Spectrum Sharing Between MIMO-MC Radars and Communication Systems

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***Abstract*—By employing sparse sampling, MIMO radars with matrix completion (MIMO-MC) achieve the performance of MIMO radars but with significantly fewer data samples. Further, due to the way the sparse sampling modulates the interference channel between radar and communication systems, MIMO-MC radars are good candidates for spectrum sharing. In this paper, we design a scenario in which a MIMO-MC radar optimally co-exists with a MIMO wireless communication system in the presence of clutter. Both the radar and the communication systems employ transmit precoding. The precoders and the radar sub-sampling scheme are jointly designed to maximize the receive signal-to-interference ratio of the radar while meeting certain rate and power constraints for the communication system. Efficient optimization algorithms are provided. Simulation results demonstrate that MIMO-MC radars offer significant improvement as compared to MIMO radars in the co-existence scenario considered here.**

***Index Terms*—MIMO radar, matrix completion, spectrum sharing, precoding, alternating optimization**

I. Introduction

As high-data-rate applications become prevalent, spectrum congestion in commercial wireless communications is a growing problem. On the other hand, recent government studies have shown that huge chunks of spectrum held by federal agencies are underutilized in urban areas [1]. In an effort to relieve spectrum congestion, the Federal Communications Commission (FCC) and the National Telecommunications and Information Administration (NTIA) have proposed to make available 150 MHz of spectrum in the 3.5 GHz band, previously used by federal radar systems for surveillance and air defense, to be shared by both radar and communication applications [2], [3]. When communication and radar systems overlap in the spectrum, they exert interference to each other. Spectrum sharing targets at enabling radar and communication systems to share the spectrum efficiently by minimizing interference effects.

This paper investigates spectrum sharing of a MIMO communication system and a matrix completion (MC) based, collocated MIMO radar (MIMO-MC) system [4]–[6]. MIMO radars transmit different waveforms from their transmit (TX) antennas, and their receive (RX) antennas forward their measurements to a fusion center for further processing. Based on the forwarded data, the fusion center populates a matrix, referred to as the "data matrix", which is subsequently used by standard array processing methods for target detection. For a relatively small number of targets, the data matrix is low-rank [4], thus allowing one to fully reconstruct it (under certain conditions) based on a small, uniformly sampled set of its entries. This observation is the basis of MIMO-MC radars; the RX antennas forward to the fusion center a small number of pseudo-randomly sub-Nyquist samples of the target returns, along with their sampling scheme, with each RX antenna partially filling a row of the data matrix. The full data matrix can be recovered via MC techniques. The subsampling at the antennas avoids the need for high rate analog-to-digital converters, and the reduced amount of samples translates into power and bandwidth savings in the antenna-fusion center link. Further, due to the fact that the full data matrix is recovered there is no loss of SNR due to undersampling.

Most of the existing multiple-input-multiple-output (MIMO) radar-communication spectrum sharing literature addresses interference mitigation either for the communication system [7]–[11], or for the radar [12]. Spectrum sharing between traditional MIMO radars and communication systems was initially considered in [7]–[11], where the radar interference to the communication system was eliminated by projecting the radar waveforms onto the null space of the interference channel between the radar and the communication system. However, projection-type techniques might miss targets lying in the row space of the interference channel. In addition, the interference from the communication system to the radar was not considered in [7]–[11]. Spatial filtering at the radar receiver was proposed in [12] to reduce interference from the communication systems. To the best of our knowledge, co-design of radar and communication systems for spectrum sharing was proposed in [13]–[16] for the first time. Compared to radar design approaches of [7]–[12], the joint design has the potential to improve spectrum utilization due to increased number of design degrees of freedom.

Spectrum sharing between a MIMO-MC radar and a MIMO communication system was considered in [13] and [14], where the radar interference at the communication receiver was estimated and then subtracted from the received signal at the communication receiver. However, this approach might not work well in cases in which the radar power is so high that it saturates the communication receiver. However, even if the radar power does not cause saturation, due to random phase offsets between the radar transmitter and the communication receiver, direct subtraction of interference will always leave residual interference, which can degrade the communication system performance. The coexistence of traditional MIMO radars and a MIMO communication system was studied in [15]–[17], where precoding was used both at the radar and the communication system, and the precoders were jointly designed to maximize the SINR at the radar receiver while meeting certain communication system rate and power constraints. It was shown that radar TX precoding can effectively reduce the interference towards the communication receiver and maximize the radar SINR. Further, it can mitigate clutter.

In this paper, we propose a new spectrum sharing method for the co-existence of MIMO-MC radars and MIMO communication systems, by extending the work in [15], [17]. Due to way the sparse sampling modulates the interference channel between radar receivers and and communication system transmitters, MIMO-MC radars provide the opportunity to the communication system to design its transmission in a way that the interference to the radar is significantly reduced. In that sense, MIMO-MC radars are better candidates than traditional MIMO radars for spectrum sharing. We use precoding both at the radar and the communication system, jointly design precoding matrices, and the radar sub-sampling scheme in order to maximize the radar signal-to-interference-plus-noise ratio (SINR) subject to constraints on communication rate and power.

The paper is organized as follows. Section II provides some

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background on MIMO-MC radars. Section III introduces a signal model to describe the the MIMO-MC radar and communication systems coexistence. The problem of a MIMO communication system sharing spectrum with a MIMO-MC radar is studied in Sections IV. Numerical results and conclusions are provided in Sections V-VI. *Notation:* $\mathcal{CN}(\mu, \mathbf{\Sigma})$ denotes the circularly symmetric complex Gaussian distribution with mean μ and covariance matrix $\mathbf{\Sigma}$. $|\cdot|$ and $\mathrm{Tr}(\cdot)$ denote the matrix determinant and trace respectively. The set $\mathbb{N}_L^+$ is defined as $\{1, \ldots, L\}$. δ_{ij} denotes the Kronecker delta. $\lfloor x \rfloor$ denotes the largest integer not larger than x. $\mathbf{A}^T$ and $\mathbf{A}^H$ respectively denote the transpose and Hermitian transpose of $\mathbf{A}$.

II. MIMO-MC Radar Revisited

Consider a collocated MIMO radar system with $M_{t,R}$ TX antennas and $M_{r,R}$ RX antennas, arranged as uniform linear arrays (ULA) with inter-element spacing d_t and d_r, respectively. The radar is pulse based with pulse repetition interval T_{PRI} and carrier wavelength λ_c. The K far-field targets are with distinct angles $\{\theta_k\}$, target reflection coefficients $\{\beta_k\}$ and Doppler shifts $\{\nu_k\}$ and are assumed to fall in the same range bin. Following the clutter-free model of [5], [6], [18], the data matrix at the fusion center can be formulated as

$$\mathbf{Y}_R = \mathbf{V}_r \mathbf{\Sigma} \mathbf{V}_t^T \mathbf{P} \mathbf{S} + \mathbf{W}_R, \quad = DPS + W_R = M + W_R \tag{1}$$

where the m-th row of $\mathbf{Y}_R \in \mathbb{C}^{M_{r,R} \times L}$ contains the L samples forwarded by the m-th antenna; the waveforms are given in $\mathbf{S} = [\mathbf{s}(1), \cdots, \mathbf{s}(L)]$, with $\mathbf{s}(l) = [s_1(l), \cdots, s_{M_{t,R}}(l)]^T$ being the l-th snapshot across the transmit antennas; the transmit waveforms are assumed to be orthogonal, *i.e.*, it holds that $\mathbf{S}\mathbf{S}^H = \mathbf{I}_{M_{t,R}}$ [6]; $\mathbf{W}_R$ denotes additive noise; and $\mathbf{P} \in \mathbb{C}^{M_{t,R} \times M_{t,R}}$ denotes the transmit precoding matrix. $\mathbf{V}_t \triangleq [\mathbf{v}_t(\theta_1), \ldots, \mathbf{v}_t(\theta_K)]$ and $\mathbf{V}_r \triangleq [\mathbf{v}_r(\theta_1), \ldots, \mathbf{v}_r(\theta_K)]$ respectively denote the transmit and receive steering matrix and $\mathbf{v}_r(\theta) \in \mathbb{C}^{M_{r,R}}$ is the receive steering vector defined as

$$\mathbf{v}_r(\theta) \triangleq \left[e^{-j2\pi 0 \vartheta^r}, \ldots, e^{-j2\pi(M_{r,R}-1)\vartheta^r} \right]^T, \tag{2}$$

where $\vartheta^r = d_r \sin(\theta)/\lambda_c$ denotes the spatial frequency w.r.t. the receive array. $\mathbf{v}_t(\theta) \in \mathbb{C}^{M_{t,R}}$ is the transmit steering vector and is respectively defined. Matrix $\mathbf{\Sigma}$ is defined as $\mathbf{\Sigma} \triangleq \mathrm{diag}([\beta_1 e^{j2\pi\nu_1}, \ldots, \beta_K e^{j2\pi\nu_K}])$. $\mathbf{D} \triangleq \mathbf{V}_r \mathbf{\Sigma} \mathbf{V}_t^T$ is also called the target response matrix. After matched filtering at the fusion center, target estimation can be performed based on $\mathbf{Y}_R$ via standard array processing schemes [19].

When K is smaller than $M_{r,R}$ and L, the noise-free data matrix $\mathbf{M} \triangleq \mathbf{DPS}$ is low-rank and, under certain conditions, can be provably recovered based on a subset of its entries. This observation gave rise to MIMO-MC radars [5], [6], [18], where each RX antenna sub-samples the target returns and forwards the samples to the fusion center. The partially filled data matrix at the fusion center can be mathematically expressed as follows (see [6] Scheme I):

$$\mathbf{\Omega} \circ \mathbf{Y}_R = \mathbf{\Omega} \circ (\mathbf{M} + \mathbf{W}_R), \tag{3}$$

where $\circ$ denotes the Hadamard product; $\mathbf{\Omega}$ is the sub-sampling matrix containing 0's and 1's. The sub-sampling rate p equals $\|\mathbf{\Omega}\|_0/(LM_{r,R})$. When $p = 1$, the $\mathbf{\Omega}$ matrix is filled with 1's, and the MIMO-MC radar is identical to the traditional MIMO radar. At the fusion center, the completion of $\mathbf{M}$ can be achieved by the following nuclear norm minimization problem [20] $Y_R = M + W_R$

$$\min_{\mathbf{M}} \|\mathbf{M}\|_* \quad \text{s.t. } \|\mathbf{\Omega} \circ \mathbf{M} - \mathbf{\Omega} \circ \mathbf{Y}_R\|_F \leq \delta, \tag{4}$$

where $\delta > 0$ is a parameter determined by the sampled entries of the noise matrix, *i.e.*, $\mathbf{\Omega} \circ \mathbf{W}_R$. When $\mathbf{S}$ is chosen as any random unitary matrix [21], it was shown in [22] that the data matrix $\mathbf{M}$ can be stably reconstructed with high accuracy and retaining all the received target echo power. This means that the radar waveform

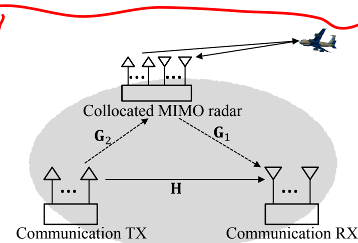

Fig. 1: A MIMO communication system sharing spectrum with a colocated MIMO radar system

can be changed periodically, without affecting the matrix completion performance. Changing the waveform would be good for security reasons. Furthermore, it was shown in [22] the matrix completion performance is independent of $\mathbf{P}$. This means that we can design $\mathbf{P}$, without affecting the incoherence property of $\mathbf{M}$, for the purpose of transmit beamforming and interference suppression. This key observation validates the feasibility of radar precoding based spectrum sharing approaches for MIMO-MC radar and communication systems in the sequel.

III. System Model

We consider the coexistence scenario in [14], as shown in Fig. 1, where a MIMO-MC radar system and a MIMO communication system operate using the same carrier frequency. Note that the coexistence model is general, because when full sampling is adopted the MIMO-MC radar turns to be a traditional MIMO radar.

Suppose that the two systems use narrowband waveforms with the same symbol rate and are synchronized in sampling time (see [14] for the case of mismatched symbol rates). Consider the same target scene in a particular range bin as in Section II, but also containing clutter. The signal received by the radar and communication RX antennas during L symbol durations can be respectively expressed as

Radar fusion center:

$$\mathbf{\Omega} \circ \mathbf{Y}_R = \mathbf{\Omega} \circ \big(\underbrace{\mathbf{DPS}}_{\text{signal}} + \underbrace{\mathbf{CPS} + \mathbf{G}_2 \mathbf{X} \mathbf{\Lambda}_2}_{\text{interference}} + \underbrace{\mathbf{W}_R}_{\text{noise}}\big), \tag{5a}$$

Communication receiver:

$$\mathbf{Y}_C = \underbrace{\mathbf{HX}}_{\text{signal}} + \underbrace{\mathbf{G}_1 \mathbf{PS} \mathbf{\Lambda}_1}_{\text{interference}} + \underbrace{\mathbf{W}_C}_{\text{noise}}, \tag{5b}$$

where $\mathbf{Y}_R$, $\mathbf{D}$, $\mathbf{P}$, $\mathbf{S}$, $\mathbf{W}_R$, and $\mathbf{\Omega}$ are defined in Section II. The waveform-dependent interference $\mathbf{CPS}$ contains interferences from point scatterers (clutter or interfering objects). Suppose that there are K_c point clutters with angles $\{\theta_k^c\}$, reflection coefficients $\{\beta_k^c\}$ in the same range bin as the targets. $\mathbf{C} \triangleq \sum_{k=1}^{K_c} \beta_k^c \mathbf{v}_r(\theta_k^c) \mathbf{v}_t^T(\theta_k^c)$ is the clutter response matrix. $\mathbf{Y}_C$ and $\mathbf{W}_C$ denote the received signal and additive noise at the communication RX antennas, respectively. The columns of $\mathbf{X} \triangleq [\mathbf{x}(1), \ldots, \mathbf{x}(L)]$ are codewords from the codebook of the communication system. We assume that $\mathbf{W}_{R/C}$ contains i.i.d random entries distributed as $\mathcal{CN}(0, \sigma_{R/C}^2)$. $\mathbf{H} \in \mathbb{C}^{M_{r,C} \times M_{t,C}}$ denotes the communication channel, where $M_{r,C}$ and $M_{t,C}$ denote respectively the number of RX and TX antennas of the communication system; $\mathbf{G}_1 \in \mathbb{C}^{M_{r,C} \times M_{t,R}}$ and $\mathbf{G}_2 \in \mathbb{C}^{M_{r,R} \times M_{t,C}}$ denote the interference channels between the communication and radar systems. All channels are assumed to be flat fading and remain the same over L symbol intervals [7], [8], [10], [23]. The flat fading assumption might be not valid for the communication and interference channels as the communication signal bandwidth increases. If the model needs to be treated as frequency selective, then one could consider OFDM type of radar transmissions and communication signals. In that scenario, the formulation discussed above, would apply on each carrier. Phase synchronization is assumed for the radar and communication systems separately. However, the random phase jitters of the oscillators at the transmitter and the receiver PLLs may result in time-varying phase offsets between the MIMO-MC radar and the communication system [14]. We model such phase offsets in the diagonal matrix $\mathbf{\Lambda}_i, i \in \{1,2\}$, where its diagonal contains the random phase offset $e^{j\alpha_{il}}$ between the MIMO-MC radar and the communication system at the l-th symbol.

IV. The Proposed Spectrum Sharing Method

In this section, we first derive the communication rate and radar SINR in terms of communication and radar waveforms and formulate the MIMO-MC radar and MIMO communication spectrum sharing problem. In Section IV-A, an optimization algorithm is proposed using alternating optimization. We briefly discuss the spectrum sharing formulation for traditional MIMO radars in Section IV-B.

For the communication system, the covariance of interference plus noise is given by

$$\mathbf{R}_{\text{Cin}} = \mathbf{G}_1 \mathbf{\Phi} \mathbf{G}_1^H + \sigma_C^2 \mathbf{I} \tag{6}$$

where $\mathbf{\Phi} \triangleq \mathbf{P}\mathbf{P}^H/L$ is positive semidefinite. For $l \in \mathbb{N}_L^+$, the *instaneous* information rate is unknown because the interference plus noise is not necessarily Gaussian due to the random phase offset $\alpha_1(l)$. Instead, we are interested in a lower bound of the rate, which is given by [24]

$$\underline{C}(\mathbf{R}_{xl}, \mathbf{\Phi}) \triangleq \log_2 \left| \mathbf{I} + \mathbf{R}_{\text{Cin}}^{-1} \mathbf{H} \mathbf{R}_{xl} \mathbf{H}^H \right|,$$

which is achieved when the codeword $\mathbf{x}(l)$, $l \in \mathbb{N}_L^+$ is distributed as $\mathcal{CN}(0, \mathbf{R}_{xl})$. The average communication rate over L symbols is as follows:

$$C_{\text{avg}}(\{\mathbf{R}_{xl}\}, \mathbf{\Phi}) \triangleq \frac{1}{L} \sum_{l=1}^{L} \underline{C}(\mathbf{R}_{xl}, \mathbf{\Phi}), \tag{7}$$

where $\{\mathbf{R}_{xl}\}$ denotes the set of all $\mathbf{R}_{xl}$'s.

The MIMO-MC radar only partially samples $\mathbf{Y}_R$. Therefore, only the sampled target signal and sampled interference determine the matrix completion performance. Based on this observation, we define the *effective signal power* (ESP) and *effective interference power* (EIP) at the radar RX node as follows:

$$\begin{aligned} \text{ESP} &\triangleq \mathbb{E}\left\{ \text{Tr}\left(\mathbf{\Omega} \circ (\mathbf{DPS}) \left(\mathbf{\Omega} \circ (\mathbf{DPS})^H \right) \right) \right\} \\ &= pLM_{r,R} \text{Tr}\left(\mathbf{\Phi} \mathbf{D}_t \right), \\ \text{EIP} &\triangleq pLM_{r,R} \text{Tr}\left(\mathbf{\Phi} \mathbf{C}_t \right) + \sum\nolimits_{l=1}^{L} \text{Tr}\left(\mathbf{G}_{2l} \mathbf{R}_{xl} \mathbf{G}_{2l}^H \right), \end{aligned}$$

where $\mathbf{D}_t = \sum_{k=1}^{K} \sigma_{\beta_k}^2 \mathbf{v}_t^*(\theta_k) \mathbf{v}_t^T(\theta_k)$, $\mathbf{C}_t = \sum_{k=1}^{K_c} \sigma_{\beta_k^c}^2 \mathbf{v}_t^*(\theta_k^c) \mathbf{v}_t^T(\theta_k^c)$, σ_{β_k} and $\sigma_{\beta_k^c}$ denote the standard deviation of β_k and β_k^c, respectively; $\mathbf{G}_{2l} \triangleq \mathbf{\Delta}_l \mathbf{G}_2$ and $\mathbf{\Delta}_l = \text{diag}(\mathbf{\Omega}_{\cdot l})$. The derivation is omitted for brevity.

Remark 1. The sub-sampling at the radar receiver effectively modulates the interference channel $\mathbf{G}_2$ from the communication transmitter to the radar receiver. At sampling time l, only the interference at radar RX antennas corresponding to 1's in $\mathbf{\Omega}_{\cdot l}$ are sampled. Equivalently, the effective interference channel during the l-th symbol duration is $\mathbf{G}_{2l}$. Therefore, adaptive communication transmission with symbol dependent covariance matrix $\mathbf{R}_{xl}$ is used in order to match the variation of the effective interference channel $\mathbf{G}_{2l}$ [14]. The disadvantage is high computational cost. A sub-optimal alternative is constant rate communication transmission, *i.e.*, $\mathbf{R}_{xl} \equiv \mathbf{R}_x, \forall l \in \mathbb{N}_L^+$. □

Incorporating the expressions for effective target signal, interference and additive noise, the *effective* radar SINR is given as

$$\text{ESINR} = \frac{\text{Tr}\left(\mathbf{\Phi} \mathbf{D}_t \right)}{\text{Tr}\left(\mathbf{\Phi} \mathbf{C}_t \right) + \sum_{l=1}^{L} \text{Tr}\left(\mathbf{G}_{2l} \mathbf{R}_{xl} \mathbf{G}_{2l}^H \right) / (pLM_{r,R}) + \sigma_R^2}.$$

In this paper, we consider the scenario where the radar searches in particular directions of interest given by set $\{\theta_k\}$ for targets with unknown RCS variances [25], [26]. For the unknown $\{\sigma_{\beta_k}^2\}$, we instead use the worst possible target RCS variance $\{\sigma_0^2\}$, which is the smallest target RCS variance that could be detected by the radar. In practice, the prior on $\{\theta_k\}$ could be obtained in various ways. For example, in tracking applications, the target parameters obtained from previous tracking cycles are provided to focus the transmit power onto directions of interest. We assume that $\{\sigma_{\beta_k^c}^2\}$ and $\{\theta_k^c\}$ are known. In practice, these clutter parameters could be estimated when target is absent [27].

In a cooperative fashion, the radar and the communication system will jointly design the communication TX covariance matrices $\{\mathbf{R}_{xl}\}$, the radar precoder $\mathbf{P}$ (embedded in $\mathbf{\Phi}$), and the radar sub-sampling scheme $\mathbf{\Omega}$. Based on the results in [22], the radar precoder $\mathbf{P}$ can be designed without affecting the incoherence property of $\mathbf{M}$. The sub-sampling scheme also needs to be designed to ensure that the data matrix can be completed from partial samples. In matrix completion literature, $\mathbf{\Omega}$ is either a uniformly random sub-sampling matrix [20], or a matrix with a large spectral gap[1] [28]. We will design $\mathbf{\Omega}$ with fixed sub-sampling rate p and a large spectral gap.

The above stated spectrum sharing problem can be formulated as follows

$$\begin{aligned} (\mathbf{P}_1) \quad & \max_{\{\mathbf{R}_{xl}\} \succeq 0, \mathbf{\Phi} \succeq 0, \mathbf{\Omega}} \text{ESINR}\left(\{\mathbf{R}_x\}, \mathbf{\Omega}, \mathbf{\Phi} \right), \\ & \text{s.t. } C_{\text{avg}}(\{\mathbf{R}_{xl}\}, \mathbf{\Phi}) \geq C, && \text{(8a)} \\ & \sum_{l=1}^{L} \text{Tr}\left(\mathbf{R}_{xl} \right) \leq P_C, L\text{Tr}\left(\mathbf{\Phi} \right) \leq P_R, && \text{(8b)} \\ & \text{Tr}\left(\mathbf{\Phi} \mathbf{V}_k \right) \geq \xi \text{Tr}(\mathbf{\Phi}), \forall k \in \mathbb{N}_K^+, && \text{(8c)} \\ & \mathbf{\Omega} \text{ is proper}, && \text{(8d)} \end{aligned}$$

where $\mathbf{V}_k \triangleq \mathbf{v}_t^*(\theta_k) \mathbf{v}_t^T(\theta_k)$. The constraint of (8a) restricts the communication rate to be at least C, in order to support reliable communication and avoid service outage. The constraints of (8b) restrict the total communication and radar transmit power to be no larger than P_C and P_R, respectively. The constraints of (8c) restrict that the power of the radar probing signal at interested directions must be not smaller than that achieved by the uniform precoding matrix $\frac{\text{Tr}(\mathbf{\Phi})}{M_{t,R}} \mathbf{I}$, *i.e.*, $\mathbf{v}_t^T(\theta_k) \mathbf{\Phi} \mathbf{v}_t^*(\theta_k) \geq \xi \mathbf{v}_t^T(\theta_k) \frac{\text{Tr}(\mathbf{\Phi})}{M_{t,R}} \mathbf{I} \mathbf{v}_t^*(\theta_k) = \xi \text{Tr}(\mathbf{\Phi})$. $\xi \geq 1$ is a parameter used to control the beampattern at the interested target angles.

Problem $(\mathbf{P}_1)$ is non-convex w.r.t. optimization variable triple $(\{\mathbf{R}_x\}, \mathbf{\Omega}, \mathbf{\Phi})$. We propose an algorithm to find a local solution via alternating optimization in Subsection IV-A.

A. Solution to $(\mathbf{P}_1)$ Using Alternating Optimization

Next, we discuss the alternating iterations w.r.t. $\{\mathbf{R}_{xl}\}$, $\mathbf{\Omega}$, and $\mathbf{\Phi}$.

1) **The Alternating Iteration w.r.t.** $\{\mathbf{R}_{xl}\}$*:* We first solve $\{\mathbf{R}_{xl}\}$ while fixing $\mathbf{\Omega}$ and $\mathbf{\Phi}$ to be the solution from the previous iteration:

$$\begin{aligned} (\mathbf{P_R}) \quad & \min_{\{\mathbf{R}_{xl}\} \succeq 0} \sum_{l=1}^{L} \text{Tr}\left(\mathbf{G}_{2l} \mathbf{R}_{xl} \mathbf{G}_{2l}^H \right) \\ & \text{s.t. } C_{\text{avg}}(\{\mathbf{R}_{xl}\}, \mathbf{\Phi}) \geq C, \sum_{l=1}^{L} \text{Tr}\left(\mathbf{R}_{xl} \right) \leq P_C. \end{aligned} \tag{9}$$

Problem $(\mathbf{P_R})$ is convex and involves multiple matrix variables, the joint optimization with respect to which requires high computational complexity. The semidefinite matrix variables $\{\mathbf{R}_{xl}\}$ have $LM_{t,C}^2$ real scalar variables, which will results in a complexity of $\mathcal{O}((LM_{t,C}^2)^{3.5})$ if an interior-point method [29] is used. An efficient

[1]The spectral gap of a matrix is defined as the difference between the largest singular value and the second largest singular value.

algorithm for solving the above problem can be implemented based on the Lagrangian dual decomposition [29]. Please refer to [14, Algorithm 1] for the detailed solution.

2) **The Alternating Iteration w.r.t. $\mathbf{\Omega}$***:* By simple algebraic manipulation, the EIP from the communication transmission can be reformulated as

$$\sum_{l=1}^{L} \mathrm{Tr}\left(\mathbf{G}_{2l}\mathbf{R}_{xl}\mathbf{G}_{2l}^{H}\right) \equiv \mathrm{Tr}(\mathbf{\Omega}^T\mathbf{Q}),$$

where the l-th column of $\mathbf{Q}$ contains the diagonal entries of $\mathbf{G}_2\mathbf{R}_{xl}\mathbf{G}_2^H$. Recall that the sampling matrix $\mathbf{\Omega}$ is required to have large spectral gap. However, it is difficult to incorporate such conditions in the above optimization problem. Based on the fact that row and column permutation of the sampling matrix would not affect its singular values and thus the spectral gap, our prior work [14] proposed a suboptimal approach to search the best sampling scheme by permuting rows and columns of an initial sampling matrix $\mathbf{\Omega}^0$, *i.e.*,

$$\min_{\mathbf{\Omega}} \ \mathrm{Tr}(\mathbf{\Omega}^T\mathbf{Q}) \quad \text{s.t. } \mathbf{\Omega} \in \wp(\mathbf{\Omega}^0), \tag{10}$$

where $\wp(\mathbf{\Omega}^0)$ denotes the set of matrices obtained by arbitrary row and/or column permutations. The $\mathbf{\Omega}^0$ is generated with binary entries and $\lfloor pLM_{r,R} \rfloor$ ones. One good candidate for $\mathbf{\Omega}^0$ would be a uniformly random sampling matrix, as such matrix exhibit large spectral gap with high probability [28]. Multiple trials with different $\mathbf{\Omega}^0$'s can be used to further improve the choice of $\mathbf{\Omega}$. However, the search space is very large since $|\wp(\mathbf{\Omega}^0)| = \Theta(M_{r,R}!L!)$. In this paper, we propose to reduce the search space as follows

$$\min_{\mathbf{\Omega}} \ \mathrm{Tr}(\mathbf{\Omega}^T\mathbf{Q}) \quad \text{s.t. } \mathbf{\Omega} \in \wp_r(\mathbf{\Omega}^0), \tag{11}$$

where $\wp_r(\mathbf{\Omega}^0)$ denotes the set of matrices obtained by arbitrary row permutations. The search space in (11) $|\wp_r(\mathbf{\Omega}^0)| = \Theta(M_{r,R}!)$ is greatly reduced compared to that in (10). Furthermore, the following proposition shows that such reduction of search space comes without any performance loss.

Proposition 1. *For any $\mathbf{\Omega}^0$, searching for an $\mathbf{\Omega}$ in $\wp_r(\mathbf{\Omega}^0)$ can achieve the same EIP as searching in $\wp(\mathbf{\Omega}^0)$.*

The proof can be found in [**?**] and is omitted here for brevity. To formulate (11) as a linear assignment problem, we construct a cost matrix $\mathbf{C}^r \in \mathbb{R}^{M_{r,R}\times M_{r,R}}$ with $[\mathbf{C}^r]_{ml} \triangleq \mathbf{\Omega}_{m\cdot}(\mathbf{Q}_{l\cdot})^T$. The optimal solution of (11) can obtained efficiently in polynomial time $\mathcal{O}(M_{r,R}^3)$ using the Hungarian algorithm [30].

3) **The Alternating Iteration w.r.t. $\mathbf{\Phi}$***:* For the optimization of $\mathbf{\Phi}$ with fixed $\{\mathbf{R}_{xl}\}$ and $\mathbf{\Omega}$, the constraint in (8a) is nonconvex w.r.t. $\mathbf{\Phi}$. The first order Taylor expansion of $\underline{C}(\mathbf{R}_{xl}, \mathbf{\Phi})$ at $\bar{\mathbf{\Phi}}$ is given as

$$\underline{C}(\mathbf{R}_{xl}, \mathbf{\Phi}) \approx \underline{C}(\mathbf{R}_{xl}, \bar{\mathbf{\Phi}}) - \mathrm{Tr}\left[\mathbf{A}_l(\mathbf{\Phi} - \bar{\mathbf{\Phi}})\right],$$

where $\mathbf{A}_l$ is given in (12) on the top of next page.

The sequential convex programming technique is applied to solve $\mathbf{\Phi}$ by repeatedly solve the following approximate problem

$$\begin{aligned}(\mathbf{P}_{\mathbf{\Phi}}) \ \max_{\mathbf{\Phi}\succeq 0} \ & \frac{\mathrm{Tr}(\mathbf{\Phi}\mathbf{D}_t)}{\mathrm{Tr}(\mathbf{\Phi}\mathbf{C}_t)+\rho}, \\ \text{s.t. } & \mathrm{Tr}\left(\mathbf{\Phi}\right) \le P_R/L, \mathrm{Tr}\left(\mathbf{\Phi}\mathbf{A}\right) \le \tilde{C}, \\ & \mathrm{Tr}\left(\mathbf{\Phi}\mathbf{V}_k\right) \ge \xi \mathrm{Tr}\left(\mathbf{\Phi}\right), \forall k \in \mathbb{N}_K^+,\end{aligned} \tag{13}$$

where $\tilde{C} = \sum_{l=1}^{L}(\underline{C}(\mathbf{R}_{xl}, \bar{\mathbf{\Phi}}) + \mathrm{Tr}(\bar{\mathbf{\Phi}}\mathbf{A}_l) - C)$, $\mathbf{A} = \sum_{l=1}^{L}\mathbf{A}_l$, $\rho = \sum_{l=1}^{L}\mathrm{Tr}\left(\mathbf{R}_{xl}\mathbf{G}_2^H\mathbf{\Delta}_l\mathbf{G}_2\right)/(pLM_{r,R}) + \sigma_R^2$ are real positive constants w.r.t. $\mathbf{\Phi}$, and $\bar{\mathbf{\Phi}}$ is updated as the solution of the previous repeated problem. Problem (13) could be equivalently formulated as a semidefinite programming problem (SDP) via Charnes-Cooper Transformation [27]:

$$\begin{aligned}\max_{\tilde{\mathbf{\Phi}}\succeq 0, \phi>0} \ & \mathrm{Tr}(\tilde{\mathbf{\Phi}}\mathbf{D}_t), \quad \text{s.t. } \mathrm{Tr}(\tilde{\mathbf{\Phi}}\mathbf{C}_t) = 1 - \phi\rho \\ & \mathrm{Tr}\left(\tilde{\mathbf{\Phi}}\right) \le \phi P_R/L, \mathrm{Tr}\left(\tilde{\mathbf{\Phi}}\mathbf{A}\right) \le \phi\tilde{C}, \\ & \mathrm{Tr}\left(\tilde{\mathbf{\Phi}}(\mathbf{V}_k - \xi\mathbf{I})\right) \ge 0, \forall k \in \mathbb{N}_K^+.\end{aligned} \tag{14}$$

The optimal solution of (14), denoted by $(\tilde{\mathbf{\Phi}}^*, \phi^*)$, can be obtained by using any standard interior-point method based SDP solver with a complexity of $\mathcal{O}((M_{t,R}^2)^{3.5})$. The solution of (13) is given by $\tilde{\mathbf{\Phi}}^*/\phi^*$. In each alternating iteration w.r.t. $\mathbf{\Phi}$, it is required to solve several iterations of SDP due to the sequential convex programming.

It is easy to show that the objective function, *i.e.*, ESINR, is nondecreasing during the alternating iterations of $\{\mathbf{R}_{xl}\}$, $\mathbf{\Omega}$ and $\mathbf{\Phi}$, and is upper bounded. According to the monotone convergence theorem [31], the alternating optimization is guaranteed to converge.

B. *Traditional MIMO Radars*

The traditional MIMO radars without sub-sampling can be considered as special with $p = 1$, and thus there is no need for the matrix completion. In such case, the constant-rate communication transmission becomes optimal scheme because the interference channel $\mathbf{G}_2$ stays as a constant for the period of L symbol time due to the block fading assumption. The spectrum sharing problem has the same form as $(\mathbf{P}_1')$ with the objective function being

$$\mathrm{SINR} = \frac{\mathrm{Tr}\left(\mathbf{\Phi}\mathbf{D}_t\right)}{\mathrm{Tr}\left(\mathbf{\Phi}\mathbf{C}_t\right) + \mathrm{Tr}\left(\mathbf{G}_2\mathbf{R}_x\mathbf{G}_2^H\right)/M_{r,R} + \sigma_R^2}.$$

Traditional MIMO radars can achieve approximately the same spectrum sharing performance as MIMO-MC radars when the communication system transmits at a constant rate. However, for MIMO-MC radars, the adaptive communication transmission and the radar sub-sampling matrix can be designed to achieve significant radar SINR reduction over the traditional MIMO radars. This advantageous flexibility is introduced by the sparse sensing (*i.e.* sub-sampling) in MIMO-MC radars.

V. Numerical Results

Unless otherwise stated, we use the following default values for the system parameters. The MIMO radar system consists of collocated $M_{t,R} = 16$ TX and $M_{r,R} = 16$ RX antennas, respectively forming transmit and receive half-wavelength uniform linear arrays. The radar waveforms are chosen from the rows of a random orthonormal matrix [13]. We set the length of the radar waveforms to $L = 16$. The wireless communication system consists of collocated $M_{t,C} = 4$ TX and $M_{r,C} = 4$ RX antennas, respectively forming transmit and receive half-wavelength uniform linear arrays. For the communication capacity and power constraints, we take $C = 16$ bits/symbol and $P_C = 64$ (the power is normalized by the power of the radar waveform). The radar transmit power budget is $P_R = 1000 \times P_C$, which is typical in radar systems. The additive white Gaussian noise variances are $\sigma_C^2 = \sigma_R^2 = 0.01$. There are three stationary targets with RCS variance $\sigma_{\beta 0}^2 = 0.5$, located in the far-field with pathloss 10^{-3}, and clutter is generated by four point scatterers. All scatterers RCS variances are set to be identical and are denoted by σ_β^2, which is decided by the prescribed clutter to noise ratio (CNR) $10\log\sigma_\beta^2/\sigma_R^2$. The channel $\mathbf{H}$ is modeled as Rayleigh fading, *i.e.*, it contains independent entries, distributed as $\mathcal{CN}(0, 1)$. The interference channels $\mathbf{G}_1$ and $\mathbf{G}_2$ are modeled as Rician fading. The power in the direct path is 0.1, and the variance of Gaussian components contributed by the scattered paths is 10^{-3}. The performance metrics considered here include the radar *effective* SINR, the matrix completion relative recovery error, defined as $\|\mathbf{M} - \hat{\mathbf{M}}\|_F/\|\mathbf{M}\|_F$, where $\hat{\mathbf{M}}$ is the

$$\mathbf{A}_l \triangleq -\left(\frac{\partial \underline{C}(\mathbf{R}_{xl}, \mathbf{\Phi})}{\partial \Re(\mathbf{\Phi})}\right)^T_{\mathbf{\Phi}=\bar{\mathbf{\Phi}}} = \mathbf{G}_1^H[(\mathbf{G}_1\mathbf{\Phi}\mathbf{G}_1^H + \sigma_C^2\mathbf{I})^{-1} - (\mathbf{G}_1\mathbf{\Phi}\mathbf{G}_1^H + \sigma_C^2\mathbf{I} + \mathbf{H}\mathbf{R}_{xl}\mathbf{H}^H)^{-1}]\mathbf{G}_1\big|_{\mathbf{\Phi}=\bar{\mathbf{\Phi}}}. \tag{12}$$

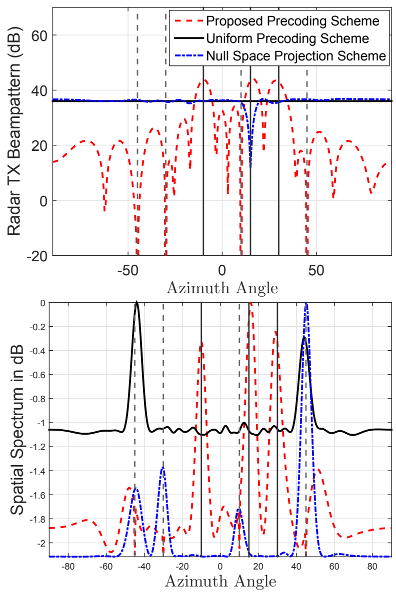

Fig. 2: Radar transmit beampattern and MUSIC spatial pseudo-spectrum for MIMO-MC radar and MIMO communication system spectrum sharing. $M_{t,R} = M_{r,R} = 16, M_{t,C} = M_{r,C} = 4$. The true positions of the targets and clutters are labeled using solid and dashed vertical lines, respectively. CNR=30 dB.

completed data matrix at the radar fusion center, and the radar transmit beampattern, *i.e.*, the transmit power for different azimuth angles $\mathbf{v}_t^T(\theta)\mathbf{P}\mathbf{v}_t^*(\theta)$. Monte Carlo simulations with 100 independent trials are carried out to get an average performance.

A. The Radar Transmit Beampattern and MUSIC Spectrum

In this subsection, we present an example to show the advantages of the proposed radar precoding scheme as compared to the trivial uniform precoding, *i.e.*, $\mathbf{P} = \sqrt{LP_R/M_{t,R}}\mathbf{I}$, and null space projection (NSP) precoding, *i.e.*, $\mathbf{P} = \sqrt{LP_R/M_{t,R}}\mathbf{V}\mathbf{V}^H$, where $\mathbf{V}$ contains the basis of the null space of $\mathbf{G}_1$ [10]. For the proposed joint-design based scheme in (8), we choose $\xi = \lfloor \xi_{\max} \rfloor$. The target angles w.r.t. the array are respectively $-10°$, $15°$, and $30°$; the four point scatterers are at angles $-45°$, $-30°$, $10°$, and $45°$. The CNR is 30 dB. In this simulation, the direct path in $\mathbf{G}_1$ is generated as $\sqrt{0.1}\mathbf{v}_t(\phi)\mathbf{v}_t^H(\phi)$, where $\phi = 15°$, with $\mathbf{v}_t(\phi)$ is defined in (2). In other words, the communication receiver is taken at the same azimuth angle as the second target.

Precoding schemes	ESINR	MC Relative Recovery Errors	Relative RCS Est. RMSE
Joint-design precoding	31.3dB	0.038	0.028
Uniform precoding	-44.3dB	1.00	1.000
NSP based precoding	-46.3dB	1.00	0.995

TABLE I: Radar ESINR, MC relative recovery errors, and relative target RCS estimation RMSE for MIMO-MC radar and MIMO communication system spectrum sharing. The simulation setting is the same as that for Fig. 2.

The radar transmit beampattern and the spatial pseudo-spectrum obtained using the MUSIC algorithm are shown in Fig. 2. The corresponding ESINR, MC relative recovery error, and relative target RCS estimation RMSE are listed in Table I. From Fig. 2, we observe that the proposed joint-design based precoding scheme successfully focuses the transmit power towards the three targets and nullifies the power towards the point scatterers. The three targets can be accurately estimated from the pseudo-spectrum obtained by the proposed scheme. As expected, the uniform precoding scheme just spreads the transmit power uniformly in all directions. The NSP precoding scheme results in a similar beampattern as the uniform precoding scheme except the deep null at the direction of the communication receiver. This means that the transmit power towards the second target is severely attenuated by the NSP precoding scheme. It is highly possible that the second target will be missed. In addition, both the uniform and NSP precoding schemes have no capability of clutter mitigation. As shown in Fig. 2 and Table I, the proposed joint-design based precoding scheme achieves significant improvement in ESINR, MC relative recovery error, and target RCS estimation accuracy.

B. MIMO-MC Radars and Traditional MIMO Radars

In this subsection, we present a simulation to show the advantages of MIMO-MC radars compared to the traditional full-sampled MIMO radars. The parameters are the same as those in the previous simulation but with fixed $\sigma^2_{G_1} = 0.3$ and $\sigma^2_{G_2} = 1$, which indicates strong mutual interference, especially the interference from the communication transmitter to the radar receiver. The radar transmit power budget P_R is taken to be equal to $10 \times P_C$. We consider two targets; one is randomly located and the other is taken to be $25°$ away. We also consider 4 randomly located point scatterers. Fig. 3 shows the results under different MIMO-MC sub-sampling rates p. Note that full sampling is used for the traditional MIMO radar. The MC relative recover error for the traditional radar is actually the output distortion-to-signal ratio. A smaller distortion-to-signal ratio corresponds to a larger output SNR. For ease of comparison, a black dashed line is used for the traditional MIMO radar. We observe that the MIMO-MC radar achieves better performance in ESINR than the traditional radar. This is due to the fact that the communication system can effectively prevent its transmission from interfering the radar system when the number of actively sampled radar RX antennas is small, *i.e.*, sub-sampling is small. In addition, the larger ESINR of the MIMO-MC radar results in a larger output SNR than that of the traditional radar. Furthermore, the MIMO-MC radar achieves better target RCS estimation accuracy than the traditional radar if its sub-sampling rate is between 0.4 and 0.7. For p larger than 0.7, the target RCS estimation accuracy achieved by the MIMO-MC radar is worse than that achieved by the traditional radar because small ESINRs for $p \geq 0.7$ introduce high distortion in the completed data matrix. We conclude that MIMO-MC radars can coexist with communication systems and achieve better target RCS estimation than traditional radars while saving up to 60% data samples. Such significant advantage is introduced by the sparse sensing (*i.e.* sub-sampling) in MIMO-MC radars as discussed in Section IV-B.

VI. Conclusions

In this paper, we have considered the co-existence of a MIMO-MC radar and a wireless MIMO communication system by sharing a common carrier frequency. The radar transmit precoder, the radar sub-sampling scheme, and the communication transmit covariance matrix

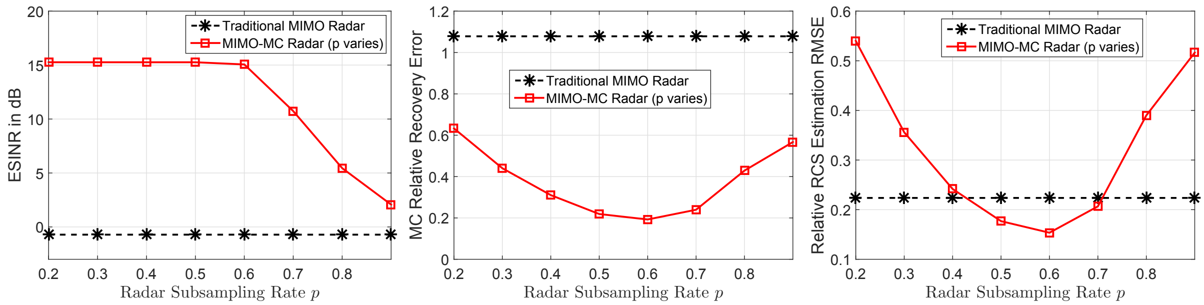

Fig. 3: Comparison of spectrum sharing with traditional MIMO radars and MIMO-MC radars with different subsampling rates p. $M_{t,R} = 16, M_{r,R} = M_{t,C} = 8, M_{r,C} = 2$.

have been jointly designed to maximize the radar SINR while meeting certain rate and power constraints for the communication system. We have observed that the MIMO-MC radar achieves better performance in ESINR and output SNR than the traditional radar. MIMO-MC radars can coexist with communication systems and achieve better target RCS estimation than traditional radars while saving up to 60% in data samples. The cost for these advantages is the additional computation for matrix completion. 以增加计算量为代价

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