Presentation, Derivation and Numerical Experiments of a Group of Extrapolation Formulas

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***Abstract*—In this paper, Taylor expansion and Vandermonde determinant are used to derive the coefficients of the group of extrapolation formulas. In the derivation of the coefficients, we first use Taylor expansion to expand the formulas, then the expansion is transformed into Vandermonde matrix. The general formula of coefficients is obtained by using the characteristics of Vandermonde determinant and Cramer's rule. After careful observation, we find that the rule of the coefficients is similar to that of Pascal's triangle. Three functions are used in numerical experiments. We first evaluate the effect of different step size values on error using order-3 formula, and then we evaluate the effect of different order formulas on error with some fixed step size value.**

***Keywords*—Extrapolation; Taylor expansion; Vandermonde matrix; Cramer's rule; Pascal's triangle**

I. Introduction

In scientific computing, it always occurs that the information of a function $y = f(t)$ is known only at some given points. However, we sometimes need to estimate the values of some other points [1]. In order to solve this problem, experts have put forward many methods [1], [2]. For example, using Taylor expansion [2] to approximate a function $f(t)$ is a method, which only needs the function information of a certain point. However, the shortcoming of the method is that the high-order derivatives must be known, but they are often unavailable or hard to compute.

Suppose that the values of function $f(t)$ are known at $n+1$ points, i.e., $(t_0, y_0), (t_1, y_1), \cdots, (t_n, y_n)$, and these points satisfy

$$\alpha \le t_0 < t_1 < \cdots < t_n \le \beta,$$

where α and β are real numbers known as interval bounds. The estimated value y can be termed as interpolation value or extrapolation value based on its corresponding abscissa value t. If $t \in [t_0, t_n]$, the estimated value y is called an interpolation value. If $t \notin [t_0, t_n]$, the estimated value y is called an extrapolation value [1].

In the mathematical field of numerical analysis, interpolation is a method of constructing new data points within the domain of a discrete set of known data points. Extrapolation is similar to interpolation which estimates variables between known points, and it is a method for estimating the value of a variable outside of the original observation domain. Although the uncertainty of extrapolation method is great and the risk of producing meaningless results is high, extrapolation is a good way of predicting (especially using regular data), and it has many applications in real life [3], [4]. A polynomial curve is always created from the known data in interpolation [5]. The common methods are Lagrange interpolation [1], [6], [7] and Newton polynomial [1], [8], [9], but these two methods are mainly oriented to interpolation problems, and the error terms of these methods can only reflect the error magnitude of interpolation.

This paper presents a group of formulas based on the fact that the predicted value is strongly correlated with previously known data. Based on Taylor expansion, we expand this group of formulas and get a set of equations which could be transformed into Vandermonde matrix. According to the property of Vandermonde determinant [10], [11], [12], [13], the general formula of coefficients is obtained by using Cramer's rule [14]. The coefficients of the formulas of the first ten orders are obtained by using the general formula of coefficients. The rule of coefficients is similar to that of Pascal's triangle [15]. In the section of numerical experiments, we divide each experiment into two parts to evaluate the errors caused by different step size values and different formulas, respectively. In the first part, we use the same formula to calculate the errors caused by different step size values, and, in the second part, we fix the step size value and calculate the errors caused by different order formulas.

II. Presentation of general formula group

We suppose that the value we need to predict has strong correlation with the previously known data. Let the value y_k denote $y_k = y(t_k)$, where $t_k = kh$ and the step size $h \in (0, 1)$ with $k = 1, 2, 3 \cdots$.

$$\begin{cases} y_{k+1} = a_0 y_k + O(h^1), \\ y_{k+1} = a_0 y_k + a_1 y_{k-1} + O(h^2), \\ y_{k+1} = a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} + O(h^3), \\ y_{k+1} = a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} \\ \qquad + a_3 y_{k-3} + O(h^4), \\ \vdots \\ y_{k+1} = \sum_{j=0}^{n-1} a_j y_{k-j} + O(h^n), \end{cases} \tag{1}$$

TABLE I
VALUES OF SPECIFIC COEFFICIENTS

Order	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	1									
2	2	-1								
3	3	-3	1							
4	4	-6	4	-1						
5	5	-10	10	-5	1					
6	6	-15	20	-15	6	-1				
7	7	-21	35	-35	21	-7	1			
8	8	-28	56	-70	56	-28	8	-1		
9	9	-36	84	-126	126	-84	36	-9	1	
10	10	-45	120	-210	252	-210	120	-45	10	-1

that is,

$$\begin{cases} p_{k+1} \doteq a_0 y_k,\ p_{k+1} = y_{k+1} + O(h^1); \\ p_{k+1} \doteq a_0 y_k + a_1 y_{k-1},\ p_{k+1} = y_{k+1} + O(h^2); \\ p_{k+1} \doteq a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2},\ p_{k+1} = y_{k+1} + O(h^3); \\ p_{k+1} \doteq a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} \\ \qquad +a_3 y_{k-3},\ p_{k+1} = y_{k+1} + O(h^4); \\ \vdots \\ p_{k+1} \doteq \sum_{j=0}^{n-1} a_j y_{k-j},\ p_{k+1} = y_{k+1} + O(h^n); \end{cases}$$

where "$\doteq$" denotes the operation of assigning the computation result in the right-hand side of the equation to the variable in the left-hand side of the equation.

The coefficients $a_0, a_1, \cdots, a_{n-1}$ of this group of formulas are derived, and the general formula is obtained in the rest of this paper. The coefficients of the formulas of the first ten orders are given in TABLE I, and their formula form is shown in TABLE II. In the following sections, we show the derivation of this formula group, as well as numerical results.

III. Derivation of Specific Formula Group

The derivation of these formulas is based on Taylor expansion, and the general formula of the coefficients is obtained by Vandermonde matrix and Cramer's rule. In this section, in order to introduce our derivation with profundity and also for better understanding, we first introduce the derivation of the order-2 formula and the order-3 formula, then we introduce the derivation of the general formula of the coefficients. Besides, the order-1 formula is easy to obtain and prove, thus with details omitted.

A. Derivation of Order-2 Formula

Taylor expansion of an infinitely differentiable function $f(t)$ at $t = \alpha$ is as follows:

$$\begin{aligned} f(t) =& f(\alpha) + f'(\alpha)(t-\alpha) + \frac{f''(\alpha)}{2!}(t-\alpha)^2 + \dots \\ &+ \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k + \dots . \end{aligned}$$

Consider the order-2 extrapolation formula:

$$y_{k+1} = a_0 y_k + a_1 y_{k-1} + O(h^2). \tag{2}$$

In order to calculate the values of coefficients a_0 and a_1, we need to expand the function $y(t)$ using Taylor expansion at $t = t_{k+1}$. The expansion is as follows:

$$\begin{aligned} y(t) =& y(t_{k+1}) + y'(t_{k+1})(t - t_{k+1}) \\ &+ O((t - t_{k+1})^2). \end{aligned} \tag{3}$$

Now we use formula (3) to expand y_k and y_{k-1} at $t = t_{k+1}$. The expansion group is as follows:

$$\begin{cases} y_k = y_{k+1} - y'_{k+1} \cdot h + O(h^2), \\ y_{k-1} = y_{k+1} - 2y'_{k+1} \cdot h + O(h^2). \end{cases} \tag{4}$$

We first multiply the first equation in (4) by a_0 and the second equation in (4) by a_1, and then add them up:

$$\begin{aligned} (a_0 + a_1) y_{k+1} =& a_0 y_k + a_1 y_{k-1} \\ &+ (a_0 + 2a_1) y'_{k+1} \cdot h + O(h^2). \end{aligned} \tag{5}$$

In order to make equation (5) and formula (2) have the same form, we eliminate the derivative term in equation (5), and the following results can be obtained.

$$\begin{cases} a_0 + a_1 = 1, \\ a_0 + 2a_1 = 0, \end{cases} \Rightarrow \begin{cases} a_0 = 2, \\ a_1 = -1. \end{cases}$$

Now we have the coefficients of the order-2 formula, i.e., $y_{k+1} = 2y_k - y_{k-1} + O(h^2)$.

B. Derivation of Order-3 Formula

Consider the order-3 extrapolation formula:

$$y_{k+1} = a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} + O(h^3). \tag{6}$$

In order to calculate the values of coefficients a_0, a_1 and a_2, we need to expand the function $y = f(t)$ using Taylor expansion at $t = t_{k+1}$. The corresponding expansion is as follows:

$$\begin{aligned} y(t) =& y(t_{k+1}) + y'(t_{k+1})(t - t_{k+1}) \\ &+ y''(t_{k+1})(t - t_{k+1})^2/2 + O((t - t_{k+1})^3). \end{aligned} \tag{7}$$

Now we use formula (7) to expand y_k, y_{k-1} and y_{k-2} at $t = t_{k+1}$, and the expansion group is as follows:

$$\begin{cases} y_k = y_{k+1} - y'_{k+1} \cdot h + y''_{k+1} \cdot h^2/2 + O(h^3), \\ y_{k-1} = y_{k+1} - 2y'_{k+1} \cdot h + y''_{k+1} \cdot (2h)^2/2 + O(h^3), \\ y_{k-2} = y_{k+1} - 3y'_{k+1} \cdot h + y''_{k+1} \cdot (3h)^2/2 + O(h^3). \end{cases} \tag{8}$$

We first multiply the first equation in (8) by a_0, the second equation by a_1 and the third equation by a_2, and then add them up:

$$\begin{aligned} (a_0 + a_1 + a_2) y_{k+1} =& a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} \\ &+ (a_0 + 2a_1 + 3a_2) y'_{k+1} \cdot h/2 \\ &+ (a_0 + 4a_1 + 9a_2) y''_{k+1} \cdot h^2/2 \\ &+ O(h^3). \end{aligned} \tag{9}$$

TABLE II
EXTRAPOLATION FORMULAS

Order	Formula
1	$y_{k+1} = y_k + O(h^1)$
2	$y_{k+1} = 2y_k - y_{k-1} + O(h^2)$
3	$y_{k+1} = 3y_k - 3y_{k-1} + y_{k-2} + O(h^3)$
4	$y_{k+1} = 4y_k - 6y_{k-1} + 4y_{k-2} - y_{k-3} + O(h^4)$
5	$y_{k+1} = 5y_k - 10y_{k-1} + 10y_{k-2} - 5y_{k-3} + y_{k-4} + O(h^5)$
6	$y_{k+1} = 6y_k - 15y_{k-1} + 20y_{k-2} - 15y_{k-3} + 6y_{k-4} - y_{k-5} + O(h^6)$
7	$y_{k+1} = 7y_k - 21y_{k-1} + 35y_{k-2} - 35y_{k-3} + 21y_{k-4} - 7y_{k-5} + y_{k-6} + O(h^7)$
8	$y_{k+1} = 8y_k - 28y_{k-1} + 36y_{k-2} - 70y_{k-3} + 56y_{k-4} - 28y_{k-5} + 8y_{k-6} - y_{k-7} + O(h^8)$
9	$y_{k+1} = 9y_k - 36y_{k-1} + 84y_{k-2} - 126y_{k-3} + 126y_{k-4} - 84y_{k-5} + 36y_{k-6} - 9y_{k-7} + y_{k-8} + O(h^9)$
10	$y_{k+1} = 10y_k - 45y_{k-1} + 120y_{k-2} - 210y_{k-3} + 252y_{k-4} - 210y_{k-5} + 120y_{k-6} - 45y_{k-7} + 10y_{k-8} - y_{k-9} + O(h^{10})$

In order to make equation (9) and formula (6) have the same form, we eliminate the derivative terms in equation (9), and the following equation groups can be obtained.

$$\begin{cases} a_0 + a_1 + a_2 = 1, \\ a_0 + 2a_1 + 3a_2 = 0, \\ a_0 + 4a_1 + 9a_2 = 0, \end{cases} \Rightarrow \begin{cases} a_0 = 3, \\ a_1 = -3, \\ a_2 = 1. \end{cases}$$

Now we have the coefficients of the order-3 formula, i.e., those of the following formula:

$$y_{k+1} = 3y_k - 3y_{k-1} + y_{k-2} + O(h^3).$$

This formula has been proposed by another paper [16], which presents a set of data smoothing extrapolation formulas equivalent to the least squares method. However, this paper mainly focuses on the derivation of a group of extrapolation formulas, and the above order-3 formula is a special case of it.

C. Derivation of Order-n Formula

The order-n formula has the following form:

$$y_{k+1} = \sum_{j=0}^{n-1} a_j y_{k-j} + O(h^n), \tag{10}$$

where a_j denotes the coefficient of y_{k-j}. In order to obtain the specific formula of a_j, theoretical derivation is carried out on the basis of Taylor expansion technique. Taylor expansions of $y_{k-n+1}, \ldots, y_{k-1}$ and y_k are listed below:

$$\begin{cases} y_k = y_{k+1} - y'_{k+1} \cdot h + y''_{k+1} \cdot (-h)^2/2! \\ \qquad + \cdots + y_{k+1}^{(n-1)} \cdot (-h)^{n-1}/(n-1)! + O(h^n), \\ y_{k-1} = y_{k+1} - y'_{k+1} \cdot 2h + y''_{k+1} \cdot (-2h)^2/2! \\ \qquad + \cdots + y_{k+1}^{(n-1)} \cdot (-2h)^{n-1}/(n-1)! + O(h^n), \\ y_{k-2} = y_{k+1} - y'_{k+1} \cdot 3h + y''_{k+1} \cdot (-3h)^2/2! \\ \qquad + \cdots + y_{k+1}^{(n-1)} \cdot (-3h)^{n-1}/(n-1)! + O(h^n), \\ \vdots \\ y_{k-n+1} = y_{k+1} - y'_{k+1} \cdot nh + y''_{k+1} \cdot (-n)^2 h^2/2! \\ \qquad + \cdots + y_{k+1}^{(n-1)} \cdot (-nh)^{n-1}/(n-1)! + O(h^n). \end{cases} \tag{11}$$

We first multiply the $(j+1)$th equation in (11) by a_j with $j = 0, 1, \cdots, n-1$, and then add them up. In order to make the resultant equation satisfy the form of formula (10), all the derivative terms should be eliminated. Therefore, we have the following equation group:

$$\begin{cases} a_0 + a_1 \cdots + a_{n-1} = 1, \\ a_0 + 2a_1 + \cdots + na_{n-1} = 0, \\ \vdots \\ a_0 + 2^{n-1}a_1 + \cdots + (n)^{n-1}a_{n-1} = 0. \end{cases}$$

We convert the above equations into a matrix-vector form such as $A\mathbf{x} = \mathbf{b}$, where A represents the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 3 & \ldots & n \\ 1 & 4 & 9 & \ldots & n^2 \\ \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \ldots & n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

After observation, we find that the coefficient matrix is an $n \times n$ Vandermonde matrix which has the following form:

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{bmatrix},$$

and its determinant is as follows:

$$\det(A) = \prod_{1 \le q < p \le n} (x_p - x_q),$$

where x_p and x_q are factors in the $n \times n$ Vandermonde matrix A. According to Cramer's rule, we can calculate the value of a_j, $j = 0, 1, 2, \cdots, n-1$, by the following formula:

$$a_j = \frac{\det(A_j)}{\det(A)} = \frac{\prod_{1 \le q < p \le n} (z_p - z_q)}{\prod_{1 \le q < p \le n} (x_p - x_q)}, \tag{12}$$

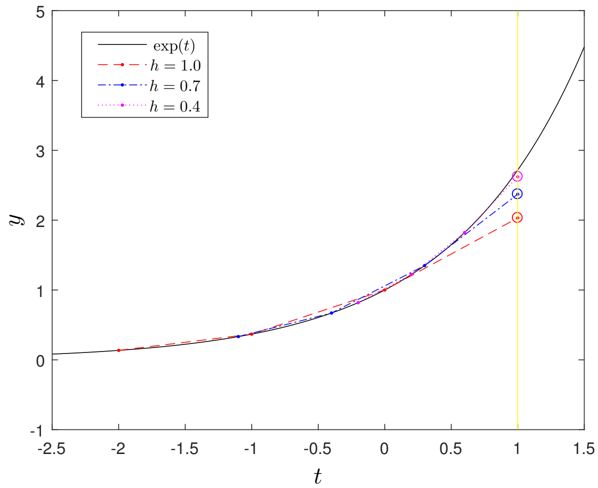

Fig. 1. Extrapolation of $\exp(t)$ using order-3 formula with different step size h values

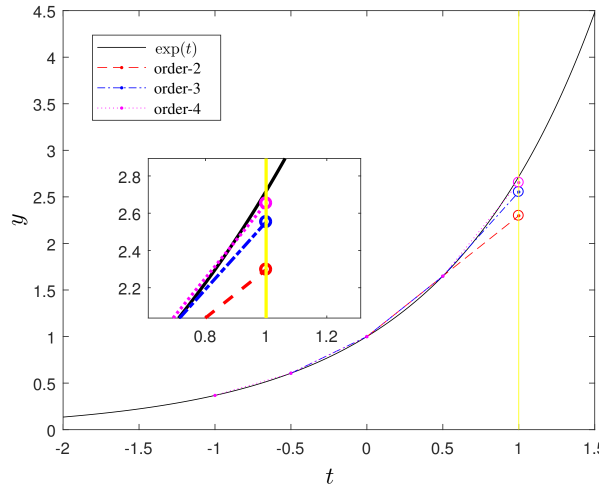

Fig. 2. Extrapolation of $\exp(t)$ using different order formulas with fixed step size $h = 0.5$

TABLE III
ERRORS OF EXTRAPOLATION USING ORDER-3 FORMULA WITH DIFFERENT STEP SIZE VALUES

Step size	Extrapolation value	Error
1	2.031696959722	0.6866
0.1	2.715939255464	0.0023
0.01	2.718279150614	2.6778×10^{-6}
0.001	2.718281825745	2.7142×10^{-9}
0.0001	2.718281828456	2.7187×10^{-12}

where z_p and z_q are factors of the $n \times n$ Vandermonde matrix A_j, which is a matrix formed by replacing the $(j+1)$th column of matrix A with the column vector $\mathbf{b}$. The $n \times n$ matrix A_j has the following form:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_j & 0 & \dots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_j^2 & 0 & \dots & x_n^2 \\ \vdots & & & & & & & \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \dots & x_j^{n-1} & 0 & \dots & x_n^{n-1} \end{bmatrix}.$$

Simplifying the formula (12), we can get the general formula to calculate coefficients:

$$\begin{aligned} a_j &= \frac{(-1)^j j!(j+2)(j+3)\dots n}{j!(n-j-1)!} \\ &= \frac{(-1)^j (j+2)(j+3)\dots n}{(n-j-1)!} \\ &= \frac{(-1)^j (j+1)!(j+2)(j+3)\dots n}{(j+1)!(n-j-1)!} \\ &= (-1)^j \binom{n}{j+1}, \end{aligned} \tag{13}$$

where n is the formula order. Using formula (13), we can get the values of a_j, with $j = 0, 1, 2, \cdots, n-1$, which are shown in TABLE I. Besides, the extrapolation formulas are shown in TABLE II.

TABLE IV
ERRORS OF EXTRAPOLATION USING DIFFERENT ORDER FORMULAS WITH FIXED STEP SIZE $h = 0.5$

Order	Extrapolation value	Error
1	1.6487212707	1.0696
2	2.6936652938	0.0246
3	2.7159392555	0.0023
4	2.7180589032	2.2293×10^{-4}
5	2.7182606143	2.1214×10^{-5}
6	2.7182798097	2.0188×10^{-6}
7	2.7182816363	1.9211×10^{-7}
8	2.7182818102	1.8282×10^{-8}

D. Rules of Coefficients

Actually, we can find that TABLE I has some relationship with the Pascal's triangle in terms of the coefficients. In Pascal's triangle, the number in the nth row and kth column is denoted by $\binom{n}{k}$. The rules of Pascal's triangle can be described as the following formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This means that, in Pascal's triangle, each number equals the sum of its upper left and its upper right numbers. Similarly, in the triangle of TABLE I, we can find the rules that each number equals the number above it minus the number above its left neighbor.

Note that the number in the nth row and kth column are equal to the number in $(n-1)$th row and kth column minus

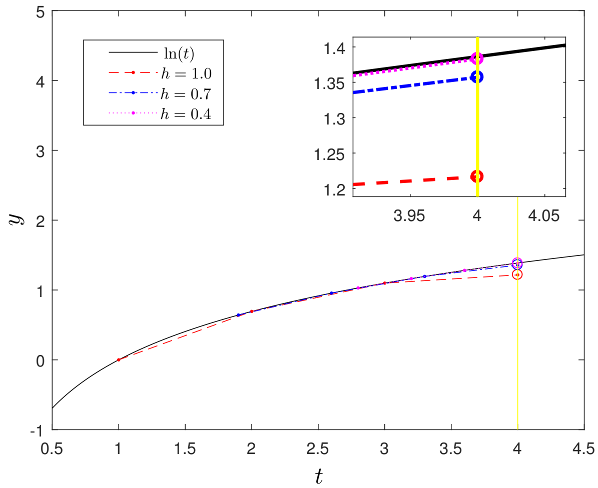

Fig. 3. Extrapolation of $\ln(t)$ using order-3 formula with different step size h values

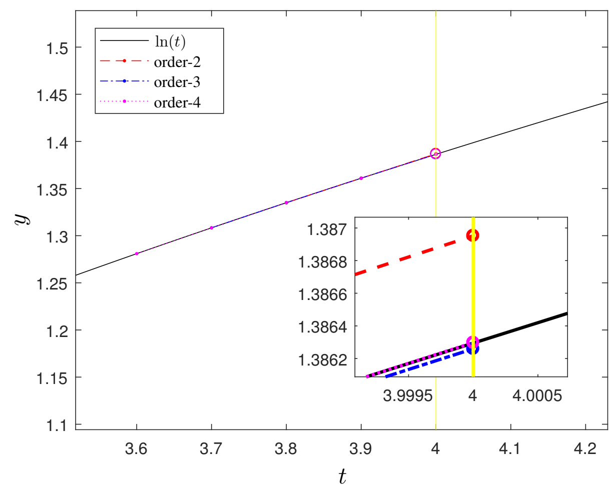

Fig. 4. Extrapolation of $\ln(t)$ using different order formulas with fixed step size $h = 0.1$

TABLE V
ERRORS OF EXTRAPOLATION USING ORDER-3 FORMULA WITH DIFFERENT STEP SIZE VALUES

Step size	Extrapolation value	Error
1	1.216395324324	0.1699
0.1	1.386259278860	3.5082×10^{-5}
0.01	1.386294329515	3.1605×10^{-8}
0.001	1.386294361089	3.1285×10^{-11}
0.0001	1.386294361120	3.1974×10^{-14}

TABLE VI
ERRORS OF EXTRAPOLATION USING DIFFERENT ORDER FORMULAS WITH FIXED STEP SIZE $h = 0.1$

Order	Extrapolation value	Error
1	1.3609765531	0.0253
2	1.3869520395	6.5768×10^{-4}
3	1.3862592789	3.5082×10^{-5}
4	1.3862972453	2.8842×10^{-6}
5	1.3862940360	3.2508×10^{-7}
6	1.3862944083	4.7131×10^{-8}
7	1.3862943527	8.4455×10^{-9}
8	1.3862943629	1.8201×10^{-9}

the number $(n-1)$th row and $(k-1)$th column. The following formula is used to describe the above rule:

$$a_{ij} = -a_{(i-1)(j-1)} + a_{(i-1)j},$$

where, again, a_{ij} denotes the number of column j in row i of TABLE I.

IV. Numerical Experiments

In this section, we make some numerical experiments to justify our formula group. We mainly use three functions, i.e., $\exp(t)$, $\ln(t)$ and $\sin(t)$, to test the effect of our extrapolation formulas. The main purpose of these experiments is to show the validity of our extrapolation formulas, i.e., the correctness of the error terms. In the experiment, we first fix the order of the extrapolation formula and change the step size values to compute the error, then we fix the step size value and change the order of extrapolation formulas to compute the error.

We first use the common function $\exp(t)$ to carry out numerical experiments. In the first part of our experiments, we compare the effects of different step size h values on errors by using order-3 formula to justify the error term. In the second part of our experiments, we compare the effect of different order formulas on errors by fixing the value of step size h.

Finally, we use the function $\ln(t)$ and the function $\sin(t)$ to carry out numerical experiments, and then the results of these experiments are compared with the results of the experiments using the function $\exp(t)$. These experimental results are consistent with the error terms in our extrapolation formulas.

A. Effect of Step Size Values

According to the error terms of the formulas in TABLE II, the magnitude of error is relevant to step size h value. In this subsection, we use order-3 formula as follows to evaluate the effect of different step size values in terms of error.

$$y_{k+1} = 3y_k - 3y_{k-1} + y_{k-2} + O(h^3).$$

In order to show the variation of errors with different step size values more intuitively, we use the step size h values of 1, 0.7 and 0.4 to compute the value of $\exp(t)$ at $t = 1$, respectively, and show them on Fig. 1. It is easy to see from the figure that the predicted value tends to the true value as the step size h value decreases. The exact value of $\exp(t)$ at $t = 1$ is $\exp(1) = 2.718281828459046\cdots$

In order to evaluate the effect of step size values more precisely, we use the step size values of 1, 0.1, 0.01, 0.001 and

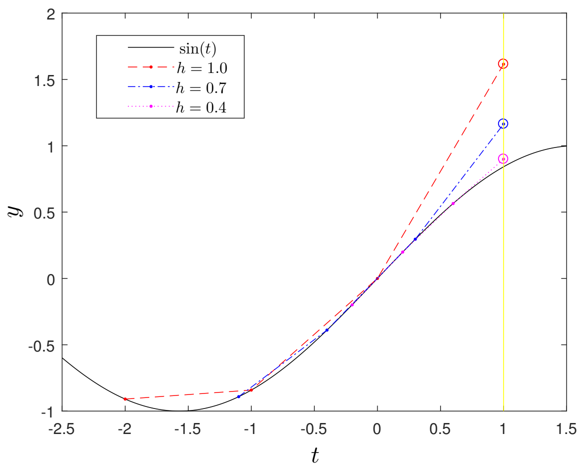

Fig. 5. Extrapolation of sin(t) using order-3 formula with different step size h values

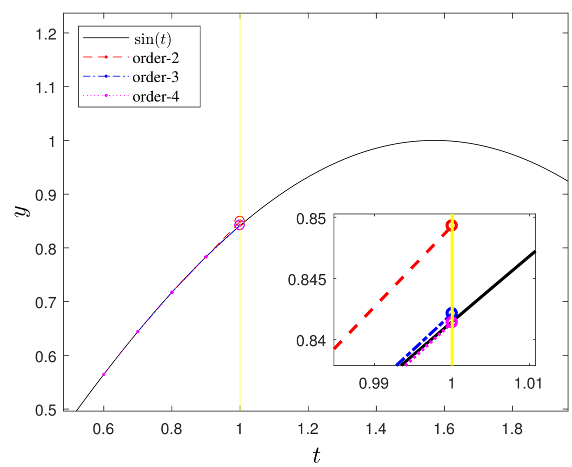

Fig. 6. Extrapolation of sin(t) using different order formulas with fixed step size $h = 0.1$

TABLE VII
ERRORS OF EXTRAPOLATION USING ORDER-3 FORMULA WITH DIFFERENT STEP SIZE VALUES

Step size	Extrapolation value	Error
1	1.615115527598	0.7736
0.1	0.842130143422	6.5916×10^{-4}
0.01	0.099834413024	9.9638×10^{-7}
0.001	0.841470985349	5.4156×10^{-10}
0.0001	0.841470984808	5.4046×10^{-13}

0.0001 to compute the value of error at $t = 1$, respectively, and the results are shown in TABLE III.

We can find that the variation of errors is consistent with the error term $O(h^3)$ in order-3 formula.

B. Effect of Different Order Formulas

According to the error terms of the formulas in TABLE II, different order formulas have different error terms, and the order of error term increases with the order of the formula. In this subsection, we fix the step size value at $h = 0.5$ and compute the extrapolation value of the target function using different order formulas at $t = 1$.

In order to show the variation of errors with different formulas more intuitively, we use the 2nd, 3rd and 4th-order formulas to compute the the value of function $\exp(t)$ at $t = 1$, respectively, and then show the results on Fig. 2. Note that the extrapolation points in the figure are too close to be observed by the naked eye, and thus we partially enlarge the corresponding figure area. If the following figures have the same problem, they will also be partially enlarged. From Fig. 2, we can find that the predicted value tends to the true value as the formula order increases.

In order to evaluate the effect of different order formula more precisely, we use the 2nd, 3rd, $\cdots$, 8th-order formulas to compute the error of the target function at $t = 1$, respectively.

TABLE VIII
ERRORS OF EXTRAPOLATION USING DIFFERENT ORDER FORMULAS WITH FIXED STEP SIZE $h = 0.1$

Order	Extrapolation value	Error
1	0.7833269096	0.0581
2	0.8492977284	0.0078
3	0.8421301434	6.5916×10^{-4}
4	0.8413993687	7.1616×10^{-5}
5	0.8414636831	7.3017×10^{-6}
6	0.8414716274	6.4261×10^{-7}
7	0.8414710642	7.9377×10^{-8}
8	0.8414709792	5.6276×10^{-9}

The results are shown in TABLE IV. Comparing the extrapolation values with the exact value of $\exp(1)$, we can find that the magnitude variation of extrapolation errors is consistent with that of error terms of different order formulas.

C. Extrapolation Results of Other Functions

We use the target functions $\ln(t)$ and $\sin(t)$ to repeat the experiments above-mentioned. We compute the error when using different step size values and the error when using different order formulas, respectively. In the experiments, the extrapolation node of function $\ln(t)$ is $t = 4$, and the extrapolation node of function $\sin(t)$ is $t = 1$. The exact value of $\ln(t)$ at $t = 4$ is $\ln(4) = 1.3862943611199\cdots$, and the exact value of $\sin(t)$ at $t = 1$ is $\sin(1) = 0.8414709848079\cdots$.

Correspondingly, Fig. 3 is the extrapolation image of the function $\ln(t)$ at $t = 4$ by using order-3 formula with step size values of 0.1, 0.7 and 0.4. Besides, Fig. 5 is the extrapolation image of the function $\sin(t)$ in the same cases.

TABLE V shows the error of function $\ln(t)$ at $t = 4$ by using step size values of $1, 0.1, 0.01, 0.001$ and 0.0001 of the order-3 formula, and TABLE VII is about the errors of the target function $\sin(t)$ in the same cases. From TABLE V and

TABLE IX
SPECIFIC VALUES OF COEFFICIENTS OF ORDER-11 FORMULA THROUGH ORDER-18 FORMULA

Order	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
11	11	-55	165	-330	462	-462	330	-165	55	-11	1							
12	12	-66	220	-495	792	-924	792	-495	220	-66	12	-1						
13	13	-78	286	-715	1287	-1716	1716	-1287	715	-286	78	-13	1					
14	14	-91	364	-1001	2002	-3003	3432	-3003	2002	-1001	364	-91	14	-1				
15	15	-105	455	-1365	3003	-5005	6435	-6435	5005	-3003	1365	-455	105	-15	1			
16	16	-120	560	-1820	4368	-8008	11440	-12870	11440	-8008	4368	-1820	560	-120	16	-1		
17	17	-136	680	-2380	6188	-12376	19448	-24310	24310	-19448	12376	-6188	2380	-680	136	-17	1	
18	18	-153	816	-3060	8568	-18564	31824	-43758	48620	-43758	31824	-18564	8568	-3060	816	-153	18	-1

TABLE VII, it can be seen that the magnitude variation of errors is consistent with the variation of the error terms in the formulas.

Correspondingly, Fig. 4 is the extrapolation image of the function $\ln(t)$ using different order formulas with step size value $h = 0.1$. Besides, Fig. 6 is the extrapolation image of the function $\sin(t)$ in the same cases.

TABLE VI shows the error of function $\ln(t)$ at $t = 1$ by using different order formulas with step size value $h = 0.1$, and TABLE VIII shows the error of the function $\sin(t)$ in the same cases. From TABLE VI and TABLE VIII, it is also easy for us to find that the magnitude variation of errors is consistent with the variation of the error terms in the formulas.

V. Conclusion

In this paper, a group of extrapolation formulas have been presented. To do so, Taylor expansion, Vandermonde matrix, Vandermonde determinant, and Cramer's rule have been used to derive the coefficients of the group of formulas. A general formula for computing the coefficients has been proposed. Moreover, the variation rule of coefficients have been analyzed, and we have found that it is similar to that of Pascal's triangle. In the section of numerical experiments, three target functions have been used to substantiate the theoretical results. The effect of step size values on error and the effect of different order formulas on error have been evaluated. The results of numerical experiments have shown that the group of extrapolation formulas has good properties.

Appendix

In addition to TABLE I and TABLE II, we present the coefficients of order-11 formula through order-18 formula in TABLE IX.

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