Signal Tracking Approach for Simultaneous Estimation of Phase and Instantaneous Frequency

Rahul G Waghmare*, Sai Subrahmanyam G R K †,Deepak Mishra*
*Dept. of Avionics IIST, Thiruvananthapuram Kerala, India - 695547
Email: deepak.mishra@iist.ac.in
†Dept. of Earth & space Science, IIST, Thiruvananthapuram Kerala, India - 695547

***Abstract*—Phase estimation plays an important role in various signal processing areas like Radar, Sonar, power systems, speech analysis, communications and many others. The phase of the analytic form of the non stationary signals can be used to find instantaneous frequency. This paper addresses the problem of simultaneous phase and instantaneous frequency estimation from polynomial phase signals embedded in Gaussian noise. Here we have introduced the modified signal tracking approach which is then realized using unscented Kalman filter. The state space model is derived using Taylor series expansion of the phase of polynomial phase signal as process model while Polar to Cartesian conversion as measurement model. Proposed method, compared with state-of-the-art, performs better for signals with higher order polynomial phase variations at lower Signal-to-Noise-Ratio (0-5dB). We also present the simulation results for phase estimation.**

Keywords*—Non-stationary phase signals, Signal tracking approach, Phase estimation, Kalman filters

I. Introduction

Non-stationary signals are common in many signal processing areas like Radar, Sonar [1], communications [2], speech analysis [3], power system [4]. Classical spectral analysis methods are not appropriate for analysis of such signals. Therefore it becomes important to develop technique that can provide a platform for analysis of these signals.

Non-stationary signals are first converted into it's analytic form using Hilbert transform filter. The phase of these signals are then modelled as polynomial approximation of appropriate order according to Weirstrass theorem [5]. These signals are called as Polynomial Phase Signals (PPS). The estimation of parameters of these signals in noisy environment has gained considerable interest in signal processing studies.

The phase estimation approach from complex signal can be divided into two categories, (1) Parametrize the time-variations and use parameter estimation algorithm directly, (2) use tracking algorithms to track the signal for estimating the relevant parameters [6]. Former approach is realized using various methods such as Maximum Likelihood Estimation (MLE), High-order Ambiguity function (HAF) [7], Cubic polynomial phase function (CPF) [8], Discrete Chirp-Fourier Transform (DCFT) [9], Discrete Polynomial Phase Transform (DPT) [10], [11] and others [12], [13]. Whereas later approach is also discussed extensively in literature like Kalman-Tretter Filter [14], Kalman filter for chirp signal parameter estimation [15], robust EKF [16], [17]. These filters are optimal as long as the signal is either linear or quadratic, but are poor when the phase distribution requires higher order polynomials for approximation and when the signal is noisy. All these methods are based on parameter estimation approach, which models the non-stationary signals as PPS. This approach works efficiently when the degree of the polynomials used for phase approximation is below 4. If polynomials of higher degrees are used for approximation of phase, small error in coefficients of the higher order terms gives larger estimation error, making this approach unreliable.

The purpose of this paper is two fold. First is to propose an efficient phase estimation approach from one dimensional complex phase modulated signal through UKF based parameter estimation or signal tracking, similar to [18], and compare the performance of the proposed approach with the state-of-the-art. Second is to extend the state vector to include higher order derivatives so that the same approach can be used for simultaneous estimation of phase and its derivatives without additional computational requirement. We note that the state model used in [18], which uses only 2 terms of the Taylor series, can not be efficiently used for phase derivative estimation/ instantaneous frequency estimation as there is no support for derivative terms of the state vector in the state space model. Through simulation study it is found that the proposed approach gives better performance when the phase is having larger dynamic range and when it is severely corrupted by noise (esp. for SNR values of 0-5 dB) where classical parameter estimation approach fails to estimate the phase.

We used unscented Kalman filter as a signal tracking algorithms. As Kalman filters are known for their tuning issues, it is observed that the problem of tuning of the parameters matrices Q and R can be brought down to the tuning of scalars by first choosing appropriate matrices and then by multiplying them with scalars as a tuning parameters (K_R & K_Q). The rational for defining Q and R given in section III.

The outline of the paper is as follows. In section II, we describe the state space model derivation and theoretical foundation for phase estimation. Section III gives the simulation study demonstrating performance of the proposed method and finally we conclude paper in section IV.

II. Theory

A. State Space Model

A non-stationary signal corrupted by Additive White Gaussian Noise (AWGN) in its analytic form given as:

$$\mathbf{f}(n) = a(n)e^{i\phi(n)} + \eta(n) \tag{1}$$

where $a(n)$ and $\phi(n)$ are the amplitude and the phase of the PPS and $\eta(n)$ represents the AWGN noise.

As phase of the PPS signals are modelled as polynomial of higher degree, we can write the Taylor series expansion of the phase function $\phi(n)$ in equation 1 as:

$$\phi(n+1) = \phi(n) + \frac{1}{1!}\phi^{(1)}(n) + .. + \frac{1}{M!}\phi^{(M)}(n) + w(n) \quad (2)$$

Here, $w(n)$ represents unmodelled process noise along with higher order terms of Taylor series. The amplitude is modelled as random walk so that we can write,

$$a(n+1) = a(n) + w_a(n) \quad (3)$$

Now, using equation 3 we can write the set of equations in matrix form as:

$$\mathbf{x}(n+1) = \mathbf{F}\mathbf{x}(n) + \mathbf{w}(n) \quad (4)$$

where,

$$\begin{aligned}
\mathbf{x}(n) &= \begin{bmatrix} a(n) & \phi(n) & \phi^{(1)}(n) & \cdots & \phi^{(M)}(n) \end{bmatrix}^T \\
\mathbf{F} &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{M!} \\ 0 & 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(M-1)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \\
\mathbf{w}(n) &= \begin{bmatrix} w_a(n) & w_0(n) & w_1(n) & \cdots & w_M(n) \end{bmatrix}^T
\end{aligned}$$

Here, the vector $\mathbf{x}(n)$ denotes the state vector consisting of phase and its derivative terms. State vector at $n+1$ instant can be predicted using state transition matrix $\mathbf{F}$. $\mathbf{w}(n)$ is the unmodelled process noise.

Measurement signal can be predicted using the present state vector using the relation:

$$\Re[\mathbf{f}(n)] = a(n)cos(\phi(n)) \quad (5)$$
$$\Im[\mathbf{f}(n)] = a(n)sin(\phi(n)) \quad (6)$$

Thus the measurement prediction function $\mathbf{h}(\cdot)$ can be written as:

$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix} x_1cos(x_2) \\ x_1sin(x_2) \end{bmatrix} \quad (7)$$

The i^{th} element of the state vector $\mathbf{x}$ is denoted byx_i. We assume the measurement (observation) noise to be additive white Gaussian noise with zero mean and variance of σ_η^2. The final observation model becomes

$$\mathbf{z}(n) = \mathbf{h}(\mathbf{x}(n)) + \eta(n) \quad (8)$$

B. Unscented Kalman Filter

UKF uses a deterministic sampling technique known as the unscented transform to pick a minimal set of sample points (called sigma points) around the mean such that these points capture mean and covariance of a prior random variable exactly, while approximating the mean and covariance of the transformed random variable up to the third order in Taylor series [19]. Consider, propagation of a random variable $\mathbf{x}$ (dimension L), a state vector in our case, having mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P_x}$, through a non-linear function $\mathbf{y} = \mathbf{h}(\mathbf{x})$. To calculate the statistics of $\mathbf{y}$, we form a matrix of sigma points as follows:

$$\boldsymbol{\chi}_0 = \bar{\mathbf{x}} \quad (9)$$
$$\boldsymbol{\chi}_i = \bar{\mathbf{x}} + (\sqrt{(L+\lambda)\mathbf{P_x}})_i, \forall i = 1, \ldots, L \quad (10)$$
$$\boldsymbol{\chi}_i = \bar{\mathbf{x}} - (\sqrt{(L+\lambda)\mathbf{P_x}})_i, \forall i = L+1, \ldots, 2L \quad (11)$$
$$\mathbf{W}_0^{(m)} = \frac{\lambda}{(L+\lambda)} \quad (12)$$
$$\mathbf{W}_0^{(c)} = \frac{\lambda}{(L+\lambda)} + (1 - \alpha^2 + \beta) \quad (13)$$
$$\mathbf{W}_i^{(m)} = \mathbf{W}_i^{(c)} = \frac{1}{2(L+\lambda)}, \forall i = 1, \ldots, 2L \quad (14)$$

where, $\boldsymbol{\chi}_i$ are the Sigma Points, $\mathbf{W}_i^{(m)}$ and $\mathbf{W}_i^{(c)}$ are the weights to calculate mean and covariance respectively. $\lambda = \alpha^2(L+k) - L$, is a scaling factor, α determines the spread of the sigma points around $\mathbf{x}$ and is usually set to a small positive value. β is used to incorporate the prior knowledge about distribution of $\mathbf{x}$. $(\sqrt{(L+\lambda)P_x})_i$ is the i^{th} row of the matrix square root.

These sigma vectors are then passed through transformation function to get

$$\boldsymbol{\gamma}_i = h(\boldsymbol{\chi}_i), \forall i = 0, \ldots, 2L \quad (15)$$

Corresponding mean and covariance of the posterior random variable $\mathbf{y}$ are approximated using above sigma points as

$$\bar{\mathbf{y}} = \sum_{i=0}^{2L} W_i^{(m)} \boldsymbol{\gamma}_i \quad (16)$$
$$\mathbf{P_y} = \sum_{i=0}^{2L} W_i^{(c)} [(\gamma_i - \bar{\mathbf{y}})(\gamma_i - \bar{\mathbf{y}})^T] \quad (17)$$

As the non-linearity involved in the state-space model is the Polar to Cartesian conversion (i.e., converting the complex field from $re^{i\phi}$ to $a+ib$), as shown by Julier *et. al.* [19], the unscented transform performs better than linearization and close to the Monte Carlo method. This provides motivation for considering and proposing a Unscented Kalman Filter (UKF) framework rather than to extend the EKF based approach [16].

The UKF is a straightforward extension of the UT to the recursive estimation of Equations 4 and 8, where the augmented state vector is redefined as the concatenation of the original state and noise variables. The UT sigma point selection scheme (Equations 9 through 14) is applied to this new augmented state RV to calculate the corresponding sigma points. Then the state and noise sigma points are instantiated through state and observation equations, while the statistics for the KF equations are determined using corresponding sigma points. Finally the Kalman update is employed.

Note that, unlike EKF, no explicit calculation of Jacobian or Hessian are necessary to implement this algorithm. Furthermore, the overall number of computations are the same order as that of EKF. Through simulation study, it is found that with proper initialization of the algorithm, we can explicitly calculate the unwrapped phase, instantaneous frequency and even the higher derivatives of the phase from the state vector. Also fine tuning of the parameters P_0 -the initial error co-variance, Q -the process noise co-variance, and R -the measurement noise co-variance matrix is required for this algorithm. We can also adopt polynomial approximation method proposed in [15], [16] for this algorithm, in such cases the same filter can be used for parameter estimation, but such formulation requires extra computational complexity of estimating parameters and then fitting the polynomial using those parameters. We have employed UKF for parameter estimation also and demonstrated its efficacy for both direct signal estimation and parameter estimation followed by phase polynomial approximation in the section III.

III. Simulation Results

To verify the proposed UKF based method, complex signal was simulated using MATLAB on Windows 7 operating system based workstation with intel(R) Xeon(R) CPU at 3.20GHz with 16GB of primary memory. It is found that the proposed algorithm takes almost same time as that of the Kalman filter based approaches. It takes 0.17 seconds to process the 1024 samples of the polynomial phase signal.

A. Quadratic Phase

To compare the performance of UKF based algorithm with robust extended Kalman filter (EKF) algorithm presented by Janos Gal *et al*. [15], [16], we used the phase parameters as depicted in [16]. The coefficients of the quadratic polynomial phase are: $a_0 = \pi/2, a_1 = 0.0785, a_2 = 1.309 \times 10^{-3}$. The polynomial phase signal is then constructed using

$$P(n) = a_0 + a_1 n + a_2 n^2 \quad n \in [1, 1024] \tag{18}$$

This phase signal was used to generate the test signal using the equation 1. The noise was added to the complex signal using MATLAB's *awgn* function at various SNR values. The performance was compared at SNR values from 0 to 20dB.

The relative error in the estimation of parameters of phase polynomial against SNR values are depicted in figures 1 and RMSE in phase estimation with respect to SNR values are presented in figure 2. From the figure 1 and 2, it is clear that both methods perform similar for quadratic phase, rather it outperforms for lower SNR values (0-5 dB). But when we

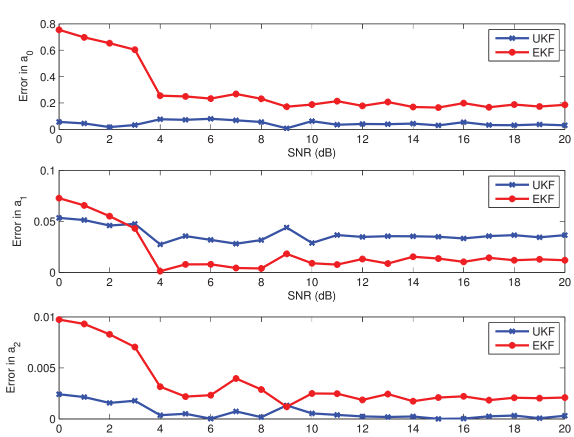

Fig. 1. Error in parameter Estimation

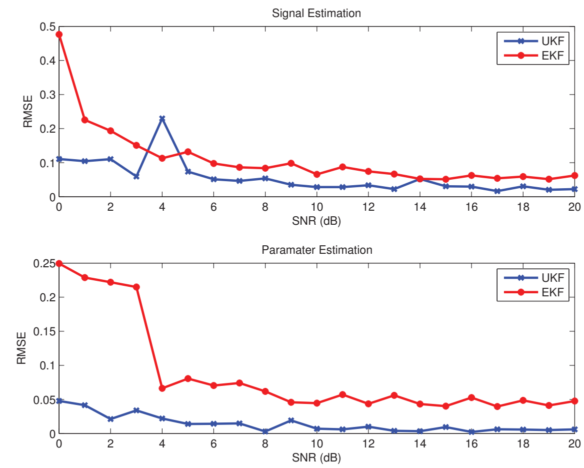

Fig. 2. RMSE in phase estimation against SNR(dB)

compare these methods for phase signals with higher dynamic distribution of phase which are difficult to approximate by polynomials, the EKF based methods cannot estimate phase accurately.

As shown in the section II, UKF based method can directly estimate the phase signal as compared to parameter estimation, hence polynomial approximation of the phase signal is not required. Besides we can use polynomial approximation to find the coefficients of polynomial using the relation given by J. Gal et al. [15], [16].

$$\boldsymbol{\theta}(n) = \mathbf{C}\mathbf{F}^{-n}\mathbf{x}(n) \tag{19}$$

where, the vector $\boldsymbol{\theta}(n)$=$[\mathbf{A}(n), a_2, a_1, a_0]^T$ contains the amplitude $(\mathbf{A}(n))$ and the coefficients of phase polynomial (a_2, a_1, a_0) of polynomial phase signal, matrix $\mathbf{C}$ is a diagonal with elements 1, 1, 1, 0.5. Also as higher order terms of the Taylor series expansion are considered as a process noise, higher order polynomial phase signals can be represented by proper selection of the process covariance matrix.

B. Phase with higher dynamic range

To simulate the Phase with larger dynamic range and more variation which can be approximated by higher order polynomials, we used *peak* function of MATLAB. The peak signal of size 1024×1024 was generated and arbitrary row was selected as a phase function. The true phase and its derivatives generated by these steps are shown in figure 3

Real part of simulated signal represents the chirp signal. Figure 4 shows the chirp signal simulated using the phase shown in figure 3, at SNR value of 10dB. In practice, the complex signal can be generated from real valued chirp signal by using Hilbert transform.

To estimate the phase signal from the complex chirp signal, whole signal was divided into segments of 16 samples each and then parameter estimation algorithm was applied to each segments. Third order polynomial was used in EKF based algorithm. Parameters of polynomials were extracted for every segment and the whole signal was reconstructed using those parameters.

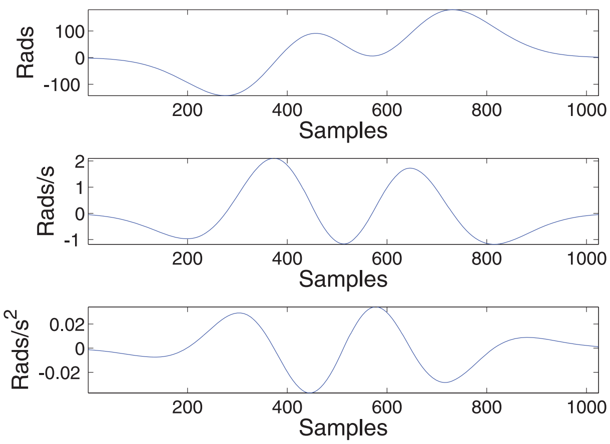

Fig. 3. True phase and its derivatives simulated using MATLAB

We used UKF as a signal tracking algorithm. Formulation proposed in theory section was kept intact for UKF based approach. The process noise covariance ($\mathbf{Q}$) is taken to be $k_Q\ diag\left[10^{-2} \quad 10^{-2} \quad 10^{-4} \quad \cdots \quad 10^{-(2M+2)}\right]$. The rational behind choice of $\mathbf{Q}$ is that the range of m^{th} derivative of phase is generally larger by magnitude of order 1 or 2 than that of $(m{+}1)^{th}$ derivative. The measurement noise covariance matrix is taken to be $\mathbf{R} = k_R\ diag\left(\left[\sigma_{re}^2 \quad \sigma_{im}^2\right]\right)$ where, σ_{re}^2 and σ_{im}^2 are the variances estimated form uniform region of the fringe pattern. This reduces the parameter tuning problem to just two scalar tuning parameters, namely, k_Q & k_R. For initialization of UKF, we used $arctan$ of the first sample of the signal. In signal estimation, the values of phase from state vector are stored directly. In parameter estimation, parameters (coefficients of polynomials) are generated for each segment using state vectors and then the phase polynomial segments are constructed using these parameters.

Error in Phase, first and second derivatives of the phase of the simulated signal are shown in figure 5.

The RMSE for whole signal reconstruction with respect to different SNR values are depicted in figure 6. From figure 6,

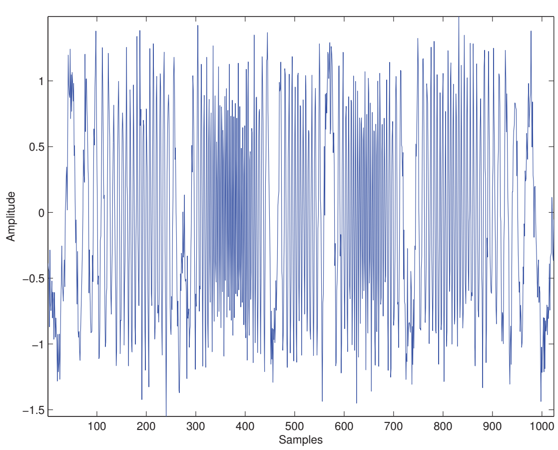

Fig. 4. Real Part of Complex Chirp Signal at SNR of 10dB

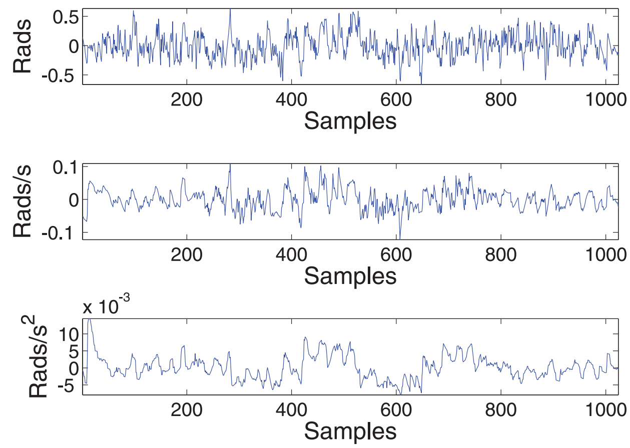

Fig. 5. Error in Estimation of Phase and derivatives at larger dynamic range at 5 dB using proposed signal tracking approach

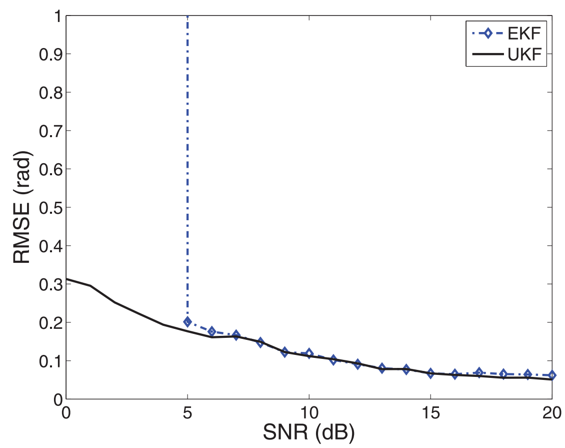

Fig. 6. RMSE in phase estimation using EKF as parameter estimator [16] and UKF as signal estimator against different SNR values.

it is clear that the EKF based method diverges as SNR value falls below 5dB, owing to linearisation used in measurement updates equation. UKF based method still works upto SNR value of 0dB.

IV. Conclusion

The paper proposes a novel approach for simultaneous estimation of phase, instantaneous frequency and even higher derivatives from the non-stationary phase signal. A combination of signal tracking approach with UKF used in this paper extends the performance of parameter estimation approach based on Kalman filters from SNR of 5 dB to 0 dB. This method provides good platform for higher order polynomial phase signal estimation as a signal estimator as well as parameter estimator.

The major issue in implementation of Kalman filter is initialization and tuning of parameter. We have given intuitive ideas about initialization of parameter tuning in the manuscript.

Since the UKF still is a Gaussian approximation, whereas nonlinear observation leads to non-Gaussian posterior and we plan to use Particle Filter to tackle it efficiently. We are looking after the potential applications of the same method for multiple component estimation from the given chirp signal.

ACKNOWLEDGEMENT

This work is funded by IIST/ISRO Fellowship by Department of Space, Govt. of India. Authors would like to thank Dr. Sai Siva Gorthi, Department of Instrumentation & Applied Physics, Indian Institute of Science, Bangalore, India 560012, for his invaluable support and help in the development of this work.

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