Practical Normalization methods in the Digital Computation of the Fractional Fourier Transform

Zhao Xinghao, Tao Ran, Deng Bing
(Dept. of Electronics Engineering, Information Science and Technology,
Beijing Institute of Technology, Beijing, 100081)
Email: zhaoxinghao03@sina.com

Abstract: Two practical Normalization methods are presented in the digital computation of the fractional Fourier transform (FRFT). One is called as discrete scaling method. The other is called as data zero-padding/interception method. Furthermore, the paper analyses their effect on parameter estimation of the chirp signal. Finally, these methods are verified by the simulation examples. The paper solves an important problem in the practical application of the FRFT digital computation, and makes the FRFT more practical in digital signal processing.

Key words: Fractional Fourier Transform; Dimensional Normalization; Parameter Estimation of Chirp signal.

Fractional Fourier transform (FRFT) has become a powerful tool for time varying signal analysis. Since a chirp signal has the best concentration in certain fractional Fourier domain, the FRFT can be used to detect the chirp-like signal and estimate the parameters [1].

From the definition[2], we can know that the FRFT expression is much more complicated than the classical FT. Fast digital computation of the FRFT is required urgently in its actual application. People have been doing research with great enthusiasm in fast digital computation methods of the FRFT[3][4][5][6]. The literature [3] presents an algorithm based on the expression decomposition, which decomposes the FRFT into a succession of simple operation. This algorithm is well known as the fastest digital computation method of the FRFT for now, which almost has the same computation speed as the FFT. However, it must be particularly noticed that computation mechanism of the algorithm determines that the original signal must be normalized before the FRFT digital computation. The normalization method mentioned in the literature [3] is just a theoretical method toward the abstract continuous time signal, which is not practical in engineering. Since the signal we can get in practice is usually discrete signal sampled at certain sample rate, so, if we want to apply the FRFT fast algorithm to practical engineering computation, the problem of discrete signal normalization must be solved.

This paper specializes in this problem and presents two practical normalization methods: One is called discrete scaling method, the other is called data zero-padding/interception method, and analyses the effect of two methods on chirp signal parameter estimation. The paper solves a practical problem on FRFT fast digital computation, which makes FRFT more practical.

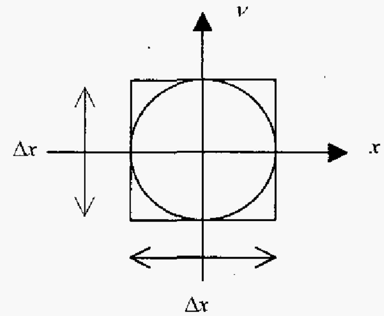

Fig.1 Circular support of normalized signal in the time-frequency space

1. Basic principle of normalization

The original signal $f(t)$ must be normalized before FRFT digital computation. Firstly, let's have a general introduction of the meaning of normalization as given in the literature [3]. Let's assume that, the original continuous signal is compact along the axis of time and frequency, the time domain representation of the signal is confined to the interval $[-\Delta t/2, \Delta t/2]$ and its frequency domain representation is confined to the interval $[-\Delta f/2, \Delta f/2]$. We then define the time-bandwidth product $N \equiv \Delta t \Delta f$, which is always greater than unity because of the uncertainty relation. Let us now introduce the scaling parameter S with the dimension of time and define new scaled coordinates:

$$x = t/S\ , \quad \nu = f \cdot S \qquad (1)$$

The new coordinates (x, ν) are clearly dimensionless. The signal is confined to the interval $[-\Delta t/(2S), \Delta t/(2S)]$ and $[-\Delta f \cdot S/2, \Delta f \cdot S/2]$ respectively under the new coordinates. In order to have the two intervals equal to each other, we choose $S = \sqrt{\Delta t/\Delta f}$ so that the length of both intervals are equal to the dimensionless quantity $\Delta x = \sqrt{\Delta t \Delta f}$. Each interval is normalized as $[-\Delta x/2, \Delta x/2]$. The Wigner distribution of normalized signal is confined within a circle of diameter Δx as shown in Fig.1. Finally, we can sample the normalized signal according to the sampling theorem, and get

$N = \Delta x^2$ samples with sampling interval $1/\Delta x$.

2. Two practical normalization methods

In practical application, what we can get is a group of discrete observation data sampled from the original continuous signal. The observation time T and the sampling frequency f_s are known values. How to normalize such discrete data is a key link between FRFT fast algorithm and practical application. This paper presents two practical normalization methods: one is called discrete scaling method, the other is called data zero-padding/interception method. A detail introduction of the two methods is given below.

2.1 Discrete scaling method

The key of discrete data normalization by scaling transform is to select proper time-width Δt, bandwidth Δf, scaling factor S and normalized width Δx, so that the scaled discrete data equal to samples of the normalized continuous signal at a sampling interval of $1/\Delta x$. It is easy to ascertain time-width as observation time T, i.e., $\Delta t = T$. We take the midpoint of signal as time origin. The time domain representation of signal is confined within the range $[-T/2, T/2]$. We do not know the exact value of signal bandwidth, however, we can know the signal's sampling frequency f_s in practice. According to the sampling theorem, the sampling frequency must be more than twice as much as the highest signal frequency. It is unnecessary that bandwidth Δf must select the lowest value, as long as signal energy is confined to the interval. It is reasonable that we have sampling frequency as the bandwidth, i.e., $\Delta f = f_s$. The signal's frequency domain is confined within the range of $[-f_s/2, f_s/2]$. By known time-width and bandwidth, we can deduce the scaling factor S and normalized width Δx as follow

$$S = \sqrt{\Delta t/\Delta f} = \sqrt{T/f_s} \tag{2}$$

$$\Delta x = \sqrt{\Delta t \Delta f} = \sqrt{T \cdot f_s} \tag{3}$$

The sampling interval of the original discrete data is $T_s = 1/f_s$. After being scaled by formula (1), the sampling interval is

$$T_s' = 1/\sqrt{T \cdot f_s} = 1/\Delta x \tag{4}$$

The former time domain ranges within $[-T/2, T/2]$. After being scaled, the range changes to $[-\Delta x/2, \Delta x/2]$. Therefore, discrete scaling method is a scaling process to the original discrete data by formula (1) and formula (2) with sampling frequency as bandwidth and observation time as time-width. Also, the scaled discrete data equal to samples of the normalized continuous signal at a sampling interval of $1/\Delta x$.

2.2 Data zero-padding/interception method

By discrete scaling method, the discrete data can be normalized through its scaling along time domain. The scaling of signal will surely lead to deformity of the signal characteristic. For example, the scaling of a chirp signal will change its chirp rate. Data zero-padding/interception method can normalize the original signal without deforming it. The key is also to select proper time-width Δt, bandwidth Δf, scaling factor S and normalized width Δx. Firstly, we set the data midpoint as the time origin. In order to avoid the distortion of the signal, we had to set scaling factor $S = 1$. We have observation time as the time-width, i.e., $\Delta t = T$, and sampling frequency as bandwidth, i.e., $\Delta f = f_s$. As for normalized width Δx, there are two circumstances as follow.

(1) If the bandwidth is bigger than time-width ($f_s > T$), we select the bigger one as Δx, i.e., $\Delta x = f_s$. The sampling interval of the original data is $1/f_s$, and its time ranges between $[-T/2, T/2]$. After normalization, the sampling interval is required to be still $1/f_s$, and its time range increases to $[-f_s/2, f_s/2]$. Therefore, the time-width of the data can be increased artificially by padding zeros in the interval $[-f_s/2, -T/2]$ and $[f_s/2, T/2]$ with the same sampling interval, so as to realize the normalization of time-width and bandwidth. This is the principle of data zero-padding method.

(2) If the time-width is bigger than the bandwidth ($T > f_s$), we select the smaller one as Δx, i.e., $\Delta x = f_s$. The sampling interval of the original data is $1/f_s$, and its time ranges between $[-T/2, T/2]$. After normalization, the sampling interval is required to be still $1/f_s$, and its time range decreases to $[-f_s/2, f_s/2]$. Therefore, we need to intercept the original data, selecting only the data within the range between $[-f_s/2, f_s/2]$, so as to realize the normalization of time-width and bandwidth. This is the principle of data interception method.

3. The effect of normalization methods on chirp signal parameters estimation

Firstly, I'd like to introduce the chirp signal parameter estimation based on FRFT. As we know, a chirp signal with given chirp rate have the best energy concentration on the corresponding fractional domain. With this characteristic, chirp signal can be detected and estimated by the FRFT. The detail procedure is that, firstly, make FRFT of the signal in every fractional order $p \in [0,2]$ respectively, and obtain the two dimensional distribution of signal energy on the parameters plane (p,u) consisting of fractional domain u and fractional order p, then chirp signal can be detected by two dimensional search of peak on the plane (p,u) with a preset threshold, meanwhile, fractional order $\hat{p}_0$ and fractional domain $\hat{u}_0$ corresponding to the peak can be estimated. If the observation model of chirp signal with noise is expressed as follow:

$$f(t)=a_0\exp\left(j\phi_0+j2\pi f_0 t+\pi\mu_0 t^2\right)+W(t)$$
$$-T/2\le t\le T/2 \qquad (5)$$

where $W(t)$ is Gaussian white noise. Then there is a relationship between chirp rate $\hat{\mu}_0$ and central frequency $\hat{f}_0$ of the chirp signal and fractional order $\hat{p}_0$ and fractional domain $\hat{u}_0$ corresponding to the peak as follow:

$$\begin{cases}\hat{\mu}_0=-\cot\left(\hat{p}_0\pi/2\right)\\ \hat{f}_0=\hat{u}_0\csc\left(\hat{p}_0\pi/2\right)\end{cases} \qquad (6)$$

By formula (6), we can get chirp rate $\hat{\mu}_0$ and central frequency $\hat{f}_0$ of the chirp signal.

When estimating chirp signal based on the FRFT, many technicians make the FRFT digital computation to the discrete observation data directly. However, they find that, although the chirp signal can be detected, the chirp rate and central frequency estimated are not the same as the theoretic values. This is because the effect of normalization on chirp parameter estimation is ignored. As we know, original observation data must be normalized before FRFT digital computation. If we make FRFT digital computation directly to original observation data, this corresponds to original data having been normalized through discrete scaling method. After being normalized, the chirp signal parameters are bound to change. The chirp rate and central frequency estimated are normalized parameters, instead of the actual parameters. This is the reason why there is a difference between calculated value and theoretic value. Therefore, we must calculate the actual chirp rate and central frequency according to the relationship between parameters before and after normalization.

Now let's deduce the relationship between chirp signal parameters before and after normalization. Let's suppose that the chirp rate and central frequency before normalization is μ_0 and f_0 respectively, and the chirp rate and central frequency after normalization is μ_0' and f_0' respectively. By formula (1) and (2), we can have:

$$\begin{cases}\mu_0'=\dfrac{v}{x}=\dfrac{f\bullet S}{t/S}=\mu_0\bullet S^2=\mu_0 T/f_s\\ f_0'=f_0\bullet S=f_0\sqrt{T/f_s}\end{cases} \qquad (7)$$

So, the relationship between chirp signal parameters before and after normalization is

$$\begin{cases}\mu_0=\mu_0' f_s/T\\ f_0=f_0'\sqrt{f_s/T}\end{cases} \qquad (8)$$

By discrete scaling method, the original signal deforms after being normalized. While by data zero-padding/interception method, no deformity will occur during normalization. If the observation signal includes chirp signal, its parameter before normalization equals to that after normalization. Thus, the chirp signal parameters estimated is rightly the original chirp parameters.

In conclusion, for chirp signal parameter estimation, each of the above two normalization methods has its strong point. Data zero-padding/interception method is simple in normalization without coordinate scaling transform. Meanwhile, chirp signal will not deform during normalization. The parameter value worked out by this method is just the actual parameter. However, only when there is no too much difference between T and f_s is it proper to adopt data zero-padding/interception method. This is because if f_s is much bigger than T, a lot of zeros had to be padded for normalization, which is a great load to FRFT computation. On the contrary, when T is much bigger than f_s, much data has to be intercepted, which surely causes loss of accuracy. Therefore, only when there is no much difference between T and f_s, is it simple and convenient adopting data zero- padding/interception method. As for discrete scaling method, we can make the FRFT digital computation directly to original data. No matter what T and f_s is, computation load only depends on the data length. However, the chirp rate and central frequency worked out are not the true value, which had to be calculated according to the relationship between parameters before and after normalization. So, discrete scaling method can be said to be an efficient normalization method when there is much difference between T and f_s.

4. Simulations

In this section, two examples are given to show the realization approach of two normalization methods and their effect to chirp parameter estimation. The first example is a group of discrete observation data with observation time $T=2s$ and sampling frequency $f_s=800Hz$, which includes a chirp signal with chirp rate $\mu_0=100Hz/s$ and central frequency $f_0=100\text{Hz}$ and SNR=0dB. Because f_s is much bigger than T, we should use discrete scaling method as follows. We take the central point of observation data as the time origin. The time interval is $[-1,1]s$. The frequency interval is $[-400,400]Hz$. So, the scaling factor $S=0.05s$. The dimensionless interval after normalization is $[-20,20]$. According to the formula (7), the chirp rate after normalization is $\mu_0'=0.25$, the central frequency after normalization is $f_0'=5$. We make FRFT of the normalized data in every fractional order $p\in[1.0,1.4]$ with sampling interval $\Delta p=0.01$ respectively, and plot the simulation results of the chirp signal detection as Fig.2.

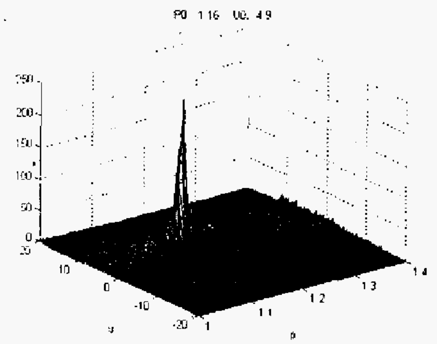

Fig .2 chirp signal detection result by discrete scaling method

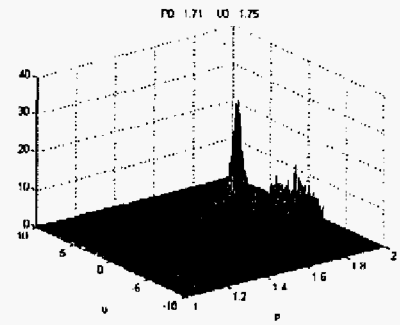

Fig.3 chirp signal detection result by data zeros-padding/interception method

In Fig.2, two-dimensional plane coordinates of the peak are $\hat{p}_0 = 1.16$ and $\hat{u}_0 = 4.9$. From the formula (6), we can work out the chirp rate $\hat{\mu}_0' = 0.2568$ and the central frequency $\hat{f}_0' = 5.0589$. This is the parameter estimation result of chirp signal after scaling normalization. Finally, according to formula (8), we can calculate the actual chirp parameters before normalization: $\hat{\mu}_0 = 102.72Hz/s$ and $\hat{f}_0 = 101.178$.

The second example is a group of discrete observation data with observation time $T = 4s$ and sampling frequency $f_s = 20Hz$, which includes a chirp signal with chirp rate $\mu_0 = 2Hz/s$ and central frequency $f_0 = 4Hz$ and SNR=0dB. Although f_s is bigger than T, there is no much difference. So, we should use data zeros-padding/interception method as follow: let's choose that the time interval is $[-2,2]s$, the time width $\Delta t = 4s$, the bandwidth $\Delta f = 20Hz$, scaling factor $S = 1s$, normalized width $\Delta x = 20$. The original time interval is $[-2,2]s$. After normalization, the time interval should increase to $[-10,10]$. The increased part $[-10,-2]$ and $[2,10]$ must be padded with zeros at the sampling interval $1/\Delta x = 0.05$. We make FRFT of the zero-padded data in every fractional order $p \in [1,2]$ with sampling interval $\Delta p = 0.01$ respectively, and plot the simulation results as shown in Fig.3.

In Fig.3, two-dimensional plane coordinate of the peak are $\hat{p}_0 = 1.71$ and $\hat{u}_0 = 1.75$. From the formula (6), we can work out the chirp rate $\hat{\mu}_0 = 2.0413$ and the central frequency $\hat{f}_0 = 3.9778$. We can see that the parameter value calculated by the zero-padding method is rightly that of the original signal.

5. Conclusions

After thorough research in the practical application of FRFT, aiming at solving the problem of difference between the normalization required by FRFT digital computation and practical engineering, This paper sets up a bridge between FRFT digital computation and its application in practice, which makes it possible for FRFT to be used in DSP.

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