \mathbf{x}^T(t_0)\mathbf{H}(t_0)\mathbf{x}(t_0)}$$

Characteristics of H

$$\boxed{\dot{\mathbf{H}}(t)+\mathbf{A}^T\mathbf{H}(t)+\mathbf{H}(t)\mathbf{A}=\mathbf{H}(t)\mathbf{B}\mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{H}(t)-\mathbf{Q}(t), \quad \text{where} \quad \mathbf{H}(t_f)=\mathbf{S}}$$

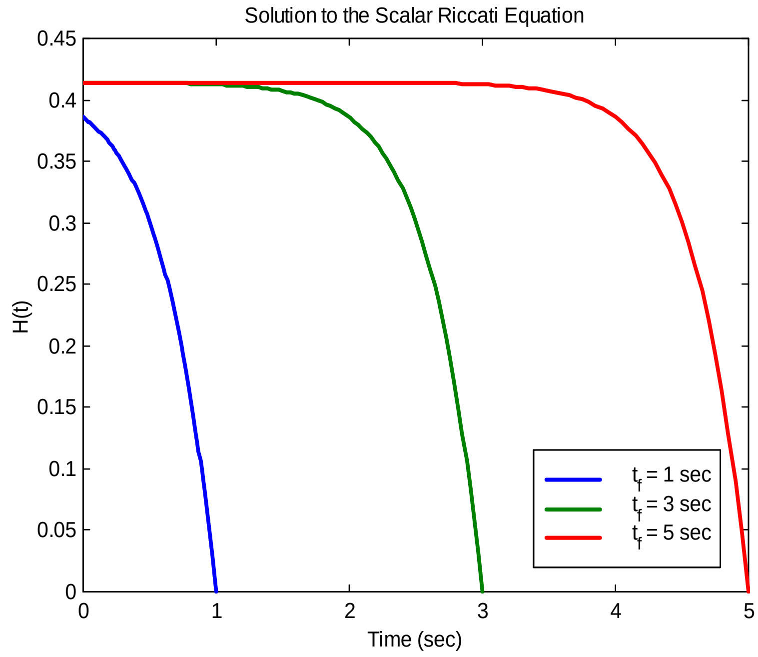
$$J^* = \mathbf{x}^T(t_0)\mathbf{H}(t_0)\mathbf{x}(t_0)$$

$$J=\mathbf{x}^T(t_f)\cdot\mathbf{S}\cdot\mathbf{x}(t_f)+\int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\cdot\mathbf{Q}(t)\cdot\mathbf{x}(t)+\mathbf{u}^T(t)\cdot\mathbf{R}(t)\cdot\mathbf{u}(t)\right]dt$$

H is positive semi-definite

H converges to a steady-state solution

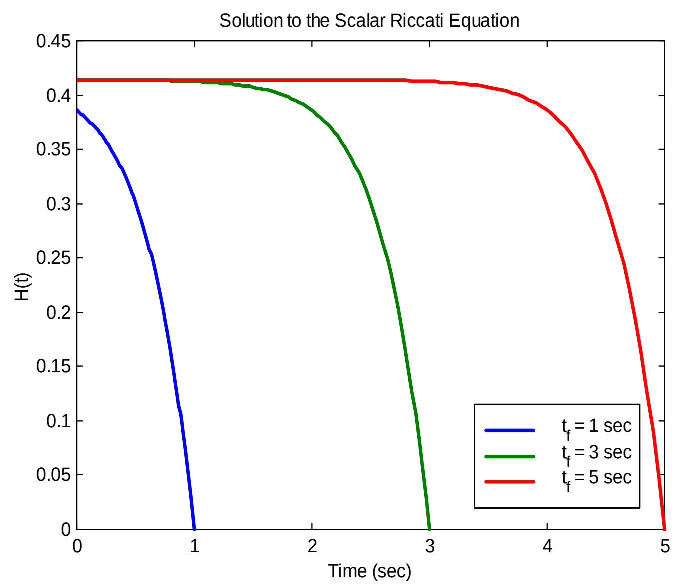
Ex.9.2:



Stationary (Infinite Horizon) LQ Problem

$$J = \mathbf{x}^T(t_f)\cdot\mathbf{S}\cdot\mathbf{x}(t_f) + \int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\cdot\mathbf{Q}(t)\cdot\mathbf{x}(t) + \mathbf{u}^T(t)\cdot\mathbf{R}(t)\cdot\mathbf{u}(t)\right]dt$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}(t)\cdot\mathbf{x}(t) = -\mathbf{K}_{LQ}(t)\cdot\mathbf{x}(t)$$



$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS}\cdot\mathbf{x}(t) = -\mathbf{K}_{LQ}\cdot\mathbf{x}(t)$$

CT Stationary (Infinite Horizon) LQ Problem

Linear system (controllable)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t) \qquad \mathbf{x}(t_o) = \mathbf{x}_o$$

Quadratic performance index ($\mathbf{Q} = \mathbf{Q}_1^T \cdot \mathbf{Q}_1$, (**A, Q$_1$**) is observable)

$$J = \int_{t_0}^{\infty} \left(\mathbf{x}^T(t)\cdot\mathbf{Q}\cdot\mathbf{x}(t) + \mathbf{u}^T(t)\cdot\mathbf{R}\cdot\mathbf{u}(t) \right) dt$$

Optimal control law (full-state feedback form)

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS}\cdot\mathbf{x}(t) = -\mathbf{K}_{LQ}\cdot\mathbf{x}(t)$$

where $\mathbf{H}_{SS}$ is the unique, positive definite solution of the CT Algebraic Riccati Equation (ARE)

$$\boxed{\mathbf{A}^T\mathbf{H}_{SS} + \mathbf{H}_{SS}\mathbf{A} - \mathbf{H}_{SS}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} + \mathbf{Q} = \mathbf{0}}$$

Optimal cost

$$J^* = \mathbf{x}^T(t_0)\mathbf{H}_{SS}\mathbf{x}(t_0)$$

Closed-loop system (Asymptotically stable)

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K}_{LQ})\cdot\mathbf{x}(t) \qquad \mathbf{x}(t_o) = \mathbf{x}_o$$

MATLAB lqr() Command

» help lqr

LQR Linear-quadratic regulator design for continuous-time systems.

[K,S,E] = LQR(A,B,Q,R,N) calculates the optimal gain matrix K such that the state-feedback law u = -Kx minimizes the cost function

J = Integral {x'Qx + u'Ru + 2*x'Nu} dt

subject to the state dynamics $\dot{x} = Ax + Bu$.

The matrix N is set to zero when omitted. Also returned are the Riccati equation solution S and the closed-loop eigenvalues E:

$$SA + A'S - (SB+N)R^{-1}(B'S+N') + Q = 0 , \quad E = EIG(A-B*K) .$$

MichiganEngineering

LQ Feedback System is Asymptotically Stable

Since H_{SS} is positive definite, select the Lyapunov function

$$V = \mathbf{x}^T \cdot \mathbf{H}_{SS} \cdot \mathbf{x}$$

$$\begin{aligned}
\Rightarrow \quad \dot{V} &= \dot{\mathbf{x}}^T \mathbf{H}_{SS} \mathbf{x} + \mathbf{x}^T \mathbf{H}_{SS} \dot{\mathbf{x}} \\
&= \mathbf{x}^T \cdot \left(\mathbf{A}^T - \mathbf{H}_{SS} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \right) \mathbf{H}_{SS} \cdot \mathbf{x} + \mathbf{x}^T \cdot \mathbf{H}_{SS} \left(\mathbf{A} - \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{H}_{SS} \right) \cdot \mathbf{x} \\
&= \mathbf{x}^T \cdot \left(\mathbf{A}^T \mathbf{H}_{SS} - \mathbf{H}_{SS} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{H}_{SS} + \mathbf{H}_{SS} \mathbf{A} - \mathbf{H}_{SS} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{H}_{SS} \right) \cdot \mathbf{x} \\
&= \mathbf{x}^T \cdot \left(-\mathbf{Q} - \mathbf{H}_{SS} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{H}_{SS} \right) \cdot \mathbf{x}
\end{aligned}$$

Since **Q** is positive semi-definite and **R** is positive definite, $\dot{V}$ is negative definite. This proves that the LQ optimal closed-loop system is asymptotically stable.

Selection of Q and R

- If there is not a good idea for the structure of **Q** and **R**, start with diagonal **Q** and **R**.

- Relative size of Q and R indicate whether you want high gain or low gain, and the relative importance of individual states (outputs) and inputs.

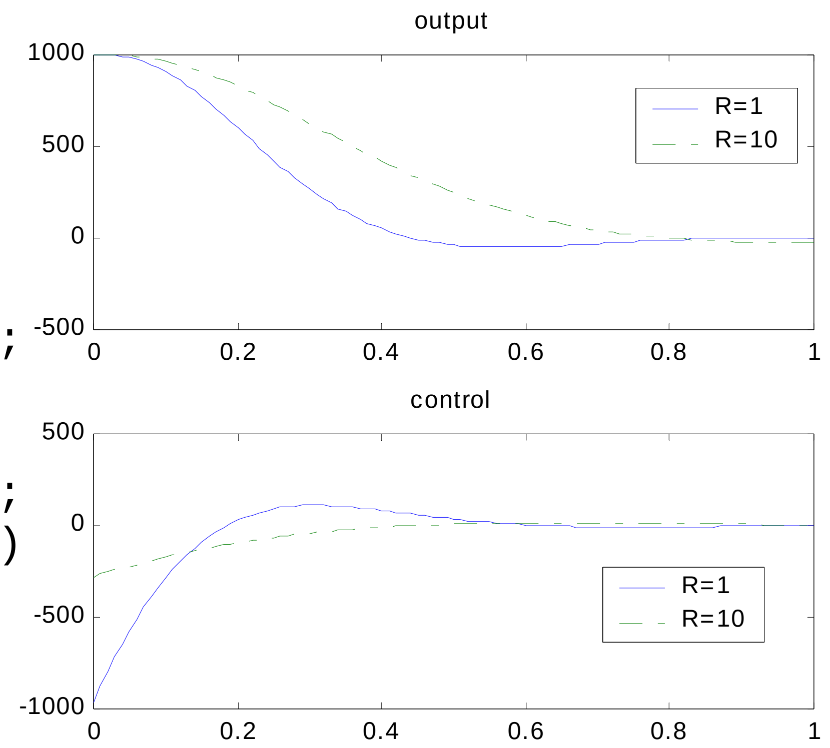
- If tolerable maximum values of states and inputs are known. Call them $x_{i,MAX}$ (i = 1,.., n) and $u_{i,MAX}$ (i = 1,.., r). Set the diagonal elements of **Q** and **R** to be inverse proportional to $\|x_{i,MAX}\|^2$ and $\|u_{i,MAX}\|^2$.

CT LQR example

LQR for a plant described by $G(s) = \dfrac{1000}{(s+1)(s+4)(s+10)}$

```
% Ex9_extra_LQ
num=1000; den=[1 15 54 40];
[A,B,C,D]=tf2ss(num,den);
Q=C'*C; R=1;
[K1,S,E]=lqr(A,B,Q,R);
X0=[0;0;1];
T=0:0.01:1;
[y1,x1,t1]=initial(A-B*K1,B,C,D,X0,T);
R=10;
[K2,S,E]=lqr(A,B,Q,R);
[y2,x2,t2]=initial(A-B*K2,B,C,D,X0,T);
subplot(211), plot(T,y1,'-',T,y2,'-.')
title('output')
legend('R=1', 'R=10')

subplot(212),plot(T,-K1*x1','-',T,-
K2*x2','-.')
title('control')
legend('R=1', 'R=10')
```



Ex.9.3 1st Order Stationary LQR Problem

Plant: $\dot{x} = a \cdot x + b \cdot u$ $\quad b \neq 0$

Cost function: $J = \int_0^\infty \left(q \cdot x^2 + r \cdot u^2 \right) dt \qquad q \geq 0 \quad \text{and} \quad r > 0$

ARE:

$$\frac{b^2}{r} H_{SS}^2 - 2aH_{SS} - q = 0$$

$$\mathbf{A}^T \mathbf{H}_{SS} + \mathbf{H}_{SS}\mathbf{A} - \mathbf{H}_{SS}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} + \mathbf{Q} = \mathbf{0}$$

$$H_{SS} = \bar{H}_2 = \frac{a + \sqrt{a^2 + \frac{b^2}{r} q}}{\frac{b^2}{r}} \geq 0$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} \cdot \mathbf{x}(t)$$

Optimal control:

$$u^*(t) = -\frac{b}{r} H_{SS} \cdot x(t) = -K_{LQ} \cdot x(t)$$

$$\boxed{K_{LQ} = \frac{a + \sqrt{a^2 + b^2 \frac{q}{r}}}{b}}$$

Closed-loop:

$$\boxed{\dot{x} = -\sqrt{a^2 + b^2 \frac{q}{r}} \cdot x}$$

Asymptotically stable!

Ex9.3 (cont.)

Open-loop	Control gain	Closed-loop
$\dot{x} = a \cdot x + b \cdot u$	$\boxed{K_{LQ} = \dfrac{a + \sqrt{a^2 + b^2 \frac{q}{r}}}{b}}$	$\boxed{\dot{x} = -\sqrt{a^2 + b^2 \dfrac{q}{r}} \cdot x}$

Case 1 (cheap control)

$$\frac{q}{r} \to \infty$$

$$K_{LQ} \to sign(b)\sqrt{\frac{q}{r}}$$

$$\dot{x} = -\left(abs(b)\sqrt{\frac{q}{r}}\right) \cdot x$$

Case 2 (expensive control)

$$\frac{q}{r} \to 0$$

$$K_{LQ} \to \frac{a + |a|}{b}$$

$$a \le 0 \qquad K_{LQ} \to 0$$

$$a > 0 \qquad K_{LQ} \to \frac{2a}{b}$$

$$a \le 0 \qquad \dot{x} = a \cdot x$$

$$a > 0 \qquad \dot{x} = -a \cdot x$$

Discrete-time LQ control

Linear system

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k) \qquad \mathbf{x}(0) = \mathbf{x}_o$$

Quadratic performance index (same PD and PSD requirement)

$$J = \frac{1}{2}\mathbf{x}^T(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}\sum_{i=0}^{N-1}\left[\mathbf{x}^T(i)\mathbf{Q}\mathbf{x}(i) + \mathbf{u}^T(i)\mathbf{R}\mathbf{u}(i)\right]$$

Optimal control law

$$\mathbf{u}^*(k) = -\left[\mathbf{R} + \mathbf{B}^T\mathbf{H}(k+1)\mathbf{B}\right]^{-1}\mathbf{B}^T\mathbf{H}(k+1)\mathbf{A}\cdot\mathbf{x}(k) = -\mathbf{K}_{LQ}(k)\cdot\mathbf{x}(k)$$

where H is solved from the (discrete-time) Riccati equation (backwards)

$$\boxed{\mathbf{H}(k) = \mathbf{Q} + \mathbf{A}^T\mathbf{H}(k+1)\mathbf{A} - \mathbf{A}^T\mathbf{H}(k+1)\mathbf{B}\left[\mathbf{R} + \mathbf{B}^T\mathbf{H}(k+1)\mathbf{B}\right]^{-1}\mathbf{B}^T\mathbf{H}(k+1)\mathbf{A}}$$

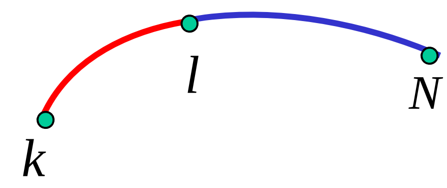
$$\mathbf{H}(N) = \mathbf{S}$$

Optimal cost

$$J_0^N{}^* = \frac{1}{2}\mathbf{x}^T(0)\mathbf{H}(0)\mathbf{x}(0)$$

Principle of Optimality

An optimal control sequence has the property that whatever the initial state and initial control are, the remaining controlling inputs must constitute an optimal sequence with respect to the state resulting from the first control.



For a problem with the cost function

$$J_k = \phi(x(N),N) + \sum_{i=k}^{N-1} f(x(i),u(i),i)$$

Objective: to find optimal $U_k = \{u(k), u(k+1), \cdots, u(N-1)\}$

$$\begin{aligned}
J_k^*(x(k)) &= \min_{u(k)\cdots u(N-1)} \left[\phi(x(N),N) + \sum_{i=k}^{N-1} f(x(i),u(i),i) \right] \\
&= \min_{u(k)\cdots u(l-1)} \left\{ \min_{u(l)\cdots u(N-1)} \left[\phi(x(N),N) + \sum_{i=k}^{l-1} f(x(i),u(i),i) + \sum_{i=l}^{N-1} f(x(i),u(i),i) \right] \right\} \\
&= \min_{u(k)\cdots u(l-1)} \left\{ \sum_{i=k}^{l-1} f(x(i),u(i),i) + \min_{u(l)\cdots u(N-1)} \left[\phi(x(N),N) + \sum_{i=l}^{N-1} f(x(i),u(i),i) \right] \right\} \\
&= \min_{u(k)\cdots u(l-1)} \left\{ \sum_{i=k}^{l-1} f(x(i),u(i),i) + J_l^*(x(l)) \right\}
\end{aligned}$$

Recursive Formula

$$\boxed{J_k^*\left(x(k)\right) = \min_{u(k)}\left[f\left(x(k), u(k), k\right) + J_{k+1}^*\left(x(k+1)\right)\right]}$$

$$J = \frac{1}{2}\mathbf{x}^T(N)\mathbf{S}\mathbf{x}(N) + \frac{1}{2}\sum_{i=0}^{N-1}\left[\mathbf{x}^T(i)\mathbf{Q}\mathbf{x}(i) + \mathbf{u}^T(i)\mathbf{R}\mathbf{u}(i)\right]$$

Final step (step N):

$$J_N^* = \frac{1}{2}\mathbf{x}^T(N)\cdot\mathbf{S}\cdot\mathbf{x}(N) \equiv \frac{1}{2}\mathbf{x}^T(N)\cdot\mathbf{H}(N)\cdot\mathbf{x}(N)$$

Previous step (step N-1):

$$J_{N-1}^* = \min_{u(k-1)}\left[\frac{1}{2}\mathbf{x}^T(N-1)\cdot\mathbf{Q}\cdot\mathbf{x}(N-1) + \frac{1}{2}\mathbf{u}^T(N-1)\cdot\mathbf{R}\cdot\mathbf{u}(N-1) + J_N^N{}^*\right]$$

$$= \min_{u(k-1)}\frac{1}{2}\left\{\begin{array}{l}\mathbf{x}^T(N-1)\mathbf{Q}\mathbf{x}(N-1) + \mathbf{u}^T(N-1)\mathbf{R}\mathbf{u}(N-1) \\ \qquad +\left[\mathbf{A}\cdot\mathbf{x}(N-1) + \mathbf{B}\cdot\mathbf{u}(N-1)\right]^T\cdot\mathbf{H}(N)\cdot\left[\mathbf{A}\cdot\mathbf{x}(N-1) + \mathbf{B}\cdot\mathbf{u}(N-1)\right]\end{array}\right\}$$

Optimality:

$$\frac{\partial J_{N-1}^N{}^*}{\partial\,\mathbf{u}(N-1)} = \mathbf{0}$$

Solution for the Discrete-Time LQ Control

$$J_{N-1}{}^{*} = \min_{u(k-1)}\left[\frac{1}{2}\mathbf{x}^T(N-1)\cdot\mathbf{Q}\cdot\mathbf{x}(N-1)+\frac{1}{2}\mathbf{u}^T(N-1)\cdot\mathbf{R}\cdot\mathbf{u}(N-1)+J_N^N{}^{*}\right]$$

$$= \min_{u(k-1)}\frac{1}{2}\begin{Bmatrix}\mathbf{x}^T(N-1)\mathbf{Q}\mathbf{x}(N-1)+\mathbf{u}^T(N-1)\mathbf{R}\mathbf{u}(N-1) \\ +\left[\mathbf{A}\cdot\mathbf{x}(N-1)+\mathbf{B}\cdot\mathbf{u}(N-1)\right]^T\cdot\mathbf{H}(N)\cdot\left[\mathbf{A}\cdot\mathbf{x}(N-1)+\mathbf{B}\cdot\mathbf{u}(N-1)\right]\end{Bmatrix}$$

$$\frac{\partial J_{N-1}^{N}{}^{*}}{\partial\,\mathbf{u}^T(N-1)} = \frac{1}{2}\left(\mathbf{R}\mathbf{u}+\mathbf{R}^T\mathbf{u}+\mathbf{B}^T\mathbf{H}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{H}\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{H}^T\mathbf{B}\mathbf{u}+\mathbf{B}^T\mathbf{H}\mathbf{B}\mathbf{u}\right)$$

$$= (\mathbf{R}+\mathbf{B}^T\mathbf{H}\mathbf{B})\mathbf{u}+\mathbf{B}^T\mathbf{H}\mathbf{A}\mathbf{x} = \mathbf{0}$$

```
We dropped the time index
to simplify the eq.
```

$$\Rightarrow \quad \left(\mathbf{R}+\mathbf{B}^T\mathbf{H}(N)\mathbf{B}\right)\cdot\mathbf{u}^*(N-1)+\mathbf{B}^T\mathbf{H}(N)\mathbf{A}\cdot\mathbf{x}(N-1) = \mathbf{0}$$

$$\Rightarrow \quad \mathbf{u}^*(N-1) = -[\mathbf{R}+\mathbf{B}^T\mathbf{H}(N)\mathbf{B}]^{-1}\mathbf{B}^T\mathbf{H}(N)\mathbf{A}\cdot\mathbf{x}(N-1)$$

$$\equiv -\mathbf{K}_{LQ}(N-1)\cdot\mathbf{x}(N-1)$$

Solution for the DTLQ (cont.)

$$J_{N-1}{}^* = \min_{u(k-1)}\left[\frac{1}{2}\mathbf{x}^T(N-1)\cdot\mathbf{Q}\cdot\mathbf{x}(N-1)+\frac{1}{2}\mathbf{u}^T(N-1)\cdot\mathbf{R}\cdot\mathbf{u}(N-1)+J_N^N{}^*\right]$$

$$= \min_{u(k-1)}\frac{1}{2}\left\{\begin{array}{l}\mathbf{x}^T(N-1)\mathbf{Q}\mathbf{x}(N-1)+\mathbf{u}^T(N-1)\mathbf{R}\mathbf{u}(N-1)\\ \qquad +\left[\mathbf{A}\cdot\mathbf{x}(N-1)+\mathbf{B}\cdot\mathbf{u}(N-1)\right]^T\cdot\mathbf{H}(N)\cdot\left[\mathbf{A}\cdot\mathbf{x}(N-1)+\mathbf{B}\cdot\mathbf{u}(N-1)\right]\end{array}\right\}$$

$$\boxed{\mathbf{u}^*(N-1) = -\mathbf{K}_{LQ}(N-1)\cdot\mathbf{x}(N-1)}$$

$$= \frac{1}{2}\mathbf{x}^T(N-1)\left[\begin{array}{l}\mathbf{Q}+\mathbf{K}_{LQ}^T(N-1)\mathbf{R}\mathbf{K}_{LQ}(N-1)\\ \qquad +\left[\mathbf{A}-\mathbf{B}\mathbf{K}_{LQ}(N-1)\right]^T\mathbf{H}(N)\left[\mathbf{A}-\mathbf{B}\mathbf{K}_{LQ}(N-1)\right]\end{array}\right]\mathbf{x}(N-1)$$

$$\equiv \frac{1}{2}\mathbf{x}^T(N-1)\mathbf{H}(N-1)\mathbf{x}(N-1)$$

$$\mathbf{H}(N-1) = \mathbf{Q}+\mathbf{A}^T\mathbf{H}(N)\mathbf{A}-\mathbf{A}^T\mathbf{H}(N)\mathbf{B}\left[\mathbf{R}+\mathbf{B}^T\mathbf{H}(N)\mathbf{B}\right]^{-1}\mathbf{B}^T\mathbf{H}(N)\mathbf{A}$$

$$J_k{}^* = \frac{1}{2}\mathbf{x}^T(k)\cdot\mathbf{H}(k)\cdot\mathbf{x}(k) \qquad k = N-1,\ N$$

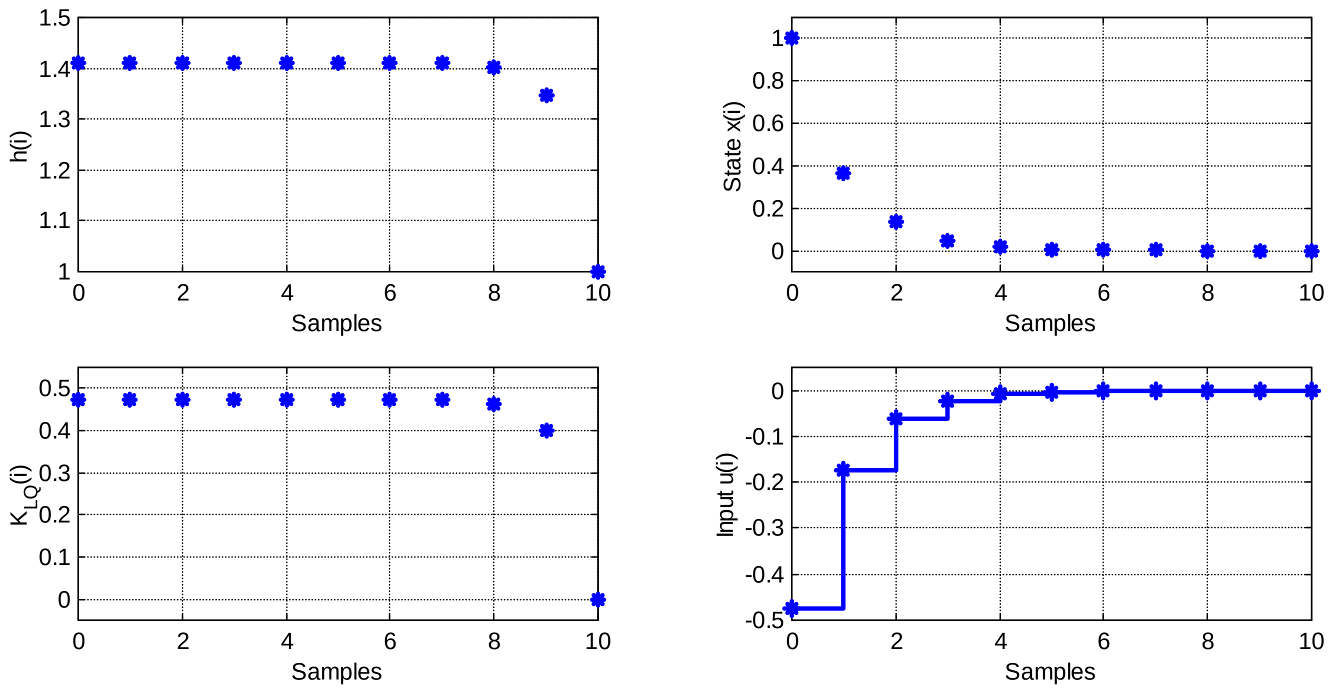
Ex.9.5 1st Order Discrete-Time LQR Problem

$$x(k+1) = 0.8 \cdot x(k) + 0.92 \cdot u(k)$$

$$J = \frac{1}{2} x^2(10) + \frac{1}{2} \sum_{i=0}^{9} \left[x^2(i) + u^2(i) \right]$$

$$h(k) = 1 + 0.8^2 \cdot h(k+1) - \frac{0.8^2 \cdot 0.92^2 \cdot h^2(k+1)}{1 + 0.92^2 \cdot h(k+1)} \qquad K_{LQ}(k) = \frac{0.8 \cdot 0.92 \cdot h(k+1)}{1 + 0.92^2 \cdot h(k+1)}$$

$$= 1 + 0.64 \cdot h(k+1) - \frac{0.5417 \cdot h^2(k+1)}{1 + 0.8464 \cdot h(k+1)}$$



DT Stationary (Infinite Horizon) LQ Problem

Linear system (controllable)

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k) \qquad \mathbf{x}(0) = \mathbf{x}_o$$

Quadratic performance index ($\mathbf{Q} = \mathbf{Q}_1^T \cdot \mathbf{Q}_1$, (**A, Q$_1$**) is observable)

$$J = \frac{1}{2}\sum_{i=0}^{\infty}\left[\mathbf{x}^T(i)\mathbf{Q}\mathbf{x}(i) + \mathbf{u}^T(i)\mathbf{R}\mathbf{u}(i)\right]$$

Optimal control law

$$\mathbf{u}^*(k) = -\left[\mathbf{R} + \mathbf{B}^T\mathbf{H}_{SS}\mathbf{B}\right]^{-1}\mathbf{B}^T\mathbf{H}_{SS}\mathbf{A}\cdot\mathbf{x}(k) = -\mathbf{K}_{LQ}\cdot\mathbf{x}(k)$$

where $\mathbf{H}_{SS}$ is the positive definite solution of the DT Algebraic Riccati Equation (ARE)

$$\mathbf{H}_{SS} = \mathbf{Q} + \mathbf{A}^T\mathbf{H}_{SS}\mathbf{A} - \mathbf{A}^T\mathbf{H}_{SS}\mathbf{B}\left[\mathbf{R} + \mathbf{B}^T\mathbf{H}_{SS}\mathbf{B}\right]^{-1}\mathbf{B}^T\mathbf{H}_{SS}\mathbf{A}$$

Optimal cost

$$J^* = \mathbf{x}^T(0)\mathbf{H}_{SS}\mathbf{x}(0)$$

DLQR problems

» help dlqr

DLQR Linear-quadratic regulator design for discrete-time systems.

[K,S,E] = DLQR(A,B,Q,R,N) calculates the optimal gain matrix K such that the state-feedback law u[n] = -Kx[n] minimizes the cost function

J = Sum {x'Qx + u'Ru + 2*x'Nu}

subject to the state dynamics   x[n+1] = Ax[n] + Bu[n].

The matrix N is set to zero when omitted. Also returned are the Riccati equation solution S and the closed-loop eigenvalues E:

$$A'SA - S - (A'SB+N)(R+B'SB)^{-1}(B'SA+N') + Q = 0, \quad E = EIG(A-B*K).$$

See also DLQRY, LQRD, LQGREG, and DARE.

Explained in notes

LQRD Problems

>> help lqrd

LQRD Discrete linear-quadratic regulator design from continuous cost function.

[K,S,E] = LQRD(A,B,Q,R,Ts) calculates the optimal gain matrix K
such that the discrete state-feedback law u[n] = -K x[n]
minimizes a discrete cost function equivalent to the continuous
cost function

J = Integral {x'Qx + u'Ru} dt

subject to the discretized state dynamics x[n+1] = Ad x[n] + Bd u[n]
where [Ad,Bd] = C2D(A,B,Ts). Also returned are the discrete Riccati
equation solution S and the closed-loop eigenvalues E = EIG(Ad-Bd*K).

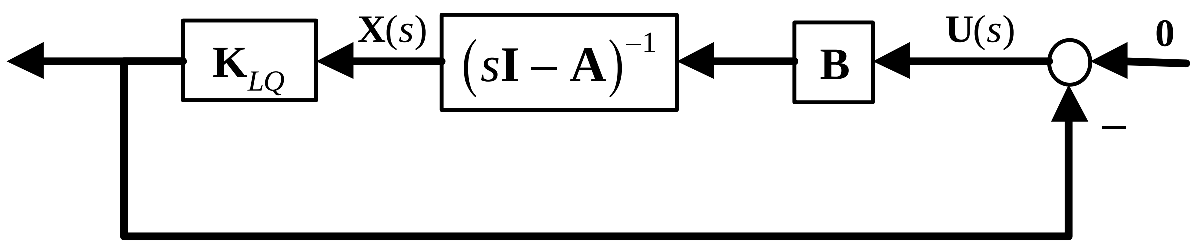
[K,S,E] = LQRD(A,B,Q,R,N,Ts) handles the more general cost function
J = Integral {x'Qx + u'Ru + 2*x'Nu} dt .

Algorithm: the continuous plant (A,B,C,D) and continuous weighting
matrices (Q,R,N) are discretized using the sample time Ts and
zero-order hold approximation. The gain matrix K is then calculated
using DLQR.

CT Return Difference Equality of LQR Systems

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} \cdot \mathbf{x}(t) = -\mathbf{K}_{LQ} \cdot \mathbf{x}(t)$$



$$\underbrace{[\mathbf{I}_r + \mathbf{L}(-s)]^T}_{\text{return difference function matrix}} \mathbf{R}[\mathbf{I}_r + \mathbf{L}(s)] = \mathbf{R} + \mathbf{G}^T(-s)\mathbf{G}(s)$$

Return Difference Equality

where

$$\mathbf{L}(s) = \mathbf{K}_{LQ}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$\mathbf{G}(s) = \mathbf{Q}_1(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \qquad \mathbf{Q} = \mathbf{Q}_1^T \cdot \mathbf{Q}_1$$

Proof of CT Return Difference Equality

ARE $$\boxed{\mathbf{A}^T\mathbf{H}_{SS} + \mathbf{H}_{SS}\mathbf{A} - \mathbf{H}_{SS}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} + \mathbf{Q} = \mathbf{0}}$$

$$\mathbf{A}^T\mathbf{H}_{SS} + \mathbf{H}_{SS}\mathbf{A} - \mathbf{H}_{SS}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} + \mathbf{Q}_1^T\mathbf{Q}_1 + s\mathbf{H}_{SS} - s\mathbf{H}_{SS} = \mathbf{0}$$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS}\cdot\mathbf{x}(t) = -\mathbf{K}_{LQ}\cdot\mathbf{x}(t)$$

$$\mathbf{H}_{SS}(s\mathbf{I}-\mathbf{A}) + (-s\mathbf{I}-\mathbf{A}^T)\mathbf{H}_{SS} + \mathbf{H}_{SS}\mathbf{B}\mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{H}_{SS} = \mathbf{Q}_1^T\mathbf{Q}_1$$

$$\boxed{\mathbf{B}^T(-s\mathbf{I}-\mathbf{A})^{-1}} \Rightarrow \quad \mathbf{H}_{SS}(s\mathbf{I}-\mathbf{A}) + (-s\mathbf{I}-\mathbf{A}^T)\mathbf{H}_{SS} + \mathbf{K}_{LQ}^T\mathbf{R}\mathbf{K}_{LQ} = \mathbf{Q}_1^T\mathbf{Q}_1 \quad \boxed{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}}$$

$$\mathbf{B}^T(-s\mathbf{I}-\mathbf{A})^{-1}\mathbf{H}_{SS}\mathbf{B} + \mathbf{B}^T\mathbf{H}_{SS}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{B}^T(-s\mathbf{I}-\mathbf{A})^{-1}\mathbf{K}_{LQ}^T\mathbf{R}\mathbf{K}_{LQ}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$
$$= \mathbf{B}^T(-s\mathbf{I}-\mathbf{A})^{-1}\mathbf{Q}_1^T\mathbf{Q}_1(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

$$\Big[\mathbf{I}_r + \underbrace{\mathbf{B}^T(-s\mathbf{I}-\mathbf{A})^{-1}\mathbf{K}_{LQ}}_{\mathbf{L}^T(-s)}\Big]\mathbf{R}\Big[\mathbf{I}_r + \underbrace{\mathbf{K}_{LQ}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}}_{\mathbf{L}(s)}\Big] = \mathbf{R} + \underbrace{\mathbf{B}^T(-s\mathbf{I}-\mathbf{A})^{-1}\mathbf{Q}_1^T}_{\mathbf{G}^T(-s)}\underbrace{\mathbf{Q}_1(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}}_{\mathbf{G}(s)}$$

Robust Stability of CT Single-Input LQR

Single input: return difference equality becomes

$$[1+L(-s)]R[1+L(s)] = R + G^T(-s)G(s)$$

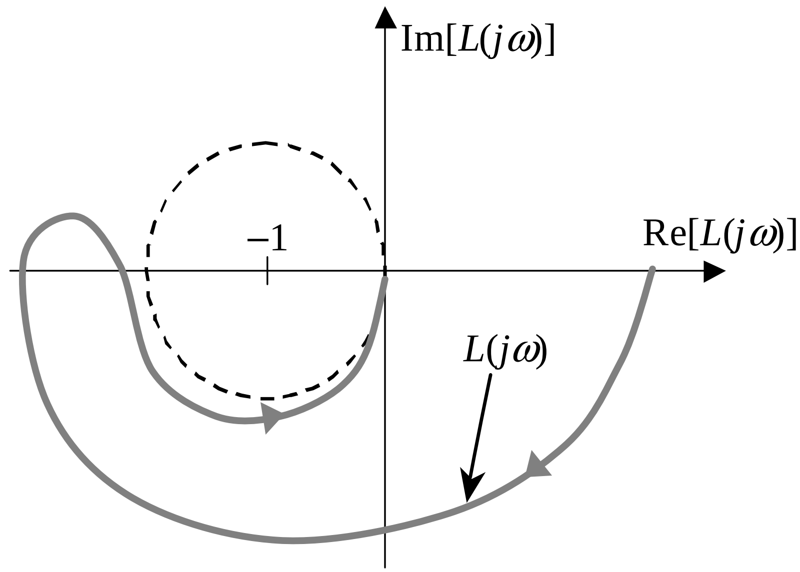
$$\Rightarrow \quad [1+L(-s)][1+L(s)] = 1+\frac{1}{R}G^T(-s)G(s)$$

$$[1+L(-j\omega)][1+L(j\omega)] = 1+\frac{1}{R}G^T(-j\omega)G(j\omega)$$

$$\Rightarrow \quad |1+L(j\omega)|^2 = 1+\frac{1}{R}G^T(-j\omega)G(j\omega)$$

$$\boxed{|1+L(j\omega)|^2 \geq 1}$$

(1) Phase margin: PM > 60°.
(2) Gain margin: $\mathrm{GM} \to \infty$.
(3) Stability for up to 50% reduction in loop gain.



Robust Stability of DT Single-Input LQR

Return Difference Equality (starting from discrete ARE)

$$\left[\mathbf{I}_r + \mathbf{L}(z^{-1})\right]^T \left(\mathbf{R} + \mathbf{B}^T \mathbf{H}_{SS} \mathbf{B}\right)\left[\mathbf{I}_r + \mathbf{L}(z)\right] = \mathbf{R} + \mathbf{G}^T(z^{-1})\mathbf{G}(z)$$

Single input cases:

$$\left[1 + L(z^{-1})\right]\left(R + B^T H_{SS} B\right)\left[1 + L(z)\right] = R + G^T(z^{-1})G(z)$$

$$\Rightarrow \quad \left[1 + L(z^{-1})\right]\left[1 + L(z)\right] = \frac{R}{R + B^T H_{SS} B} + \frac{1}{R + B^T H_{SS} B} G^T(z^{-1})G(z)$$

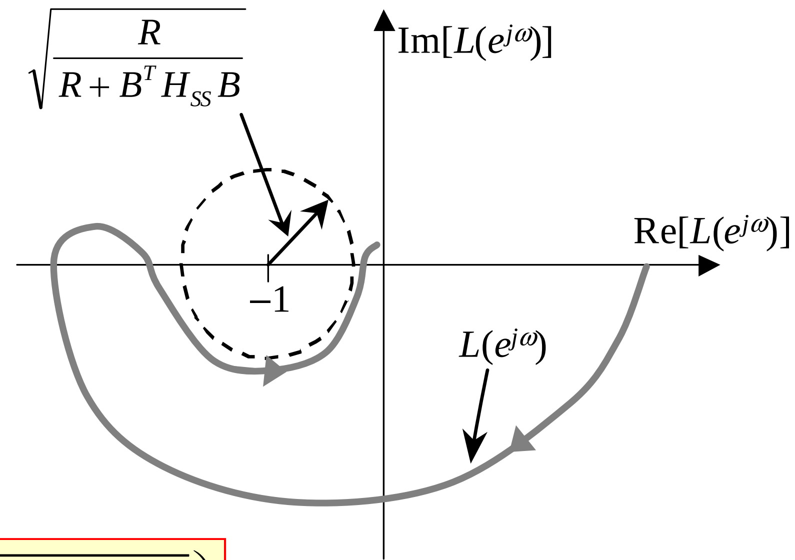
$$\left[1 + L(e^{-j\omega})\right]\left[1 + L(e^{j\omega})\right] = \frac{R}{R + B^T H_{SS} B} + \frac{1}{R + B^T H_{SS} B} G^T(e^{-j\omega})G(e^{j\omega})$$

$$\Rightarrow \quad \left|1 + L(e^{j\omega})\right|^2 = \frac{R}{R + B^T H_{SS} B} + \frac{1}{R + B^T H_{SS} B} G^T(e^{-j\omega})G(e^{j\omega})$$

$$\boxed{\left|1 + L(e^{j\omega})\right|^2 \geq \sqrt{\frac{R}{R + B^T H_{SS} B}}}$$

Robust Stability of DT (cont.)

$$\left|1+L(e^{j\omega})\right|^2 \geq \sqrt{\frac{R}{R+B^T H_{SS} B}}$$



Phase margin:

$$\mathrm{PM} > 2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{R}{R+B^T H_{SS} B}}\right)$$

Gain margin:

$$\mathrm{GM} > \frac{1}{1-\sqrt{R/\left(R+B^T H_{SS} B\right)}}$$

Loop gain changes:

$$\frac{100}{1+\sqrt{R/\left(R+B^T H_{SS} B\right)}} < \text{\% Loop Gain Change} < \frac{100}{1-\sqrt{R/\left(R+B^T H_{SS} B\right)}}$$

Linear Quadratic with Integrator (LQI)

The steady-state error of the LQR system may not be zero when the system is subject to a constant disturbance or a constant reference input. Solution: LQI

Original plant:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t)$$

Assumptions:

The plant is controllable and observable

The system matrix **A** is non-singular (i.e. no open-loop poles at the origin).

$\mathbf{CA}^{-1}\mathbf{B}$ is non-singular.

Basic idea: penalize $\boxed{\dot{\mathbf{u}}}$

$$\boxed{\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{r}}$$

LQI (cont.)

Define a new performance index

$$J = \int_{t_0}^{\infty} \left(\left[\mathbf{y}(t) - \mathbf{r}\right]^T \cdot \mathbf{Q}_{\mathrm{y}} \cdot \left[\mathbf{y}(t) - \mathbf{r}\right] + \mathbf{v}^T(t) \cdot \mathbf{R} \cdot \mathbf{v}(t) \right) dt$$

$$\boxed{\mathbf{e} = \mathbf{y} - \mathbf{r}}$$

$$\boxed{\dot{\mathbf{u}} = \mathbf{v}}$$

where r is the (constant) set-point. Define new state vector

$$\frac{d}{dt} \underbrace{\begin{bmatrix} \mathbf{e} \\ \dot{\mathbf{x}} \end{bmatrix}}_{\mathbf{x}_{\mathrm{E}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}}_{\mathbf{A}_{\mathrm{E}}} \begin{bmatrix} \mathbf{e} \\ \dot{\mathbf{x}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}}_{\mathbf{B}_{\mathrm{E}}} \cdot \mathbf{v}(t) \quad \Rightarrow \quad \dot{\mathbf{x}}_{\mathrm{E}} = \mathbf{A}_{\mathrm{E}} \cdot \mathbf{x}_{\mathrm{E}} + \mathbf{B}_{\mathrm{E}} \cdot \mathbf{v}$$

$$J = \int_{t_0}^{\infty} [\mathbf{x}_{\mathrm{E}}^T(t) \cdot \underbrace{\begin{bmatrix} \mathbf{Q}_{\mathrm{y}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{Q}_{\mathrm{E}}} \cdot \mathbf{x}_{\mathrm{E}}(t) + \mathbf{v}^T(t) \cdot \mathbf{R} \cdot \mathbf{v}(t)] dt$$

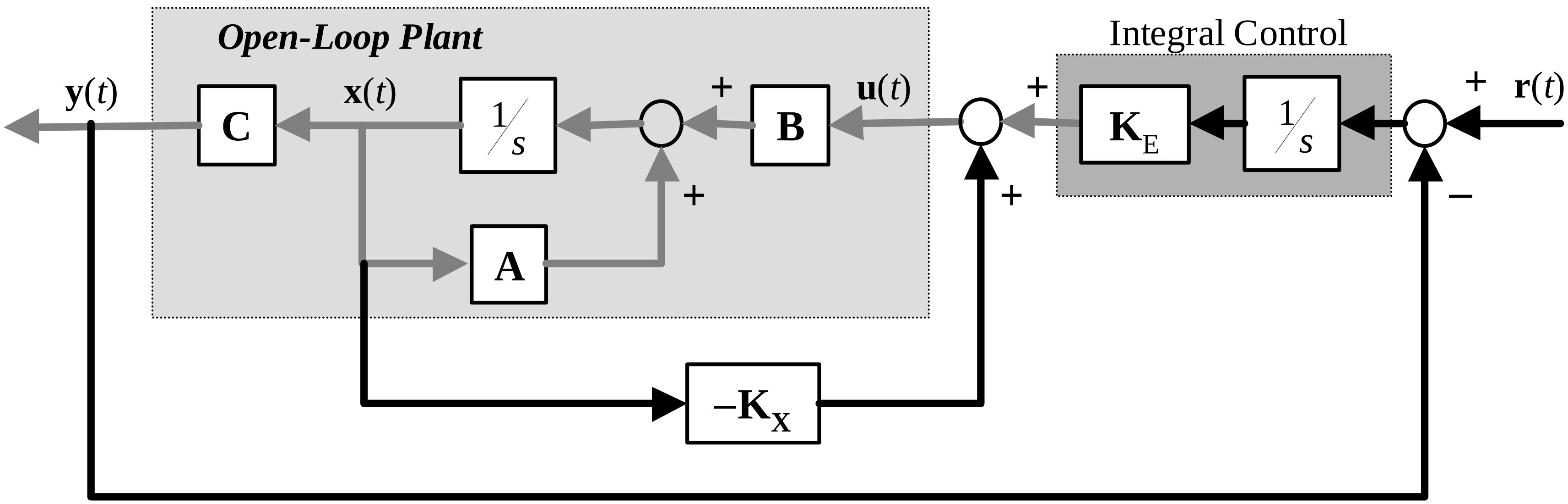
$$\mathbf{v}(t) = -\mathbf{R}^{-1}\mathbf{B}_{\mathrm{E}}^T\mathbf{H}_{SS} \cdot \mathbf{x}_{\mathrm{E}}(t) = -\left[\mathbf{K}_{\mathrm{E}} \quad \mathbf{K}_{\mathrm{X}}\right]\mathbf{x}_{\mathrm{E}}(t) = -\mathbf{K}_{\mathrm{E}} \cdot \mathbf{e}(t) - \mathbf{K}_{\mathrm{X}} \cdot \dot{\mathbf{x}}(t)$$

$$\mathbf{A}_{\mathrm{E}}^T\mathbf{H}_{SS} + \mathbf{H}_{SS}\mathbf{A}_{\mathrm{E}} - \mathbf{H}_{SS}\mathbf{B}_{\mathrm{E}}\mathbf{R}^{-1}\mathbf{B}_{\mathrm{E}}^T\mathbf{H}_{SS} + \mathbf{Q}_{\mathrm{E}} = \mathbf{0}$$

$$\boxed{\mathbf{u}(t) = \mathbf{K}_{\mathrm{E}} \cdot \int_0^t [\mathbf{r} - \mathbf{y}(t)] dt - \mathbf{K}_{\mathrm{X}} \cdot \mathbf{x}(t)}$$

LQI (cont.)

$$\mathbf{u}(t) = \mathbf{K}_{\mathrm{E}} \cdot \int_0^t [\mathbf{r} - \mathbf{y}(t)]\, dt - \mathbf{K}_{\mathrm{X}} \cdot \mathbf{x}(t)$$



Discrete-Time LQI

Basic idea: penalize $\boxed{\Delta\mathbf{u}(k) = \mathbf{u}(k) - \mathbf{u}(k-1)}$

$\boxed{\mathbf{e}(k) = \mathbf{y}(k) - \mathbf{r}}$

Define $\Delta\mathbf{x}(k) = \mathbf{x}(k) - \mathbf{x}(k-1)$

$$\Delta\mathbf{x}(k+1) = \mathbf{A}\cdot\Delta\mathbf{x}(k) + \mathbf{CB}\cdot\Delta\mathbf{u}(k)$$

$$\mathbf{e}(k+1) = \mathbf{e}(k) + \mathbf{C}\cdot\Delta\mathbf{x}(k+1) = \mathbf{e}(k) + \mathbf{CA}\cdot\Delta\mathbf{x}(k) + \mathbf{CB}\cdot\Delta\mathbf{u}(k)$$

$$\Rightarrow \begin{bmatrix} \mathbf{e}(k+1) \\ \Delta\mathbf{x}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{CA} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}}_{\mathbf{A}_\mathrm{E}} \underbrace{\begin{bmatrix} \mathbf{e}(k) \\ \Delta\mathbf{x}(k) \end{bmatrix}}_{\mathbf{x}_\mathrm{E}} + \underbrace{\begin{bmatrix} \mathbf{CB} \\ \mathbf{B} \end{bmatrix}}_{\mathbf{B}_\mathrm{E}} \cdot\Delta\mathbf{u}(k) \quad \text{or} \quad \mathbf{x}_\mathrm{E}(k+1) = \mathbf{A}_\mathrm{E}\cdot\mathbf{x}_\mathrm{E}(k) + \mathbf{B}_\mathrm{E}\cdot\Delta\mathbf{u}(k)$$

$$J = \sum_{i=0}^{\infty}\left[\mathbf{e}^T(i)\cdot\mathbf{Q}_\mathbf{y}\cdot\mathbf{e}(i) + \Delta\mathbf{u}^T(i)\cdot\mathbf{R}\cdot\Delta\mathbf{u}(i)\right]$$

$$= \sum_{i=0}^{\infty}\left\{\mathbf{x}_\mathrm{E}^T(t)\cdot\underbrace{\begin{bmatrix} \mathbf{Q}_\mathbf{y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{Q}_\mathrm{E}}\cdot\mathbf{x}_\mathrm{E}(t) + \Delta\mathbf{u}^T(t)\cdot\mathbf{R}\cdot\Delta\mathbf{u}(t)\right\}$$

Discrete-Time LQI (cont.)

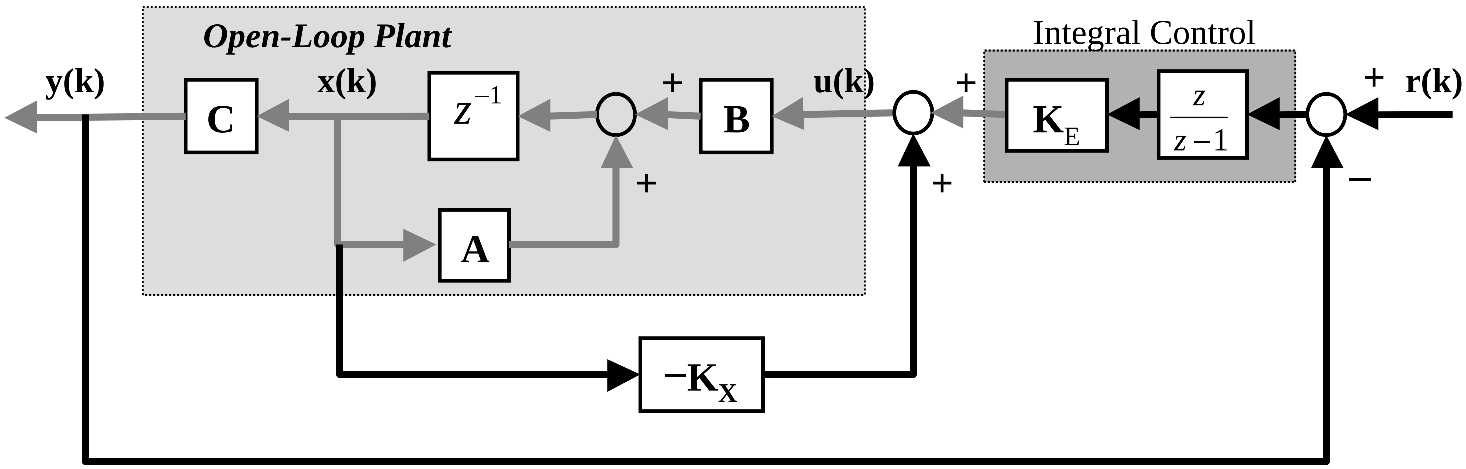
Solve the DARE (dlqr())

$$\mathbf{H}_{SS} = \mathbf{Q}_{\mathrm{E}} + \mathbf{A}_{\mathrm{E}}^{T}\mathbf{H}_{SS}\mathbf{A}_{\mathrm{E}} - \mathbf{A}_{\mathrm{E}}^{T}\mathbf{H}_{SS}\mathbf{B}_{\mathrm{E}}\left[\mathbf{R} + \mathbf{B}_{\mathrm{E}}^{T}\mathbf{H}_{SS}\mathbf{B}_{\mathrm{E}}\right]^{-1}\mathbf{B}_{\mathrm{E}}^{T}\mathbf{H}_{SS}\mathbf{A}_{\mathrm{E}}$$

$$\Delta\mathbf{u}(k) = -\left(\mathbf{R} + \mathbf{B}_{\mathrm{E}}^{T}\mathbf{H}_{SS}\mathbf{B}_{\mathrm{E}}\right)^{-1}\mathbf{B}_{\mathrm{E}}^{T}\mathbf{H}_{SS}\mathbf{A}_{\mathrm{E}}\cdot\mathbf{x}_{\mathrm{E}}(k) = -\mathbf{K}_{\mathrm{E}}\cdot\mathbf{e}(k) - \mathbf{K}_{\mathrm{X}}\cdot\Delta\mathbf{x}(k)$$

$$\mathbf{u}(k) = \sum_{i=0}^{k}\Delta\mathbf{u}(i) \qquad (\mathrm{u}(0)=0)$$

$$\mathbf{u}(t) = -\mathbf{K}_{\mathrm{E}}\cdot\sum_{i=0}^{k}\mathbf{e}(i) - \mathbf{K}_{\mathrm{X}}\cdot\sum_{i=0}^{k}\Delta\mathbf{x}(i) = \mathbf{K}_{\mathrm{E}}\cdot\sum_{i=0}^{k}\left[\mathbf{r} - \mathbf{y}(i)\right] - \mathbf{K}_{\mathrm{X}}\cdot\mathbf{x}(k)$$



Frequency Shaped LQ

Parseval's Theorem (Parseval's Identity)

$$\int_{-\infty}^{\infty} f^T(t) f(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^T(-j\omega) F(j\omega)\, d\omega$$

where $f(t)$ and $F(j\omega)$ are Fourier transform pairs

Proof: see Chap 9 notes, page 32.

Frequency Shaped LQ

Original cost function:

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T(t)\cdot\mathbf{Q}\cdot\mathbf{x}(t)+\mathbf{u}^T(t)\cdot\mathbf{R}\cdot\mathbf{u}(t)\right)dt$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\mathbf{X}^T(-j\omega)\cdot\mathbf{Q}\cdot\mathbf{X}(j\omega)+\mathbf{U}^T(-j\omega)\cdot\mathbf{R}\cdot\mathbf{U}(j\omega)\right)d\omega$$

In general, a frequency-dependent weighting can be used:

$$J = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\mathbf{X}^T(-j\omega)\cdot\mathbf{Q}(j\omega)\cdot\mathbf{X}(j\omega)+\mathbf{U}^T(-j\omega)\cdot\mathbf{R}(j\omega)\cdot\mathbf{U}(j\omega)\right)d\omega$$

$$\boxed{\begin{aligned}\mathbf{Q}(j\omega) &= \mathbf{Q}_F^T(-j\omega)\cdot\mathbf{Q}_F(j\omega)\\ \mathbf{R}(j\omega) &= \mathbf{R}_F^T(-j\omega)\cdot\mathbf{R}_F(j\omega)\end{aligned}}$$

Assume: $\mathbf{Q}_F(s) = \mathbf{C}_1(s\mathbf{I}-\mathbf{A}_1)^{-1}\mathbf{B}_1+\mathbf{D}_1$

$$\boxed{\begin{aligned}\dot{\mathbf{z}}_1(t) &= \mathbf{A}_1\cdot\mathbf{z}_1(t)+\mathbf{B}_1\cdot\mathbf{x}(t)\\ \mathbf{x}_F(t) &= \mathbf{C}_1\cdot\mathbf{z}_1(t)+\mathbf{D}_1\cdot\mathbf{x}(t)\end{aligned}}$$

filtered states

$$\mathbf{X}^T(-j\omega)\cdot\mathbf{Q}(j\omega)\cdot\mathbf{X}(j\omega) = \mathbf{X}_F^T(-j\omega)\mathbf{X}_F^T(j\omega)$$

FSLQ

Assume: $\mathbf{U}_F(j\omega) = \mathbf{R}_F(j\omega)\cdot\mathbf{U}(j\omega)$

$$\dot{\mathbf{z}}_2(t) = \mathbf{A}_2\cdot\mathbf{z}_2(t) + \mathbf{B}_2\cdot\mathbf{u}(t)$$
$$\mathbf{u}_F(t) = \mathbf{C}_2\cdot\mathbf{z}_2(t) + \mathbf{D}_2\cdot\mathbf{u}(t)$$

filtered inputs

$$J = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\mathbf{X}^T(-j\omega)\cdot\mathbf{Q}(j\omega)\cdot\mathbf{X}(j\omega) + \mathbf{U}^T(-j\omega)\cdot\mathbf{R}(j\omega)\cdot\mathbf{U}(j\omega)\right)d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\mathbf{X}_F^T(-j\omega)\cdot\mathbf{X}_F(j\omega) + \mathbf{U}_F^T(-j\omega)\cdot\mathbf{U}_F(j\omega)\right)d\omega$$
$$= \int_0^{\infty}\left(\mathbf{x}_F^T(t)\cdot\mathbf{x}_F(t) + \mathbf{u}_F^T(t)\cdot\mathbf{u}_F(t)\right)dt$$

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{z}_1(t)\\ \mathbf{z}_2(t)\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{A} & \mathbf{0} & \mathbf{0}\\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2\end{bmatrix}}_{\mathbf{A}_E}\underbrace{\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{z}_1(t)\\ \mathbf{z}_2(t)\end{bmatrix}}_{\mathbf{x}_E} + \underbrace{\begin{bmatrix}\mathbf{B}\\ \mathbf{0}\\ \mathbf{B}_2\end{bmatrix}}_{\mathbf{B}_E}\cdot\mathbf{u}(t), \quad \text{or} \quad \dot{\mathbf{x}}_E(t) = \mathbf{A}_E\cdot\mathbf{x}_E(t) + \mathbf{B}_E\cdot\mathbf{u}(t)$$

Interpretation of FSLQ

$$\mathbf{u}(t) = -\mathbf{R}_E^{-1}\left[\mathbf{B}_E^T\mathbf{H}_{SS} + \mathbf{N}_E^T\right]\cdot\mathbf{x}_E(t) = -\mathbf{K}\cdot\mathbf{x}(t) - \mathbf{K}_1\cdot\mathbf{z}_1(t) - \mathbf{K}_2\cdot\mathbf{z}_2(t)$$