LECTURE 3

Z-TRANSFORM

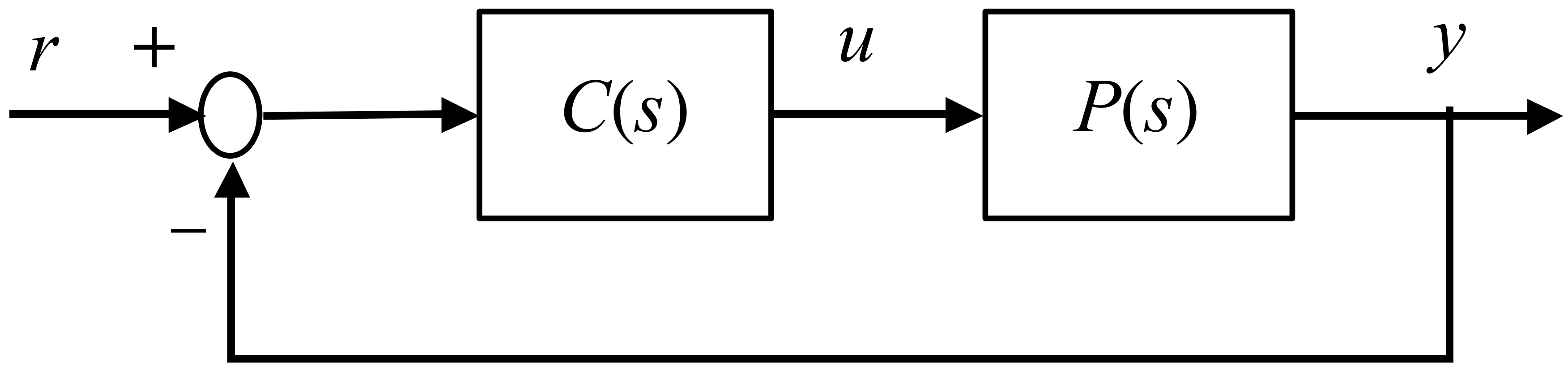
OUTLINE

- Definition and important properties of z-Transform
- Inverse z-Transform
- Solving linear difference equations using z-Transform
- Pulse transfer function and impulse response sequence
- Frequency response of discrete-time systems

Z-TRANSFORM

- The counter-part of Laplace transform (used in continuous-time domain) in the discrete-time domain
- Why Laplace transform
  - Differentiation/integrations -> algebraic operations
  - Convolution relationships between signals are transformed into multiplication/divisions

WHY TRANSFORMED SIGNAL/SYSTEMS



$$y(t) = \int_0^t u(t-\tau_1)p(\tau_1)d\tau_1$$

$$u(t) = \int_0^t [r(t-\tau_2) - y(t-\tau_2)]c(\tau_2)d\tau_2$$

$$\Rightarrow y(t) = \int_0^t \{\int_0^{t-\tau_1} [r(t-\tau_1-\tau_2) - y(t-\tau_1-\tau_2)]c(\tau_2)d\tau_2\} p(\tau_1)d\tau_1$$

$$Y(s) = \frac{P(s)C(s)}{1+P(s)C(s)} R(s)$$

DEFINITION OF Z-TRANSFORM

- The *z*-transform of a sampled sequence *x*(*kT*) or *x*(*k*), where *k* is non-negative integers and *T* is the sampling period, is defined by

$$X(z) = \mathrm{Z}[x^*(t)] = \mathrm{Z}[x(kT)] = \mathrm{Z}[x(k)] = \sum_{k=-\infty}^{\infty} x(kT)z^{-k} = \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

- In the one-sided *z*-transform, we assume *x*(*kT*) = *x*(*k*) = 0 for *k*<0.

$$X(z) = \mathrm{Z}[x(k)] = \sum_{k=0}^{\infty} x(k)z^{-k}$$

EXAMPLES OF COMPUTING X(Z)

Example 3.3 *Unit Step Function*

$$u(k)=\begin{cases}1 & k\geq 0\\ 0 & k<0\end{cases}$$

$$U(z)=Z\left[u(k)\right]=\sum_{k=0}^{\infty}1\cdot z^{-k}=1+z^{-1}+z^{-2}+\cdots$$

$$=\frac{1}{1-z^{-1}}=\frac{z}{z-1}\quad \text{for } \left|z^{-1}\right|<1\left(\text{or } \left|z\right|>1\right)$$

"Integrator"
(off by a factor of T)

UNIT RAMP FUNCTION

$$u(k)=\begin{cases} kT & k\geq 0 \\ 0 & k<0 \end{cases}$$

$$U(z)=Z\left[u(k)\right]=\sum_{k=0}^{\infty} kT\cdot z^{-k}=T\sum_{k=0}^{\infty} k\cdot z^{-k}$$

$$=T(z^{-1}+2z^{-2}+3z^{-3}+\cdots)$$

$$=T(z^{-1}+z^{-2}+z^{-3}+\cdots$$

$$+z^{-2}+z^{-3}+z^{-4}\cdots)$$

$$+z^{-3}+z^{-4}+z^{-5}\cdots)$$

$$=T(\frac{z^{-1}}{1-z^{-1}}+\frac{z^{-2}}{1-z^{-1}}+\frac{z^{-3}}{1-z^{-1}}+\ldots)$$

$$=T\frac{z^{-1}}{(1-z^{-1})^2}=\frac{Tz}{(z-1)^2}$$

POLYNOMIAL FUNCTION

$$x(k)=\begin{cases} a^k & k\geq 0 \\ 0 & k<0 \end{cases}$$

$$X(z)=Z\left[x(k)\right]=\sum_{k=0}^{\infty}a^k\cdot z^{-k}=\sum_{k=0}^{\infty}\left(a^{-1}z\right)^{-k}$$

$$=\frac{1}{1-\left(a^{-1}z\right)^{-1}}=\frac{z}{z-a}\quad \text{for } \left|z\right|>a$$

MULTIPLICATION BY a^k

- If X(z) is the z transform of x(k), then the z transform of a^k x(k) is given by $X(a^{-1}z)$

$$Z\left[a^k x(k)\right] = \sum_{k=0}^{\infty} a^k x(k) \cdot z^{-k} = \sum_{k=0}^{\infty} x(k)\left(a^{-1}z\right)^{-k} = X(a^{-1}z)$$

EXPONENTIAL FUNCTIONS

$$x(k) = \begin{cases} e^{-akT} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$X(z) = Z\left[x(k)\right] = \sum_{k=0}^{\infty} e^{-akT} \cdot z^{-k} = \sum_{k=0}^{\infty} \left(e^{aT} z\right)^{-k}$$

$$= \frac{1}{1-\left(e^{aT} z\right)^{-1}} = \frac{z}{z - e^{-aT}} \quad \text{for } \left|z\right| > e^{-aT}$$

SINUSOIDAL FUNCTION

$$x(k) = \begin{cases} \sin(\omega kT) & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$X(z) = \mathrm{Z}\left[x(k)\right] = \sum_{k=0}^{\infty} \sin(\omega kT) \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{e^{j\omega kT} - e^{-j\omega kT}}{2j} \cdot z^{-k}$$

$$= \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega T} z^{-1}} - \frac{1}{1 - e^{-j\omega T} z^{-1}}\right) = \frac{1}{2j} \frac{\left(e^{j\omega T} - e^{-j\omega T}\right) z^{-1}}{1 - \left(e^{j\omega T} + e^{-j\omega T}\right) z^{-1} + z^{-2}}$$

$$= \frac{z^{-1} \sin(\omega T)}{1 - 2z^{-1} \cos(\omega T) + z^{-2}} = \frac{z \cdot \sin(\omega T)}{z^2 - 2z \cdot \cos(\omega T) + 1} \quad \text{for } |z| > 1$$

Z-TRANSFORM TABLE

TABLE 2–1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

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TABLE 2–1 (continued)

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$	$\sin\omega kT$	$\frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos\omega t$	$\cos\omega kT$	$\frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$	$e^{-akT}\sin\omega kT$	$\frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$	$e^{-akT}\cos\omega kT$	$\frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k\cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
23.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

$x(t) = 0$, for $t < 0$.
$x(kT) = x(k) = 0$, for $k < 0$.
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$.

IMPORTANT PROPERTIES OF Z-TRANSFORM -1

- Linearity
  - *z*-transform is a linear transformation

$$\boxed{\mathrm{Z}\left[a \cdot x(k) + b \cdot y(k)\right] = a \cdot \mathrm{Z}\left[x(k)\right] + b \cdot \mathrm{Z}\left[y(k)\right] = a \cdot X(z) + b \cdot Y(z)}$$

Proof: $\boxed{X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}}$

$$\begin{aligned}
\mathrm{Z}\left[a \cdot x(k) + b \cdot y(k)\right] &= \sum_{k=0}^{\infty} a \cdot x(k) \cdot z^{-k} + \sum_{k=0}^{\infty} b \cdot y(k) \cdot z^{-k} \\
&= a \cdot \left(\sum_{k=0}^{\infty} x(k) \cdot z^{-k} \right) + b \cdot \left(\sum_{k=0}^{\infty} y(k) \cdot z^{-k} \right) \\
&= a \cdot \mathrm{Z}\left[x(k)\right] + b \cdot \mathrm{Z}\left[y(k)\right] = a \cdot X(z) + b \cdot Y(z)
\end{aligned}$$

IMPORTANT PROPERTIES OF Z-TRANSFORM - 2

- Time Shift
  - If $x(k) = 0$ for $k < 0$ and $x(k)$ has the z-transform $X(z)$, then

$$Z\left[x(k-d)\right] = z^{-d} \cdot X(z)$$

$$Z\left[x(k+d)\right] = z^{d} \cdot \left[X(z) - \sum_{j=0}^{d-1} x(j) \cdot z^{-j}\right] = z^{d} \cdot X(z) - \sum_{i=1}^{d} x(d-i) \cdot z^{i}$$

$$= z^{d} X(z) - z^{d} x(0) - z^{d-1} x(1) - z^{d-2} x(2) - \cdots - z \cdot x(d-1)$$

$$Z[f(k+d)] = z^{d} F(z) - \sum_{j=1}^{d} z^{j} f(d-j)$$

Z: time advance, Z^{-1}: time delay

PROOF

Proof:

$$\mathcal{Z}\left[x(k-d)\right]=\sum_{k=0}^{\infty}x(k-d)\cdot z^{-k}\text{ , let }k-d=j\text{ , then}$$

$$Z\left[x(k-d)\right]=\sum_{k=0}^{\infty}x(k-d)\cdot z^{-k}=\sum_{j=-d}^{\infty}x(j)\cdot z^{-d-j}=z^{-d}\sum_{j=-d}^{\infty}x(j)\cdot z^{-j}$$

$$=z^{-d}\sum_{j=0}^{\infty}x(j)\cdot z^{-j}=z^{-d}\cdot X(z)$$

$$\mathcal{Z}\left[x(k+d)\right]=\sum_{k=0}^{\infty}x(k+d)\cdot z^{-k}\text{ , let }k+d=j\text{ , then}$$

$$\mathcal{Z}\left[x(k+d)\right]=\sum_{k=0}^{\infty}x(k+d)\cdot z^{-k}=\sum_{j=d}^{\infty}x(j)\cdot z^{d-j}=z^{d}\sum_{j=d}^{\infty}x(j)\cdot z^{-j}$$

$$=z^{d}\left[\sum_{j=0}^{\infty}x(j)\cdot z^{-j}-\sum_{j=0}^{d-1}x(j)\cdot z^{-j}\right]=z^{d}\cdot\left[\mathcal{Z}\left[x(j)\right]-\sum_{j=0}^{d-1}x(j)\cdot z^{-j}\right]$$

$$=z^{d}\cdot\left[X(z)-\sum_{j=0}^{d-1}x(j)\cdot z^{-j}\right]=z^{d}\cdot X(z)-\sum_{i=1}^{d}x(d-i)\cdot z^{i}\quad\text{where }i=d-j$$

IMPORTANT PROPERTIES OF Z-TRANSFORM - 3

- Initial Value Theorem (IVT)
  - If the *z*-transform of *x*(*k*) is *X*(*z*) and if $\lim_{z\to\infty} X(z)$ exists, then the initial value of *x*(*k*) (i.e., *x*(0)) is

$$x(0) = \lim_{z\to\infty} X(z)$$

Proof:

$$\boxed{X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}}$$

IMPORTANT PROPERTIES OF Z-TRANSFORM - 4

- Final Value Theorem (FVT)
  - If the z-transform of $x(k)$ is $X(z)$ and if exists, then the value of $x(k)$ as is given

$$\boxed{x(\infty) = \lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(z-1)\cdot X(z)\right]}$$

Proof:

$$\mathrm{Z}\left[x(k+1)-x(k)\right] = z\cdot X(z) - z\cdot x(0) - X(z) = (z-1)X(z) - z\cdot x(0)$$

$$= \sum_{k=0}^{\infty} x(k+1)\cdot z^{-k} - \sum_{k=0}^{\infty} x(k)\cdot z^{-k} = \sum_{k=0}^{\infty}\left[x(k+1)-x(k)\right]\cdot z^{-k}$$

$$\lim_{z\to 1}\left[(z-1)X(z) - z\cdot x(0)\right] = \lim_{z\to 1}\left[(z-1)X(z)\right] \cancel{- x(0)}$$

$$= \lim_{z\to 1}\sum_{k=0}^{\infty}\left[x(k+1)-x(k)\right]\cdot z^{-k} = \sum_{k=0}^{\infty}\left[x(k+1)-x(k)\right]$$

$$= \lim_{k\to\infty} x(k) \cancel{- x(0)}$$

IMPORTANT PROPERTIES OF Z-TRANSFORM - 5

- Convolution
  - Discrete convolution in the time domain is equivalent to multiplication in the *z* domain, i.e., if

$$g(k) \otimes u(k) \equiv \sum_{j=0}^{k} g(k-j) \cdot u(j) = \sum_{j=0}^{k} g(j) \cdot u(k-j)$$

then

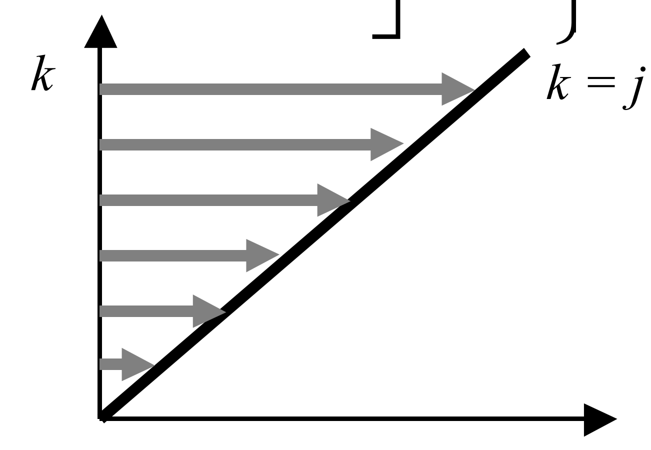
$$\boxed{\mathrm{Z}\left[g(k) \otimes u(k)\right] = G(z) \cdot U(z)}$$

where $\mathrm{Z}\left[g(k)\right] = G(z)$ and $\mathrm{Z}\left[u(k)\right] = U(z)$

PROOF FOR THE CONVOLUTION PROPERTY

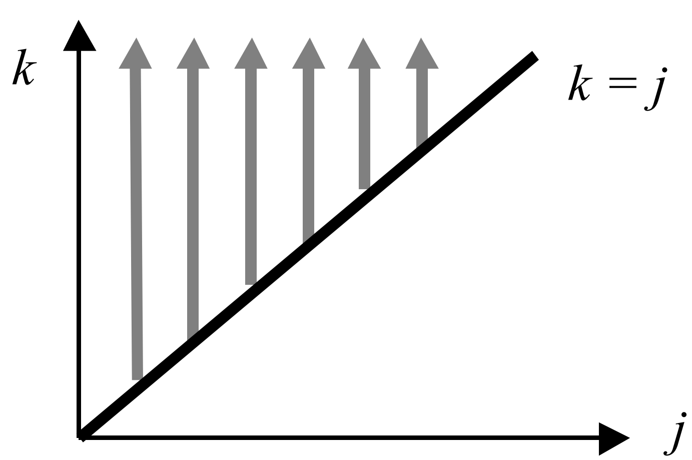
$$Z[g(k)\otimes u(k)] = Z\left[\sum_{j=0}^{k} g(k-j)\cdot u(j)\right] = \sum_{k=0}^{\infty}\left\{\left[\sum_{j=0}^{k} g(k-j)\cdot u(j)\right]\cdot z^{-k}\right\}$$

$$= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} g(k-j)\cdot z^{-k}\cdot u(j)\right]$$



$$= \sum_{j=0}^{\infty}\left[\sum_{k=j}^{\infty} g(k-j)\cdot z^{-k}\right]\cdot u(j) = \sum_{j=0}^{\infty}\left[\sum_{i=0}^{\infty} g(i)\cdot z^{-i-j}\right]\cdot u(j)$$

where $k-j=i$



$$= \left[\sum_{i=0}^{\infty} g(i)\cdot z^{-i}\right]\cdot\left[\sum_{i=0}^{\infty} u(j)\cdot z^{-j}\right]$$

$$= G(z)\cdot U(z)$$

IMPORTANCE OF CONVOLUTION PROPERTY

Given a linear discrete-time system described by its impulse transfer function

$$G(z) = \frac{Y(z)}{U(z)} = \mathrm{Z}\left[g(k)\right]$$

Given an input sequence

$$u(k) = u(0)\cdot\delta_0(k) + u(1)\cdot\delta_0(k-1) + u(2)\cdot\delta_0(k-2) + \cdots = \sum_{j=0}^{\infty} u(j)\cdot\delta_0(k-j)$$

$$\Rightarrow \quad y(k) = \sum_{j=0}^{k} g(k-j)\cdot u(j) = g(k)\otimes u(k)$$

$$\Rightarrow \quad Y(z) = Z\left[g(k)\otimes u(k)\right] = G(z)\cdot U(z)$$

INVERSE Z-TRANSFORM

- Given a *z*-transform function *X*(*z*), the corresponding time domain sequence *x*(*k*) can be obtained using the *inverse z-transform*. The inverse *z*-transform is defined to be

$$x(k) = Z^{-1}\left[X(z)\right]$$

In practice, the inverse z-transform can be obtained from

- Cauchy Residue Theorem
- Direct Long Division
- Partial Fraction Expansion
- Computation method (e.g., impulse response)

EX: INVERSE Z-TRANSFORM USING THE CAUCHY RESIDUE THEOREM

$$x(k) = \mathrm{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi j}\oint_C X(z)\cdot z^{k-1}\,dz$$

the contour integration can be evaluated using the Cauchy Residue Theorem, e.g.

$$X(z) = \frac{z}{(z-1)(z-2)}$$

$$\begin{aligned}
x(k) &= \frac{1}{2\pi j}\cdot 2\pi j\cdot\left(\text{sum of the residue of the integral}\right)\\
&= \frac{1}{2\pi j}\cdot 2\pi j\cdot\left(\sum (z-p_i)X(z)z^{k-1}\Big|_{z=p_i}\right) = \left(\frac{z}{z-1}\cdot z^{k-1}\Bigg|_{z=2} + \frac{z}{z-2}\cdot z^{k-1}\Bigg|_{z=1}\right)\\
&= 2^k - 1
\end{aligned}$$

EX: INVERSE Z-TRANSFORM USING LONG DIVISION

$$X(z) = \frac{z^2 + z}{z^2 - 3z + 4} = \frac{1 + z^{-1}}{1 - 3z^{-1} + 4z^{-2}}$$

$$\begin{array}{r l}
 & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} \\ \cline{2-2}
1 - 3z^{-1} + 4z^{-2} & \big) \; 1 + z^{-1} \\
 & \underline{1 - 3z^{-1} + 4z^{-2}} \\
 & \quad 4z^{-1} - 4z^{-2} \\
 & \quad \underline{4z^{-1} - 12z^{-2} + 16z^{-3}} \\
 & \qquad 8z^{-2} - 16z^{-3} \\
 & \qquad \underline{8z^{-2} - 24z^{-3} + 32z^{-4}} \\
 & \qquad\quad 8z^{-3} - 32z^{-4} \\
 & \qquad\qquad \vdots
\end{array}$$

$$x(0) = 1, \quad x(1) = 4, \quad x(2) = 8, \quad x(3) = 8, \cdots$$

Main Problem: No closed-form solution

PARTIAL FRACTION EXPANSION + TABLE LOOKUP

- The procedure is very similar to the one used in solving the inverse Laplace transform.

$$X(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)} = A_0 + A_1\frac{z}{z-p_1} + A_2\frac{z}{z-p_2} + \cdots + A_n\frac{z}{z-p_n}$$

$$A_0 = X(0)$$

where

$$A_i = \frac{z-p_i}{z}X(z)\bigg|_{z=p_i}, \quad i = 1,\ 2,\ 3, \cdots$$

$$x(k) = Z^{-1}\left[X(z)\right] = A_0\cdot\delta_0(k) + A_1\cdot(p_1)^k + A_2\cdot(p_2)^k + \cdots + A_n\cdot(p_n)^k$$

EX: INVERSE Z-TRANSFORM USING PARTIAL FRACTION EXPANSION

$$X(z)=\frac{0.5(1-e^{-T})^2(z^2+e^{-T}z)}{(z-1)(z-e^{-T})(z-e^{-2T})}=A_1\frac{z}{z-1}+A_2\frac{z}{z-e^{-T}}+A_3\frac{z}{z-e^{-2T}}$$

$$A_1=\frac{z-1}{z}X(z)\bigg|_{z=1}=\frac{0.5(1-e^{-T})^2(1+e^{-T})}{(1-e^{-T})(1-e^{-2T})}=0.5\frac{(1-e^{-T})(1+e^{-T})}{(1-e^{-2T})}=0.5$$

$$A_2=\frac{z-e^{-T}}{z}X(z)\bigg|_{z=e^{-T}}=-\frac{0.5(1-e^{-T})(e^{-2T}+e^{-2T})}{e^{-T}(e^{-T}-e^{-2T})}=-1$$

$$A_3=\frac{z-e^{-2T}}{z}X(z)\bigg|_{z=e^{-2T}}=\frac{0.5(1-e^{-T})^2(e^{-4T}+e^{-3T})}{e^{-2T}(e^{-2T}-1)(e^{-2T}-e^{-T})}=0.5$$

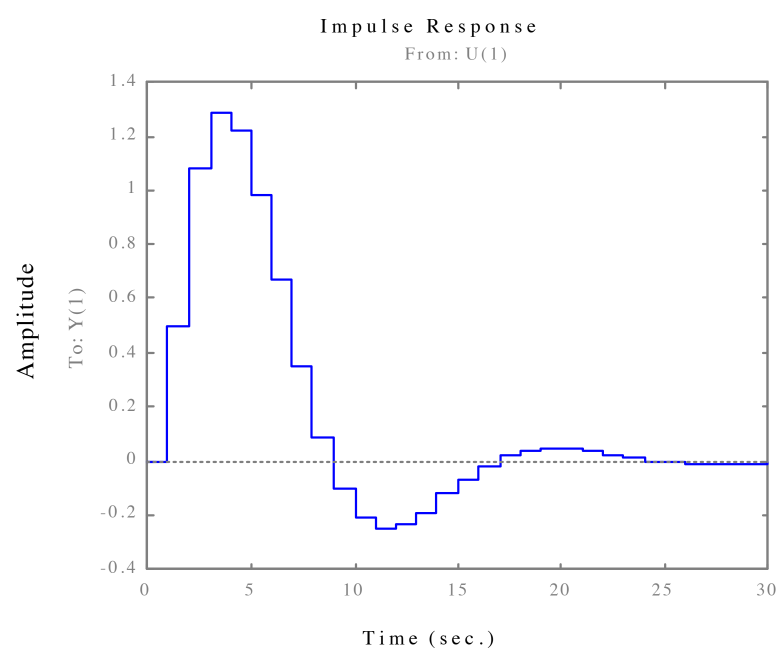
$$x(k)=Z^{-1}\left[X(z)\right]=0.5-\left(e^{-T}\right)^k+0.5\left(e^{-2T}\right)^k=0.5-e^{-kT}+0.5e^{-2kT}$$

INVERSE Z-TRANSFORM USING COMPUTERS

$$X(z) = \frac{0.5z^{-1} + 0.33z^{-2}}{1 - 1.5z^{-1} + 0.66z^{-2}} = \frac{0.5z + 0.33}{z^2 - 1.5z + 0.66}$$

MATLAB command

```
num = [0.5 0.33];
den = [1 -1.5 0.66];
x = dimpulse(num,den);
```

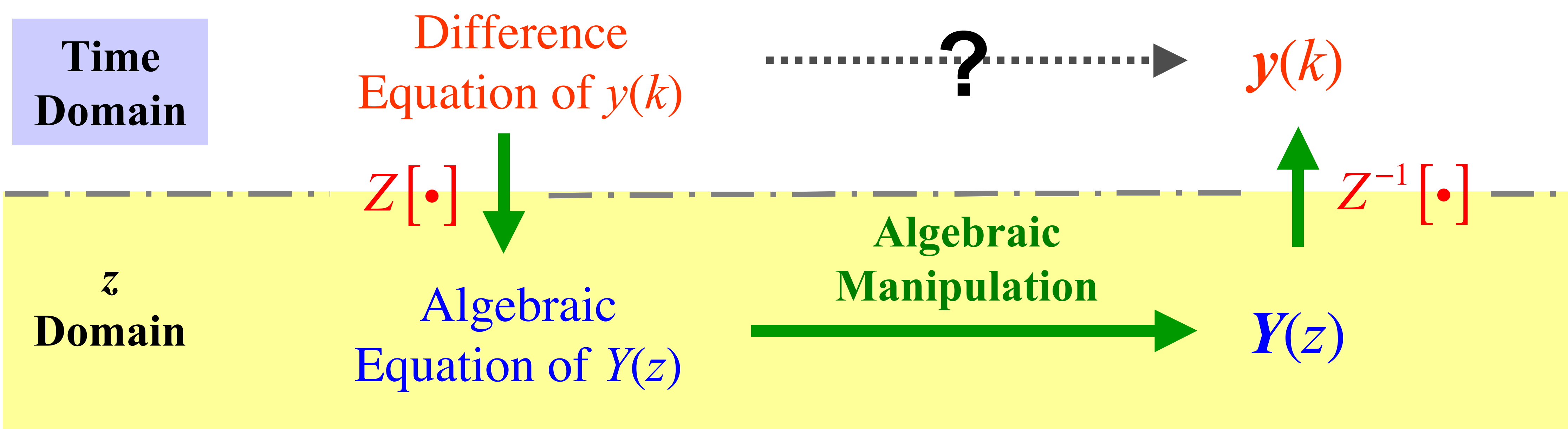



```
x =
        0
   0.5000
   1.0800
   1.2900
   1.2222
   0.9819
   0.6662
   0.3512
   0.0872
  -0.1011
  -0.2091
  -0.2470
  -0.2325
```

SOLVING LINEAR DIFFERENCE EQUATIONS

Difference equation

$$y(k)+a_1 y(k-1)+a_2 y(k-2)+\cdots+a_n y(k-n) = b_0 u(k)+b_1 u(k-1)+b_2 u(k-2)+\cdots+b_n u(k-n)$$



Use the time shift property

$$Z\left[x(k-d)\right]=z^{-d}\cdot X(z)$$

$$Z\left[x(k+d)\right]=z^{d}X(z)-z^{d}x(0)-z^{d-1}x(1)-z^{d-2}x(2)-\cdots-z\cdot x(d-1)$$

EX: SOLVING DIFFERENCE EQUATION USING Z-TRANSFORM

Free response

$$x(k+2)+3x(k+1)+2x(k)=0, \quad x(0)=0, \quad x(1)=1$$

$$\boxed{\begin{aligned} &\mathrm{Z}\left[x(k-d)\right]=z^{-d}\cdot X(z) \\ &\mathrm{Z}\left[x(k+d)\right]=z^{d}X(z)-z^{d}x(0)-z^{d-1}x(1)-z^{d-2}x(2)-\cdots-z\cdot x(d-1) \end{aligned}}$$

$$\Rightarrow \quad z^2X(z)-z^2x(0)-z\cdot x(1)+3z\cdot X(z)-3z\cdot x(0)+2X(z)=0$$

$$X(z)=\frac{z}{z^2+3z+2}=\frac{z}{(z+1)(z+2)}=\frac{z}{z+1}-\frac{z}{z+2}$$

$$x(k)=Z^{-1}\left[X(z)\right]=Z^{-1}\left[\frac{z}{z-(-1)}\right]-Z^{-1}\left[\frac{z}{z-(-2)}\right]=(-1)^k-(-2)^k$$

$f(kT), k\geq 0$	$F(z)$
a^k	$\frac{z}{z-a}$

SOLVING DIFFERENCE EQUATION (CONTD.)

Forced response

$$x(k+2)+0.4\cdot x(k+1)-0.32\cdot x(k)=u(k)$$

$$u(k)=\begin{cases}1, & k\geq 0\\ 0, & \text{otherwise}\end{cases}$$

where $x(0) = 0$ and $x(1) = 1$ and $u(k)$ is a unit step input

$$\Rightarrow z^2X(z)-z^2x(0)-z\cdot x(1)+0.4\cdot z\cdot X(z)-0.4\cdot z\cdot x(0)-0.32\cdot X(z)=\frac{z}{z-1}$$

$$\Rightarrow X(z)=\frac{z^2}{(z-1)(z^2+0.4z-0.32)}=\frac{z^2}{(z-1)(z+0.8)(z-0.4)}$$

$$\Rightarrow X(z)=0.926\frac{z}{z-1}-0.3704\frac{z}{z+0.8}-0.5556\frac{z}{z-0.4}$$

$$\Rightarrow x(k)=0.926-0.3704\cdot(-0.8)^k-0.5556\cdot(0.4)^k$$

PULSE TRANSFER FUNCTION

- The transfer function for the continuous-time system relates the Laplace transform of the continuous-time output to that of the continuous-time input.

- For discrete-time systems, the *pulse transfer function* relates the *z*-transform of the output at the sample instants to that of the sampled input.

PULSE TRANSFER FUNCTION (CONT.)

$$y(k)+a_1\cdot y(k-1)+a_2\cdot y(k-2)+\cdots+a_n\cdot y(k-n)$$
$$=b_0\cdot u(k)+b_1\cdot u(k-1)+b_2\cdot u(k-2)+\cdots+b_n\cdot u(k-n)$$

$$\Rightarrow\ Y(z)+a_1z^{-1}Y(z)+a_2z^{-2}Y(z)+\cdots+a_nz^{-n}Y(z)$$
$$=b_0U(z)+b_1z^{-1}U(z)+b_2z^{-2}U(z)+\cdots+b_nz^{-n}U(z)$$

$$\Rightarrow\ \left(1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}\right)\cdot Y(z)=\left(b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_nz^{-n}\right)\cdot U(z)$$

$$\Rightarrow\quad G(z)=\frac{Y(z)}{U(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}\quad\left(=\frac{N(z)}{D(z)}\right)$$

Pulse transfer function

PULSE TRANSFER FUNCTION

- Why the name?

$$u(k)=\delta_0(k)=\begin{cases}1 & \text{for } k=0\\ 0 & \text{for } k\neq 0\end{cases} \Rightarrow \qquad U(z)=\mathrm{Z}\left[u(k)\right]=\mathrm{Z}\left[\delta_0(k)\right]=1$$

$$Y(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}=G(z)$$

$G(z)$ is the z-transform of the response of the system under the Kronecker delta function (pulse) input, *i.e. $g(k)$ is the unit impulse response.*

IMPULSE RESPONSE FUNCTION

$$g(k) = Z^{-1}[G(z)] = Z^{-1}\left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\right]$$

The inverse transform of $G(z)$

$$y(k) = u(0) \cdot g(k) + u(1) \cdot g(k-1) + u(2) \cdot g(k-2) + \cdots$$

$$\Rightarrow \quad y(k) = \sum_{j=0}^{k} u(j) \cdot g(k-j) = u(k) \otimes g(k)$$

$$Y(z) = Z[g(k) \otimes u(k)] = Z[g(k)] \cdot Z[u(k)] = G(z) \cdot U(z)$$

EX: IMPULSE RESPONSE FUNCTION

$$y(k+3) = 2u(k+3) - u(k+2) + 4u(k+1) + u(k)$$

$$G(z) = \frac{2z^3 - z^2 + 4z + 1}{z^3} = 2 - z^{-1} + 4z^{-2} + z^{-3}$$

$$g(k) = Z^{-1}[G(z)] = Z^{-1}\left[2 - z^{-1} + 4z^{-2} + z^{-3}\right]$$

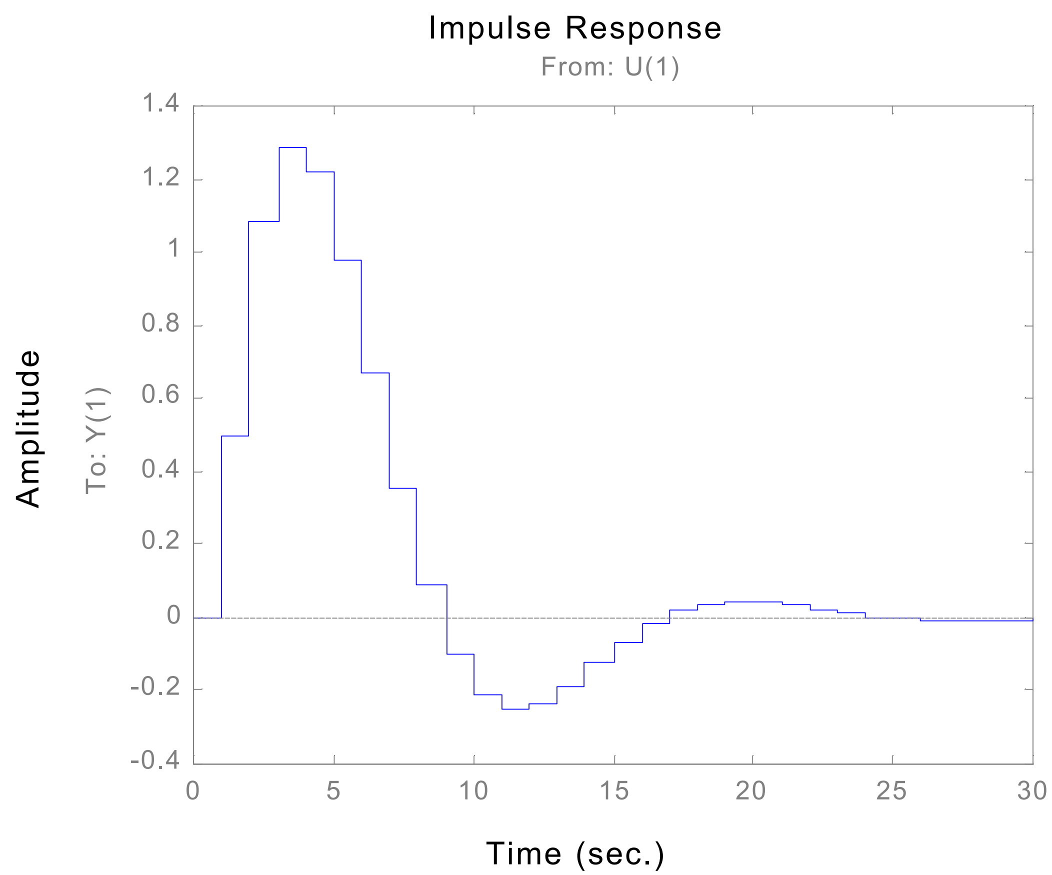
$$g(0) = 2, \quad g(1) = -1, \quad g(2) = 4, \quad g(3) = 1, \quad \boxed{g(k) = 0, \quad \text{for } k > 3}$$

The impulse response of a system with all of its poles at the origin will have finite non-zero terms. Impulse response of this type is often called *finite impulse response* (FIR) and the system (digital filter) that have all its poles at the origin is often call *finite impulse response* (*FIR*) *filter*.

EXAMPLE OF IIR SYSTEMS

A system that is not FIR is called to have infinite impulse response (IIR).

$$X(z) = \frac{0.5z^{-1} + 0.33z^{-2}}{1 - 1.5z^{-1} + 0.66z^{-2}} = \frac{0.5z + 0.33}{z^2 - 1.5z + 0.66}$$



x =

0
0.5000
1.0800
1.2900
1.2222
0.9819
0.6662
0.3512
0.0872
-0.1011
-0.2091
-0.2470
-0.2325

FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

- (Steady-state) Response of (stable) system under sinusoidal inputs. For a plant

$$G(z) = \frac{Y(z)}{U(z)} = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

where $|p_i| < 1$ for all i. Sinusoidal (cosine) input

$$u(k) = A\cos(\omega kT) = \frac{A}{2}\left(e^{j\omega kT} + e^{-j\omega kT}\right)$$

$$\Rightarrow \quad U(z) = \frac{A}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right)$$

$$Y(z) = G(z)\cdot U(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}\cdot\frac{A}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right)$$

FREQUENCY RESPONSE (CONT.)

$$Y(z) = G(z)\cdot U(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}\cdot\frac{A}{2}\left(\frac{z}{z-e^{j\omega T}}+\frac{z}{z-e^{-j\omega T}}\right)$$

$$\Rightarrow\quad Y(z) = B\frac{z}{z-e^{j\omega T}} + C\frac{z}{z-e^{-j\omega T}} + \cancel{\sum_{i=1}^{n} D_i \frac{z}{z-p_i}}$$

Steady state

$$B = \frac{z-e^{j\omega T}}{z}Y(z)\bigg|_{z=e^{j\omega T}} = \frac{A}{2}\left[1+\frac{z-e^{j\omega T}}{z-e^{-j\omega T}}\right]G(z)\bigg|_{z=e^{j\omega T}} = \frac{A}{2}G\left(e^{j\omega T}\right)$$

$$C = \frac{z-e^{-j\omega T}}{z}Y(z)\bigg|_{z=e^{-j\omega T}} = \frac{A}{2}\left[\frac{z-e^{-j\omega T}}{z-e^{j\omega T}}+1\right]G(z)\bigg|_{z=e^{-j\omega T}} = \frac{A}{2}G\left(e^{-j\omega T}\right)$$

$\Rightarrow$ Steady State Response

$$Y_{SS}(z) = \frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z-e^{j\omega T}} + G\left(e^{-j\omega T}\right)\frac{z}{z-e^{-j\omega T}}\right]$$

STEADY-STATE RESPONSE

$$Y_{SS}(z) = \frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z-e^{j\omega T}} + G\left(e^{-j\omega T}\right)\frac{z}{z-e^{-j\omega T}} \right]$$

$$G\left(e^{j\omega T}\right) = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{j\angle G\left(e^{j\omega T}\right)} = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{j\phi}$$

$$G\left(e^{-j\omega T}\right) = \left|G\left(e^{-j\omega T}\right)\right| \cdot e^{j\angle G\left(e^{-j\omega T}\right)} = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{-j\phi}$$

$$Y_{SS}(z) = \frac{A}{2} \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \left[e^{j\phi}\frac{z}{z-e^{j\omega T}} + e^{-j\phi}\frac{z}{z-e^{-j\omega T}} \right]$$

$$y_{ss}(k) = \frac{A}{2} \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \left[e^{j\phi}\left(e^{j\omega T}\right)^k + e^{-j\phi}\left(e^{-j\omega T}\right)^k \right] = A \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \frac{1}{2}\left(e^{j(\omega kT+\phi)} + e^{-j(\omega kT+\phi)} \right)$$

$$\Rightarrow \boxed{y_{ss}(k) = A \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \cos(\omega kT + \phi)} \quad \text{where} \quad \phi = \angle G\left(e^{j\omega T}\right)$$

STEADY-STATE RESPONSE (CONT.)

$$y(k) = \frac{A}{2}\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\left[e^{j\phi}\left(e^{j\omega T}\right)^{k} + e^{-j\phi}\left(e^{-j\omega T}\right)^{k}\right] = A\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\frac{1}{2}\left(e^{j(\omega kT+\phi)} + e^{-j(\omega kT+\phi)}\right)$$

note $\quad u(k) = A\cos(\omega kT) = \frac{A}{2}\left(e^{j\omega kT} + e^{-j\omega kT}\right)$

$$y(k) = A\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\cos(\omega kT+\phi), \quad \text{where} \quad \phi = \angle G\left(e^{j\omega T}\right)$$

Similar to the continuous-time case, the steady-state response of the system $G(z)$ under a sinusoidal input is also sinusoidal with the same frequency but scaled in amplitude and shifted in phase.

The amplitude of the steady-state response is scaled by a factor of $\left|G\left(e^{j\omega T}\right)\right|$, which will be referred to as the system gain associated with $G(z)$ at frequency ω, and shifted in time by $\angle G\left(e^{j\omega T}\right)$, the phase of the system at frequency ω .

FREQUENCY RESPONSE

- The frequency response function of a discrete system can be obtained by replacing the *z*-transform complex variable *z* with $e^{j\omega T}$, i.e.

$$\boxed{G\left(e^{j\omega T}\right) = G(z)\big|_{z=e^{j\omega T}} = G\left(\cos(\omega T) + j\sin(\omega T)\right)}$$

This is because $\boxed{z = e^{Ts}}$

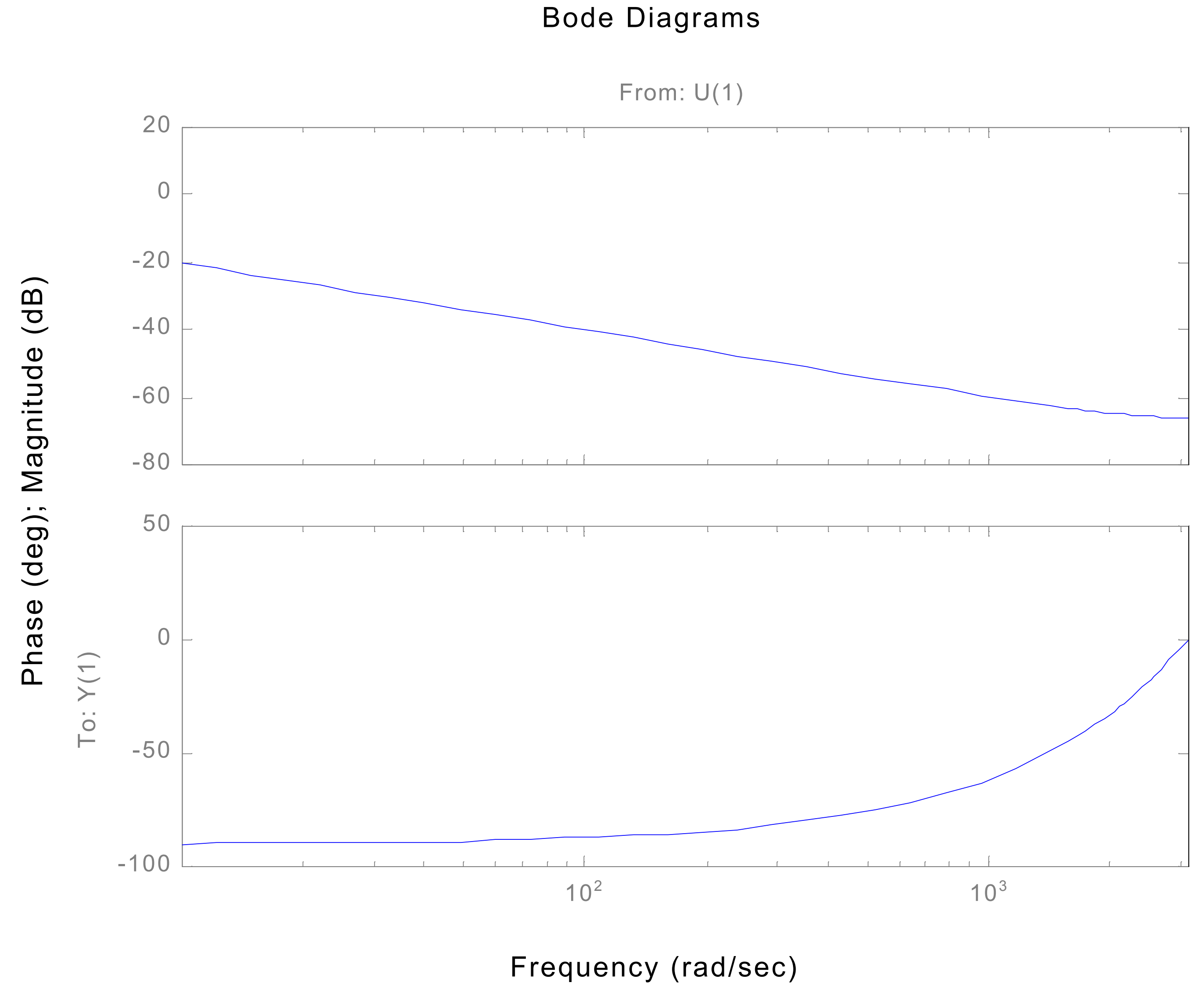
- Steady State Gain (DC gain)
The steady state gain of a discrete-time system can be obtained by letting $\omega = 0$, i.e.

$$\boxed{\text{DC Gain} = G\left(e^{j\omega T}\right)\big|_{\omega=0} = G(z)\big|_{z=1} = G(1)}$$

EXAMPLE: INTEGRATOR

$$G(s) = \frac{1}{s} \quad \Longrightarrow \quad G(z) = \frac{Tz}{z-1}$$

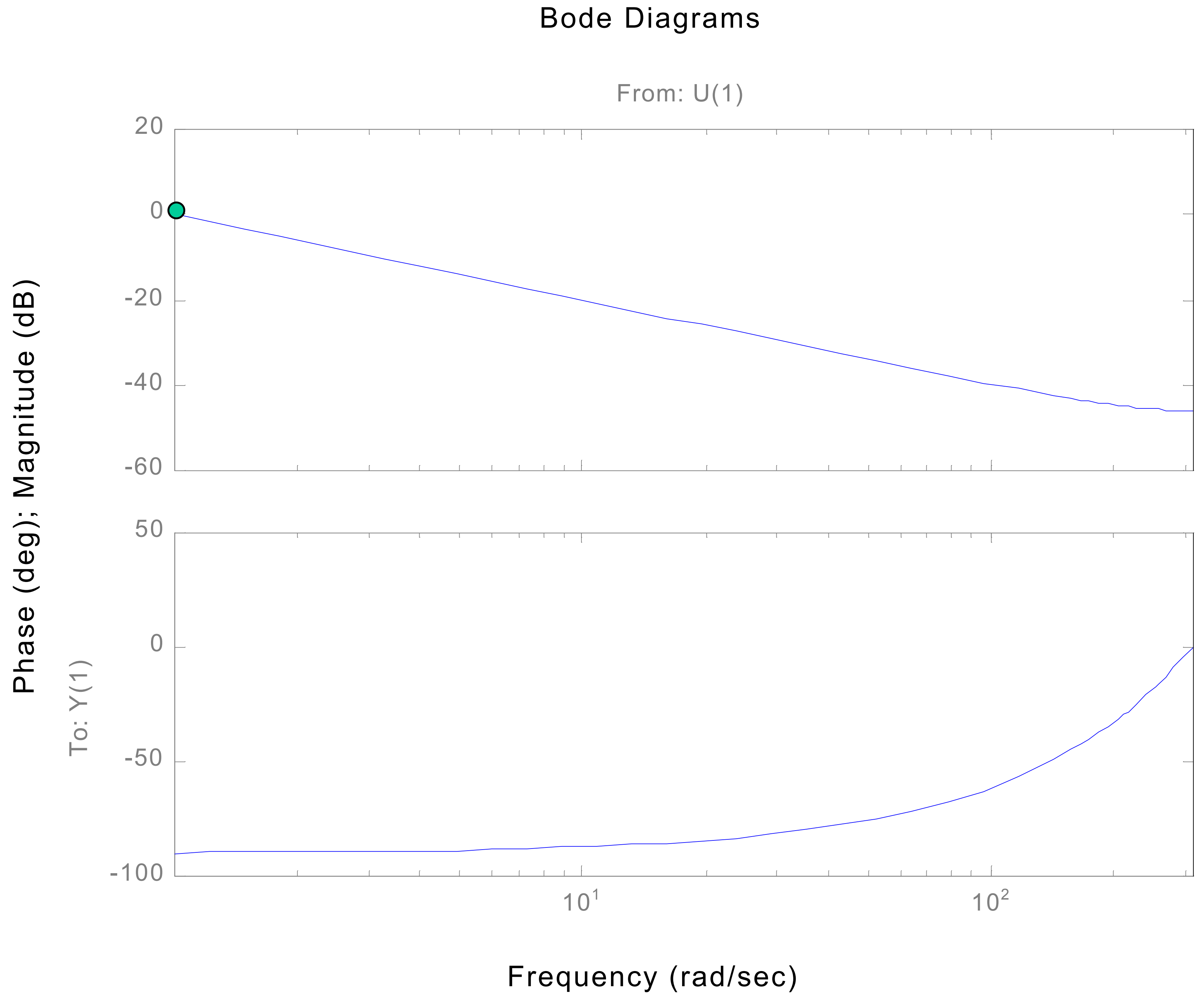
```
» dbode([0.001 0], [1 -1], 0.001);  % sampling frequency = 1 kHz
```



Nyquist freq.

EXAMPLE: INTEGRATOR (CONT.)

T=0.01

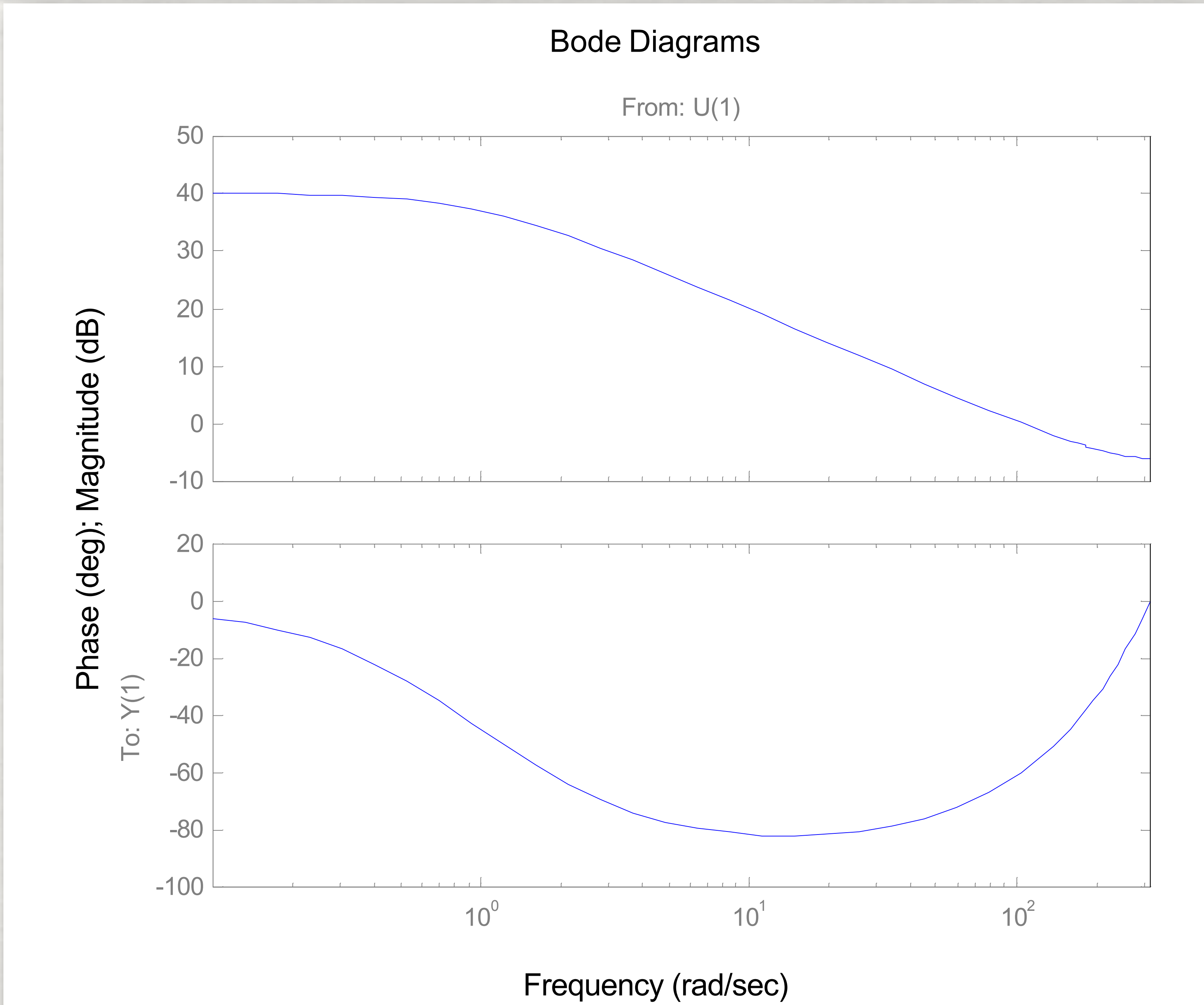


Nyquist freq.

EXAMPLE: SIMPLE POLE

4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$

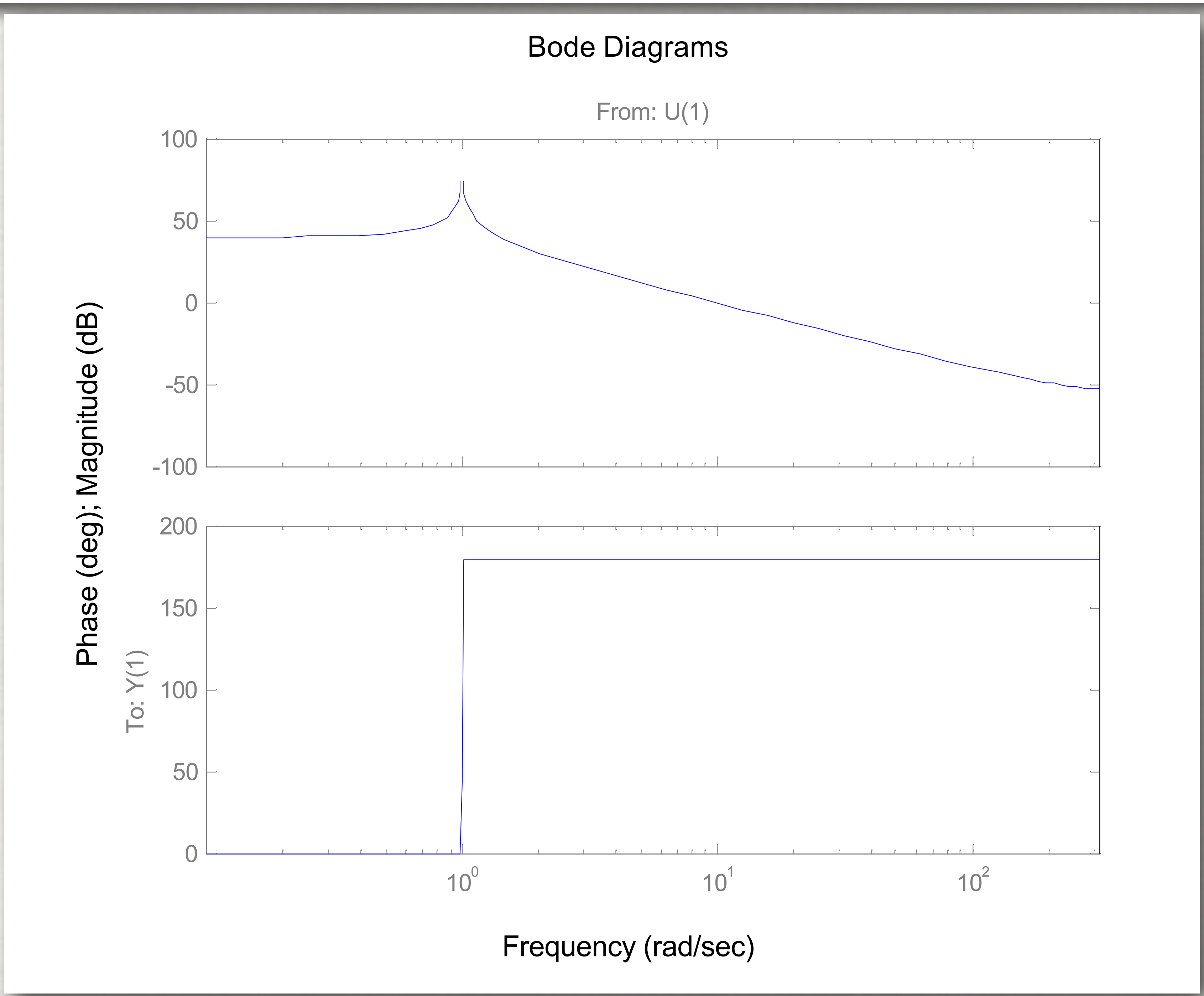
```
» a = 1; T = 0.01;
» dbode([1 0], [1 -exp(-a*T)], T);
```



EXAMPLE: HARMONIC OSCILLATION

14.	$\dfrac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\dfrac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$

```
» w = 1; T = 0.01;
» dbode([sin(w*T) 0], [1 -1\2*cos(w*T) 1], T);
```



DISCRETE-TIME FREQUENCY RESPONSE IS PERIODIC

$$\boxed{G\left(e^{j\omega T}\right) = G(z)\Big|_{z=e^{j\omega T}} = G\left(\cos(\omega T) + j\sin(\omega T)\right)}$$

Obviously, will repeat itself every sample frequency $\omega_S = 2\pi/T$ rad/sec.

$$\left|G\left(e^{j\omega T}\right)\right| = \left|G\left(e^{j(\omega_S-\omega)T}\right)\right| = \left|G\left(e^{j(\omega-\omega_S)T}\right)\right|$$

$$\angle G\left(e^{j\omega T}\right) = -\angle G\left(e^{j(\omega_S-\omega)T}\right) = \angle G\left(e^{j(\omega-\omega_S)T}\right)$$

EXAMPLE: FREQUENCY RESPONSE

$$y(k) = e^{-2T} y(k-1) + u(k), \quad \text{where } T = \frac{\pi}{5}$$

$$\Rightarrow \quad Y(z) = e^{-2T} z^{-1} Y(z) + U(z)$$

$$\Rightarrow \quad G(z) = \frac{z}{z - e^{-2T}}$$

$$\Rightarrow \quad \boxed{G(e^{j\omega T}) = \frac{e^{j\omega T}}{e^{j\omega T} - e^{-2T}}}$$

$$\left|G(e^{j\omega T})\right| = \frac{\left|e^{j\omega T}\right|}{\left|e^{j\omega T} - e^{-2T}\right|} = \frac{1}{\sqrt{\left(\cos(\omega T) - e^{-2T}\right)^2 + \sin^2(\omega T)}}$$

$$\angle G(e^{j\omega T}) = \angle\left(e^{j\omega T}\right) - \angle\left(e^{j\omega T} - e^{-2T}\right) = \omega T - \tan^{-1}\left(\frac{\sin(\omega T)}{\cos(\omega T) - e^{-2T}}\right)$$

EXAMPLE (CONT.)

$$G(z)=\frac{z}{z-e^{-2T}} \quad \Rightarrow \quad G\left(e^{j\omega T}\right)=\frac{e^{j\omega T}}{e^{j\omega T}-e^{-2T}}$$

```
T = pi/5;
G = tf([1 0],[1 -exp(-2*T)],T);

% Set up frequency vector:
w = linspace(0,50,200);
out = freqresp(G,w);

for i = 1:length(w)
    fr(i,1) = out(:,:,i);
end

subplot(211);plot(w,abs(fr));
subplot(212);
plot(w,180/pi*angle(fr));
```