Complete Characterization of the Solvability of PAPR Reduction for OFDM by Tone Reservation

Holger Boche, Ullrich J. Mönich, Ezra Tampubolon
Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

***Abstract*—In this paper we analyze the peak-to-average power ratio (PAPR) reduction by tone reservation for orthogonal frequency division multiplexing (OFDM) schemes. In addition to the strong solvability of the PAPR reduction problem, where the PAPR has to be bounded by some constant, we consider a weaker form of solvability, where only the boundedness of the peak value of the signal is required. We show that for OFDM both forms of solvability are equivalent. Further, we show that in the case where the PAPR problem is not solvable, the set of input signals that lead to an unbounded OFDM signal is a residual set. As a consequence, if the upper density of the carriers, used for information transmission, is positive, the set of input signals that lead to a bounded OFDM signal is a meager set.**

I. Introduction

In modern communication systems, orthogonal transmission schemes, e.g., orthogonal frequency division multiplexing (OFDM), are widely used. Mathematically the transmit signal s of an orthogonal transmission scheme with duration T_s has the form

$$s(t) = \sum_{k\in\mathbb{Z}} c_k \phi_k(t), \quad t \in [0, T_s],$$

where $\{\phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal system (ONS) of functions, and $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ are the information bearing coefficients. Each ϕ_k is called carrier.

Although orthogonal transmission schemes enable high data rates and have other desirable properties [1], one drawback is the large peak-to-average power ratio (PAPR) of such systems [2]. Large PAPRs are problematic for power amplifiers and generally impair the quality of transmission [3].

Several methods have been proposed for reducing the PAPR, among them the tone reservation method [4]–[9], which we consider in this paper. In this method, the set of available carriers $\{\phi_k\}_{k\in\mathbb{Z}}$ is partitioned into two sets, the first of which is used to carry the information, and the second of which to reduce the PAPR.

II. Notation

By $L^p[0,1]$, $1 \le p \le \infty$, we denote the usual L^p-spaces on the interval $[0,1]$, equipped with the norm $\|\cdot\|_p$. For an index set $\mathcal{I} \subset \mathbb{Z}$, we denote by $\ell^2(\mathcal{I})$ the set of all square summable sequences $c = \{c_k\}_{k\in\mathcal{I}}$ indexed by $\mathcal{I}$. The norm is given by $\|c\|_{\ell^2(\mathcal{I})} = (\sum_{k\in\mathcal{I}}|c_k|^2)^{1/2}$. A subset $\mathcal{M}$ of a metric space X is said to be nowhere dense in $\mathcal{M}$ if the closure $[\mathcal{M}]$ does not contain a non-empty open set of X. $\mathcal{M}$ is said to be meager (or of the first category) if $\mathcal{M}$ is the countable union of sets each of which is nowhere dense in X. $\mathcal{M}$ is said to be nonmeager (or of the second category) if it is not meager. The complement of a meager set is called a residual set. Meager sets may be considered as "small". One property that shows the richness of residual sets is the following: the countable intersection of residual sets is always a residual set.

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III. Problem Formulation and Basic Properties

A. The Peak To Average Power Ratio (PAPR)

Without loss of generality, we restrict to signals with a duration $T_s = 1$. For a signal $s \in L^2[0,1]$, we define

$$\mathrm{PAPR}(s) = \frac{\|s\|_{L^\infty[0,1]}}{\|s\|_{L^2[0,1]}},$$

i.e., PAPR is the ratio between the peak value of the signal and the square root of the power of the signal. Note that the PAPR is usually defined as the square of this value. This difference however is irrelevant for the results in this paper. In the case of an orthogonal transmission scheme, using the ONS $\{\phi_k\}_{k\in\mathbb{Z}} \subset L^2[0,1]$, the PAPR of the transmit signal

$$s(t) = \sum_{k\in\mathbb{Z}} c_k \phi_k(t), \quad t \in [0,1],$$

is given by $\mathrm{PAPR}(s) = \|\sum_{k\in\mathbb{Z}} c_k\phi_k\|_{L^\infty[0,1]}/\|c\|_{\ell^2(\mathbb{Z})}$ because $\{\phi_k\}_{k\in\mathbb{Z}}$ is an ONS, implying $\|s\|_{L^2[0,1]} = \|c\|_{\ell^2(\mathbb{Z})}$.

For an orthogonal transmission scheme, the peak value of the signal s, and hence the PAPR, can become large, as the following result shows. Given any system $\{\phi_n\}_{n=1}^N$ of N orthonormal functions in $L^2[0,1]$, there exist a sequence $\{c_n\}_{n=1}^N \subset \mathbb{C}$ of coefficients with $\sum_{n=1}^N |c_n|^2 = 1$, such that $\|\sum_{n=1}^N c_n\phi_n\|_{L^\infty[0,1]} \ge \sqrt{N}$.

B. Tone Reservation and Solution Concepts

Tone reservation is one approach to reduce the PAPR. Let $\{\phi_k\}_{k\in\mathbb{Z}} \subset L^2[0,1]$ be an ONS. We additionally assume that $\|\phi_k\|_\infty < \infty$, $k \in \mathbb{Z}$, to avoid problems with unbounded carriers. In the tone reservation method, the index set $\mathbb{Z}$ is partitioned in two disjoint sets $\mathcal{K}$ and $\mathcal{K}^\complement$. Note that the set $\mathcal{K}$ can be finite or infinite. For a given sequence $a \in \ell^2(\mathcal{K})$, the goal is to find a sequence $b \in \ell^2(\mathcal{K}^\complement)$ such that the peak value of the signal

$$s(t) = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k(t)}_{=:A(t)} + \underbrace{\sum_{k\in\mathcal{K}^\complement} b_k\phi_k(t)}_{=:B(t)}, \quad t \in [0,1],$$

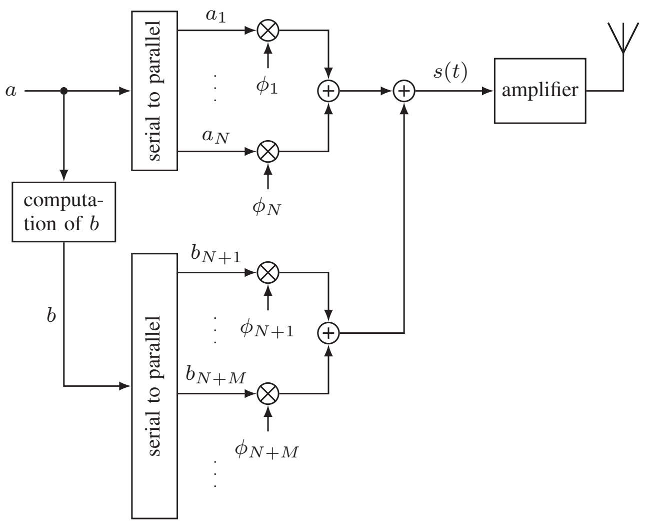

Fig. 1. Block diagram of an OFDM transmission scheme with tone reservation. In this example we have $\mathcal{K} = \{1, \dots, N\}$ and $\mathcal{K}^{\complement} = \mathbb{N} \setminus \mathcal{K}$.

is as small as possible. For notational convenience we introduced the abbreviations A and B, which will be used throughout the paper. A denotes the signal part which contains the information and B the part which is used to reduce the PAPR.

Note that we allow infinitely many carriers to be used for the compensation of the PAPR. This is also of practical interest, since the solvability of the PAPR problem in this setting is a necessary condition for the solvability of the PAPR problem in the setting with finitely many carriers.

Definition 1 (Strong solvability of the PAPR problem)**.** For an ONS $\{\phi_k\}_{k\in\mathbb{Z}}$ and a set $\mathcal{K} \subset \mathbb{Z}$, we say that the PAPR problem is strongly solvable with constant C_{EX}, if there exists a constant $C_{\mathrm{EX}} < \infty$ such that for all $a \in \ell^2(\mathcal{K})$ there exists a $b \in \ell^2(\mathcal{K}^{\complement})$ such that

$$\|A + B\|_{L^\infty[0,1]} \le C_{\mathrm{EX}} \|a\|_{\ell^2(\mathcal{K})}. \tag{1}$$

If the PAPR reduction problem is strongly solvable, condition (1) immediately implies that $\|b\|_{\ell^2(\mathcal{K}^{\complement})} \le C_{\mathrm{EX}} \|a\|_{\ell^2(\mathcal{K})}$, because

$$\begin{aligned}\left(\sum_{k\in\mathcal{K}^{\complement}} |b_k|^2\right)^{\frac{1}{2}} &\le \left(\sum_{k\in\mathcal{K}} |a_k|^2 + \sum_{k\in\mathcal{K}^{\complement}} |b_k|^2\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 \left|\sum_{k\in\mathcal{K}} a_k\phi_k(t) + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \operatorname*{ess\,sup}_{t\in[0,1]} \left|\sum_{k\in\mathcal{K}} a_k\phi_k(t) + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k(t)\right|. \end{aligned} \tag{2}$$

Definition 2 (Weak solvability of the PAPR problem)**.** For an ONS $\{\phi_k\}_{k\in\mathbb{Z}}$ and a set $\mathcal{K} \subset \mathbb{Z}$, we say that the PAPR problem is weakly solvable if for all $a \in \ell^2(\mathcal{K})$ we have

$$F(a) := \inf_{b\in\ell^2(\mathcal{K}^{\complement})} \left\| \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k \right\|_{L^\infty[0,1]} < \infty.$$

This is a weaker form of solvability compared to strong solvability (Definition 1). The peak value of the transmit signal is only required to be bounded and not to be controlled by the norm of the sequence $a = \{a_k\}_{k\in\mathcal{K}}$ as in (1). If the PAPR problem is strongly solvable, it is also weakly solvable.

C. Basic Properties and Equivalent Formulation of the PAPR Problem

Since the functional F plays an important role in the analysis, we state some elementary properties next.

Observation 1.

1) *We have $F(a) \ge 0$ for all $a \in \ell^2(\mathcal{K})$, and $F(a) = 0$ if and only if $a = 0$.*
2) *For all $a^{(1)}, a^{(2)} \in \ell^2(\mathcal{K})$ we have $F(a^{(1)} + a^{(2)}) \le F(a^{(1)}) + F(a^{(2)})$.*
3) *For all $a \in \ell^2(\mathcal{K})$ and all $\lambda \in \mathbb{C}$ we have $F(\lambda a) = |\lambda| F(a)$.*

The proof of these properties is straightforward and omitted due to space constraints.

According to Observation 1, the mapping $F\colon \ell^2(\mathcal{K}) \to \mathbb{R}$ satisfies all properties of a norm. However, as a functional mapping from $(\ell^2(\mathcal{K}), \|\cdot\|_{\ell^2(\mathcal{K})})$ to $\mathbb{R}$, F is not necessarily bounded nor continuous. A functional $F\colon \ell^2(\mathcal{K}) \to \mathbb{R}$ is called bounded if there exists a constant C such that $|F(a)| \le C\|\cdot\|_{\ell^2(\mathcal{K})}$ for all $a \in \ell^2(\mathcal{K})$. Note that the PAPR problem is strongly solvable if and only if the functional $F\colon \ell^2(\mathcal{K}) \to \mathbb{R}$ is bounded. Thus, an important problem is to find all ONS for which F is bounded or continuous.

The condition $F(a) < \infty$ for all $a \in \ell^2(\mathcal{K})$ of course implies that for every $a \in \ell^2(\mathcal{K})$ there exists a $b \in \ell^2(\mathcal{K}^{\complement})$ such that $\|A + B\|_{L^\infty[0,1]} < \infty$. Since it is possible that $A \notin L^\infty[0,1]$ this indeed poses a restriction on the set $\mathcal{K}$. For example in the OFDM case, where the ONS is given by the system of complex exponentials, we have for $\mathcal{K} = \{2^k\}_{k\in\mathbb{N}\cup\{0\}}$ and $a_k = (k+1)^{-1/2-10^{-4}}$, $k \in \mathbb{N}\cup\{0\}$ that

$$A(t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\frac{1}{2}+10^{-4}}} \mathrm{e}^{i2\pi 2^k t} \tag{3}$$

is unbounded on every interval $[t_1, t_2]$, $0 \le t_1 < t_2 \le 1$. Nevertheless, there exists a $b \in \ell^2(\mathcal{K}^{\complement})$ such that $\|A + B\|_{L^\infty[0,1]} < \infty$. Three partial sums of the series in (3) are visualized in Fig. 2.

For $a \in \ell^2(\mathcal{K})$ we define the set

$$\mathcal{M}(a) = \{b \in \ell^2(\mathcal{K}^{\complement})\colon \|A + B\|_{L^\infty[0,1]} < \infty\}. \tag{4}$$

It is easy to see that $F(a) < \infty$ for all $a \in \ell^2(\mathcal{K})$ if and only if $\mathcal{M}(a) \neq \emptyset$ for all $a \in \ell^2(\mathcal{K})$. Further, $\mathcal{M}(a)$ is a convex set. The convexity of the set $\mathcal{M}(a)$ implies that finding $\inf_{b\in\mathcal{M}(a)} \|A + B\|_{L^\infty[0,1]}$ is a convex optimization problem as soon as $\mathcal{M}(a)$ is known.

In [7] a complete characterization of the strong solvability of the PAPR problem was given for the case that the ONS

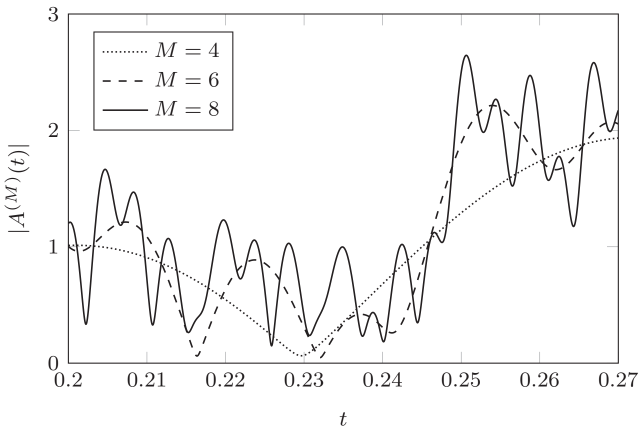

Fig. 2. Plot of a segment of the absolute value of the signal $A^{(M)}(t) = \sum_{k=0}^{M} 1/(k+1)^{1/2+10^{-4}} \, \mathrm{e}^{i2\pi 2^k t}$ for $M = 4, 6, 8$.

$\{\phi_k\}_{k\in\mathcal{K}}$ is complete. To state the theorem, we introduce the set

$$L^1(\mathcal{K}) = \overline{\left\{ \sum_{k\in\mathcal{K}\cap\mathcal{M}} a_k\phi_k \colon a_k \in \mathbb{C},\ \mathcal{M} \subset \mathbb{Z} \text{ finite} \right\}}^{L^1[0,1]},$$

which is the closed linear span of all finite linear combinations of functions in $\{\phi_k\}_{k\in\mathcal{K}}$. $L^1(\mathcal{K})$ is a closed subspace of $L^1[0,1]$.

Theorem 1. *Let $\{\phi_k\}_{k\in\mathbb{Z}}$ be a bounded complete ONS in $L^2[0,1]$, $\mathcal{K} \subset \mathbb{Z}$, and $C_{EX} > 0$. The PAPR problem is strongly solvable with constant C_{EX} if and only if we have $\|f\|_{L^2[0,1]} \le C_{EX}\|f\|_{L^1[0,1]}$ for all $f \in L^1(\mathcal{K})$.*

We will use this characterization in Sections IV and V.

IV. PAPR Reduction for OFDM

In the following we consider the special case of an OFDM transmission scheme. Here the ONS is given by $\{\phi_k\}_{k\in\mathbb{Z}} = \{\mathrm{e}^{i2\pi k\,\cdot}\}_{k\in\mathbb{Z}}$. We will show that for an OFDM system weak solvability implies strong solvability. That is, if there exists a subset $\mathcal{K} \subset \mathbb{Z}$ such that $F(a) < \infty$ for all $a \in \ell^2(\mathcal{K})$ then there exists a constant C_{EX} such that $F(a) \le C_{\mathrm{EX}}\|a\|_{\ell^2(\mathcal{K})}$ for all $a \in \ell^2(\mathcal{K})$.

For the proof we need several auxiliary results, which require some notation. Let $\mathcal{K} \subset \mathbb{Z}$ be such that the PAPR problem can be weakly solved in the sense of Definition 2. Further, let $a \in \ell^2(\mathcal{K})$ be arbitrary. For $N \in \mathbb{N}$ we set

$$a_N(k) = \begin{cases} a_k\left(1 - \frac{|k|}{N}\right), & |k| < N, \\ 0, & |k| \ge N, \end{cases} \qquad k \in \mathcal{K}. \tag{5}$$

Then

$$A_N(t) = \sum_{\substack{k\in\mathcal{K} \\ |k|<N}} a_k\left(1 - \frac{|k|}{N}\right)\mathrm{e}^{i2\pi kt}, \quad t \in [0,1],$$

is a trigonometric polynomial, satisfying $A_N \in L^\infty[0,1]$. Further, let

$$F_N(a) = \inf_{b\in\ell^2(\mathcal{K}^{\complement})} \|A_N + B\|_{L^\infty[0,1]}. \tag{6}$$

The next lemma shows that the functionals F_N are continuous.

Lemma 1. *Let $\mathcal{K} \subset \mathbb{Z}$. Then, for all $N \in \mathbb{N}$, $F_N \colon \ell^2(\mathcal{K}) \to \mathbb{R}$ is a continuous functional. Further, for all $a^{(1)}, a^{(2)} \in \ell^2(\mathcal{K})$ we have $|F_N(a^{(1)}) - F_N(a^{(2)})| \le \sqrt{2N-1}\|a^{(1)} - a^{(2)}\|_{\ell^2(\mathcal{K})}$.*

The proof of Lemma 1 uses the finite dimensionality of the underlying sequence space, which is limited by N. Due to space constraints the proof is not presented here.

The following lemma reflects the essential property of OFDM, and also shows why the coefficients a_N have been selected according to (5).

Lemma 2. *Let $\mathcal{K} \subset \mathbb{Z}$ be such that the PAPR problem is weakly solvable. For all $a \in \ell^2(\mathcal{K})$ and all $N \in \mathbb{N}$ we have $F_N(a) \le F(a)$.*

Proof. Let $a \in \ell^2(\mathcal{K})$ and $N \in \mathbb{N}$ be arbitrary but fixed. Further let $\epsilon > 0$ be arbitrary. There exists a $b^{(\epsilon)} \in \ell^2(\mathcal{K}^{\complement})$ such that

$$\|A + B^{(\epsilon)}\|_{L^\infty[0,1]} \le F(a) + \epsilon. \tag{7}$$

We have

$$\begin{aligned} &A_N(t) + B_N^{(\epsilon)}(t) \\ &= \sum_{\substack{k\in\mathcal{K} \\ |k|<N}} a_k\left(1 - \frac{|k|}{N}\right)\mathrm{e}^{i2\pi kt} + \sum_{\substack{k\in\mathcal{K}^{\complement} \\ |k|<N}} b_k^{(\epsilon)}\left(1 - \frac{|k|}{N}\right)\mathrm{e}^{i2\pi kt} \\ &= \int_0^1 (A(\tau) + B^{(\epsilon)}(\tau))K_N(t-\tau)\,\mathrm{d}\tau, \end{aligned}$$

where

$$K_N(t) = \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N}\right)\mathrm{e}^{i2\pi kt} = \frac{1}{N}\left(\frac{\sin(\frac{Nt}{2})}{\sin(\frac{t}{2})}\right)^2$$

denotes the Fejér kernel. It follows that

$$\begin{aligned} |A_N(t) + B_N^{(\epsilon)}(t)| &\le \int_0^1 |A(\tau) + B^{(\epsilon)}(\tau)|K_N(t-\tau)\,\mathrm{d}\tau \\ &\le (F(a) + \epsilon)\int_0^1 K_N(t-\tau)\,\mathrm{d}\tau = F(a) + \epsilon, \end{aligned}$$

where we used (7) and the fact that $\int_0^1 K_N(t-\tau)\,\mathrm{d}\tau = 1$ for all $t \in \mathbb{R}$. Further we have

$$F_N(a) \le \|A_N + B_N^{(\epsilon)}\|_{L^\infty[0,1]} \le F(a) + \epsilon.$$

Since this inequality is valid for all $\epsilon > 0$, it follows that $F_N(a) \le F(a)$. $\square$

Our goal is to show that weak solvability of the PAPR problem implies strong solvability. Therefore, we need to show the boundedness of the functional F. To this end, we first define a suitable functional $\overline{F}$ in (8) and prove its boundedness. Note that convex and homogeneous functionals that are unbounded can be found for every infinite dimensional Banach space. Hence, the boundedness of $\overline{F}$ is not obvious.

Lemma 3. *Let $\mathcal{K} \subset \mathbb{Z}$ be such that the PAPR problem is weakly solvable. Then there exists a constant $0 < C_1 < \infty$ such that*

$$\overline{F}(a) := \sup_{N\in\mathbb{N}} F_N(a) \leq C_1 \|a\|_{\ell^2(\mathcal{K})} \tag{8}$$

for all $a \in \ell^2(\mathcal{K})$.

Proof. From Lemma 1, we know that $\{F_N\}_{N\in\mathbb{N}}$ is a sequence of continuous functionals on $\ell^2(\mathcal{K})$, and, according to Lemma 2, we have $\overline{F}(a) \leq F(a) < \infty$ for all $a \in \ell^2(\mathcal{K})$. Thus, we can apply the generalized uniform boundedness principle [10], which gives that there exists a constant C_2 and a neighborhood

$$U_\delta(\hat{a}) = \{a \in \ell^2(\mathcal{K})\colon \|a-\hat{a}\|_{\ell^2(\mathcal{K})} < \delta\},$$

$\hat{a} \in \ell^2(\mathcal{K})$, $\delta > 0$, such that, for all $a \in U_\delta(\hat{a})$, we have $F_N(a) \leq C_2$, and consequently $\overline{F}(a) \leq C_2$. $\overline{F}(a)$ is a non-negative, convex, and positively homogeneous functional. Next, we want to infer the global behavior of $\overline{F}$ from its local behavior on $U_\delta(\hat{a})$. Let $\tilde{a} \in U_\delta(0)$ be arbitrary but fixed. Then we have $\tilde{a} + \hat{a} \in U_\delta(\hat{a})$, and it follows that

$$\begin{aligned}\overline{F}(\tilde{a}) = \overline{F}(\tilde{a}+\hat{a}-\hat{a}) &\leq \overline{F}(\tilde{a}+\hat{a}) + \overline{F}(-\hat{a})\\ &\leq C_2 + \overline{F}(\hat{a}) \leq 2C_2\end{aligned}$$

for all $\tilde{a} \in U_\delta(0)$. Let $a \in \ell^2(\mathcal{K})$, $a \neq 0$, be arbitrary but fixed and $a_\delta = (\delta/2)(a/\|a\|_{\ell^2(\mathcal{K})})$. Then we have $\|a_\delta\|_{\ell^2(\mathcal{K})} = \delta/2$, which implies that $a_\delta \in U_\delta(0)$. It follows that

$$\frac{\delta}{2\|a\|_{\ell^2(\mathcal{K})}} \overline{F}(a) = \overline{F}(a_\delta) \leq 2C_2,$$

or, equivalently $\overline{F}(a) \leq (4C_2/\delta)\|a\|_{\ell^2(\mathcal{K})}$. □

We need another auxiliary result, namely the boundedness of the functional F for sequences a with only finitely many non-zero elements if the PAPR problem is weakly solvable.

Lemma 4. *Let $\mathcal{K} \subset \mathbb{Z}$ be such that the PAPR problem is weakly solvable, and let $a = \{a_k\}_{k\in\mathcal{K}} \in \ell^2(\mathcal{K})$ be such that $a_k \neq 0$ for only finitely many $k \in \mathcal{K}$. Then we have $F(a) \leq C_1\|a\|_{\ell^2(\mathcal{K})}$.*

Proof. Let $a \in \ell^2(\mathcal{K})$ be such that $a_k \neq 0$ for only finitely many $k \in \mathcal{K}$. We assume that a is concentrated on the set $\mathcal{K} \cap [-M, M]$. Let $N > M$ and $\epsilon > 0$ be arbitrary but fixed. For $b \in \ell^2(\mathcal{K})$ with $B \in L^\infty[0,1]$ we have

$$\|A+B\|_{L^\infty[0,1]} \leq \|A - A_N\|_{L^\infty[0,1]} + \|A_N + B\|_{L^\infty[0,1]}.$$

It follows that $F(a) \leq \|A - A_N\|_{L^\infty[0,1]} + F_N(a)$. We further have $F_N(a) \leq \sup_{N\in\mathbb{N}} F_N(a) \leq C_1\|a\|_{\ell^2(\mathcal{K})}$, where we used Lemma 3 in the last inequality. Since

$$\begin{aligned}|A(t) - A_N(t)| &= \left| \sum_{\substack{k\in\mathcal{K}\\|k|<M}} \left[a_k - a_k\left(1 - \frac{|k|}{N}\right)\right] \mathrm{e}^{i2\pi k t} \right|\\ &\leq \sum_{\substack{k\in\mathcal{K}\\|k|<M}} |a_k| \frac{|k|}{N} = \frac{1}{N} \sum_{\substack{k\in\mathcal{K}\\|k|<M}} |a_k||k|,\end{aligned}$$

we further have $\lim_{N\to\infty}\|A - A_N\|_{L^\infty[0,1]} = 0$. Combining all partial results we see that $F(a) \leq C_1\|a\|_{\ell^2(\mathcal{K})}$. □

Now follows our first main result, the boundedness of the functional F if the PAPR problem is weakly solvable.

Theorem 2. *Let $\mathcal{K} \subset \mathbb{Z}$. If we have $F(a) < \infty$ for all $a \in \ell^2(\mathcal{K})$ then there exists a constant $0 < C_1 < \infty$ such that $F(a) \leq C_1\|a\|_{\ell^2(\mathcal{K})}$ for all $a \in \ell^2(\mathcal{K})$.*

Theorem 2 shows that for OFDM, the weak solvability of the PAPR problem implies the strong solvability. Thus, for OFDM, weak solvability and strong solvability are equivalent.

Proof of Theorem 2. For all $a \in \ell^2(\mathcal{K})$ with $\|a\|_{\ell^2(\mathcal{K})} = 1$ and $a_k \neq 0$ for only finitely many $k \in \mathcal{K}$, we have according to Lemma 4 that $F(a) \leq C_1$. Hence, for every $\epsilon > 0$ there exists a $b^{(\epsilon)} \in \ell^2(\mathcal{K}^\complement)$ such that $\|A + B^{(\epsilon)}\|_{L^\infty[0,1]} \leq C_1 + \epsilon$. Let

$$L^1(\mathcal{K}^{\mathrm{F}}) = \left\{ \sum_{k\in\mathcal{K}\cap\mathcal{M}} a_k \,\mathrm{e}^{i2\pi k\,\cdot} : a_k \in \mathbb{C},\ \mathcal{M} \subset \mathbb{Z} \text{ finite} \right\}$$

and $f \in L^1(\mathcal{K}^{\mathrm{F}})$ be arbitrary but fixed. We have

$$\begin{aligned}\left|\sum_{k\in\mathcal{K}} c_k \overline{a}_k\right| &= \left|\int_0^1 f(t)\overline{A(t) + B^{(\epsilon)}(t)}\,\mathrm{d}t\right|\\ &\leq \int_0^1 |f(t)||A(t) + B^{(\epsilon)}(t)|\,\mathrm{d}t \leq (C_1+\epsilon)\int_0^1 |f(t)|\,\mathrm{d}t,\end{aligned}$$

where we used that $\int_0^1 f(t)\overline{B^{(\epsilon)}(t)}\,\mathrm{d}t = 0$. We choose $a_k = c_k/\|c\|_{\ell^2(\mathcal{K})}$. Then it follows that

$$\left(\sum_{k\in\mathcal{K}} |c_k|^2\right)^{\frac{1}{2}} = \|f\|_{L^2[0,1]} \leq (C_1+\epsilon)\|f\|_{L^1[0,1]}.$$

Since this inequality is valid for all $\epsilon > 0$, we obtain

$$\|f\|_{L^2[0,1]} \leq C_1\|f\|_{L^1[0,1]} \tag{9}$$

for all $f \in L^1(\mathcal{K}^{\mathrm{F}})$. The next step is to show that (9) is also true for all $f \in L^1(\mathcal{K})$. This short proof, which uses that $L^1(\mathcal{K}^{\mathrm{F}})$ is dense in $L^1(\mathcal{K})$ and basic properties of the spaces $L^1[0,1]$ and $L^2[0,1]$, is omitted due to space constraints. Finally, application of Theorem 1 completes the proof. □

Our definition of weak solvability is practically the weakest meaningful requirement that can be imposed. Therefore, it is interesting to see that both definitions for the solvability of the PAPR problem—weak and strong solvability—are equivalent for OFDM.

If the PAPR problem cannot be solved in the strong sense then it is unclear how the peak value of the transmit signal behaves for certain sequences. One may ask if there are only very few sequences for which the peak value cannot be controlled, i.e., for which inequality (1) in Definition 1 is not satisfied, or if the set of those sequences is large. We will give an answer to this question in Theorem 4.

First, we note that we do not have to require the weak solvability for all sequences $a \in \ell^2(\mathcal{K})$, it suffices to have weak solvability for a nonmeager set. This already implies the strong solvability for all sequences $a \in \ell^2(\mathcal{K})$.

Theorem 3. *Let $\mathcal{K} \subset \mathbb{Z}$. If there exists a nonmeager set $\mathcal{E}$ in $\ell^2(\mathcal{K})$ such that $F(a) < \infty$ for all $a \in \mathcal{E}$, then there exists a constant $C_1 < \infty$ such that $F(a) \leq C_1 \|a\|_{\ell^2(\mathcal{K})}$ for all $a \in \ell^2(\mathcal{K})$.*

Proof. We use the generalized uniform boundedness principle [10]. It follows that $\overline{F}(a) < \infty$ for a nonmeager set, and consequently $\overline{F}(a) \leq C_1 \|a\|_{\ell^2(\mathcal{K})}$ for all $a \in \ell^2(\mathcal{K})$. □

Based on Theorem 3 we can now show that if the PAPR is not strongly solvable, then the set of sequences $a \in \ell^2(\mathcal{K})$ for which the peak value is infinite is a residual set, i.e., large in a topological sense. Or, to phrase it differently, the set of sequences $a \in \ell^2(\mathcal{K})$ such that $\mathcal{M}(a)$, as defined in (4), is non-empty is a meager, i.e., small in a topological sense.

Theorem 4. *Let $\mathcal{K} \subset \mathbb{Z}$ such that the strong PAPR problem is not solvable. Then $\{a \in \ell^2(\mathcal{K}) \colon F(a) = \infty\}$ is a residual set.*

Proof. According to Theorem 3 there exists no nonmeager set $\mathcal{E}$ in $\ell^2(\mathcal{K})$ such that $F(a) < \infty$ for all $a \in \mathcal{E}$. Hence, the set of all $a \in \ell^2(\mathcal{K})$ such that $F(a) < \infty$ is a meager set, and $\{a \in \ell^2(\mathcal{K}) \colon F(a) = \infty\}$ is a residual set. □

V. Application

In this section we consider specific sets $\mathcal{K}$, and study the solvability of the PAPR problem for these sets. For a subset $\mathcal{K}$ of the integers, we call

$$\overline{d}(\mathcal{K}) := \limsup_{N \to \infty} \frac{|\mathcal{K} \cap \{-N, \ldots, N\}|}{2N+1}$$

the upper density of $\mathcal{K}$.

Theorem 5. *Let $\mathcal{K} \subset \mathbb{Z}$ be an arbitrary set such that $\overline{d}(\mathcal{K}) > 0$. Then, for the set $\mathcal{K}$ and all constants $C_{EX} < \infty$, the PAPR problem is not strongly solvable.*

Proof. Let $\mathcal{K} \subset \mathbb{Z}$ be such that $\overline{d}(\mathcal{K}) = \beta > 0$. Let $0 < \epsilon < \beta$ be arbitrary. There exists a sequence $\{N_r\}_{r \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{N \to \infty} N_r = \infty$ such that

$$\frac{|\mathcal{K} \cap \{-N_r, \ldots, N_r\}|}{2N_r + 1} \geq \beta - \epsilon$$

for all $r \in \mathbb{N}$. We use Szemerédi's theorem, which can be stated as follows: For every number $m \in \mathbb{N}$ there exists a number $N_{\text{SZ}} = N_{\text{SZ}}(m, \beta - \epsilon)$ such that for all $N \geq N_{\text{SZ}}$ we have that every set $\mathcal{M} \subset \{-N, \ldots, N\}$ with $|\mathcal{M}| \geq (2N+1)(\beta - \epsilon)$ contains an arithmetic progression of length m [11].

Suppose that the assertion of the theorem is false. Then the PAPR problem is strongly solvable for some constant $C_{\text{EX}} < \infty$. We choose $m \in \mathbb{N}$ arbitrary. From Szemerédi's theorem it follows that, for all elements of the sequence $\{N_r\}_{r \in \mathbb{N}}$ with $N_r \geq N_{\text{SZ}}(m, \beta - \epsilon)$, the set $\mathcal{K} \cap \{-N_r, \ldots, N_r\}$ contains an arithmetic progression of length m. Using the condition from Theorem 1, it can be shown that a necessary condition for the strong solvability of the PAPR problem with constant C_{EX} is

$$C_{\text{EX}} > \frac{\sqrt{m}}{\frac{4}{\pi^2} \log(m) + 3 + \frac{2}{24 - \pi^2} + \frac{4}{\pi^2}}. \tag{10}$$

Since $m \in \mathbb{N}$ was arbitrary, inequality (10) shows that $C_{\text{EX}} = \infty$, which is a contradiction. □

Theorem 5 gives us a necessary condition for the strong solvability of the PAPR problem: $\overline{d}(\mathcal{K}) = 0$ needs to be satisfied, otherwise the PAPR problem cannot be strongly solvable. However, the condition $\overline{d}(\mathcal{K}) = 0$ is not sufficient.

Next, we present two corollaries for the OFDM case.

Corollary 1. *Let $\{\phi_k\}_{k \in \mathbb{Z}} = \{\mathrm{e}^{i 2\pi k \cdot}\}_{k \in \mathbb{Z}}$, and let $\mathcal{K} \subset \mathbb{Z}$ be such that $\overline{d}(\mathcal{K}) > 0$. Then $\{a \in \ell^2(\mathcal{K}) \colon F(a) < \infty\}$ is a meager set.*

Proof. We have $\overline{d}(\mathcal{K}) > 0$, i.e, for $\mathcal{K}$ and all constants C_{EX}, the PAPR problem is not strongly solvable, according to Theorem 5. Therefore, Theorem 2 shows that the PAPR problem is also not weakly solvable. Moreover, Theorem 3 implies that there exists no nonmeager set $\mathcal{E}$ in $\ell^2(\mathcal{K})$ such that $F(a) < \infty$ for all $a \in \mathcal{E}$. Consequently, $\{a \in \ell^2(\mathcal{K}) \colon F(a) < \infty\}$ is a meager set. □

Corollary 2. *Let $\{\phi_k\}_{k \in \mathbb{Z}} = \{\mathrm{e}^{i 2\pi k \cdot}\}_{k \in \mathbb{Z}}$, and let $\mathcal{K} \subset \mathbb{Z}$ be such that $\overline{d}(\mathcal{K}) > 0$. Then $\mathcal{S} = \{a \in \ell^2(\mathcal{K}) \colon \exists b \in \ell^2(\mathcal{K}^{\complement})$ s.t. $\|A + B\|_{L^\infty[0,1]} < \infty\}$ is a meager set.*

Proof. For all $a \in \ell^2(\mathcal{K})$ we have $F(a) < \infty$ if and only if $s \in \mathcal{S}$. The assertion follows from Corollary 1. □

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