

THE FILTER BANK APPROACH FOR THE FRACTIONAL FOURIER TRANSFORM

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ABSTRACT

In this work, we develop an equivalent filter bank structure for the computation of the fractional Fourier transform (FrFT). The purpose of this work is to provide an unified approach to the computation of the FrFT via the filter bank approach.

1. INTRODUCTION

The concepts of Fourier analysis and synthesis are fundamental for signal processing. Recently, the fractional Fourier transform, a generalization of the Fourier transform with a transform parameter arises many applications such as phase retrieval, signal detection, radar, tomography, and data compression [1]. And it is known as one of the most important tools for time-varying or nonstationary signal processing [2].

The fractional Fourier transform was implemented by the optical system, see [3]. There are also several numerical algorithms for computation of the continuous fractional Fourier transform (CFFT), especially elegantly developed by Ozaktas et al. in [1]. We show all these algorithms can be implemented by a multirate filter bank system.

There are several discrete forms of the fractional Fourier transform, for example, [1] provided an algorithm for the computation of the discrete transformation by sampling the fractional Fourier-transformed signal. This approach is similar to the derivation of the discrete Fourier transform (DFT). The second approach proposed in [4], considered the discrete rotation Fourier transform (DRFT) as an operator from C^M to C^M . And the DRFT of a finite signal can be easily computed directly. In this work we show the equivalent structure to the traditional filter bank structure.

From the two sections, we have the similar structure to implement the computation of the fractional Fourier transform, we define two structures for further study of this approach.

The multirate filter bank system to implement the CFFT from M samples in time domain to M samples in the α th domain is introduced in section 2. In section 3, we first review the definition of the DRFT, and introduce the DRFT via fast Fourier transform (FFT) directly, and then we derive the DRFT of a periodic signal and its equivalent filter bank representation. We develop a similar structure for these implementations, both for CFFT and DRFT in section 4. In section 5, we have a simple comparison with the FFT-based approach and the proposed approach. We conclude and introduce further development in section 6.

2. THE CONTINUOUS FRACTIONAL FOURIER TRANSFORM

Recall that the CFFT of a function $x(t)$ is defined as [1]

$$\mathfrak{F}^\alpha x(u) = A_\phi \int_{-\infty}^{\infty} B_\alpha(u, t) x(t) dt \quad (1)$$

where $B_\alpha(u, t) = A_\phi \exp[j\pi\{u^2 \cot \phi - 2ut \csc \phi + t^2 \cot \phi\}]$ with $A_\phi = |\sin \phi|^{-\frac{1}{2}} e^{-i\pi \text{sgn}(\sin \phi)/4 + i\phi/2}$ and $\phi = \frac{\alpha\pi}{2}$. Further, $B_0(u, t) = \delta(u - t)$ and $B_{\pm 2}(u, t) = \delta(u + t)$.

For a signal with compact Wigner distribution, suppose there are M samples with sample period Δt , we can evaluate (1) with the digital signal processing technique without employing the direct numerical integration. The following two algorithms [1] allows us to obtain the samples of the α th transform domain in terms of the original function.

2.1. Method 1:

The direct computation of multiply by a chirp convolution followed by another chirp multiplication. The CFFT for $x(t)$ can be represented by

$$\mathfrak{F}^\alpha x(u) = A_\phi e^{-j\pi u^2 \tan \frac{\phi}{2}} (e^{j\pi \csc \phi u^2} * g(u)) \quad (2)$$

where $g(u) = e^{-j\pi u^2 \tan \frac{\alpha}{2}} x(u)$.

The CFFT is computed by the following

$$\mathbf{x}_\alpha = F_I^\alpha \mathbf{x} \quad (3)$$

$$F_I^\alpha = D \Lambda H_{lp} \Lambda J = D H^\alpha J \quad (4)$$

where D and J are matrices representing the decimation and the interpolation operations. Λ is a diagonal matrix that corresponds to the chirp multiplication, and H_{lp} corresponds to the convolution operation. Now we define

$$\mathbf{H}^\alpha(z) = \Lambda H_{lp} \Lambda \quad (5)$$

which is viewed as the polyphase matrix of the system. This operation can be easily implemented in the filter bank structure [5][6] shown in Fig. 1.

2.2. Method 2:

The CFFT can also be expressed as

$$\mathfrak{F}^\alpha x(u) = A_\phi e^{j\pi \cot \phi u^2} \int_{-\infty}^{\infty} e^{j2\pi \csc \phi uv} [e^{j\pi \cot \phi v} x(v)] dv$$

By sampling theorem, the integral is now calculated by

$$\mathbf{x}^\alpha = F_{II}^\alpha \mathbf{x} \quad (6)$$

$$F_{II}^\alpha = \begin{cases} DK_\alpha J, & 0.5 \leq |\alpha| \leq 1.5, \\ F_{II}^\alpha = F_{II}^{\alpha-1} F, & 0.5 \leq |\alpha-1| \leq 1.5 \end{cases} \quad (7)$$

where

$$K_\alpha(m, n) = \frac{A_\phi}{2\Delta t} \exp(j\pi [\frac{\alpha m^2}{4\Delta t^2} - \csc \phi \frac{mn}{2\Delta t^2} + \frac{\alpha n^2}{4\Delta t^2}])$$

and F is the DFT matrix for $|m|$ and $|n| \leq N$. Now the polyphase matrix

$$\mathbf{H}^\alpha(z) = \begin{cases} K_\alpha, & 0.5 \leq |\alpha| \leq 1.5, \\ K_\alpha F, & 0.5 \leq |\alpha-1| \leq 1.5 \end{cases} \quad (8)$$

will be used to implement (7). The two algorithms can be essentially the well-known multirate filter bank structure, therefore, the samples of the fractional Fourier transform can be easily obtained by the filter bank structure shown in Fig. 1.

2.3. The CFFT filter bank

The computation of the CFFT of a signal can be obtained via the similar multirate filter bank (see Fig. 1) with polyphase transfer matrix (5) and (8), respectively.

3. THE DISCRETE FRACTIONAL FOURIER TRANSFORM

There are two main approaches to implement the discrete fractional Fourier transform (DRFT).

3.1. The CFFT approach:

This approach was proposed by [1] and described in the preceding section. For the rest of this work, we discuss the alternative method.

3.2. The functional power of the DFT matrix:

In [7][4], the DRFT of a signal is defined by taking the α th power of the DFT matrix. The operator is a suitable candidate for the computation of the DRFT. Before any more suitable approach appears, we present this approach and show that the filter bank structure can also implement this work. In [4], the M -point DRFT of a sequence of signal $\{x(n)\}_{n=1}^M$ is defined as

$$x^\alpha(m) = \sum_{n=0}^{M-1} K_\alpha(m, n) x(n), \quad m = 0, 1, 2, \dots, M-1 \quad (9)$$

or in vector form

$$\mathbf{x}^\alpha = \mathbf{K}_\alpha \mathbf{x}, \quad 0 \leq \alpha < 2\pi \quad (10)$$

where $\mathbf{x}^\alpha = (x^\alpha(0), x^\alpha(1), \dots, x^\alpha(M-1))^T$ and $\mathbf{x} = (x(0), x(1), \dots, x(M-1))^T$. The above form can be seen as a block version of the scalar sequences $\{x(n)\}_{n=0}^{M-1}$ and $\{x^\alpha(n)\}_{n=0}^{M-1}$, respectively.

The α -DRFT kernel \mathbf{K}_α is a polyphase matrix with entries

$$K_\alpha(m, n) = a_0(\alpha) \delta[(m-n)_M] + a_2(\alpha) \delta[(m+n)_M] + \frac{a_1(\alpha)}{\sqrt{M}} e^{-j\frac{2\pi}{M}mn} + \frac{a_3(\alpha)}{\sqrt{M}} e^{j\frac{2\pi}{M}mn} \quad (11)$$

where $a_0(\alpha) = \frac{1}{2}(1+e^{j\alpha}) \cos \alpha$, $a_1(\alpha) = \frac{1}{2}(1-j e^{j\alpha}) \sin \alpha$, $a_2(\alpha) = \frac{1}{2}(-1+e^{j\alpha}) \cos \alpha$, $a_3(\alpha) = \frac{1}{2}(-1-j e^{j\alpha}) \sin \alpha$.

3.3. The implementation of DRFT via FFT

According to (11), we need to define the time reversal operator $\overline{W}(x(n)_M) = x((-n)_M)$. In fact,

$$\overline{W} = W^2$$

where W is the DFT matrix with entries $W_{nk} = \frac{e^{-j\frac{2\pi}{N}nk}}{\sqrt{N}}$ for $0 \leq n, k \leq N-1$. The DRFT operator with parameter α defined in (11) can be written as

$$K_\alpha x = a_0(\alpha) x(n) + a_2(\alpha) x(-n) + a_1(\alpha) X(n) + a_3(\alpha) X(-n) \quad (12)$$

This means that the DRFT of $x(n)$ is the linear combination of $x(n)$ and $x(-n)$ as well as the DFT of

$x(n)$ and $x(-n)$, respectively. With suitable arrangement, the DRFT of $x(n)$ can be computed by an efficient method via FFT. However, with the advantage of computation efficiency, this approach is lacking in a view of system science. The direct implementation of this approach is shown in Fig. 2.

3.4. The DRFT filter bank

Suppose there are M samples in one period of a periodic signal $x(t)$, denoted by $\{x(n)\}_{n=0}^{M-1}$. We generate the vector $(s_0(n), s_1(n), \dots, s_{M-1}(n))^T$ by passing $x(n)$ through a serial to parallel mechanism, so that $s_i(n) = x(n-i)$. From (9), we have

$$x_m^\alpha(n) = \sum_{i=0}^{M-1} s_i(n) K_\alpha(m, i).$$

The z -transform of

$$\begin{aligned} X_m^\alpha(z) &= \sum_{i=0}^{M-1} S_i(z) K_\alpha(m, i) \\ &= \sum_{i=0}^{M-1} K_\alpha(m, i) z^{-i} X(z). \end{aligned}$$

where $X(z)$ is the z -transform of $x(n)$. Then we can write

$$X_k^\alpha(z) = H_k^\alpha(z) X(z)$$

where $H_k^\alpha(z) = \sum_{i=0}^{M-1} K_\alpha(m, i) z^{-i}$.

Remark: The M -point DRFT operator is periodic in both time and frequency, that is,

$$\begin{aligned} K_\alpha(n, k) &= K_\alpha(n+M, k) \\ &= K_\alpha(n, k+M) \\ &= K_\alpha(n+M, k+M) \end{aligned}$$

Hence, the DRFT of a periodic signal is also periodic.

4. THE POLYPHASE REPRESENTATION OF FRFT OPERATOR

It is well known that the structure shown in Fig. 3 is equivalent to the polyphase structure which appears in the theory of filter bank frequently. For convenience, we define two types of structures to implement the FrFT filter bank. The first one is the type 1 FrFT filter bank with the associated filter bank (see Fig. 4)

$$H_k^\alpha(z) = \sum_{l=0}^{M-1} z^{-l} H_{kl}^\alpha(z) = \sum_{l=0}^{M-1} z^{-l} K_\alpha(k, l),$$

for $0 \leq k \leq M-1$, or in the vector equivalent form

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \mathbf{H}^\alpha(z) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix} =: \mathbf{H}^\alpha(z) \mathbf{e}(z)$$

where $\mathbf{H}^\alpha(z)$ is known as a polyphase matrix. The second is the type 2 FrFT filter bank (see Fig. 5)

$$H_k^\alpha(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} H_{kl}^\alpha(z^M)$$

for $0 \leq k \leq M-1$, or

$$\begin{aligned} & \begin{bmatrix} H_0^\alpha(z) & H_1^\alpha(z) & \cdots & H_{M-1}^\alpha(z) \end{bmatrix} \\ &= \begin{bmatrix} z^{-(M-1)} & z^{-(M-2)} & \cdots & 1 \end{bmatrix} \mathbf{H}^\alpha(z). \end{aligned}$$

Evidently the equivalent representation of the two types of filter banks is an M -input M -output (MIMO) linear time-invariant (LTI) system. More precisely, this MIMO system can be characterized by an $M \times M$ transfer matrix $\mathbf{H}^\alpha(z) = [K_\alpha(m, n)]_{0 \leq m, n \leq M-1}$. The "blocking mechanism" (see Fig. 3) for type 1 polyphase matrix can be considered to be a serial to parallel converter of data. Similarly, the "unblocking mechanism" used in the type 2 polyphase transform is a parallel to serial converter.

5. PERFORMANCE COMPARISON

The appearance of large-scale integrated circuits has reduced the emphasis on minimizing the number of multiplications and is causing the considered structure using many parallel devices rather than a few high-speed devices. The computation of the FFT-based algorithms is about $O(N \log N)$ for a sample of number N . However, there are N multiplications and N additions in each filter. With increasing N , the filter bank structure can have a large amount of computation saving with the expense of hardware.

6. CONCLUSION

We have presented an equivalent filter bank structure which implements the fractional Fourier transform. With this approach, we can unify the implementation via the equivalent filter bank system. This filter bank structure can also be used to process the short time fractional Fourier transform. This framework not only provides a novel approach to compute the FrFT of a given function, but also reveal a new generalization of the conventional DFT filter bank for digital signal processing.

It is known that over different angles (i.e. α) the signal energy is squeezed into different frequencies, hence we can find applications of the FrFT operation as well as the filter bank.

7. REFERENCES

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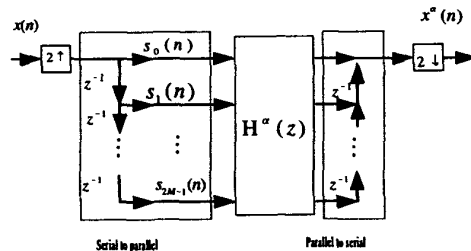


Fig. 1 The polyphase representation to implement the CFFT.

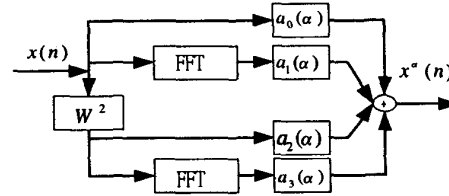


Fig. 2 The direct implementation of the DRFT via FFT.

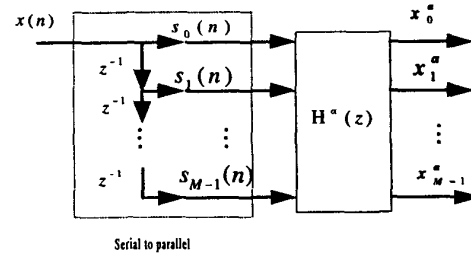


Fig. 3 The direct implementation of the DRFT.

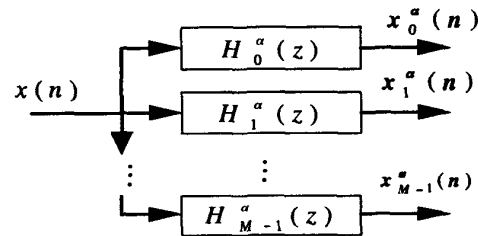


Fig. 4 The type 1 implementation of the DRFT bank.

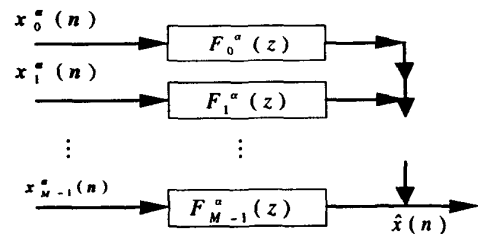


Fig. 5 The type 2 implementation of DRFT bank.