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# Selection of Dimensional Normalization Parameter in Digital Computation of the Fractional Fourier Transform

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***Abstract:*** **In the digital computation of the fractional Fourier Transform, the dimensional normalization is needed. The initial frequency and chirp rate of a chirp signal are changed by the dimensional normalization parameter, and the position of the chirp signal's peak on the two-dimensional parameter (p, u) plane is also changed. This influences the detection of the multi-component chirp signals. In order to decrease the influence, a method is presented to choose the dimensional normalization parameter, based on the distance between two chirp signals' peaks. First, the coordinates of the chirp signal's peak is educed on the (p, u) plane. And the relationships between the dimensional normalization parameter and the distances between two chirp signals' peaks on p axis and on u axis are analyzed. It is discovered that the distances varies with the dimensional normalization parameter and they both have their own maximums. By changing signals' observation time and sampling frequency, selecting a reasonable dimensional normalization parameter can increase the distances between two signals' peaks. It can improve multi-component chirp signals' resolution ability and reduce the shading effect between strong signals and weak signals. The effectiveness of the method is verified by the computer simulations.**

***Keywords:*** **fractional Fourier transform; dimensional normalization parameter; selection**

## I. INTRODUCTION

The fractional Fourier transform (FRFT), the generalization of the Fourier transform (FT), can be interpreted as a rotation by an angle $\alpha$ in the time-frequency plane. It is a uniform time-frequency transform, and it has been a new theory of time- frequency signal analysis. And the FRFT of a signal can also be interpreted as a decomposition of the signal in terms of chirps. So FRFT is a good tool for the detection and parameter estimation of chirp signals. 1980, Namias first introduced the mathematical definition of the FRFT[1]. And the integral form of the FRFT operator was given by McBride et al[2]. However, the FRFT was applied in signal processing very slowly. Until 1994, Almeida proved the relationship between the FRFT and these time-frequency representations such as the Wigner Vill distribution(WVD), the ambiguity function and the short-time Fourier transform, and interpreted the FRFT as a rotation of the time-frequency plane[3]. 1996, an fast algorithm for digital computation of the FRFT was given based on Fast Fourier Transform (FFT) by Ozaktas et al[4]. Its computational quantity is $O\left(N\log_2 N\right)$, and N is the quantity of the signal's sampling pointers. These establish the foundation of the FRFT in signal processing. Several algorithms have been given for the detection and parameter estimation of chirp signals based on the FTFT[5~7].

But the original signal must be normalized before digital computation in the fast algorithm for computation of the FRFT[4]. Some features of the signal are changed by the dimensional normalization. The changes influence the detection of multi-component chirp signals. In this paper, take two chirp signals as objects of study to analyze the effect of the dimensional normalization parameter to the detection of multi-component chirp signals. It is discovered that adjusting the parameter by changing observation time and sampling frequency of the signal can enlarge the distance of two peaks of two chirp signals on the (p, u) plane. This can reduce the mutual effect among signals, and improve the detection ability and efficiency of the FRFT. So we present a method to select signal observation time and sampling frequency appropriately.

## II. DEFINITION OF THE FRACTIONAL FOURIER TRANSFORM

The FRFT of a signal $x\left(t\right)$ can be defined as

$$X_p\left(u\right)=F_p\left[x\right]\left(u\right)=\int_{-\infty}^{\infty}x(t)K_p(t,u)dt \qquad (1)$$

where $p$ is a real number and called the order of the FRFT, $\alpha = p\pi/2$ is an rotated angle of the FRFT, and $K_p(t,u)$ is the kernel of the FRFT

$$K_p(t,u)=\begin{cases}A_\alpha \exp\left(j\pi\dfrac{t^2+u^2}{2}\cot\alpha - j\pi tu\csc\alpha\right), & \alpha\neq n\pi \\ \delta(t-u), & \alpha = 2n\pi \\ \delta(t+u), & \alpha=(2n\pm1)\pi\end{cases} \qquad (2)$$

where $A_\alpha = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}$. The inverse FRFT is

$$x(t) = \int_{-\infty}^{\infty} X_p(u) K_{-p}(t,u) du \quad (3)$$

Eq.(3) indicates that signal $x(t)$ can be interpreted as a decomposition to a basis formed by the orthonormal chirp functions in the $u$ domain, and the $u$ domain is usually called the fractional Fourier domain, in which the time and frequency domains are its special cases.

## III. PRINCIPLE OF NORMALIZATION

In the fast algorithm of the FRFT digital computation[4], the original signal $x(t)$ must be normalized. In the literature [4], the method of normalization is introduced below. It assumes that, the original continuous signal is compact along the axes of time and frequency, the time-domain representation of the signal is confined to the interval $[-\Delta t/2, \Delta t/2]$ and its frequency-domain representation is confined to the interval $[-\Delta f/2, \Delta f/2]$. So the signal's time-bandwidth product is $N = \Delta t \Delta f$, which is always greater than unity because of the uncertainty relation. Because time-domain and frequency-domain have different dimensions. In order to compute the FRFT, time-domain and frequency-domain must be transformed to the domain whose dimension is "1". It introduces the scaling parameter $s$ with the dimension of time and introduces scaled coordinates

$$x = t/s, \quad v = fs \quad (4)$$

The new coordinates $(x, v)$ are clearly dimensionless. The signal is confined to the interval $[-\Delta t/(2s),\ \Delta t/(2s)]$ and $[-\Delta fs/2, \Delta fs/2]$ respectively under the new coordinates. In order to have the two intervals equal to each other, it chooses $s = (\Delta t/\Delta f)^{1/2}$, so that the length of both intervals are equal to the dimensionless quantity $\Delta x = (\Delta t \Delta f)^{1/2}$. Both intervals are normalized as $[-\Delta x/2, \Delta x/2]$. In the newly defined coordinates, according to the sampling theorem, the signal can be represented in both domains with $N = \Delta x^2$ samples spaced $1/\Delta x = 1/\sqrt{N}$ apart.

However, in practical application, what we can get is a group of discrete observation data sampled from the original continuous signal. The observation time $t_0$ and the sampling frequency $f_s$ are known values. How to normalize such discrete data is a key linking the FRFT fast algorithm and practical application methods. In the literature [8] two practical normalization methods are presented. One is called discrete scaling method, and the other is called data zero-padding/interception method. Referring literatures [4,8,9], this paper uses the discrete scaling method. Its detail introduction is given as follows. The key of discrete data normalization by scaling transform is to select proper time-width $\Delta t$, bandwidth $\Delta f$, scaling factor $s$ and normalized width $\Delta x$, so that the scaled discrete data equal to samples of the normalized continuous signal at a sampling interval of $1/\Delta x$. The observation time $t_0$ is as the time-band, i.e., $\Delta t = t_0$. The midpoint of signal is taken as the time origin. The time domain representation of signal is confined within the range $[-t_0/2, t_0/2]$. The exact value of signal bandwidth is not known. However, the signal sampling frequency $f_s$ is known in practice. According to the sampling theorem, the sampling frequency must be more than twice as much as the highest signal frequency. It is unnecessary that bandwidth $\Delta f$ must select the lowest value, as long as signal energy is confined to the interval. It is reasonable that the sampling frequency is used as the bandwidth, i.e., $\Delta f = f_s$. The signal's frequency domain is confined within the range of $[-f_s/2, f_s/2]$. By known time-band and bandwidth, the scaling factor $s$ and normalized width $\Delta x$ can be deduced as follow

$$s = (\Delta t/\Delta f)^{1/2} = (t_0/f_s)^{1/2} \quad (5)$$

$$\Delta x = (\Delta t \Delta f)^{1/2} = (t_0 f_s)^{1/2} \quad (6)$$

The sampling interval of the original discrete data is $t_s = 1/f_s$, After being scaled by formula (4), the sampling interval is

$$t_s' = (t_0 f_s)^{-1/2} = 1/\Delta x \quad (7)$$

The former time domain ranges within $[-t_0/2, t_0/2]$. After being scaled, the range is changed to $[-\Delta x/2, \Delta x/2]$. Therefore, discrete scaling method is a scaling process to the original discrete data by formula (4) and formula (5) with sampling frequency as bandwidth and observation time as time-width. Also, the scaled discrete data equal to samples of the normalized continuous signal at a sampling interval of $1/\Delta x$.

## Ⅳ. THE COORDINATES OF A CHIRP SIGNAL'S PEAK ON (P, U) PLANE

The model of a single chirp signal $x(t)$ is assumed as

$$x(t) = A\exp\left(j2\pi f_0 t + j\pi \mu_0 t^2\right) \quad (8)$$

where $f_0$ is the signal's initial frequency, and $\mu_0$ is the signal's chirp rate. The signal's observation time and sampling frequency are respective $t_0$ and $f_s$. The time-frequency distribution of the signal $x(t)$ after normalization is shown in fig.1.

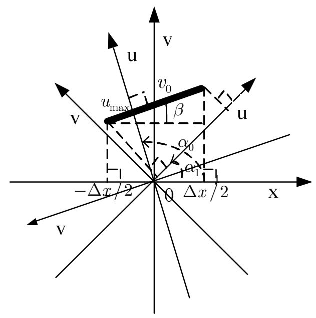


Fig.1 time-frequency distribution of chirp signal $x(t)$ after dimensional normalization

In fig.1, the thick line indicates the time-frequency distribution line of signal $x(t)$ after normalization. The angle $\beta$ ( $\beta \in (0, \pi/2)$ or $\beta \in (\pi/2, \pi)$ ) is the angle between the time-frequency distribution line and the $x$ axis, and the value of $\beta$ is chosen as $\beta \in (0, \pi/2)$ in this paper. The $u \perp v$ axes of coordinates indicates the time-frequency axes rotated by an arbitrary angle $\alpha$ counterclockwise, and the value of angle $\alpha$ is $\alpha \in (0, \pi)$ . The angles $\alpha_0$ and $\alpha_1$ are two rotation angles of the FRFT, and $\alpha_0$ ( $\alpha_0 = p_0\pi/2$ , $p_0$ is the optimal fractional order) is the optimal fractional angle of chirp signal $x(t)$ . The relationship between the angle $\alpha_0$ and the angle $\beta$ is $\alpha_0 = \beta + \pi/2$ . The pointer $v_0$ is the intersection of signal's time-frequency distribution line and the $v$ axis. The pointer $u_{\max}$ is the intersection of the signal's time-frequency distribution line and the $u$ axis in the optimal fractional Fourier domain.

Assume the chirp rate and initial frequency of signal $x(t)$ after dimensional normalization are respective $\mu_0'$ and $v_0$ . In fig.1, the relationship between $\beta$ and $\mu_0'$ is $\tan\beta = \mu_0'$ , and the relationship between $\alpha_0$ and $\beta$ is $\alpha_0 = \beta + \pi/2$ . By formula (4) and (5), we can have

$$\begin{cases} \mu_0' = \dfrac{v}{x} = \dfrac{fs}{t/s} = \mu_0 s^2 = \mu_0 t_0 / f_s \\ v_0 = f_0 s = f_0 \left(t_0 / f_s\right)^{1/2} \end{cases} \tag{9}$$

$$\alpha_0 = \operatorname{arccot}(-\mu_0 t_0 / f_s) \tag{10}$$

So the optimal fractional order of signal $x(t)$ is

$$p_0 = \operatorname{arccot}(-\mu_0 t_0 / f_s) / (\pi/2) \tag{11}$$

In fig.1, the projection coordinates of the signal's time-frequency distribution line's middle pointer $v_0$ on the $u$ axis in the arbitrary fractional Fourier transform domain is

$$u_m = v_0 \sin\alpha \tag{12}$$

where $\alpha$ is the arbitrary rotation angle of the FRFT. By formula (12), the middle pointer of signal's spectrum $u_m$ varies with $\alpha$ on the $u$ axis in arbitrary fractional Fourier transform domain. When the rotation angle is the optimal fractional angle $\alpha_0$ , in the corresponding fractional Fourier domain, the chirp signal's spectrum forms a peak, and the coordinate of the peak on the $u$ axis is

$$u_{\max} = v_0 \sin\alpha_0 \tag{13}$$

By formula (11) and (13), we can get the coordinates of signal $x(t)$ 's peak on the $(p,u)$ plane. And by formula (9), (11) and (13), we can discover that the coordinates of chirp signal's peak on the $(p,u)$ plane is changed by the dimensional normalization parameter $s$ . So, we can change the coordinates of chirp signal's peak on the $(p,u)$ plane by changing signal's observation time $t_0$ and sampling frequency $f_s$ .

## V. THE EFFECT OF DIMENSIONAL NORMALIZATION ON DETECTION OF MULTI-COMPONENT CHIRP SIGNALS

As we know, a finite length chirp signal has different spectrum distribution characteristic in various fractional Fourier transform domain, and it has the best energy concentration only in the corresponding optimal fractional domain and forms a peak. With this characteristic, chirp signals can be detected and estimated by the FRFT. The detail procedure is that, firstly, make FRFT of the signals in every fractional order $p \in [0,2]$ respectively, and obtain the three-dimensional distribution of signal energy spectrum on the parameters $(p,u)$ plane consisting of fractional domain $u$ and fractional order $p$ , then chirp signal can be detected by two-dimensional search of peak on $(p,u)$ plane with a preset threshold, meanwhile, fractional order $\hat{p}_0$ and fractional domain $\hat{u}_{\max}$ coordinates corresponding to the peak can be estimated. Choose two chirp signals $x_r$ and $x_l$ , and their models are same to equation (8). Assume their initial frequencies are respective $f_r = 10\mathrm{Hz}$ and $f_l = 11\mathrm{Hz}$ , their chirp rates are respective $\mu_r = 10\mathrm{Hz/s}$ and $\mu_l = 12$ $\mathrm{Hz/s}$ , their optimal fractional rotation angle are respective $\alpha_r$ and $\alpha_l$ ( $\alpha_r < \alpha_l$ , $\alpha_r = p_r\pi/2$ , $\alpha_l = p_l\pi/2$ ), and their peaks' coordinates on the $u$ axis are respective $u_{r\max}$ and $u_{l\max}$ in their each optimal fractional domain. So their peaks' coordinates are respective ( $p_r, u_{r\max}$ ) and ( $p_l, u_{l\max}$ ) on the ( $p,u$ ) plane. Below, we analyze the effect of dimensional normalization parameter $s = \left(t_0 / f_s\right)^{1/2}$ on the distance between pointer ( $p_r, u_{r\max}$ ) and pointer $(p_l, u_{l\max})$ .

By formula (11), on ( $p,u$ ) plane, the distance between the two peaks of chirp signals $x_r$ and $x_l$ on the $p$ axis is

$$\Delta R_p = |p_l - p_r| = \left|\operatorname{arccot}\left(-\mu_l t_0 / f_s\right) - \operatorname{arccot}\left(-\mu_r t_0 / f_s\right)\right| / (\pi/2) \tag{14}$$

And by formula (13), on ( $p,u$ ) plane, the distance between the two peaks of chirp signals $x_r$ and $x_l$ on $u$ axis is

$$\Delta R_u = |u_{r\max} - u_{l\max}| = \left(t_0 / f_s\right)^{1/2} |f_l \sin\alpha_l - f_r \sin\alpha_r|$$

$$= \left(t_0 / f_s\right)^{1/2} \left| \frac{f_l}{\sqrt{1+\mu_l^2\left(t_0 / f_s\right)^2}} - \frac{f_r}{\sqrt{1+\mu_r^2\left(t_0 / f_s\right)^2}} \right| \quad (15)$$

Formula (14) and formula (15) respectively offer a method to compute the distances between two peaks on u axis and p axis. By formula (14) and (15), we discover that when the chirp rates and initial frequencies of signal $x_r$ and signal $x_l$ are fixed, we can change the distances between two peaks of signal $x_r$ and signal $x_l$ on u axis and p axis on $(p,u)$ plane by changing their observation time and sampling frequency. In the previous sections we have assumed $f_r = 10\text{Hz}$ , $f_l = 11\text{Hz}$ , $\mu_r = 10\text{Hz/s}$ and $\mu_l = 12\text{Hz/s}$ , and we study the variational curve of the distance between two peaks of signal $x_r$ and signal $x_l$ with the dimensional normalization parameter $s = \left(t_0 / f_s\right)^{1/2}$ . As shown in fig.2, $\Delta R_p$ , the distance between two peaks of signal $x_r$ and signal $x_l$ on p axis, varies with $t_0 / f_s$ . From fig.2, we observe that when $t_0 / f_s$ is equal to 0.0915, $\Delta R_p$ gets to its maximum 0.0580, when $t_0 / f_s$ increases from 0 to 0.0915, $\Delta R_p$ increases progressively, and when $t_0 / f_s$ is more than 0.0915, $\Delta R_p$ decrease progressively. As shown in fig.3, $\Delta R_u$ , the distance between two peaks of signal $x_r$ and signal $x_l$ on u axis, varies with $t_0 / f_s$ . From fig.3, we observe that when $t_0 / f_s$ is equal to 0.0296, $\Delta R_u$ gets its first maximum 0.1336, when $t_0 / f_s$ increases from 0 to 0.0296, $\Delta R_u$ increases progressively, when $t_0 / f_s$ increases from 0.0296 to 0.0954, $\Delta R_u$ decrease progressively, when $t_0 / f_s$ increases from 0.0954 to 0.3056, $\Delta R_u$ increases progressively and gets its second maximum 0.1195, and when $t_0 / f_s$ is more than 0.3056, $\Delta R_u$ decrease progressively again.

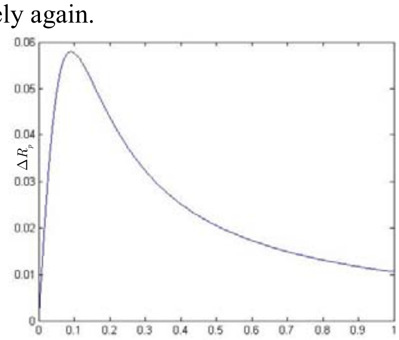

Fig2. the curve of $\Delta R_p$ varying with $s = t_0 / f_s$

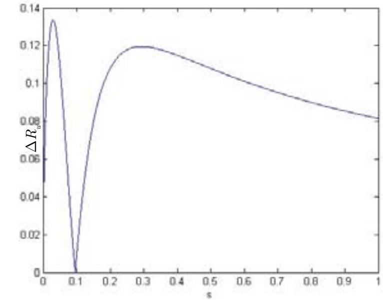

Fig3. the curve of $\Delta R_u$ varying with $s = t_0 / f_s$

The distance between two signals' peaks influences the detection and parameter estimation of multi-component chirp signals. So, when the sampling theorem is satisfied, we can choose appropriate observation time $t_0$ and sampling frequency $f_s$ to augment the distance between two signals' peaks on p axis and on u axis, in order to reduce the effect among multi-component chirp signals.

## VI. SIMULATIONS

In this section, signal $x_r$ and signal $x_l$ are still used as the research objects. The effect of the dimensional normalization parameter on the detection of multi-component chirp signals is analyzed in the three dimensional images of the FRFT. The dimensional normalization parameter $s$ is adjusted by changing signals' observation time $t_0$ and sampling frequency $f_s$ . Same to the previous sections, the initial frequencies of signal $x_r$ and signal $x_l$ are also respective $f_r = 10\text{Hz}$ and $f_l = 11\text{Hz}$ , and their chirp rates are also respective $\mu_r = 10\text{Hz/s}$ and $\mu_l = 12\text{Hz/s}$ . The simulations are classed two cases: the first case, the amplitudes of signal $x_r$ and signal $x_l$ are equal, namely, $A_r = A_l$ , analyze the distinguishability of signal $x_r$ and signal $x_l$ in the three dimensional images of the FRFT. The second case, the amplitudes of signal $x_r$ and signal $x_l$ are different, assuming $A_r = 0.3A_l$ , analyze the shading intensity between signal $x_r$ and signal $x_l$ in the three dimensional images of the FRFT. For the first case, when the observation time and sampling frequency of the signals are respective $t_0 = 1\text{s}$ , $f_s = 180\text{Hz}$ , the three-dimensional images of the FRFT is shown in fig.4, and, when the observation time and sampling frequency are respective $t_0 = 1\text{s}$ , $f_s = 50\text{Hz}$ , the three-dimensional images of the FRFT is shown in fig.5. For the second case, when the observation time and sampling frequency of the signals are respective $t_0 = 1\text{s}$ , $f_s = 180\text{Hz}$ , the three-dimensional images of the FRFT is shown in fig.6, and, when the observation time and sampling frequency are respective $t_0 = 1\text{s}$ , $f_s = 50\text{Hz}$ , the three-dimensional images of the FRFT is shown in fig.7.

Comparing fig.4 and fig.5, we can discover that two peaks of signal $x_r$ and signal $x_l$ can't be distinguished in fig.4, but in fig.5 the peaks can be distinguished clearly. When $t_0$ is equal to 1s and $f_s$ is equal to 180Hz, the dimensional normalization parameter $s$ is equal to 0.0056. When $t_0$ is equal to 1s and $f_s$ is equal to 50 Hz, the dimensional normalization parameter $s$ is equal to 0.02. From fig.2 and fig.3, when $s$ is equal to 0.02, the distance between two peaks is much bigger. Comparing fig.6 and fig.7, we can also discover that in fig.6 the peak of signal $x_r$ is shaded by the peak of signal $x_l$, and the signal $x_r$ can't be detected. But in fig.7, the peak of signal $x_r$ can be clearly shown. So when the distance between two peaks is much bigger, the mutual effect among multi-component chirp signals is less.

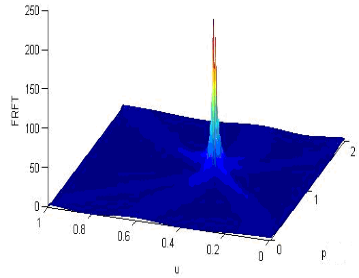

Fig4. 3-D images of the chirp signals' fractional Fourier transform, $A_r = A_l$, $t_0 = 1\text{s}$, $f_s = 180\text{Hz}$

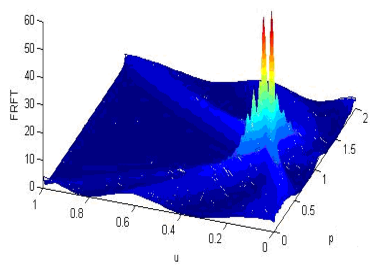

Fig5. 3-D images of the chirp signals' fractional Fourier transform, $A_r = A_l$, $t_0 = 1\text{s}$, $f_s = 50\text{Hz}$

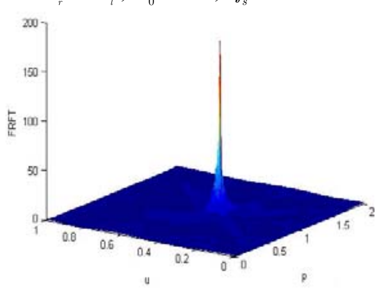

Fig6. 3-D images of the chirp signals' fractional Fourier transform, $A_r = 0.3$, $A_l = 1$, $t_0 = 1\text{s}$, $f_s = 180\text{Hz}$

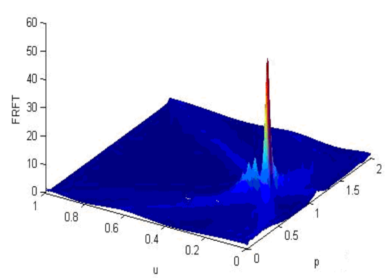

Fig7. 3-D images of the chirp signals' fractional Fourier transform, $A_r = 0.3$, $A_l = 1$, $t_0 = 1\text{s}$, $f_s = 50\text{Hz}$

From the previous sections, we know that when the chirp rate and initial frequency are fixed, we can choose suitable observation time and sampling frequency to augment the distance between two peaks of two signals, and the sampling frequency must satisfy the sampling theorem. This method can reduce the mutual effects among multi-component chirp signals and improve the detection ability of the FRFT.

## Ⅶ. CONCLUSION

In this paper, the selection of dimensional Normalization parameter in digital computation of the fractional Fourier transform is investigated. The dimensional normalization parameter can adjust the distance between two peaks of two chirp signals. According to the principle of dimensional normalization, we can select a suitable dimensional normalization parameter to augment the distance between two peaks by changing signals' observation time and sampling frequency. Using this phenomenon, augmenting the distance can decrease the mutual effects among multi-component chirp signals. So, when the fractional Fourier transform is used to detection multi-component chirp signals, we can't blindly extend observation time and add sampling frequency, and we should select a suitable observation time and sampling frequency to improve the detection ability of the FRFT, according to the relationship between the distance between peaks of chirp signals and the dimensional normalization parameter. This paper presents a method to select suitable observation time and sampling frequency.

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