Instructor's Solution Manual
for
Numerical Methods: Using MATLAB

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Contents

Chapter 1

Preliminaries

1.1 Review of Calculus

1. (a) $L = \lim_{n\to\infty} \frac{4n+1}{2n+1} = 2$
   $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty}(2 - \frac{4n+1}{2n+1}) = 2 = \frac{4}{2} = 0$

   (b) $\lim_{n\to\infty} \frac{2n^2+6n-1}{4n^2+2n+1} = \frac{2}{4} = \frac{1}{2}$
   $\lim_{n\to\infty} \epsilon_n = (\frac{1}{2} - \frac{2n^2+6n-1}{4n^2+2n+1}) = \frac{1}{2} - \frac{1}{2} = 0$

2. (a) $\lim_{n\to\infty} sin(x_n) = sin(\lim_{n\to\infty} x_n) = sin(2)$

   (b) $\lim_{n\to\infty} ln(x_n^2) = ln(\lim_{n\to\infty} x_n^2) = ln(4)$

3. (a) Since f is continuous on $[-1, 0]$; solve

$$\begin{aligned} -x^2 + 2x + 3 &= 2 \\ x^2 - 2x - 1 &= 0 \\ x &= \tfrac{2\pm\sqrt{2^2-4(1)(-1)}}{2} \\ c &= 1 - \sqrt{2} \in [-1, 0] \end{aligned}$$

   (b) Since f is continuous on $[6, 8]$; solve

$$\begin{aligned} \sqrt{x^2 - 5x - 2} &= 3 \\ x^2 - 5x - 11 &= 0 \\ x &= \tfrac{5\pm\sqrt{5^2-4(1)(-11)}}{2} \\ c &= \tfrac{5+\sqrt{69}}{2} \in [6, 8] \end{aligned}$$

4. (a) $f'(x) = 2x - 3 = 0$, thus the critical points are $c = \pm 1$. Thus
   $min\{f(-1), f(1), f(2)\} = min\{5, -1, -1\} = -1$ and
   $max\{f(-1), f(1), f(2)\} = max\{5, -1, -1\} = 5$

   (b) $f'(x) = -2\cos(x)\sin(x) - \cos(x) = -\cos(x)(2\sin(x) + 1) = 0$, thus the critical points are $c = \pi, 7\pi/6, 11\pi/6$. Thus

$min\{f(0), f(\pi), f(7\pi/6), f(11\pi/6), f(2\pi)\} = min\{1, 1, 5/4, 5/4, 1\} =$ 1 and
$max\{f(0), f(\pi), f(7\pi/6), f(11\pi/6), f(2\pi)\} = max\{1, 1, 5/4, 5/4, 1\} =$ 5/4

5. (a) $f'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 0$, thus $c = 0, \pm\sqrt{2} \in [-2, 2]$

   (b)

$$\begin{aligned} f'(x) &= \cos(x) + 2\cos(2x) \\ &= \cos(x) + 2(2\cos^2(x) - 1) \\ &= 4\cos^2(x) + \cos(x) - 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} x &= (-1 \pm \sqrt{33})/8 \\ c &= \cos^{-1}((-1 \pm \sqrt{33})/8), 2\pi - \cos^{-1}((-1 \pm \sqrt{33})/8) \end{aligned}$$

6. (a) $f'(x) = \frac{1}{2\sqrt{x}}$ and $\frac{f(4)-f(0)}{4-0} = \frac{1}{2}$. Solving $\frac{1}{2\sqrt{x}} = \frac{1}{2}$ yields $c = 1$.

   (b) $f'(x) = (x^2 + 2x)/(x + 1)^2$ and $\frac{f(1)-f(0)}{1-0} = \frac{1}{2}$. Solving $f'(x) = (x^2 + 2x)/(x + 1)^2 = \frac{1}{2}$ yields $c = -1 + \sqrt{2}$

7. The given function satisfies the hypotheses of the Generalized Rolle's Theorem. Since $f(0) = f(1) = f(3) = 0$, there exista a $c \in (0, 3)$ such that $f''(c) = 0$. Solve $6c - 8 = 0$ to find $c = 4/3$.

8. (a) $\int_0^2 xe^x dx = xe^x - e^x|_0^2 = e^2 + 1$

   (b) $\int_{-1}^1 \frac{3x}{x^2+1} dx = \frac{3}{2}\ln(x^2 + 1)|_{-1}^1 = 0$ (The integrand is an odd function)

9. (a) $\frac{d}{dx}\int_0^x t^2\cos(t)dt = x^2\cos(x)$

   (b) $\frac{d}{dx}\int_1^{x^3} e^{t^2} dt = e^{(x^3)^2}(3x^2) = 3x^2e^{x^6}$

10. (a) $\frac{1}{4-(-3)}\int_{-3}^4 6x^2dx = \frac{2}{7}x^3|_{-3}^4 = 52$. Solving $6x^2 = 52$ yields $c = \pm\sqrt{26/3} \in [-3, 4]$.

   (b) $\frac{2}{3\pi}\int_0^{3\pi/2} x\cos(x)dx = \frac{2}{3\pi}(x\sin(x) + \cos(x))_0^{3\pi/2} = -(1 + \frac{2}{3\pi})$. Use a calculator to approximate the solution(s): $x\cos(x) = -(1 + \frac{2}{3\pi})$; $c \approx 2.16506, 4.43558 \in [0, 3\pi/2]$.

11. (a) $\frac{1}{1-\frac{1}{2}} = 2$

   (b) $\frac{1}{1-\frac{2}{3}} = 3$

   (c) $\sum_{n=1}^\infty \frac{3}{n(n+1)} = 3\sum_{n=1}^\infty(\frac{1}{n} - \frac{1}{n+1}) = 3\lim_{k\to\infty}\sum_{n=1}^k(\frac{1}{n} - \frac{1}{n+1}) = 3$

   (d) $\sum_{k=1}^\infty \frac{1}{4k^2-1} = \frac{1}{2}\sum_{k=1}^\infty(\frac{1}{2k-1} - \frac{1}{2k+1})$
   $= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^n(\frac{1}{2k-1} - \frac{1}{2k+1}) = \frac{1}{2}$

12. (a) $-\frac{5}{128}(x-1)^4 + \frac{1}{16}(x-1)^3 - \frac{1}{8}(x-1)^2 + \frac{1}{2}(x-1) + 1$
    (b) $4x^2 + 3x + 1$
    (c) $\frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$

13. The Taylor polynomial of degree $n = 4$ expanded about $x_0 = 0$ for $f(x) = \sin(x)$ is $P(x)$.

14. (a) $P(3) = -24$
    (b) $P(-1) = 20$

15. The average area is given by; $\frac{1}{3-1}\int_1^3 \pi r^2 dr = \frac{\pi}{2}(\frac{r^3}{3})_1^3 = \frac{13\pi}{3}$.

16. Any polynomial $P(x)$ satisfies the hypotheses of Rolle's Theorem on the interval $[a, b]$. Thus $P'(x)$ has at least $n-1$ real roots in the interval $[a, b]$, $P''(x)$ has at least $n-2$ real roots in the interval $[a, b], \ldots$, and $P^{(n-1)}$ has at least $n-(n-1) = 1$ real root in the interval $[a, b]$.

17. If f, f' and f'' are defined on the interval $[a, b]$, then f is continuous on the interval $[a, b]$ and f is differentiable on the interval (a, b). By Theorem 1.6 (Mean Value Theorem) there exists numbers $c_1 \in (a, c)$ and $c_2 \in (c, b)$ such that:

$$f'(c_1) = \frac{f(c) - f(a)}{c - a} \text{ and } f'(c_2) = \frac{f(b) - f(c)}{b - c}$$

. But, since $f(a) = f(b) = 0$ it follows that $f'(c_1) = f(c)/(c-a)$ and $f'(c_2) = f(c)/(c-b)$. Given that f' and f'' are defined in the interval $[a, b]$, it follows that f' also satisfies the hypotheses of Theorem 1.6. Thus there exists a number $d \in (a, b)$ such that:

$$f''(d) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{\frac{f(c)}{c-b} - \frac{f(c)}{c-a}}{c_2 - c_1} = \frac{f(c)(b-a)}{(c_2 - c_1)(c-b)(c-a)} < 0,$$

since $f(c) > 0$.

1.2 Binary Numbers

1. Answers will depend on specific platform.

2. (a) 21 (b) 56 (c) 254 (d) 519

3. (a) 0.75 (b) 0.65625 (c) 0.6640625 (d) 0.85546875

4. (a) 1.4140625 (b) 3.1416015625

5. (a) $\sqrt{2} - 1.4140625 = 0.00015109\ldots$
   (b) $\pi - 3.1416015625 = -0.000008908\ldots$

6. (a) $23 = 10111_{two}$

$$\begin{aligned} 23 &= 2(11)+1 & b_0 &= 1 \\ 11 &= 2(5)+1 & b_1 &= 1 \\ 5 &= 2(2)+1 & b_2 &= 0 \\ 2 &= 2(1)+0 & b_3 &= 0 \\ 1 &= 2(0)+1 & b_4 &= 1 \end{aligned}$$

(b) $87 = 1010111_{two}$

$$\begin{aligned} 87 &= 2(43)+1 & b_0 &= 1 \\ 43 &= 2(21)+1 & b_1 &= 1 \\ 21 &= 2(10)+1 & b_2 &= 1 \\ 10 &= 2(5)+0 & b_3 &= 0 \\ 5 &= 2(2)+1 & b_4 &= 1 \\ 2 &= 2(1)+0 & b_5 &= 0 \\ 1 &= 2(0)+1 & b_6 &= 1 \end{aligned}$$

(c) $378 = 101111010_{two}$

(d) $2388 = 100101010100_{two}$

7. (a) 0.0111_{two} (b) 0.1101_{two} (c) 0.10111_{two} (d) 0.1001011_{two}

8. (a) $0.0\overline{0011}_{two}$

(b) $\frac{1}{3} = 0.\overline{d_1 d_2}_{two} = 0.\overline{01}_{two}$

$$\begin{array}{lll} 2R = \frac{2}{3} & d_1 = 0 = INT(\frac{2}{3}) & F_1 = \frac{2}{3} = FRAC(\frac{2}{3}) \\ 2F_1 = \frac{4}{3} & d_2 = 1 = INT(\frac{4}{3}) & F_2 = \frac{1}{3} = FRAC(\frac{4}{3}) \\ 2F_2 = \frac{2}{3} & d_3 = 0 = INT(\frac{2}{3}) & F_3 = \frac{2}{3} = FRAC(\frac{2}{3}) \\ 2F_3 = \frac{4}{3} & d_4 = 1 = INT(\frac{4}{3}) & F_4 = \frac{1}{3} = FRAC(\frac{4}{3}) \\ \vdots & \vdots & \vdots \end{array}$$

(c) $\frac{1}{7} = 0.\overline{d_1 d_2 d_3}_{two} = 0.\overline{001}_{two}$

$$\begin{array}{lll} 2R = \frac{2}{7} & d_1 = 0 = INT(\frac{2}{7}) & F_1 = \frac{2}{7} = FRAC(\frac{2}{7}) \\ 2F_1 = \frac{4}{7} & d_2 = 0 = INT(\frac{4}{7}) & F_2 = \frac{4}{7} = FRAC(\frac{4}{7}) \\ 2F_2 = \frac{8}{7} & d_3 = 1 = INT(\frac{8}{7}) & F_3 = \frac{1}{7} = FRAC(\frac{8}{7}) \\ 2F_3 = \frac{2}{7} & d_4 = 0 = INT(\frac{2}{7}) & F_4 = \frac{2}{7} = FRAC(\frac{2}{7}) \\ \vdots & \vdots & \vdots \end{array}$$

9. (a)

$$\begin{aligned} \tfrac{1}{10} - 0.0001100_{two} &= 0.0\overline{0011}_{two} - 0.0001100_{two} \\ &= 0.00000000\overline{1100}_{two} \\ &= \tfrac{1}{160} \\ &= 0.00625 \end{aligned}$$

(b)

$$\begin{aligned}
\tfrac{1}{7} - 0.0010010_{two} &= 0.\overline{001}_{two} - 0.0010010_{two} \\
&= 0.000000001\overline{001}_{two} \\
&= \tfrac{1}{448} \\
&= 0.0022321428\ldots
\end{aligned}$$

10. In Theorem 1.14 let $c = \frac{1}{8}$ and $r = \frac{1}{8}$, then

$$\frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \cdots = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}$$

11. In Theorem 1.14 let $c = 3/16$ and $r = 1/16$, then

$$\frac{3}{16} + \frac{3}{256} + \frac{3}{4096} + \cdots = \frac{\frac{3}{16}}{1 - \frac{1}{16}} = \frac{1}{5}$$

12. $\frac{1}{2} = \frac{5}{10}$. Assume $\left(\frac{1}{2}\right)^k = \frac{5^k}{10^k}$. Then

$$\begin{aligned}
\left(\frac{1}{2}\right)^{k+1} &= \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right) \\
&= \left(\frac{5^k}{10^k}\right)\left(\frac{5}{10}\right) \\
&= \frac{5^{k+1}}{10^{k+1}}
\end{aligned}$$

Therefore, by the principle of mathematical induction, 2^{-N} can be represented as a decimal number that has N digits.

13. (a)

$$\begin{array}{rcrcr}
\frac{1}{3} & \approx & 0.1011_{two} \times 2^{-1} & = & 0.1011 \times 2^{-1} \\
\frac{1}{5} & \approx & 0.1101_{two} \times 2^{-2} & = & 0.01101 \times 2^{-1} \\
\hline
\frac{8}{15} & & & & 0.100011_{two} \times 2^{0}
\end{array}$$

$$\begin{array}{rcrcr}
\frac{8}{15} & \approx & 0.1001_{two} \times 2^{0} & = & 0.1001_{two} \times 2^{0} \\
\frac{1}{6} & \approx & 0.1011_{two} \times 2^{-2} & = & 0.001011_{two} \times 2^{0} \\
\hline
\frac{7}{10} & & & & 0.101111_{two} \times 2^{0}
\end{array}$$

Thus $(\frac{1}{3} + \frac{1}{5}) + \frac{1}{6} \approx 0.1100_{two}$

(b)

$$\begin{array}{rcrcr}
\frac{1}{10} & \approx & 0.1101_{two} \times 2^{-3} & = & 0.001101 \times 2^{-1} \\
\frac{1}{3} & \approx & 0.1011_{two} \times 2^{-1} & = & 0.101100 \times 2^{-1} \\
\hline
\frac{13}{30} & & & & 0.111001_{two} \times 2^{-1}
\end{array}$$

$$\begin{array}{rcrcl} \frac{13}{30} & \approx & 0.1110_{two} \times 2^{-1} & = & 0.011100_{two} \times 2^{0} \\ \frac{1}{5} & \approx & 0.1101_{two} \times 2^{-2} & = & 0.001101_{two} \times 2^{0} \\ \hline \frac{19}{30} & & & & 0.101001_{two} \times 2^{0} \end{array}$$

Thus $(\frac{1}{10} + \frac{1}{3}) + \frac{1}{5} \approx 0.1010_{two}$

(c)

$$\begin{array}{rcrcl} \frac{3}{17} & \approx & 0.1011_{two} \times 2^{-2} & = & 0.01011 \times 2^{-1} \\ \frac{1}{9} & \approx & 0.1110_{two} \times 2^{-3} & = & 0.001110 \times 2^{-1} \\ \hline \frac{44}{153} & & & & 0.100100_{two} \times 2^{-1} \end{array}$$

$$\begin{array}{rcrcl} \frac{44}{153} & \approx & 0.1001_{two} \times 2^{-1} & = & 0.1001_{two} \times 2^{-1} \\ \frac{1}{7} & \approx & 0.1001_{two} \times 2^{-2} & = & 0.01001_{two} \times 2^{-1} \\ \hline \frac{461}{1071} & & & & 0.11011_{two} \times 2^{-1} \end{array}$$

Thus $(\frac{3}{17} + \frac{1}{9}) + \frac{1}{7} \approx 0.1110_{two} \times 2^{-1}$

(d)

$$\begin{array}{rcrcl} \frac{7}{10} & \approx & 0.1011_{two} \times 2^{0} & = & 0.1011000 \times 2^{0} \\ \frac{1}{9} & \approx & 0.1110_{two} \times 2^{-3} & = & 0.0001110 \times 2^{0} \\ \hline \frac{73}{90} & & & & 0.1100110_{two} \times 2^{0} \end{array}$$

$$\begin{array}{rcrcl} \frac{73}{90} & \approx & 0.1101_{two} \times 2^{0} & = & 0.110100_{two} \times 2^{0} \\ \frac{1}{7} & \approx & 0.1011_{two} \times 2^{-2} & = & 0.001001_{two} \times 2^{0} \\ \hline \frac{601}{630} & & & & 0.111101_{two} \times 2^{0} \end{array}$$

Thus $(\frac{7}{10} + \frac{1}{9}) + \frac{1}{7} \approx 0.1111_{two}$

14. (a) $10 = 101_{three}$
    (b) $23 = 212_{three}$
    (c) $421 = 120121_{three}$
    (d) $1784 = 211002_{three}$

15. (a) $\frac{1}{3} = 0.1_{three}$
    (b) $\frac{1}{2} = 0.\overline{1}_{three}$
    (c) $\frac{1}{10} = 0.\overline{0022}_{three}$
    (d) $\frac{11}{27} = 0.102_{three}$

16. (a) (a) $10 = 20_{five}$
    (b) (b) $35 = 120_{five}$
    (c) (c) $721 = 1034_{five}$
    (d) (d) $734 = 10414_{five}$

17. (a) $\frac{1}{3} = 0.\overline{13}_{five}$
    (b) $\frac{1}{2} = 0.\overline{2}_{five}$
    (c) $\frac{1}{10} = 0.0\overline{2}_{five}$
    (d) $\frac{154}{625} = 0.1104_{five}$

1.3 Error Analysis

1. (a) $x - \hat{x} = 0.00008182, \frac{x-\hat{x}}{x} = 0.0000300998\ldots$, 4- significant digits
   (b) $y - \hat{y} = 350, \frac{y-\hat{y}}{y} = 0.0355871\ldots$, 2-significant digits
   (c) $z = \hat{z} = 0.000008, \frac{z-\hat{z}}{z} = 0.117647$, 0-significant digits
2.

$$\begin{aligned}\int_0^{1/4} e^{x^2}dx &\approx \int_0^{1/4}(1 + x^2 + \tfrac{x^4}{3} + \tfrac{x^6}{3!})dx \\ &= (x + \tfrac{x^3}{3} + \tfrac{x^5}{5(2!)} + \tfrac{x^7}{7(3!)})_{x=0}^{x=1/4} \\ &= \tfrac{1}{4} + \tfrac{1}{192} + \tfrac{1}{10240} + \tfrac{1}{688128} \\ &= \tfrac{292807}{1146880} \approx 0.2553074428 = \hat{p}\end{aligned}$$

3. (a) $p_1 + p_2 = 1.414 + 0.09125 = 1.505$
   $p_1p_2 = (2.1414)(0.09125) = 0.1290$
   (b) $p_1 + p_2 = 31.415 + 0.027182 = 31.442$
   $p_1p_2 = (31.415)(0.27182) = 0.85392$
4. (a) $\frac{0.70711385222-0.70710678110}{0.00001} = \frac{0.00000707103}{0.00001} = 0.707103$ The error involves loss of significance.
   (b) $\frac{0.69317218025-0.6931478056}{0.00005} = \frac{0.00002499969}{0.00005} = 0.4999938$ The error involves loss of significance.
5. (a) $ln(\frac{x+1}{x})$
   (b) $\frac{1}{\sqrt{x^2+1}+x}$
   (c) $cos(2x)$
   (d) $cos(x/2)$
6. (a)

$$\begin{aligned}P(2.72) &= (2.72)^3 - 3(2.72)^2 + 3(2.72) - 1 \\ &= 20.12 - 3(7.398) + 8.16 - 1 \\ &= 20.12 - 22.19 + 8.16 - 1 \\ &= 5.09\end{aligned}$$

$$\begin{aligned}Q(2.72) &= ((2.72 - 3)(2.72) + 3)(2.72) - 1 \\ &= (-0.2800)(2.72) + 3)(2.72) - 1 \\ &= (-0.7616 + 3)(2.72) - 1 \\ &= (2.2384)(2.72) - 1 \\ &= 6.088 - 1 \\ &= 5.088\end{aligned}$$

$$\begin{aligned}R(2.72) &= (2.72 - 1)^3 \\ &= (1.72)^3 \\ &= 5.088\end{aligned}$$

(b)

$$\begin{aligned}
P(0.975) &= (((0.975)^3 - 3(0.975)^2) + 3(0.975)) - 1 \\
&= ((0.9268 - 3(0.9506)) + 2.925) - 1 \\
&= ((0.9268 - 2.852) + 2.925) + 1 \\
&= (-1.925 + 2.925) - 1 \\
&= 1 - 1 = 0
\end{aligned}$$

$$\begin{aligned}
Q(0.975) &= ((0.975 - 3)(0.975) + 3)(0.975) - 1 \\
&= ((-2.025)(0.975) + 3)(0.975) - 1 \\
&= (-1.9774 + 3)(0.975) - 1 \\
&= (1.026)(0.975) - 1 \\
&= 1 - 1 = 0
\end{aligned}$$

$$\begin{aligned}
R(0.975) &= (0.975 - 1)^3 \\
&= (-0.025)^3 \\
&= -0.00001562
\end{aligned}$$

7. (a) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} \approx 0.498$

   (b) $\frac{1}{729} + \frac{1}{243} + \frac{1}{81} + \frac{1}{27} + \frac{1}{9} + \frac{1}{3} \approx 0.499$

8. (a) The propagation of error is $\epsilon_p + \epsilon_q + \epsilon_r$.

   (b)

$$\frac{p}{q} = \frac{\hat{p} + \epsilon_p}{\hat{q} + \epsilon_q} = \frac{\hat{p}}{\hat{q}} + \frac{\epsilon_p + \frac{\hat{p}}{\hat{q}}\epsilon_q}{\hat{q} + \epsilon_q}$$

   Hence, if $1 < |\hat{q}| < |\hat{p}|$, then there is a possibility of magnification of the original error.

   (c)

$$\begin{aligned}
pqr &= (\hat{p} + \epsilon_p)(\hat{q} + \epsilon_q)(\hat{r} + \epsilon_r) \\
&= \hat{p}\hat{q}\hat{r} + \hat{p}\hat{r}\epsilon_q + \hat{q}\hat{r}\epsilon_p + \hat{p}\hat{q}\epsilon_r + \hat{r}\epsilon_p\epsilon_q + \hat{q}\epsilon_p\epsilon_r + \hat{p}\epsilon_q\epsilon_r + \epsilon_p\epsilon_q\epsilon_r \\
&= \hat{p}\hat{q}\hat{r} + (\hat{p}\hat{r}\epsilon_q + \hat{q}\hat{r}\epsilon_p + \hat{p}\hat{q}\epsilon_r) \\
&\quad + (\hat{r}\epsilon_p\epsilon_q + \hat{q}\epsilon_p\epsilon_r + \hat{p}\epsilon_q\epsilon_r) + \epsilon_p\epsilon_q\epsilon_r
\end{aligned}$$

   Depending on the absolute values of $\hat{p}$, $\hat{q}$, and $\hat{r}$, there is a possibility of magnification of the original errors ϵ_p, ϵ_q, and ϵ_r.

9.

$$\begin{aligned}
\tfrac{1}{1-h} + \cos(h) &= 2 + h + \tfrac{h^2}{2} + h^3 + \boldsymbol{O}(h^4) \\
(\tfrac{1}{1-h})\cos(h) &= 1 + h + \tfrac{h^2}{2} + \tfrac{h^3}{2} + \boldsymbol{O}(h^4)
\end{aligned}$$

10.

$$\begin{aligned} e^h + \sin(h) &= 1 + 2h + \tfrac{h^2}{2} + \boldsymbol{O}(h^4) \\ e^h sin(h) &= h + h^2 + \tfrac{h^3}{3} + \boldsymbol{O}(h^5) \end{aligned}$$

An intermediate computation was

$$(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!})(h - \frac{h^3}{3!}) = h + h^2 + \frac{h^3}{3} - \frac{h^5}{24} - \frac{h^6}{36} - \frac{h^7}{144}$$

11.

$$\begin{aligned} \cos(h) + \sin(h) &= 1 + h - \tfrac{h^2}{2} - \tfrac{h^3}{6} + \tfrac{h^4}{24} + \boldsymbol{O}(h^5) \\ \cos(h)\sin(h) &= h - \tfrac{2h^3}{3} + \tfrac{2h^5}{15} + \boldsymbol{O}(h^7) \end{aligned}$$

An intermediate comutation was

$$(1 - \frac{h^2}{2} + \frac{h^4}{24})(h - \frac{h^3}{6} + \frac{h^5}{120}) = h - \frac{2h^3}{3} + \frac{2h^5}{15} - \frac{h^7}{90} + \frac{h^9}{2880}$$

12.

$$\begin{aligned} x_1 &= \tfrac{-b+\sqrt{b^2-4ac}}{2a} \\ &= (\tfrac{-b+\sqrt{b^2-4ac}}{2a})(\tfrac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}}) \\ &= \tfrac{b^2-(b^2-4ac)}{2a(-b-\sqrt{b^2-4ac})} \\ &= \tfrac{-2c}{b+\sqrt{b^2-4ac}} \end{aligned}$$

The case for x_2 is handled in a similar manner.

13. (a) $x_1 = -0.001000$, $x_2 = -1000$
(b) $x_1 = -0.00100$, $x_2 = -10000$
(c) $x_1 = -0.000010$, $x_2 = -100000$
(d) $x_1 = -0.000001$, $x_2 = -1000000$

Chapter 2

The Solution of Nonlinear Equations $f(x) = 0$

2.1 Iteration for Solving $x = g(x)$

1. (a) Clearly, $g(x) \in C[0,1]$. Since $g'(x) = -x/2 < 0$ on the interval $[0,1]$, the function $g(x)$ is strictly decreasing on the interval $[0,1]$. If g is strictly decreasing on $[0,1]$, then $g(0) = 1$ and $g(1) = 0$ imply that $g([0,1]) = [0,1] \subseteq [0,1]$. Thus, by Theorem 2.2, the function $g(x)$ has a fixed point on the interval $[0,1]$.

   In addition: $|f'(x)| = |-x/2| = x/2 \leq 1/2 < 1$ on the interval $[0,1]$. Thus, by Theorem 2.2, the function $g(x)$ has a unique fixed point on the interval $[0,1]$.

   (b) Clearly, $g(x) \in C[0,1]$. Since $g'(x) = -\ln(2)2^{-x} < 0$ on the interval $[0,1]$, the function $g(x)$ is strictly decreasing on the interval $[0,1]$. If g is strictly decreasing on $[0,1]$, then $g(0) = 1$ and $g(1) = 1/2$ imply that $g([0,1]) = [1/2,1] \subseteq [0,1]$. Thus, by Theorem 2.2,the function $g(x)$ has a fixed point on the interval $[0,1]$.

   In addition: $|g'(x)| = |-\ln(2)2^{-2}| = \ln(2)2^{-2} \leq \ln(2) < \ln(e) = 1$ on the interval $[0,1]$. Thus, by Theorem 2.2, the function $g(x)$ has an unique fixed point on the interval $[0,1]$.

   (c) Clearly $g(x)$ is continuous on $[0.5,5.2]$ and $g([0.5,5.2]) \not\subseteq [0.5,5.2]$. But, $g([0.5,2]) \subseteq [0.5,2]$. Thus, the hypotheses of the first part of Theorem 2.2 are satisfied and g has a fixed point in $[0.5,2]$. While $(1,1)$ is the unique fixed point in $[0.5,2]$, $|f'(1)| = 1 \not< 1$, thus the hypothesee in part (4) of Theorem 2.2 cannot be satisfied.

2. (a)

$$\begin{aligned} g(x) &= x \\ -4 + 4x - \tfrac{1}{2}x^2 &= 0 \\ x^2 - 6x + 8 &= 0 \\ x &= 2, 4 \end{aligned}$$

and

$$\begin{aligned} g(2) &= -4 + 8 - 2 &= 2 \\ g(4) &= -4 + 16 - 8 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} p_0 &= 1.9 \\ p_1 &= 1.795 \\ p_2 &= 1.5689875 \\ p_3 &= 1.04508911 \end{aligned}$$

(c)

$$\begin{aligned} p_0 &= 3.8 \\ p_1 &= 3.98 \\ p_2 &= 3.9998 \\ p_3 &= 3.99999998 \end{aligned}$$

(d) For part (b)

$$\begin{aligned} E_0 &= 0.1 & R_0 &= 0.95 \\ E_1 &= 0.205 & R_1 &= 0.1025 \\ E_2 &= 0.43110125 & R_2 &= 0.21550625 \\ E_3 &= 0.95491089 & R_3 &= 0.477455444 \end{aligned}$$

(e) The sequence in part (b) does not converge to $P = 2$. The sequence in part (c) converges to $P = 4$.

3. (a) $p_1 = \sqrt{13}$, $p_2 = \sqrt{6 + \sqrt{13}}$, converges

(b) $p_1 = \frac{3}{2}$, $p_2 = \frac{7}{3}$, converges

(c) $p_1 = 4.083333$, $p_2 = 5.537869$, diverges

(d) $p_1 = -5.5$, $p_2 = -69.5$, diverges

4. The fixed points are $P = 2$ and $P = -2$. Since $g'(2) = 5$ and $g'(-2) = -3$, fixed- point iteration will not converge to $P = 2$ and $P = -2$, respectively.

5.

$$\begin{aligned} x &= x\cos(x) \\ x(1 - \cos(x)) &= 0 \\ x &= 2n\pi \end{aligned}$$

Thus $g(x)$ has infinitely many fixed points: $P = 2n\pi$, where $n \in Z$. Note:

$$|g'(2n\pi)| = |\cos(2n) - 2n\pi\ in(2n\pi)| = 1.$$

Thus Theorem 2.3 may not be used to find the fixed points of $g(x)$.

6. $|p_2 - p_1| = |g(p_1) - g(p_0)| = |g'(c_0)(p_1 - p_0)| < K|p_1 - p_2|$

7. $|E_1| = |P - p_1| = |g(P) - g(p_0)| = |g'(c_0)(P - p_0)| > |P - p_0| = |E_0|$

8. (a) By way of contradiction assume there exists k such that $p_{k+1} = g(p_k) \geq p_k$. It follows that:

$$\begin{aligned} -0.0001p_k^2 + p_k &\geq p_k \\ -0.0001p_k^2 &\geq 0 \\ p_k &= 0 \end{aligned}$$

Thus $p_{k-1} = 0$ or $p_{k-1} = 10,000$. Clearly, $p_{k-1} \neq 10,000$, since the maximum value of $g(x)$ is 2500. Thus, if $p_k = 0$, then $p_{k-1}, \ldots, p_1 = 0$. A contradiction to the hypothesis $p_0 = 1$. Therefore, $p_0 > p_1 > \cdots > p_n > p_{n+1} > \cdots$.

(b) By way of contradiction assume there exists k such that $p_j \leq 0$. It follows that:

$$\begin{aligned} g(p_{j-1}) &\leq 0 \\ -0.0001p_{j-1}^2 + p_{j-1} &\leq 0 \\ (-0.0001p_{j-1} + 1)\, p_{j-1} &\leq 0 \end{aligned}$$

From part (a); if $p_{j-1} = 0$, then $p_1 \neq 0$. Thus $p_{j-1} \neq 0$. If $p_{j-1} < 0$, then

$$\begin{aligned} -0.0001p_{j-1} + 1 &\geq 0 \\ p_{j-1} &\geq 10,000, \end{aligned}$$

a contradiction. If $p_{j-1} > 0$, then

$$\begin{aligned} -0.0001p_{j-1} + 1 &\geq 0 \\ p_{j-1} &\leq 10,000, \end{aligned}$$

a contradiction. Therefore, $p_n > 0$ for all n.

(c) $\lim_{n\to\infty} p_n = 0$

9. (a) $g(3) = (0.5)(3) + 1.5 = 3$

(b) $|P - p_n| = |3 - 1.5 - 0.5p_{n-1}| = |1.5 - 0.5p_{n-1}| = \frac{1}{2}|3 - p_{n-1}| = \frac{1}{2}|P - p_{n-1}|$

(c) Using mathematical induction we note that $|P - p_1| = \frac{1}{2}|P - p_0|$ and assume that $|P - p_k| = \frac{1}{2^k}|P - p_0|$. Thus

$$\begin{aligned} |P - p_{k+1}| &= \tfrac{|P-p_k|}{2} \\ &= \tfrac{|P-p_0|}{2(2^k)} \\ &= \tfrac{|P-p_0|}{2^{k+1}} \end{aligned}$$

10. (a) Note: $p_1 = p_0/2,\ p_2 = p_0/2^2, \ldots, p_{k+1} = p_0/2^{k+1}, \ldots$. Thus

$$\frac{|p_{k+1} - p_k|}{|p_{k+1}|} = \frac{|2^{-k-1} - 2^{-k}|}{|2^{-k-1}|} = \frac{2^{-k}(1 - 2^{-1})}{2^{-k}2^{-1}} = 1$$

(b) Clearly, the stopping criteria will (theoretically) never be satisfied.

11. In inequality (11): $|P - p_n| \le K^n|P - p_0|$, where $|g'(x)| \le K < 1$. Therefore, the smaller the value of K the faster fixed-point iteration converges.

2.2 Bracketing Methods for Locating a Root

1.

$$\begin{array}{lllll} I_0 = (0.11 + 0.12)/2 &= 0.115 & A(0.115) &= 254,403 \\ I_1 = (0.11 + 0.115)/2 &= 0.1125 & A(0.1125) &= 246,072 \\ I_2 = (0.1125 + .125)/2 &= 0.11375 & A(0.11375) &= 250,198 \end{array}$$

2.

$$\begin{array}{lllll} I_0 = (0.13 + 0.14)/2 &= 0.135 & A(0.135) &= 394,539 \\ I_1 = (0.135 + 0.14)/2 &= 0.1375 & A(0.1375) &= 408,435 \\ I_2 = (0.135 + 0.1375)/2 &= 0.13625 & A(0.13625) &= 401,420 \end{array}$$

3. (a) $f(-3) > 0$, $f(0) < 0$, and $f(3) > 0$; thus roots lie in the intervals $[-3, 0]$ and $[0, 3]$.

(b) $f(\pi/4) > 0$ and $f(\pi/2) < 0$; thus a root lies in the interval $[\pi/4, \pi/2]$.

(c) $f(3) < 0$ and $f(5) > 0$; thus a root lies in the interval $[3, 5]$.

(d) $f(3) > 0$, $f(5) < 0$, and $f(7) > 0$; thus roots lie in the intervals $[3, 5]$ and $[5, 7]$.

4. $[-2.4, -1.6], [-2.0, -1.6], [-2.0, -1.8], [-1.9, -1.8], [-1.85, -1.80]$

5. $[0.8, 1.6], [1.2, 1.6], [1.2, 1.4], [1.2, 1.3], [1.25, 1.30]$

6. $[3.2, 4.0], [3.6, 4.0], [3.6, 3.8], [3.6, 3.7], [3.65, 3.70]$

7. $[6.0, 6.8], [6.4, 6.8], [6.4, 6.6], [6.5, 6.5], [6.40, 6.45]$

8. (a) Starting with $a_0 < b_0$, then either $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$, or $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$. In either case we have $a_0 \leq a_1 < b_1 \leq b_0$. Now assume that the result is true for $n = 1, 2, \ldots, k$; in particular $a_0 \leq a_1 \leq \cdots \leq a_k < b_k \leq \cdots \leq b_1 \leq b_0$. Then either $a_{k+1} = a_k$ and $b_{k+1} = \frac{a_k+b_k}{2}$, or $a_{k+1} = \frac{a_k+b_k}{2}$ and $b_{k+1} = b_k$. In either case we have $a_k \leq a_{k+1} < b_{k+1} \leq b_k$. Hence $a_0 \leq a_1 \leq \cdots \leq a_k \leq a_{k+1} < b_{k+1} \leq b_k \leq \cdots \leq b_1 \leq b_0$. Thus by mathematical induction we have proven that $a_0 \leq a_1 \leq \cdots \leq a_n < b_n \leq \cdots \leq b_1 \leq b_0$ for all n.

   (b) From part (a) either $a_1 = a_0$, $b_1 = \frac{a_0+b_0}{2}$, and $b_1 = a_1 = \frac{b_0-a_0}{2}$ or $a_1 = \frac{a_0+b_0}{2}$, $b_1 = b_0$, and $b_1 - a_1 = \frac{b_0-a_0}{2}$. Now assume that the result is true for $n = 1, 2, \ldots, k$, in particular $b_k - a_k = \frac{b_0 - a_0}{2^k}$. Then either $a_{k+1} = a_k$, $b_{k+1} = \frac{a_k+b_k}{2}$, and $b_{k+1} - a_{k+1} = \frac{b_k-a_k}{2} = \frac{b_0-a_0}{2^{k+1}}$ or $a_{k+1} = \frac{a_k+b_k}{2}$, $b_{k+1} = b_k$, and $b_{k+1} - a_{k+1} = \frac{b_k-a_k}{2} = \frac{b_0-a_0}{2^{k+1}}$. Thus by mathematical induction we have proven that $b_n - a_n = \frac{b_0-a_0}{2^n}$ for all n.

   (c) Using part (c) it follows that the sequence $\{a_n\}$ is non-decreasing and bounded above by b_0, hence it is a convergent sequence and we write $\lim_{n\to\infty} a_n = L_1$. Similarly, the sequence $\{b_n\}$ is non-increasing and bounded below by a_0, hence it is a convergent sequence and we write $\lim_{n\to\infty} b_n = L_2$

   To show that the two limits are equal we observe that

$$\begin{aligned} L_2 &= \lim_{n\to\infty} b_n \\ &= \lim_{n\to\infty}(a_n + (b_n - a_n)) \\ &= \lim_{n\to\infty} a_n + \lim_{n\to\infty}(b_n - a_n) \\ &= L_1 + \lim_{n\to\infty} \tfrac{b_0-a_0}{2^n} \\ &= L_1 + 0 = L_1 \end{aligned}$$

   Since $a_n \leq c_n \leq b_n$ the squeeze principle for limits implies that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n \lim_{n\to\infty} b_n$$

9. (a) The function does not change sign on the interval $[3, 7]$.

   (b) $\lim_{n\to\infty} a_n = 2 = \lim_{n\to\infty} b_n$, but $f(x)$ is undefined at 2.

10. (a) It will converge to the zero at $x = \pi$.

   (b) $\lim_{n\to\infty} a_n = \pi/2 = \lim_{n\to\infty} b_n$, but $f(x)$ is undefined at $\pi/2$.

11. Solve:

$$\begin{aligned} \tfrac{7-2}{2^N} &< 5 \times 10^{-9} \\ \ln(5) - N\ln(2) &< \ln(5 \times 10^{-9}) \\ N &> \tfrac{]ln(5) - \ln(5\times10^{-9})}{\ln(2)} \\ N &> 29.89735 \end{aligned}$$

Thus $N = 30$.

12.

$$\begin{aligned} c_n &= b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)} \\ &= \frac{b_n(f(b_n) - f(a_n)) - f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)} \\ &= \frac{-b_n f(a_n) + a_n f(b_n)}{f(b_n) - f(a_n)} \\ &= \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)} \end{aligned}$$

13.

$$\begin{aligned} \frac{|b-a|}{2^{N+1}} &< \delta \\ \ln\left(\frac{|b-a|}{2^{N+1}}\right) &< \ln(\delta), \end{aligned}$$

since ln is a strictly increasing function. Thus

$$\begin{aligned} \ln(b-a) - (N+1)\ln(2) &< \ln(\delta) \\ \frac{\ln(b-a) - \ln(\delta)}{\ln(2)} &< N+1 \\ N &> \frac{\ln(b-a) - \ln(\delta)}{\ln(2)} - 1 \end{aligned}$$

Therefore, the smallest value of N is

$$N = int\left(\frac{\ln(b-a) - \ln(\delta)}{\ln(2)}\right)$$

14. The bisection method can't converge to $x = 2$, unless $c_n = 2$ for some $n \geq 1$.

15. We refer the reader to "Which Root Does the Bisection Algorithm Find?" by George Corliss, Mathematical Modeling: Classroom Notes in Applied Mathematics, Murray Klankin Ed., SIAM, 1987.

2.3 Initial Approximation and Convergence Criteria

1. Approximate root location -0.7. Computed root -0.7034674225.
2. Approximate root location 0.7. Computed root 0.7390851332.
3. Approximate root locations -1.0 and 0.6. Computed roots -1.002966954 and 0.6348668712.

4. Approximate root locations ± 1.8. Computed roots ± 1.807375379.

5. Approximate root locations 1.4 and 3.0. Computed roots 1.412391172 and 3.057103550.

6. Approximate root locations ± 1.2 and 0.

2.4 Newton-Raphson and Secant Methods

1. (a) $p_k = p_{k-1} - \frac{p_{k-1}^2 - p_{k-1} + 2}{2p_{k-1} - 1}$
   (b)
$$\begin{aligned} p_0 &= -1.5 \\ p_1 &= -0.0625 \\ p_2 &= 1.7743 \\ p_3 &= 0.4505 \end{aligned}$$

2. (a) $p_k = p_{k-1} - \frac{p_{k-1}^2 - p_{k-1} - 3}{2p_{k-1} - 1}$
   (b)
$$\begin{aligned} p_0 &= 1.6 \\ p_1 &= 2.52727 \\ p_2 &= 2.31521 \\ p_3 &= 2.30282 \end{aligned}$$
   (c)
$$\begin{aligned} p_0 &= 0.0 \\ p_1 &= -3.0 \\ p_2 &= -1.7143 \\ p_3 &= -1.3416 \\ p_4 &= -1.3410 \end{aligned}$$

3. (a) $p_k = p_{k-1} - \frac{1}{4}(p_{k-1} - 1)$
   (b)
$$\begin{aligned} p_0 &= 2.1 \\ p_1 &= 2.075 \\ p_2 &= 2.05625 \\ p_3 &= 2.0421875 \\ p_4 &= 2.031640625 \end{aligned}$$
   (c) Convergence is linear. The error is reduced by a factor of $\frac{3}{4}$ with each iteration.

4. (a) $p_k = p_{k-1} - \frac{p_{k-1}^3 - 3p_{k-1} - 2}{3p_{k-1}^2 - 3}$
   (b)

$$\begin{aligned} p_0 &= 2.1 \\ p_1 &= 2.00606062 \\ p_2 &= 2.00002434 \\ p_3 &= 2.00000000 \\ p_4 &= 2.00000000 \end{aligned}$$

   (c) Convergence is quadratic. The number of accurate decimal places (roughly) doubles with each iteration.

5. (a) $p_k = p_{k-1} + \frac{1}{\tan(p_{k-1})}$
   (b) No, $p_0 = 3$, $p_1 = -4.01525$. The sequence $\{p_k\}$ converges to $-\frac{3\pi}{2}$.
   (c) Yes, $p_0 = 5$, $p_1 = 4.70149$. The sequence $\{p_k\}$ converges to $\frac{3\pi}{2}$.

6. (a) $p_k = p_{k-1} - (1 + p + k - 1^2)\arctan(p_{k-1})$
   (b) i.

$$\begin{aligned} p_0 &= 1.0 \\ p_1 &= -0.570796327 \\ p_2 &= -0.116859904 \\ p_3 &= -0.001061022 \\ p_4 &= 0.000000001 \end{aligned}$$

   ii. $\lim_{n\to\infty} p_k = 0.0$
   (c) i.

$$\begin{aligned} p_0 &= 2.0 \\ p_1 &= -3.535743590 \\ p_2 &= 13.95095909 \\ p_3 &= -279.344667 \\ p_4 &= 122016.9990 \end{aligned}$$

   ii. The sequence is a case of divergent oscillation.

7. (a) $p_k = p_{k-1} - \frac{p_{k-1}}{1 - p_{k-1}}$
   (b) i.

$$\begin{aligned} p_0 &= 0.20 \\ p_1 &= -0.05 \\ p_2 &= -0.002380952 \\ p_3 &= -0.000005655 \\ p_4 &= -0.000000000 \end{aligned}$$

   ii. $\lim_{n\to\infty} p_k = 0.0$

(c) i.

$$
\begin{aligned}
p_0 &= 20.0 \\
p_1 &= 21.05263158 \\
p_2 &= 22.10250034 \\
p_3 &= 23.14988809 \\
p_4 &= 24.19503505
\end{aligned}
$$

ii. $\lim_{n\to\infty} p_k = \infty$

(d) $f(p_4) \approx 0.00000000075155$

8. $p_2 = 2.41935484$, $p_3 = 2.41436464$

9. $p_2 = 2.46371308$, $p_3 = 2.27027831$

10. $p_2 = -1.52140264$, $p_3 = -1.52137968$

11. Following the procedure outlined in Corollary 2.2, we assume that A is a real number and find the Newton-Raphson iteration function $g(x)$ for the function $f(x) = x^3 - A$. Thus

$$
\begin{aligned}
g(x) &= x - \frac{f(x)}{f'(x)} \\
&= x - \frac{x^3 - A}{3x^2} \\
&= x - \frac{x - \frac{A}{x^2}}{3} \\
&= \frac{2x + \frac{A}{x^2}}{3}
\end{aligned}
$$

Now let p_0 be an initial approximation to $\sqrt[3]{A}$. Thus the Newton- Raphson iteration is defined by

$$
p_k = \frac{2p_{k-1} + A/p_{k-1}^2}{3}
$$

for $k = 1, 2, \ldots$.

12. (a) $\sqrt[N]{A}$

(b) Following the procedure outlined in Corollary 2.2, we assume that A is an appropriate real number and find the Newton-Raphson iteration function $g(x)$ for the function $f(x) = x^N - A$. Thus

$$
\begin{aligned}
g(x) &= x - \frac{f(x)}{f'(x)} \\
&= x - \frac{x^N - A}{Nx^{N-1}} \\
&= x - \frac{x - \frac{A}{x^{N-1}}}{N} \\
&= \frac{(N-1)x + \frac{A}{x^{N-1}}}{N}
\end{aligned}
$$

Now let p_0 be an initial approximation to $\sqrt[N]{A}$. Thus the Newton-Raphson iteration is defined by

$$p_k = \frac{(N-1)p_{k-1} + A/p_{k-1}^{N-1}}{N}$$

for $k = 1, 2, \ldots$.

13. No, because $f(x)$ has no real zeros.

14. No, because $f'(x)$ is not continuous at the root $x = 0$.

15. No, because $f(x)$ is not defined on an interval about the root $x = 0$.

16. From (12) and (13) we see that (11) is the Newton-Raphson recursive rule for the function $f(x) = x^2 - A$. The zeros of f are $\pm\sqrt{A}$. It follows from Theorem 2.5 that there is a p_0 such that (11) converges to $\sqrt{A}$.

17. (a) $g(p) = p - \frac{f(p)}{f'(p)} = p$ which implies that $-\frac{f(p)}{f'(p)} = 0$, which implies that $f(p) = 0$.

    (b) $g'(p) = 1 - \frac{(f'(p))^2 - f(p)f''(p)}{(f'(p))^2} = \frac{f(p)f'(p)}{(f'(p))^2} = \frac{0}{(f'(p))^2} = 0$. Since $g'(p) = 0$ and $g'(p)$ is a continuous function, choose $\epsilon = 1$. Then there exists an interval $(p-d, p+d)$ in which $|g'(x)| < \epsilon$ or $|g'(x)| < 1$. Therefore, Theorem 2.2 implies that $\lim_{n\to\infty} p_n = p$.

18. (a) Given

$$0 = f(p_k) + f'(p_k)(p - p_k) + \frac{1}{2}f''(c_k)(p - p_k)^2$$

then

$$\begin{aligned} f(p_k) + f'(p_k)(p - p_k) &= -\frac{1}{2}f''(c_k)(p - p_k)^2 \\ (p - p_k) + \frac{f(p_k)}{f'(p_k)} &= -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2 \end{aligned}$$

    (b) The last expression in part (a) can be written as:

$$\begin{aligned} p - \left(p_k - \frac{f(p_k)}{f'(p_k)}\right) &= -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2 \\ p - p_{k+1} &= -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2 \end{aligned}$$

Assuming $f'(p_k) \approx f'(p)$ and $f''(c_k) \approx f''(p)$ when k is sufficiently large yields

$$\begin{aligned} p - p_{k+1} &\approx -\frac{f''(p)}{2f'(p)}(p - p_k)^2 \\ E_{k+1} &\approx -\frac{f''(p)}{2f'(p)}E_k \\ |E_{k+1}| &\approx \frac{|f''(p)|}{2|f'(p)|}|E_k| \end{aligned}$$

19. (a) If $1/4 \leq q < 1$, then

$$-2 \leq \log_2(q) < 0$$
$$-2 + 2m \leq log_2(q) + 2m < 2m$$
$$\frac{1}{4}(2^{2m}) \leq q \times 2^{2m} < 2^{2m}$$

By the Squeeze or Sandwich Theorem $lim_{m \to -\infty} q \times 2^{2m} = 0$ and $lim_{m \to \infty} q \times 2^{2m} = \infty$. Therefore, if $A \in \Re^+$, then there exists $m \in Z$ and $q \in [1/4, 1)$ such that $A = q \times 2^{2m}$.

(b) If $A \in \Re^+$ then $\sqrt{A} = \sqrt{q \times 2^{2m}} = q^{1/2} \times 2^m$.

20. (a)

$$\begin{aligned} p_{k+1} &= p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})} \\ &= \frac{p_k(f(p_k) - f(p_{k-1})) - f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})} \\ &= \frac{-p_k f(p_{k-1}) + p_{k-1} f(p_k)}{f(p_k) - f(p_{k-1})} \\ &= \frac{p_{k-1} f(p_k) - p_k f(p_{k-1})}{f(p_k) - f(p_{k-1})} \end{aligned}$$

(b) As the number of iterations increases the precision of the difference in the numerator can lead to a reduction in the precision of p_{k+1}.

21. If p is a root of multiplicity $M = 2$, then $f(x) = (x - p)^2 q(x)$ and $q(p) \neq 0$. Consider

$$\begin{aligned} h(x) &= x - \frac{2f(x)}{f'(x)} \\ &= x - \frac{2(x-p)^2 q(x)}{((x-p)^2 q(x))'} \\ &\approx x - \frac{2(x-p)q(x)}{(2(x-p)q(x))'} \\ &= x - \frac{k(x)}{k'(x)} \end{aligned}$$

Since p is a root of multiplicity $M = 1$ of $k(x)$ it follows that the Newton-Raphson method

$$p_k = p_{k-1} - \frac{2f(p_{k-1})}{f'(p_{k-1})}$$

converges quadratically.

22. (a) Halley's formula for finding $\sqrt{A}$ is:

$$g(x) = x - \frac{x^2 - A}{2x}\left(1 - \frac{(x^2 - A)(2)}{2(2x)^2}\right)^{-1} = \frac{x(x^2 + 3A)}{3x^2 + A}$$

When $A = 5$, Halley's iteration formula becomes

$$p_k = \frac{p_{k-1}^3 + 15p_{k-1}}{3p_{k-1}^2 + 5}$$

and $p_1 = 2.2352941176$, $p_2 = 2.2360679775$, and $p_3 = 2.2360679775$.

(b) Halley's formula for $f(x) = x^3 - 3x + 2$ is $g(x) = \frac{x^3+2x^2+4x+2}{2x^2+4x+3}$ and $p_1 = -2.0130081301$, $p_2 = -2.0000007211$, and $p_3 = -2.0000000000$.

23. (a)

$$\begin{aligned}
h(x) &= \frac{f(x)}{f'(x)} \\
&= \frac{(x-p)^M q(x)}{M(x-p)^{M-1}q(x) + (x-p)^M q'(x)} \\
&= \frac{(x-p)^M q(x)}{(x-p)^{M-1}(Mq(x) + (x-p)q'(x))} \\
&= (x-p)\left(\frac{q(x)}{Mq(x) + (x-p)q'(x)}\right) \\
&= (x-p)s(x)
\end{aligned}$$

Note

$$s(p) = \frac{q(p)}{Mq(p)} = \frac{1}{M} \neq 0.$$

Therefore, $h(x)$ has a simple root at p.

(b) From (5) the Newton-Raphson iterative function for $h(x)$ is

$$g(x) = x - \frac{h(x)}{h'(x)}.$$

Making the substitution $h(x) = f(x)/f'(x)$ yields

$$\begin{aligned}
g(x) &= x - \frac{f(x)/f'(x)}{\left(\frac{f(x)}{f'(x)}\right)'} \\
&= x - \frac{f(x)/f'(x)}{\frac{f'(x)f'(x) - f(x)f''}{(f'(x))^2}} \\
&= x - \frac{f(x)f'(x)}{(f'(x))^2 - f(x)f''(x)}
\end{aligned}$$

(c) The iteration function g in part (b) is the Newton-Raphson iterative function of a function h with a simple root at p. Therefore, by Theorem 2.6, iteration using g in part (b) converges quadratically to p

(d) $p_1 = 0.78253783237921$, $p_2 = 0.26558132223138$, $p_3 = 0.00018628551512$

24. It appears that the error in each successive iteration is proportional to the cube of the error in the previous iteration: $E_{n+1} \approx AE_n^3$, i.e.; $R = 3$. The value $A = 3/4$ is a reasonable estimate for the proportionality constant.

2.5 Aitken's Process and Steffensen's and Muller's Methods

1. (a) $\Delta p_n = 0$

   (b) $\Delta p_n = 6(n+1) + 2 - 6n - 2 = 6$

   (c) $\Delta p_n = (n+1)(n+2) - n(n+1) = 2(n+1)$

2. (a) $\Delta^2 p_n = \Delta(\Delta p_n) = \Delta(2(n+1)^2 + 1 - 2n^2 - 1) = \Delta(4n+2) = 4$

   (b) $\Delta^3 p_n = \Delta(\Delta^2 p_n) = \Delta(4) = 0$

   (c) $\Delta^4 p_n = \Delta(\Delta^3 p_n) = \Delta(0) = 0$

3.
$$\Delta p_n = \Delta(1/2^{n+1} - 1/2^n) = -1/2^{n+1}$$

and

$$\Delta^2 p_n = \Delta(\Delta p_n) = \Delta(-1/2^{n+1}) = -\frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}$$

thus

$$q_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = \frac{1}{2^n} - \frac{1/2^{2n+2}}{1/2^{n+2}} = \frac{1}{2^n} - \frac{1}{2^n} = 0$$

4.
$$\Delta p_n = \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)}$$

and

$$\begin{aligned} \Delta p_n^2 &= \Delta(\Delta p_n) \\ &= \Delta\left(-\tfrac{1}{n(n+1)}\right) \\ &= -\tfrac{1}{(n+1)(n+2)} + \tfrac{1}{n(n+1)} \\ &= \tfrac{2}{n(n+1)(n+2)} \end{aligned},$$

thus

$$q_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = \frac{1}{n} - \frac{\left(-\frac{1}{n(n+1)}\right)}{\frac{2}{n(n+1)(n+2)}} = \frac{1}{2(n+1)}$$

5.

$$\begin{aligned}
q_n &= p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} \\
&= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= \frac{p_n p_{n+2} - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= \frac{\frac{1}{2^n-1}\frac{1}{2^{n+2}-1} - \frac{1}{(2^{n+1}-1)^2}}{\frac{1}{2^{n+2}-1} - \frac{2}{2^{n+1}-1} + \frac{1}{2^n-1}} \\
&= \frac{2^n}{2^n(2^{n+1}-1)(2^{n+1}+1)} \\
&= \frac{1}{(2^2)^{n+1}-1} = \frac{1}{4^{n+1}-1}
\end{aligned}$$

6. $p_n = 1/(4^n + 4^{-n})$

n	p_n	q_n Aitken's
0	0.5	-0.26437542
1	0.23529412	-0.00158492
2	0.06225681	-0.00002390
3	0.01562119	-0.00000037
4	0.00390619	
5	0.00097656	

7. $g(x) = (6 + x)^{1/2}$

n	p_n	q_n Aitken's
0	2.5	3.00024351
1	2.91547595	3.00000667
2	2.98587943	3.00000018
3	2.99764565	3.00000001
4	2.99960758	
5	2.99993460	

8. $g(x) = \ln(x + 2)$

n	p_n	q_n Aitken's
0	3.14	3.14619413
1	3.14422280	3.14619331
2	3.14556674	3.14619323
3	3.14599408	3.14619322
4	3.14612992	
5	3.14617310	

9. $\cos(x) - 1 = 0$

n	p_n Steffensen's
0	0.5
1	0.24465808
2	0.12171517
3	0.00755300
4	0.00377648
5	0.00188824
6	0.00000003

10. In formula (4) let $p_n = S_n$ and $q_n = T_n$, then $\Delta S_n = S_{n+1} - S_n = A_{n+1}$ and $\Delta^2 S_n = \Delta(\Delta S_n) = \Delta A_{n+1} = A_{n+2} - A_{n+1}$. Substituting into formula (4) yields $T_n = S_n - \frac{A_{n+1}^2}{A_{n+2} - A_{n+1}}$.

11. The sum of the series is 99.

n	S_n	T_n
1	0.99	98.9999988
2	1.9701	99.0000017
3	2.940399	98.9999988
4	3.90099501	98.9999992
5	4.85198506	
6	5.79346521	

12. The sum is $S \approx 0.31838039$

n	S_n	T_n
1	0.23529412	0.31840462
2	0.29755093	0.31838076
3	0.31317211	0.31838039
4	0.31707830	0.31838039
5	0.31805487	
6	0.31829901	

13. The sum of the series if 4.

n	S_n	T_n
1	1.0	5.0
2	2.0	4.25
3	2.75	4.08333333
4	3.25	4.031215
5	3.5625	4.0125
6	3.75	4.00520833
7	3.859375	4.00223215
8	3.921875	4.0097656

14. The sum of the series is $\ln(2)$.

n	S_n	T_n
1	0.5	0.6875
2	0.625	0.69166667
3	0.66666667	0.69270833
4	0.68229167	0.69300595
5	0.68854167	0.69309896
6	0.69114583	0.69312996
7	0.69226191	0.69314081
8	0.69275019	0.69314476

15. $f(x) = x^3 - x - 3$.

n	p_n	$f(p_n)$
0	1.0	-2.0
1	1.2	-1.472
2	1.4	-0.656
3	1.52495614	0.02131598
4	1.52135609	-0.00014040
5	1.52137971	-0.00000001

16. $f(x) = 4x^2 - e^x$.

n	p_n	$f(p_n)$
0	4.0	9.40184997
1	4.1	6.89971240
2	4.2	3.87366896
3	4.30844335	-0.07396483
4	4.30657286	0.00047140
5	4.30658473	0.00000005

17. (a)

$$\begin{aligned}
\Delta(p_n + q_n) &= (p_{n+1} + q_{n+1}) - (p_n + q_n) \\
&= (p_{n+1} - p_n) + (q_{n+1} - q_n) \\
&= \Delta p_n + \Delta q_n
\end{aligned}$$

(b)

$$\begin{aligned}
\Delta(p_n q_n) &= p_{n+1}q_{n+1} - p_n q_n \\
&= p_{n+1}q_{n+1} - p_{n+1}q_n + p_{n+1}q_n - p_n q_n \\
&= p_{n+1}\Delta q_n + q_n \Delta p_n
\end{aligned}$$

18.

$$
\begin{aligned}
p &\approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} - p_{n+2} + \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} + \frac{p_{n+2}p_n - p_{n+1}^2 - p_{n+2}^2 + 2p_{n+2}p_{n+1} - p_n p_{n+2}}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} - \frac{p_{n+2} - 2p_{n+2}p_{n+1} + p_{n+1}}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} - \frac{(p_{n+2} - p_{n+1})^2}{p_{n+2} - 2p_{n+1} + p_n}
\end{aligned}
$$

19. (a) $E_N = K^N E_o$

(b) From part (a), if $E_N = K^N E_o$, then

$$
\begin{aligned}
|E_N| &= |K^N E_0| \\
|K|^N &< \frac{10^{-8}}{|E_0|} \\
N &> \frac{-8 - \log_{10}|E_0|}{\log_{10}|K|} \\
N &= int\left(\frac{-8 - \log_{10}|E_0|}{\log_{10}|K|}\right) + 1
\end{aligned}
$$

Chapter 3

The Solution of Linear Systems $\mathbf{AX} = \mathbf{B}$

3.1 Introduction to Vectors and Matrices

1.

	(i)	(ii)	(iii)	(iv)
(a)	$(1,4)$	$(-14,8,3)$	$(5,-20,-10)$	$(4,-7,0,2)$
(b)	$(5,-12)$	$(2,-2,1)$	$(3,4,12)$	$(-2,3,8,2)$
(c)	$(9,-12)$	$(-18,9,4)$	$(12,-24,3)$	$(3,-6,-12,6)$
(d)	5	7	9	5
(e)	$(-26,72)$	$(-32,23,-1)$	$(-9,-52,-81)$	$(17,-27,-44,-8)$
(f)	-38	65	89	-3
(g)	$\sqrt{2440}$	$\sqrt{1554}$	$\sqrt{9346}$	$\sqrt{3018}$

2. (a) $\theta = arccos(-16/21) \approx 2.437045$
   (b) $\theta = arccos(-8/117) \approx 1.639226$

3. (a) Assume that $\mathbf{X}, \mathbf{Y} \neq \mathbf{0}$. $\mathbf{X} \cdot \mathbf{Y} = 0$ iff $\cos(\theta) = 0$ iff $\theta = (2n+1)\frac{\pi}{2}$ iff $\mathbf{X}$ and $\mathbf{Y}$ are orthogonal.
   (b) Yes, $(-6,4,2) \cdot (6,5,8) = 0$
   (c) No, $(-4,8,3) \cdot (2,5,16) = 80 \neq 0$
   (d) No, $(-5,7,2) \cdot (4,1,6) = -1 \neq 0$
   (e) Any solution of $(1,2,-5) \cdot (x,y,z) = 0$

4. (a) $\mathbf{A} + \mathbf{B} = \begin{bmatrix} -5 & 18 & 6 \\ 5 & -8 & 1 \\ 8 & 6 & 1 \end{bmatrix}$
   (b) $\mathbf{A} - \mathbf{B} = \begin{bmatrix} 3 & 0 & 2 \\ -1 & 2 & -13 \\ -8 & 4 & 13 \end{bmatrix}$

(c) $3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 5 & 9 & 8 \\ 0 & 1 & -32 \\ -16 & 13 & 33 \end{bmatrix}$

5. (a) $\begin{bmatrix} -2 & 1 & 7 & 11 \\ 5 & 4 & 0 & -3 \\ 12 & -1 & 6 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$

6. (a) Yes (b) No (c) Yes (d) No

7. (20)

$$\begin{aligned} \mathbf{Y} + \mathbf{X} &= (y_1, y_2, \ldots, y_n) + (x_1, x_2, \ldots, x_n) \\ &= (y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n) \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \\ &= \mathbf{X} + \mathbf{Y} \end{aligned}$$

(24)

$$\begin{aligned} (a+b)\mathbf{X} &= (a+b)(x_1, x_2, \ldots, x_n) \\ &= ((a+b)x_1, (a+b)x_2, \ldots, (a+b)x_n) \\ &= (ax_1 + bx_1, ax_2 + bx_2, \ldots, ax_n + bx_n) \\ &= (ax_1, ax_2, \ldots, ax_n) + (bx_1, bx_2, \ldots, bx_n) \\ &= a(x_1, x_2, \ldots, x_n) + b(x_1, x_2, \ldots, x_n) \\ &= a\mathbf{X} + b\mathbf{X} \end{aligned}$$

(25)

$$\begin{aligned} c(\mathbf{X} + \mathbf{Y}) &= c((x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)) \\ &= c(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (c(x_1 + y_1), c(x_2 + y_2), \ldots, c(x_n + y_n)) \\ &= (cx_1 + cy_1, cx_2 + cy_2, \ldots, cx_n + cy_n) \\ &= (cx_1, cx_2, \ldots, cx_n) + (cy_1, cy_2, \ldots, cy_n) \\ &= c(x_1, x_2, \ldots, x_n) + c(y_1, y_2, \ldots, y_n) \\ &= c\mathbf{X} + c\mathbf{Y} \end{aligned}$$

3.2 Properties of Vectors and Matrices

1. $\mathbf{AB} = \begin{bmatrix} -11 & -12 \\ 13 & -24 \end{bmatrix}$ $\quad \mathbf{BA} = \begin{bmatrix} -15 & 10 \\ -12 & -20 \end{bmatrix}$

2. $\mathbf{AB} = \begin{bmatrix} 14 & -16 \\ 21 & -10 \end{bmatrix}$ $\quad \mathbf{BA} = \begin{bmatrix} 3 & -6 & 9 \\ 9 & 2 & 22 \\ -1 & -6 & -1 \end{bmatrix}$

3. (a) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 2 & -5 \\ -88 & -56 \end{bmatrix}$

   (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} = \begin{bmatrix} 10 & -11 \\ 4 & -8 \end{bmatrix}$

   (c) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{AB} = \begin{bmatrix} 17 & -8 \\ -10 & 2 \end{bmatrix}$

   (d) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' = \begin{bmatrix} 1 & -8 \\ 0 & -24 \end{bmatrix}$

4. $\mathbf{A}^2 = \begin{bmatrix} -34 & -7 \\ 5 & -31 \end{bmatrix}$ $\quad \mathbf{B}^2 = \begin{bmatrix} 22 & -30 & 24 \\ -19 & 45 & -34 \\ 17 & -35 & 42 \end{bmatrix}$

5. (a) 33 (b)−80 (c) not defined (d)70

6.

$$\begin{aligned}
&\mathbf{R}_x(\alpha)\mathbf{R}_y(-\alpha) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2(\alpha)+\sin^2(\alpha) & \cos(\alpha)\sin(\alpha)-\sin(\alpha)\cos(\alpha) \\ 0 & \sin(\alpha)\cos(\alpha)-\cos(\alpha)\sin(\alpha) & \sin^2(\alpha)+\cos^2(\alpha) \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}$$

7.

$$\begin{aligned}
&\mathbf{R}_x(\alpha)\mathbf{R}_y(\beta) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ \sin(\alpha)\sin(\beta) & \cos(\alpha) & -\sin(\alpha)\cos(\beta) \\ -\cos(\alpha)\sin(\beta) & \sin(\alpha) & \cos(\alpha)\sin(\beta) \end{bmatrix}
\end{aligned}$$

8.

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$

9. **(13)** We show $\mathbf{IA} = \mathbf{A}$. By hypothesis $\mathbf{A} = [a_{ij}]_{M\times N}$ and $\mathbf{I} = [\delta_{ij}]_{M\times N}$. Thus:

$$\begin{aligned}
\mathbf{IA} &= \mathbf{C} \\
&= [c_{ij}]_{M\times N} \\
&= \left[\sum_{k=1}^{M} \delta_{ij} a_{kj}\right]_{M\times N} \\
&= [\delta_{ii} a_{ij}]_{M\times N} \\
&= [a_{ij}]_{M\times N} \\
&= \mathbf{A}
\end{aligned}$$

The proof of $\mathbf{AI} = \mathbf{A}$ is similar.

(16) We will show $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B}$. Let c be a scalar, $\mathbf{A} = [a_{ij}]_{M\times N}$ and $\mathbf{B} = [b_{ij}]_{M\times N}$. Then

$$\begin{aligned}
c(\mathbf{AB}) &= c\left[\sum_{k=1}^{N} a_{ik} b_{kj}\right]_{M\times N} \\
&= \left[c\sum_{k=1}^{N} a_{ik} b_{kj}\right]_{M\times N} \\
&= \left[\sum_{k=1}^{N} c(a_{ik} b_{kj})\right]_{M\times N} \\
&= \left[\sum_{k=1}^{N} (ca_{ik}) b_{kj}\right]_{M\times N} \\
&= (c\mathbf{A})\mathbf{B}
\end{aligned}$$

Similarly, $c(\mathbf{AB}) = \mathbf{A}(c\mathbf{B})$.

10. (a) MN (b) $M(N-1)$

11.

$$\begin{aligned}
\mathbf{A}(\mathbf{B}+\mathbf{C}) &= \left[\sum_{k=1}^{N} a_{ik}(b_{kj} + c_{kj})\right]_{M\times P} \\
&= \left[\sum_{k=1}^{N}((a_{ik} b_{kj}) + (a_{ik} c_{kj}))\right]_{M\times P} \\
&= \left[\sum_{k=1}^{N} a_{ik} b_{kj} + \sum_{k=1}^{N} a_{ik} c_{kj}\right]_{M\times P} \\
&= \left[\sum_{k=1}^{N} a_{ik} b_{kj}\right]_{M\times P} + \left[\sum_{k=1}^{N} a_{ik} c_{kj}\right]_{M\times P} \\
&= \mathbf{AB} + \mathbf{AC}
\end{aligned}$$

12.

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\mathbf{C} &= \left[\sum_{k=1}^{N}(a_{ik} + b_{ik}) c_{kj}\right] \\
&= \left[\sum_{k=1}^{N}(a_{ik} c_{kj} + b_{ik} c_{kj}\right] \\
&= \left[\sum_{k=1}^{N}(a_{ik} c_{kj} + \sum_{k=1}^{N} b_{ik} c_{kj}\right] \\
&= \left[\sum_{k=1}^{N}(a_{ik} c_{kj}\right] + \left[\sum_{k=1}^{N} b_{ik} c_{kj}\right] \\
&= \mathbf{AC} + \mathbf{BC}
\end{aligned}$$

13. $\mathbf{XX}' = [1 \quad -1 \quad 2] \quad [1 \quad -1 \quad 2]' = [6]$

$$\mathbf{X}'\mathbf{X} = [1 \quad -1 \quad 2]' \quad [1 \quad -1 \quad 2] = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

14.

$$\begin{aligned}
(\mathbf{AB})' &= \mathbf{C}' = [c_{ij}]' = [c'_{ij}] = [c_{ji}] \\
&= \left[\sum_{k=1}^{N} a_{jk} b_{ki}\right] \\
&= \left[\sum_{k=1}^{N} b_{ki} a_{jk}\right] \\
&= \left[\sum_{k=1}^{N} b'_{ik} a'_{kj}\right] \\
&= \mathbf{B}'\mathbf{A}'
\end{aligned}$$

15. $(\mathbf{ABC})' = ((\mathbf{AB})\mathbf{C})' = \mathbf{C}'(\mathbf{AB})' = \mathbf{C}'(\mathbf{B}'\mathbf{A}') = \mathbf{C}'\mathbf{B}'\mathbf{A}'$

3.3 Upper-Triangular Linear Systems

1. $x_1 = 2$, $x_2 = -2$, $x_3 = 1$, $x_4 = 3$, and $det(\mathbf{A}) = 120$

2. $x_1 = 2$, $x_2 = -3$, $x_3 = 5$, $x_4 = 2$, and $det(\mathbf{A}) = 1155$

3. $x_1 = 5$, $x_2 = 4$, $x_3 = 1$, $x_4 = -6$, $x_5 = 2$, and $det(\mathbf{A}) = 48$

4. (a) $\mathbf{C} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b + 33 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}$

   (b) (We need to show that the ij *th* entry of the product $\mathbf{AB}$ is zero when $i > j$.) Given $\mathbf{A} = [a_{ij}]_{N\times N}$ and $\mathbf{B} = [b_{ij}]_{N\times N}$ are upper-triangular matrices, then by definition, $a_{ij} = 0 = b_{ij}$ when $i > j$.

   Case 1: If $i > k$ then $a_{ik} = 0$ and $a_{ik}b_{kj} = 0$.

   Case 2: If $i \not> k$, then $i \leq k$ and $i > j$ imply that $b_{kj} = 0$; and thus $a_{ik}b_{kj} = 0$.

   Thus, if $i > j$, then the ij *th* entry of the product $\mathbf{AB}$ is $\sum_{k=1}^{N} a_{ik}b_{kj} = 0$. Therefore, the product of two upper-triangular matrices is an upper-triangular matrix.

5. $x_1 = 3$, $x_2 = 2$, $x_3 = 1$, $x_4 = -1$, and $det(\mathbf{A}) = -24$

6. $x_1 = -2$, $x_2 = 2$, $x_3 = 0$, $x_4 = 3$, and $det(\mathbf{A}) = -30$

7. From Program 3.1 $x_N = b_N/a_{NN}$ and

$$x_k = \frac{b_k - \sum_{k=1}^{N} a_{kj}x_j}{a_{kk}}$$

for $k = N-1, N-2, \ldots, 1$. There are $(N-1)+1 = N$ divisions. For each value of k there are $N-(k+1)+1 = N-k$ multiplications (one for each term in the summation) and $N-(k+1)+1 = N-k$ additions/subtractions ($N-(k+1)$ from the summation and one subtraction from the numerator). Therefore, there are

$$\begin{aligned} \textstyle\sum_{k=1}^{N-1}(N-k) &= \textstyle\sum_{k=1}^{N-1} N - \sum_{k=1}^{N-1} k \\ &= (N-1)N - \tfrac{(N-1)N}{2} \\ &= \tfrac{N(N-1)}{2} \end{aligned}$$

multiplications and additions/subtractions, respectively.

3.4 Gaussian Elimination and Pivoting

1. $x_1 = -3$, $x_2 = 2$, $x_3 = 1$

2. $x_1 = 3$, $x_2 = -2$, $x_3 = 1$

3. $x_1 = 1$, $x_2 = 3$, $x_3 = 2$

4. $x_1 = 2$, $x_2 = 5$, $x_3 = 1$

5. $y = 5 - 3x + 2x^2$

6. $y = 5 + 2x - x^2$

7. $\frac{2}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

8. $x_1 = 3$, $x_2 = -1$, $x_3 = 1$, $x_4 = 2$

9. $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, $x_4 = -2$

10. $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $x_4 = -2$

11. $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $x_4 = -2$

12. $x_1 = 2$, $x_2 = 3$, $x_3 = -2$, $x_4 = 1$

13. The exact solution is $[\ x \quad y\ ]' = [\ -1 \quad 1\ ]'$. Clearly, (unexpectedly?) the MightyDo 11 solution is best: $[\ x \quad y\ ]' = [\ -0.99 \quad 1.01\ ]'$.

14. (a) Compare to $x_1 = 0.00000000$, $x_2 = 0.00000100$, $x_3 = 0.01000003$

(b) Compare to $x_1 = 0.06035701$, $x_2 = -0,0015951$, $x_3 = 0.02820363$, $x_4 = -0,25201729$

15. (a) $x_1 = 25$, $x_2 = -300$, $x_3 = 1050$, $x_4 = -1400$, $x_5 = 630$

(b) $x_1 = 28.02304$, $x_2 = -348.5887$, $x_3 = 1239.781$, $x_4 = -1666.785$, $x_5 = 753.5564$

3.5 Triangular Factorization

1. (a) $\mathbf{Y}' = [\ -4 \quad 12 \quad 3\ ]'$, $\mathbf{X}' = [\ -3 \quad 2 \quad 1\ ]'$

(b) $\mathbf{Y}' = [\ 20 \quad 39 \quad 9\ ]'$, $\mathbf{X}' = [\ 5 \quad 7 \quad 3\ ]'$

2. (a) $\mathbf{Y}' = [\ 7 \quad 9 \quad 12\ ]'$, $\mathbf{X}' = [\ 3 \quad -2 \quad 1\ ]'$

(b) $\mathbf{Y}' = [\ 25 \quad 60 \quad 42\ ]'$, $\mathbf{X}' = [\ 3 \quad -2 \quad 1\ ]'$

3. (a) $$\begin{bmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.2 & 1 & 0 \\ -0.6 & 5.5 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 & -1 \\ 0 & 0.4 & 2.8 \\ 0 & 0 & -10 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 6 \\ -5 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 20 \end{bmatrix}$$

4. (a) $$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & -2 \\ 1 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & -0.625 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 4 & -2.5 \\ 0 & 0 & 5.1875 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & -2 & 7 \\ 4 & 2 & 1 \\ 2 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0.9 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 10 & -27 \\ 0 & 0 & 8.3 \end{bmatrix}$$

5. (a) $\mathbf{Y}' = [\ 8 \quad -6 \quad 12 \quad 2\ ]'$, $\mathbf{X}' = [\ 3 \quad -1 \quad 1 \quad 2\ ]'$

(b) $\mathbf{Y}' = [\ 28 \quad 6 \quad 12 \quad 1\ ]'$, $\mathbf{X}' = [\ 3 \quad 1 \quad 2 \quad 1\ ]'$

6. $$\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -1 & -1.75 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & -7.5 \end{bmatrix}$$

7.

$$\begin{aligned} \textstyle\sum_{p=1}^{N-1}(N-p)(N-p) &= \textstyle\sum_{p=1}^{N-1}(N^2 - 2Np + p^2) \\ &= N^2(N-1) - 2N\left(\tfrac{(N-1)N}{2}\right) + \tfrac{(N-1)N(2(N-1)+1)}{6} \\ &= \tfrac{(6N^3-6N^2)-(6N^3-6N^2)+(2N^3-3N^2+N)}{6} \\ &= \tfrac{2N^3-3N^2+N}{6} \end{aligned}$$

8. Use the fact that $\mathbf{L}_1$, $\mathbf{U}_1$, $\mathbf{L}_2$, and $\mathbf{U}_2$ are all nonsingular.

9. When $r > c$:

$$\begin{aligned} d_{rc} &= \sum_{k=1}^{c} m_{rk} a_{kc}^{(k)} \\ &= \sum_{k=1}^{c-1} m_{rk} a_{kc}^{(k)} + m_{rc} a_{cc}^{(c)} \\ &= \sum_{k=1}^{c-1} (a_{rc}^{(k)} - a_{rc}^{(k+1)}) + a_{rc}^{(c)} \\ &= a_{rc}^{(1)} \end{aligned}$$

10. (a) Carry out the indicated products.

    (b) Note:

$$E_{k_j} = [e_{1j}]_{1\times N} \text{ where } e_{1j} = \begin{cases} 1 & j = k_j \\ 0 & j \neq k_j \end{cases}$$

If $P = [E'_{k_1}\ E'_{k_2}\ \cdots\ E'_{k_N}]$, then $P' = [E_{k_1}\ E_{k_2}\ \cdots\ E_{k_N}]'$. Thus $P'P = [a_{ij}]_{N\times N}$, where

$$a_{ij} = E_{k_i} E'_{k_j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Thus $P'P = I$. Similary, $PP' = I$. Therefore, $P^{-1} = P'$.

11. Let $\mathbf{A} = [a_{ij}]_{N\times N}$, where $a_{ij} = 0$, if $i > j$. Thus $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj\mathbf{A} = [a_{ij}^{-1}]$ where

$$a_{ij}^{-1} = \frac{1}{|\mathbf{A}|} A_{ji} = \frac{(-1)^{j+i}}{|\mathbf{A}|} M_{ji} = \frac{(-1)^{j+i}}{|\mathbf{A}|}(0) = 0,$$

provided $i > j$. Therefore $\mathbf{A}^{-1}$ is upper-triangular.

3.6 Iterative Methods for Linear Systems

1. (a) $\mathbf{P}_1 = (3.75, 1.8)$, $\mathbf{P}_2 = (4.2, 1.05)$, $\mathbf{P}_3 = (4.0125, 0.96)$. Iteration will converge to $\mathbf{P} = (4, 1)$.

   (b) $\mathbf{P}_1 = (3.75, 1.05)$, $\mathbf{P}_2 = (4.0125, 0.9975)$, $\mathbf{P}_3 = (3.999375, 1.000125)$. Iteration will converge to $\mathbf{P} = (4, 1)$.

2. (a) $\mathbf{P}_1 = (1.25, 1.5)$, $\mathbf{P}_2 = (1.8125, 1.8125)$, $\mathbf{P}_3 = (1.9296875, 1.953125)$. Iteration will converge to $\mathbf{P} = (2, 2)$.

   (b) $\mathbf{P}_1 = (1.25, 1.8125)$, $\mathbf{P}_2 = (1.9296875, 1.98242188)$, $\mathbf{P}_3 = (1.9934082, 1.99835205)$. Iteration will converge to $\mathbf{P} = (2, 2)$.

3. (a) $\mathbf{P}_1 = (-1, -1)$, $\mathbf{P}_2 = (-4, -4)$, $\mathbf{P}_3 = (-13, -13)$. The iteration diverges away from the solution $\mathbf{P} = (4, 1)$.

   (b) $\mathbf{P}_1 = (-1, -4)$, $\mathbf{P}_2 = (-13, -40)$, $\mathbf{P}_3 = (-121, -364)$. The iteration diverges away from the solution $\mathbf{P} = (4, 1)$.

4. (a) $\mathbf{P}_1 = (0.5, -0.5)$, $\mathbf{P}_2 = (1.25, 1.25)$, $\mathbf{P}_3 = (-1.375, 3.875)$. However, the solution is $\mathbf{P} = (0.2, 0.2)$.

   (b) $\mathbf{P}_1 = (0.5, 1.25)$, $\mathbf{P}_2 = (-1.375, -5.3125)$, $\mathbf{P}_3 = (8.46875, 29.140625)$. However, the solution is $\mathbf{P} = (0.2, 0.2)$.

5. (a)

$$\begin{aligned}
\mathbf{P}_1 &= (2, 1.375, 0.75) \\
\mathbf{P}_2 &= (2.125, 0.96875, 0.90625) \\
\mathbf{P}_3 &= (2.0125, 0.95703125, 1.0390625)
\end{aligned}$$

   Iteration will converge to $\mathbf{P} = (2, 1, 1)$

   (b)

$$\begin{aligned}
\mathbf{P}_1 &= (2, 0.875, 1.03125) \\
\mathbf{P}_2 &= (1.96875, 1.01171875, 0.989257813) \\
\mathbf{P}_3 &= (2.00449219, 0.99753418, 1.0017395)
\end{aligned}$$

   Iteration will converge to $\mathbf{P} = (2, 1, 1)$

6. (a)

$$\begin{aligned}
\mathbf{P}_1 &= (5.5, -10, 0.75) \\
\mathbf{P}_2 &= (48.875, 18.25, 4.625) \\
\mathbf{P}_3 &= (-65.1875, 224, 7.6525)
\end{aligned}$$

   However, the solution is $\mathbf{P} = (2, 1, 1)$

   (b)

$$\begin{aligned}
\mathbf{P}_1 &= (5.5, 17.5, -2.25) \\
\mathbf{P}_2 &= (-65.625, -340.375, 69.4375) \\
\mathbf{P}_3 &= (1401.71875, 7068.03125, -1415.82813)
\end{aligned}$$

   However, the solution is $\mathbf{P} = (2, 1, 1)$

7. (a)

$$\begin{aligned}
\mathbf{P}_1 &= (-8, 13, 0.333333333) \\
\mathbf{P}_2 &= (57.3333333, 45.3333333, -4.5) \\
\mathbf{P}_3 &= (214.166667, -220.83333, 11.88889)
\end{aligned}$$

   However, the solution is $\mathbf{P} = (3, 2, 1)$

(b)

$$\begin{aligned}\mathbf{P}_1 &= (-8, 45, -9.83333333)\\ \mathbf{P}_2 &= (207.16667, -825.5, 206.97222)\\ \mathbf{P}_3 &= (-3928.5278, 15934.0833, -3964.8565)\end{aligned}$$

However, the solution is $\mathbf{P} = (2, 1, 1)$

8. (a)

$$\begin{aligned}\mathbf{P}_1 &= (3.25, 1.6, 0.33333333)\\ \mathbf{P}_2 &= (2.93333, 2.183333, 1.15)\\ \mathbf{P}_3 &= (2.991667, 1.956667, 0.947222)\end{aligned}$$

Iteration will converge to $\mathbf{P} = (3, 2, 1)$

(b)

$$\begin{aligned}\mathbf{P}_1 &= (3.25, 2.25, 1.04166667)\\ \mathbf{P}_2 &= (2.9479167, 1.98125, 0.9857639)\\ \mathbf{P}_3 &= (3.0011285, 2.0030729, 0.9998640)\end{aligned}$$

Iteration will converge to $\mathbf{P} = (3, 2, 1)$

9. **(14)** If $x_k \in \Re$, then $|x_k| \geq 0$ for $k = 1, 2, \ldots, N$. Therefore, $\sum_{k=1}^{N} |x_k| \geq 0$ or $\|\mathbf{X}\|_1 \geq 0$.

(15) If $\mathbf{X} = \mathbf{0}$ then $\mathbf{X} = \underbrace{(0, 0, \ldots, 0)}_{N\,0's}$ and $\|\mathbf{X}\|_1 = \sum_{k=1}^{N} 0 = 0$. Now assume $\|\mathbf{X}\|_1 = 0$, and by way of contradiction assume $\mathbf{X} \neq \mathbf{0}$. Then there is a k such that $x_k \neq 0$ and $|x_k| > 0$. But then, $\|\mathbf{X}\|_1 = \sum_{k=1}^{N} |x_k| > 0$—a contradiction to the hypothesis $\|\mathbf{X}\|_1 = 0$. Thus, if $\|\mathbf{X}\|_1 = 0$, then $\mathbf{X} = \mathbf{0}$.

(16)

$$\begin{aligned}\|c\mathbf{X}\|_1 &= \sum_{k=1}^{N} |cx_k|\\ &= \sum_{k=1}^{N} |c||x_k|\\ &= |c| \sum_{k=1}^{N} |x_k|\\ &= |c|\|\mathbf{X}\|_1\end{aligned}$$

10. **(14)** If $x_k \in \Re$ then $x_k^2 \geq 0$ and $\|\mathbf{X}\| = \sqrt{\sum_{k=1}^{N} x_k^2} \geq 0$.

(15) If $\mathbf{X} = \mathbf{0}$ then $\mathbf{X} = \underbrace{(0, 0, \ldots, 0)}_{N\,0's}$ and $\|\mathbf{X}\| = \sqrt{\sum_{k=1}^{N} 0} = 0$. Now assume $\|\mathbf{X}\| = 0$, and by way of contradiction assume $\mathbf{X} \neq \mathbf{0}$. Then there is a k such that $x_k \neq 0$ and $x_k^2 > 0$. But then, $\|\mathbf{X}\| = \sqrt{\sum_{k=1}^{N} x_k^2} > 0$—a contradiction to the hypothesis $\|\mathbf{X}\| = 0$. Thus, if $\|\mathbf{X}\| = 0$, then $\mathbf{X} = \mathbf{0}$.

(16)

$$\begin{aligned}
\|c\mathbf{X}\| &= \sqrt{\sum_{k=1}^{N} (cx_k)^2} \\
&= \sqrt{c^2 \sum_{k=1}^{N} x_k^2} \\
&= \sqrt{c^2}\sqrt{\sum_{k=1}^{N} x_k^2} \\
&= |c|\|\mathbf{X}\|
\end{aligned}$$

(17) First note from Exercise 2 of Section 3.2 that

$$\begin{aligned}
\frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|} &= \cos(\theta) \\
\frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|} &\leq 1 \\
\mathbf{X} \cdot \mathbf{Y} &\leq \|\mathbf{X}\|\|\mathbf{Y}\|
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathbf{X} + \mathbf{Y}\|^2 &= (\mathbf{X} + \mathbf{Y}) \cdot (\mathbf{X} + \mathbf{Y}) \\
&= \mathbf{X} \cdot \mathbf{X} + \mathbf{2}(\mathbf{X} \cdot \mathbf{Y}) + \mathbf{Y} \cdot \mathbf{Y} \\
&= \|\mathbf{X}\|^2 + 2(\mathbf{X} \cdot \mathbf{Y}) + \|\mathbf{Y}\|^2 \\
&\leq \|\mathbf{X}\|^2 + 2\|\mathbf{X}\|\|\mathbf{Y}\| + \|\mathbf{Y}\|^2 \\
&= (\|\mathbf{X}\| + \|\mathbf{Y}\|)^2
\end{aligned}$$

Taking the square root of both sides of the resulting inequality yields

$$\|\mathbf{X} + \mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|$$

11. **(14)** If $x_k \in \Re$ then $|x_k| \geq 0$ and $\|\mathbf{X}\| = \max_{1 \leq k \leq N} |x_k| = \|\mathbf{X}\|_\infty \geq 0$

(15) If $\mathbf{X} = \mathbf{0}$ then $\|\mathbf{X}\|_\infty = \max_{1 \le k \le N} |x_k| = \max_{1 \le k \le N} |0| = 0$. Now assume $\|\mathbf{X}\| = 0$, and by way of contradiction assume $\mathbf{X} \neq \mathbf{0}$. Then there is a j such that $x_j \neq 0$ and $|x_j| \ge |x_k|$ for $k = 1, 2, \ldots, N$. But then, $\|\mathbf{X}\|_\infty = |x_j| \neq 0$—a contradiction of the hypothesis that $\|\mathbf{X}\| = 0$.

(16)

$$\begin{aligned} \|c\mathbf{X}\|_\infty &= \max_{1 \le k \le N} |cx_k| \\ &= \max_{1 \le k \le N} |c||x_k| \\ &= |c| \max_{1 \le k \le N} |x_k| \\ &= |c|\|\mathbf{X}\|_\infty \end{aligned}$$

(17)

$$\begin{aligned} \|\mathbf{X} + \mathbf{Y}\|_\infty &= \max_{1 \le k \le N} |x_k + y_k| \text{ say when } k = j \\ &= |x_j + y_j| \\ &\le \max_{1 \le k \le N} |x_k| + \max_{1 \le k \le N} |y_k| \\ &= \|\mathbf{X}\|_\infty + \|\mathbf{Y}\|_\infty \end{aligned}$$

3.7 Iteration for Nonlinear Systems

1. (a) $(0, 0)$
   (b) $\left(\frac{1}{3}(9 - 2\sqrt{30}), \frac{1}{3}(-6 + \sqrt{30})\right), \left(\frac{1}{3}(9 + 2\sqrt{30}), \frac{1}{3}(-6 - \sqrt{30})\right)$
   (c) $(0, n\pi)$
   (d) $(1, 2, 1)$
2. (a) $(4, -2)$
   (b) $(1, 1/2), (-1/3, 11/6)$
   (c) $(2(-1)^n, n\pi)$, $(0, (2n+1)\pi/2)$
   (d) $\left(\frac{\sqrt{-1+\sqrt{5}}}{2}, -\frac{\sqrt{-1+\sqrt{5}}}{2}, \frac{-1+\sqrt{5}}{2}\right) \left(-\frac{\sqrt{-1+\sqrt{5}}}{2}, \frac{\sqrt{-1+\sqrt{5}}}{2}, \frac{-1+\sqrt{5}}{2}\right)$
3. The system has two fixed points:

$$\left(\frac{9 - 2\sqrt{30}}{3}, \frac{6 - \sqrt{30}}{3}\right), \left(\frac{9 + 2\sqrt{30}}{3}, \frac{6 + \sqrt{30}}{3}\right)$$

Formula (16) becomes:

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = \left|\frac{2x - 1}{3}\right| + \left|-\frac{2y}{3}\right| < 1$$

and

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = \frac{1}{3} + \frac{1}{3} < 1$$

The second fixed point doen't satisfy the first inequality, thus convergence isn't guaranteed. The first fixed point satisfies both inequalities and both inequalities are satisfied for every point (x, y) in the region defined by

$$\frac{9 - 2\sqrt{30}}{3} - \frac{1}{10} < x < \frac{9 - 2\sqrt{30}}{3} + \frac{1}{10}$$

and

$$\frac{6 - \sqrt{30}}{3} - \frac{1}{10} < y < \frac{6 - \sqrt{30}}{3} + \frac{1}{10}$$

4.

$$\begin{aligned} x = g_1(x, y, z) &= \frac{1 - y - z}{6} \\ y = g_2(x, y, z) &= \frac{2 - x - z}{4} \\ z = g_3(x, y, z) &= \frac{-x - y}{5} \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| &= 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3} < 1 \\ \left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| + \left|\frac{\partial g_2}{\partial z}\right| &= \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2} < 1 \\ \left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| + \left|\frac{\partial g_3}{\partial z}\right| &= \frac{1}{5} + \frac{1}{5} + 0 = \frac{2}{5} < 1 \end{aligned}$$

Thus fixed-point iteration will converge to the fixed point $\frac{1}{107}(11, 54, -13)$ for any initial guess (p_0, q_0, r_0).

5. (a) Fixed-point iteration:

$$\begin{aligned} (p_0, q_0) &= (1.1, 2.0) \\ (p_1, q_1) &= (g_1(1.1, 2.0), g_2(1.1, 2.0)) = (1.12, 1.9975) \\ (p_2, q_2) &= (g_1(1.12, 1.9975), g_2(1.12, 1.9975)) = (1.1166, 1.9964) \\ (p_3, q_3) &= (g_1(1.1166, 1.9964), g_2(1.1166, 1.9964)) = (1.1164, 1.9966) \end{aligned}$$

(b) Seidel iteration

$$\begin{aligned} (p_0, q_0) &= (1.1, 2.0) \\ (p_1, q_1) &= (g_1(p_0, q_0), g_2(p_1, q_0)) = (1.12, 1.9964) \\ (p_2, q_2) &= (g_1(p_1, q_1), g_2(p_2, q_1)) = (1.1160, 1.9966) \\ (p_3, q_3) &= (g_1(p_2, q_2), g_2(p_3, q_2)) = (1.1166, 1.9966) \end{aligned}$$

6. (a) Fixed-point iteration:

$$\begin{aligned}(p_0, q_0) &= (-0.3, -1.3)\\ (p_1, q_1) &= (-0.2684, -1.3175)\\ (p_2, q_2) &= (-0.2694, -1.3161)\\ (p_3, q_3) &= (-0.2696, -1.3153)\end{aligned}$$

(b) Seidel iteration

$$\begin{aligned}(p_0, q_0) &= (-0.3, -1.3)\\ (p_1, q_1) &= (-0.2719, -1.3191)\\ (p_2, q_2) &= (-0.2704, -1.3139)\\ (p_3, q_3) &= (-0.2694, -1.3160)\end{aligned}$$

7.

$$\begin{aligned}\mathbf{J}(x, y) &= \begin{bmatrix} 2x-1 & -1 \\ -1 & 2y \end{bmatrix}\\ \mathbf{F}(x, y) &= \begin{bmatrix} x^2 - y - 0.2 \\ y^2 - x - 0.3 \end{bmatrix}\end{aligned}$$

(a)

$$\begin{aligned}\mathbf{P}_0 &= \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix}\\ \mathbf{P}_1 &= \mathbf{P}_0 + \mathbf{dP} = \begin{bmatrix} 1.192437 \\ 1.221849 \end{bmatrix}\\ \mathbf{P}_2 &= \mathbf{P}_1 + \mathbf{dP} = \begin{bmatrix} 1.192309 \\ 1.221601 \end{bmatrix}\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{P}_0 &= \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix}\\ \mathbf{P}_1 &= \mathbf{P}_0 + \mathbf{dP} = \begin{bmatrix} -0.2904762 \\ -0.1238095 \end{bmatrix}\\ \mathbf{P}_2 &= \mathbf{P}_1 + \mathbf{dP} = \begin{bmatrix} -0.2860634 \\ -0.1181872 \end{bmatrix}\end{aligned}$$

8. (a) Clearly $y \neq 0$. Substituting $x = 1/y$ into the first equation yields $y = \pm 1$. Thus the solutions are $(1, 1)$ and $(-1, -1)$.

(b) The values of the Jacobian determinant at the solution points are $|\mathbf{J}(1, 1)| = 0$ and $|\mathbf{J}(-1, -1)| = 0$. Newton's method depends on being able to solve a linear system where the matrix is $\mathbf{J}(p_n, q_n)$ and

(p_n, q_n) is near a solution. For this example, the system of equations is ill-conditioned and thus hard to solve with precision. In fact, for some values near a solution we have $\mathbf{J}(x_0, y_0) = 0$, for example, $\mathbf{J}(1.0001, 1.0001) = 0$.

9. Assming $a_{ii} \neq 0$, then Jacobi iteration for the linear system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

is

$$\begin{aligned} x &= g_1(x,y,z) = \frac{b_1 - a_{12}y - a_{13}z}{a_{11}} \\ y &= g_2(x,y,z) = \frac{b_2 - a_{21}x - a_{23}z}{a_{22}} \\ z &= g_3(x,y,z) = \frac{b_3 - a_{31}x - a_{32}y}{a_{33}} \end{aligned}$$

The fixed-iteration in (15) follows from these equations. If the coefficient matrix of the given linear system is strictly diagonally dominant then

$$|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{3} |a_{kj}|$$

for $k = 1, 2, 3$. In particular, for $k = 1$ we have $|a_{11}| > |a_{12}| + |a_{13}|$ and

$$\begin{aligned} \left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| &= 0 + \left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{13}}{a_{11}}\right| \\ &= \frac{|a_{12}| + |a_{13}|}{|a_{11}|} < 1 \end{aligned}$$

The conditions for $g_2(x,y,z)$ and $g_3(x,y,z)$ are derived similarly.

10. From Newton's method:

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{J}^{-1}\mathbf{F}(\mathbf{P}_k)$$

or

$$
\begin{aligned}
\begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{J}^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\mathbf{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\
&= \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} \\
&= \begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial f_2(x,y)}{\partial y} & -\frac{\partial f_1(x,y)}{\partial y} \\ -\frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \end{bmatrix} \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}
\end{aligned}
$$

Thus

$$
\begin{aligned}
g_1(x,y) &= x - \frac{\frac{\partial f_2}{\partial y} f_1 - \frac{\partial f_1}{\partial y} f_2}{|\mathbf{J}|} \\
g_2(x,y) &= y - \frac{\frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2}{|\mathbf{J}|}
\end{aligned}
$$

11. (a) From the Mean Value Theorem there are numbers a_0^* and c_0^* such that

$$
\begin{aligned}
e_1 &= \frac{\partial g_1(a_0^*, q_0)}{\partial x} e_0 + \frac{\partial g_1(p, c_0^*)}{\partial y} E_0 \\
|e_1| &= \left| \frac{\partial g_1(a_0^*, q_0)}{\partial x} e_0 + \frac{\partial g_1(p, c_0^*)}{\partial y} E_0 \right| \\
&\le \left| \frac{\partial g_1(a_0^*, q_0)}{\partial x} e_0 \right| + \left| \frac{\partial g_1(p, c_0^*)}{\partial y} E_0 \right| \\
&\le \left| \frac{\partial g_1(a_0^*, q_0)}{\partial x} \right| r_0 + \left| \frac{\partial g_1(p, c_0^*)}{\partial y} \right| r_0 \\
&= \left(\left| \frac{\partial g_1(a_0^*, q_0)}{\partial x} \right| + \left| \frac{\partial g_1(p, c_0^*)}{\partial y} \right| \right) r_0 \\
&< K r_0
\end{aligned}
$$

(b) From part (a): $|e_2| \le K r_1$ and $r_1 = max\{|e_1|, |E_1|\} \le K r_0$. Thus $|e_2| \le K(K r_0) = K^2 r_0$. Similarly, $|E_k| \le K^2 r_0$.

(c) Assume

$$
\begin{aligned}
|e_{k-1}| &\le K r_{k-2} \le K^{k-1} r_0, \\
|E_{k-1}| &\le K r_{k-2} \le K^{k-1} r_0.
\end{aligned}
$$

Then

$$\begin{aligned}|e_k| &\leq Kr_{k-1}\\ &\leq K(max\{|e_{k-1}|,|E_{k-1}|\})\\ &= K(K^{k-1}r_0) = K^k r_0\end{aligned}$$

Similarly, $|E_k| \leq Kr_{k-1} \leq K^k r_0$.

(d) Note: $\lim_{n\to\infty}|e_n| = \lim_{n\to\infty} K^n r_0 = 0(r_0) = 0$, since $0 < K < 1$. Thus

$$0 = \lim_{n\to\infty}|e_n| = \lim_{n\to\infty}|p - p_n| = \lim_{n\to\infty}(p - p_n)$$

or $\lim_{n\to\infty} p_n = p$. Similarly, $\lim_{n\to\infty} q_n = q$

12. (a) Note: As with derivativs, we have $\frac{\partial}{\partial x}(cf(x,y)) = c\frac{\partial}{\partial x}f(x,y)$. $\mathbf{F}(\mathbf{X})$ was defined as

$$\mathbf{F}(\mathbf{X}) = [f_1(x_1,\ldots,x_n)\cdots f_m(x_1,\ldots,x_n)]';$$

thus, by scalar multiplication,

$$c\mathbf{F}(\mathbf{X}) = [cf_1(x_1,\ldots,x_n)\cdots cf_m(x_1,\ldots,x_n)]'.$$

$\mathbf{J}(c\mathbf{F}(\mathbf{X})) = [j_{ik}]_{m\times n}$, where

$$j_{ik} = \frac{\partial}{\partial x_k}(cf_i(x_1,\ldots,x_n)) = c\frac{\partial}{\partial x_k}f_i(x_1,\ldots,x_n).$$

Therefore, by the definition of scalar multiplication, we have $\mathbf{J}(c\mathbf{F}(\mathbf{X})) = c\mathbf{J}(\mathbf{F}(\mathbf{X}))$.

(b) Note:

$$\begin{aligned}\mathbf{F}(\mathbf{X}) + \mathbf{G}(\mathbf{X}) &= [f_1(x_1,\ldots,x_n)\cdots f_m(x_1,\ldots,x_n)]'\\ &\quad +[g_1(x_1,\ldots,x_n)\cdots g_m(x_1,\ldots,x_n)]'\\ &= [f_1 + g_1\cdots f_m + g_m]'\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{J}(\mathbf{F}(\mathbf{X}) + \mathbf{G}(\mathbf{X})) &= \left[\frac{\partial}{\partial x_k}(f_i + g_i)\right]\\ &= \left[\frac{\partial}{\partial x_k}f_i + \frac{\partial}{\partial x_k}g_i\right]\\ &= \left[\frac{\partial}{\partial x_k}f_i\right] + \left[\frac{\partial}{\partial x_k}g_i\right]\\ &= \mathbf{J}(\mathbf{F}(\mathbf{X})) + \mathbf{J}(\mathbf{G}(\mathbf{X}))\end{aligned}$$

Chapter 4

Interpolation and Polynomial Approximation

4.1 Taylor Series and Calculation of Functions

1. (a) $P_5(x) = x - x^3/3! + x^5/5!$
   $P_7(x) = x - x^3/3! + x^5/5! - x^7/7!$
   $P_9(x) = x - x^3/3! + x^5/5! - x^7/7! + x^9/9!$

   (b) $|E_9(x)| = \left|\sin(c)x^{10}/10!\right| \le \frac{1(1^{10})}{10!} = 0.0000002755\ldots$

   (c)
$$P_5(x) = \frac{1}{\sqrt{2}}(1 + (x - \pi/4) - (x - \pi/4)^2/2 - (x - \pi/4)^3/6+ (x - \pi/4)^4/24 + (x - \pi/4)^5/120)$$

2. (a) $P_4(x) = 1 - x^2/2! + x^4/4!$
   $P_6(x) = 1 - x^2/2! + x^4/4! - x^6/6!$
   $P_8(x) = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8!$

   (b) $|E_8(x)| = \left|-\sin(c)x^9/9!\right| \le \frac{1(1^9)}{9!} = 0.0000027557\ldots$

   (c)
$$P_4(x) = \frac{1}{\sqrt{2}}(1 - (x - \pi/4) - (x - \pi/4)^2/2 + (x - \pi/4)^3/6+ (x - \pi/4)^4/24)$$

3. No, the derivatives of $f(x)$ are undefined at $x = 0$.

4. (a) $P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$

   (b) $E_5(x) - \frac{720x^6}{6!(1+c)^7)}$

5. $P_3(x) = 1 + 0x - x^2/2 + 0x^3 = 1 - x^2/2$

6. $P_3(x) = 1 + (x-1) + (x-1)^2 + (x-1)^3$. There are several ways to show $f(x) = P_3(x)$. For example, $P_3(x)$ can be expanded and simplified. The following argument makes use of the Identical Polynomial Theorem—If two polynomials of degree n agree at $n+1$ points then they are identical. Evaluate $f(x)$ and $P_3(x)$ at $x = -1, 0, 1, 2$:

$$\begin{aligned} f(-1) &= -5 = P_3(-1) \\ f(0) &= 0 = P_3(0) \\ f(1) &= 1 = P_3(1) \\ f(2) &= 4 = P_3(2) \end{aligned}$$

Thus $f(x) = P_3(x)$.

7. (a)

$$P_5(x) = 2 + \frac{(x-4)}{4} - \frac{(x-3)^2}{64} + \frac{(x-4)^3}{512} - \frac{5(x-4)^4}{16384} + \frac{7(x-4)^5}{131072}$$

(b)

$$\begin{aligned} P_5(x) &= 3 + \tfrac{1}{6}(x-9) - \tfrac{1}{216}(x-9)^2 + \tfrac{1}{3888}(x-9)^3 \\ &\quad - \tfrac{5}{279936}(x-9)^4 + \tfrac{7}{5038848}(x-9)^5 \end{aligned}$$

(c) $x_0 = 4 :\ P_5(6.5) \approx 2.55116$, $x_0 = 9 :\ P_5(6.5) \approx 2.54955$

8. (a) $f(2) = 2$, $f'(2) = 1/4$, $f''(2) = -1/32$, $f^{(3)} = 3/256$
   $P_3(x) = 2 + \frac{1}{4}(x-2) - \frac{1}{64}(x-2)^2 + \frac{1}{512}(x-2)^3$

   (b) $P_3(1) = 1.732421875$; compare with $3^{1/2} = 1.732052808\ldots$

   (c) $f^4(x) = -\frac{15}{16}(x+2)^{-7/2}$: the maximum of $|f^4(x)|$ on the interval $1 \le x \le 3$ occurs when $x = 1$ and $|f^4(x)| \le |f^4(1)| \le 3^{-7/2} \approx 0.020046$. Therefore,

$$|E_3(x)| \le \frac{(0.020046)(1^4)}{4!} = 0.00083529\ldots$$

9. $f(x) = e^x$ implies $f^{(n)}(x) = e^x$, thus $|E_n(0.1) = \left|\frac{f^{(n)}(0)(0.1-0)^n}{n!}\right| = \frac{(0.1)^n}{n!}$. $E_4(0.1) = 4.1\overline{6} \times 10^{-6}$ and $E_5(0.1) = 8.\overline{3} \times 10^{-8} < 10^{-6}$. Thus we select $n = 5$ to obtain the desired accuracy.

10. $|E_4(33\pi/32)| \le \frac{\cos(\pi)(33\pi/32-\pi)^4}{4!} < 3.8 \times 10^-6$,
    $|E_5(33\pi/32)| \le \frac{\sin(\pi)(33\pi/32-\pi)^5}{5!} < 7.5 \times 10^-8$; thus use $P_4(x)$.

11. (a) $F(x) = \int_{-1}^{x} \cos(t^2)dt$,
    $F'(x) = \cos(x^2)$,
    $F"(x) = -2x\sin(x^2)$,

$F^{(3)}(x) = -4x^2\cos(x^2) - 2\sin(x^2)$,
$F^{(4)}(x) = -12x\cos(x^2) + 8x^3\sin(x^2)$,
$F^{(5)}(x) = -12\cos(x^2) + 16x^4\cos(x^2) + 48x^2\sin(x^2)$
$F(0) = \int_{-1}^{0}\cos(t^2)dt$, $F'(0) = 1$, $F''(0) = 0$, $F'''(0) = 0$, $F^{(4)}(0) = 0$. Thus

$$P_4(x) = \int_{-1}^{0}\cos(t^2)dt + x.$$

(b) $F(0.1) \approx P_4(0.1) \approx \int_{-1}^{0}\cos(t^2)dt + 0.1 \approx 0.904524 + 0.1 = 1.00452$

(c) Note: $|E_4(x)| = \left|\frac{F^{(5)}(c)x^5}{5!}\right|$ where c is between 0 and x.
Thus $|E_4(0.1)| = \left|\frac{F^{(5)}(c)(0.1)^5}{120}\right| \le \frac{12(0.1)^5}{120} = 0.000001$

12. (a)

$$\int \frac{dx}{1+x^2} = \arctan(x)$$

and

$$\begin{aligned}
\int\left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right)dx &= \sum_{k=0}^{\infty}(-1)^k\left(\int x^{2k}dx\right)\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\end{aligned}$$

(b)

$$\begin{aligned}
\arctan\left(3^{-1/2}\right) &= 3^{-1/2} - \frac{3^{-3/2}}{3} + \frac{3^{-5/2}}{5} - \frac{3^{-7/2}}{7} + \cdots\\
\frac{\pi}{6} &= 3^{-1/2} - \frac{3^{-3/2}}{3} + \frac{3^{-5/2}}{5} - \frac{3^{-7/2}}{7} + \cdots\\
\pi &= 6\left(3^{-1/2}\right)\left(1 - \frac{3^{-1}}{3} + \frac{3^{-2}}{5} - \frac{3^{-3}}{7} + \cdots\right)\\
\pi &= 2\left(3^{1/2}\right)\left(1 - \frac{3^{-1}}{3} + \frac{3^{-2}}{5} - \frac{3^{-3}}{7} + \cdots\right)
\end{aligned}$$

(c) $2\sqrt{3}\left(1 - \frac{3^{-1}}{3} + \cdots + \frac{3^{-16}}{33}\right) \approx 3.14159265.$

13. (a)

$$f'(x) = (1+x)^{-1},\ f''(x) = -(1+x)^{-2},$$
$$f'''(x) = 2(1+x)^{-3},\ f^{(4)}(x) = -2\cdot 3(1+x)^{(-4)}, \ldots$$

Inductively we see the pattern:

$$f^k(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}$$

(b)

$$\begin{aligned} P_N(x) &= \sum_{k=0}^{N} \frac{f^{(k)}(0)x^k}{k!} \\ &= \sum_{k=0}^{N} \frac{(-1)^{k-1}(k-1)!x^k}{k!} \\ &= \sum_{k=0}^{N} \frac{(-1)^{k-1}x^k}{k} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{N-1}x^N}{N} \end{aligned}$$

(c)

$$\begin{aligned} E_N(x) &= \frac{f^{(N+1)}(c)x^{N+1}}{(N+1)!} = \frac{(-1)^{N+1-1}(N+1-1)!x^{N+1}}{(N+1)!(1+c)^{N+1}} \\ &= \frac{(-1)^N x^{N+1}}{(N+1)(1+c)^{N+1}} \end{aligned}$$

where c is between x and 0.

(d) $P_3(0.5) = 0.41666667$, $P_6(0.5) = 0.40468750$,
$P_9(0.5) = 0.40553230$, $\ln(1.5) = 0.40546511$

(e) If $0.0 \le x \le 0.5$ then

$$|E_9(x)| = \left| \frac{(-1)^9 x^{10}}{10(1+c)^{10}} \right| \le \frac{(0.5)^{10}}{10(1)} \approx 0.00009765\ldots$$

14. (a) When $k=1$, $f'(x) = p(1+x)^{p-1}$. Assume

$$f^{(n)}(x) = p(p-1)\cdots(p-n+1)(1+x)^{p-n}$$

Then

$$\begin{aligned} f^{(n+1)}(x) &= \left(f^{(n)}(x)\right)' = p(p-1)\cdots(p-n+1)(p-n)(1+x)^{p-n-1} \\ &= p(p-1)\cdots(p-n+1)(p-(n+1)+1)(1+x)^{p-(n+1)} \end{aligned}$$

Therefore, by the principle of mathematical induction, for $k \geq 1$

$$f^{(k)}(x) = p(p-1)\cdots(p-k+1)(1+x)^{p-k}$$

(b)

$$\begin{aligned} P_N(x) &= \sum_{k=0}^{N} \frac{f^{(k)}(0)x^k}{k!} = \sum_{k=0}^{N} \frac{p(p-1)\cdots(p-k+1)x^k}{k!} \\ &= 1 + px + \frac{p(p-1)x^2}{2!} + \cdots + \frac{p(p-1)\cdots(p-N+1)x^N}{N!} \end{aligned}$$

(c)

$$\begin{aligned} E_N(x) &= \frac{f^{(N+1)}(c)x^{N+1}}{(N+1)!} \\ &= \frac{p(p-1)\cdots(p-(N+1)+1)(1+c)^{p-(N+1)}x^{N+1}}{(N+1)!} \\ &= \frac{p(p-1)\cdots(p-N)x^{N+1}}{(1+c)^{N+1-p}(N+1)!} \end{aligned}$$

(d) $P_2(0.5) = 1.218750$, $P_4(0.5) = 1.224121$, $P_6(0.5) = 1.224744$, $\sqrt{1.5} = 1.224744$

(e) If $N = 5$ and $p = 1/2$ then $E_N(x)$ in part (c) becomes

$$E_5(x) = -\frac{945x^6}{2^6(720)(1+c)^{11/2}}$$

If $0.0 \leq x \leq 0.5$ and c is between 0 and 1/2, then

$$\begin{aligned} |E_5(x)| &= \left| -\frac{945x^6}{2^6(720)(1+c)^{11/2}} \right| \leq \left| \frac{945(1/2)^6}{2^6(720)(1)} \right| \\ &= \frac{21}{2^6(1024)} \approx 0.0003204\ldots \end{aligned}$$

(f) In part (a) let $p = N$:

$$\begin{aligned} P_N(x) &= 1 + Nx + \frac{N(N-1)x^2}{2!} + \cdots \\ &\quad + \frac{N(N-1)\cdots(N-N+2)x^{N-1}}{(N-1)!} \\ &\quad + \frac{N(N-1)\cdots(N-N+1)x^N}{N!} \\ &= 1 + Nx + \frac{N(N-1)x^2}{2!} + \cdots + \frac{N!x^{N-1}}{(N-1)!} + \frac{N!x^N}{N!} \\ &= 1 + Nx + \frac{N(N-1)x^2}{2!} + \cdots + Nx^{N-1} + x^N \end{aligned}$$

15. (a) For $f(x) = \cos(x)$ and $x_0 = 0$ we have

$$|E_4(x)| = \left|\frac{f^{(5)}(d)(x-0)^5}{5!}\right|$$

where d is between x and 0. Thus

$$|E_4(x)| \le \frac{(1)(c^5)}{5!} < 10^{-6}$$

or $c < (10^{-6}5!)^{1/5} \approx 0.164375\ldots$.

(b) For $f(x) = \sin(x)$ and $x_0 = \pi/2$ we have

$$|E_4(x)| = \left|\frac{f^{(5)}(d)\left(x-\frac{\pi}{2}\right)^5}{5!}\right|$$

where d is between x and $\pi/2$. Thus

$$|E_4(x)| \le \frac{(1)(c^5)}{5!} < 10^{-6}$$

or $c < (10^{-6}5!)^{1/5} \approx 0.164375\ldots$.

(c) For $f(x) = e^x$ and $x_0 = 0$ we have

$$|E_4(x)| = \left|\frac{e^d(x-0)^5}{5!}\right|$$

where d is between 0 and x. Thus

$$|E_4(x)| \le \left|\frac{e^d c^5}{5!}\right| \le \left|\frac{e^c c^5}{5!}\right|$$

or $c^5e^c < 120(10^{-6})$. The root-finding techniques from Chapter 2 can be used to show that $c < 0.158$ will satisfy the error constraint.

16. (a) $P_N(x)$ is even

(b) $P_N(x)$ is odd

17. If f is a polynomial of degree N, then f equals its Taylor polynomial of degree N expanded about x_0:

$$f(x) = P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

Consider the case where $x \ge x_0$. Since $f(x_0) > 0$ and $f^{(k)}(x_0) \ge 0$ for $k = 1, 2, \ldots N$, it follows that

$$f(x) = P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} > 0$$

for $x \ge x_0$. Therefore, the real zeros, if any, of $y = f(x)$ must be less than x_0.

18. For $f(x) = e^x$ and $x_0 = 0$: $P_N(x) = \sum_{k=0}^{N} \frac{x^k}{k!}$. Further, note that $P_N'(x) = P_{N-1}(x)$ for $N \geq 2$. By way of contradiction assume p is a root of multiplicity $M \geq 2$. Then

$$P_N(p) = \sum_{k=0}^{N} \frac{p^k}{k!} = 0$$

and

$$P_N'(p) = P_{N-1}(p) \sum_{k=0}^{N-1} \frac{p^k}{k!} = 0$$

Thus $P_N(p) - P_{N-1}(p) = \frac{p^N}{N!} = 0$ or $p = 0$. A contradiction. Therefore every real root of $P_N(x)$ has multiplicity less than or equal to one.

19. In Corollary 4.1 it was shown that $P_N'(x) = f'(x)$. Assume $P_N^{(k-1)}(x) = f^{(k-1)}(x)$. Thus

$$P_N^{(k)}(x) = \left(P_N^{(k-1)}(x)\right)' = \left(f^{(k-1)}(x)\right)' = f^{(k-1)}(x)$$

Therefore, $P_N^{(k)}(x) = f^{(k)}(x)$ for all $k \geq 1$.

20. Let $x, x_0 \in (a, b)$. Since $g(x) = 0 = g(x_0)$ and g is continuous, then by Rolle's Theorem there is a c_1 between x and x_0 such that $g'(c_1) = 0$. Inductively, there is a c_N such that $g^{(N)}(x_0) = 0 = g^{(N)}(c_N)$. Therefore, by Rolle's Theorem there is a c between x and c_N such that $g^{(N+1)}(c) = 0$.

21.

$$g^{(N+1)}(t) = f^{(N+1)}(t) - 0 - \frac{E_N(x)}{(x - x_0)^{N+1}}(N+1)!$$

$$0 = g^{(N+1)}(c) = f^{(N+1)}(c) - \frac{E_N(x)(N+1)!}{(x - x_0)^{N+1}}$$

Solving for $E_N(x)$:

$$E_N(x) = \frac{f^{(N+1)}(c)(x - x_0)^{(N+1)}}{(N+1)!}$$

4.2 Introduction to Interpolation

1. $P(x) = -0.02x^3 + 0.01x^2 - 0.2x + 1.66$

   (a) Use $x = 4$ and get $b_3 = -0.02$, $b_2 = 0.02$, $b_1 = -0.12$, $b_0 = 1.18$. Hence $P(4) = -0.36$.

(b) Use $x = 4$ and get $d_2 = -0.06$, $d_1 = -0.04$, $d_0 = -0.36$. Hence $P'(4) = -0.36$.

(c) Use $x = 4$ and get $i_4 = -0.005$, $i_3 = 0.01333333$, $i_2 = -0.4666667$, $i_1 = 1.47333333$, $i_0 = 5.89333333$. Hence $I(4) = 5.89333333$. Similarly, use $x = 1$ and get $I(1) = 1.58833333$. Thus $\int_1^4 P(x)dx = I(4) - I(1) = 5.8933333 - 1.58833333 = 4.305$.

(d) Use $x = 5.5$ and get $b_3 = -0.02$, $b_2 = -0.01$, $b_1 = -0.255$, $b_0 = 0.2575$. Hence $P(5.5) = 0.2575$.

(e) The corresponding linear system is:

$$\begin{aligned} a+b+c+d &= 1.54 \\ 8a+4b+2c+d &= 1.5 \\ 27a+9b+3c+d &= 1.42 \\ 125a+25b+5c+d &= 0.66 \end{aligned}$$

The solution is $a = -0.02$, $b = 0.1$, $c = -0.02$, and $d = 1.66$.

2. $P(x) = -0.04x^3 + 0.14x^2 - 0.16x + 2.08$

(a) Use $x = 3$ and get $b_3 = -0.04$, $b_2 = 0.02$, $b_1 = -0.10$, $b_0 = 1.78$. Hence $P(3) = 1.78$.

(b) Use $x = 3$ and get $d_2 = -0.12$, $d_1 = -0.08$, $d_0 = -0.40$. Hence $P'(3) = -0.40$.

(c) Use $x = 3$ and get $i_4 = -0.01$, $i_3 = 0.01666667$, $i_2 = -0.03$, $i_1 = 1.99$, $i_0 = 5.97$. Hence $I(3) = 5.97$. Similarly, use $x = 0$ and get $I(0) = 0$. Thus $\int_0^3 P(x)dx = I(3) - I(0) = 5.97$.

(d) Use $x = 4.5$ and get $b_3 = -0.04$, $b_2 = -0.04$, $b_1 = -0.34$, $b_0 = 0.55$. Hence $P(4.5) = 0.55$.

(e) The corresponding linear system is:

$$\begin{aligned} a+b+c+d &= 1.05 \\ 8a+4b+2c+d &= 1.10 \\ 27a+9b+3c+d &= 1.35 \\ 125a+25b+5c+d &= 1.75 \end{aligned}$$

The solution is $a = -0.029166667$, $b = 0.275$, $c = -0.570833333$, and $d = 0.1375$.

3. (a) $\frac{1.05-1.06}{1.06} \approx -0.94\%$, $\frac{1.10-1.12}{1.12} \approx -1.79\%$
$\frac{1.35-1.34}{1.34} \approx 0.75\%$, $\frac{1.75-1.78}{1.78} \approx -1.69\%$
$\frac{-0.029166667+0.02}{0.02} \approx 45.83\%$ change in a_3
$\frac{-0.570833337+4}{0.4} \approx 42.71\%$ change in a_1

(b) $P(4) = 1.625$

(c) $P'(4) = 0.229166667$

(d) $I(4) - I(1) = 4.933333333 - 1.173958333 = 3.759375$

(e) $P(5.5) = 1.70156261$

4.3 Lagrange Approximation

1. (a) $P_1(x) = (-1)\frac{x-0}{-1-0} + 0 = x + 0 = x$

   (b) $P_2(x) = (-1)\frac{(x-0)(x-1)}{(-1-0)(-1-1)} + 0 + (1)\frac{(x+1)(x-0)}{(1+1)(1-0)} = x$

   (c) $P_3(x) = (-1)\frac{x(x-1)(x-2)}{(-1)(-2)(-3)} + 0 + (1)\frac{(x+1)(x)(x-2)}{(2)(1)(-1)} = x^3$

   (d) $P_1(x) = (1)\frac{x-2}{1-2} + 8\frac{x-1}{2-1} = 7x - 6$

   (e) $P_2(x) = 0 + \frac{x(x-2)}{(1)(-1)} + 8\frac{x(x-1)}{(2)(1)} = 3x^2 - 2x$

2. $f(x) = x + 2/x$

   (a) $P_2(x) = 0.4x^2 - 1.2x + 3.8$; $P_2(1.5) = 2.9$

   (b) $P_3(x) = -0.8x^3 + 4.8x^2 - 8.8x + 7.8$; $P_3(1.5) = 2.7$

3. $f(x) = 2\sin(\pi x/6)$

   (a) $P_2(x) = (-x^2 + 7x)/6$; $P_2(2) = 5/3$

   (b) $P_3(x) = (-x^3 - 6x^2 + 67x)/60$; $P_3(2) = 1.7$

4. $f(x) = 2\sin(\pi x/6)$

   (a) $P_2(x) = (-x^2 + 7x)/6$; $P_2(4) = 2.0$

   (b) $P_3(x) = (-x^3 - 6x^2 + 67x)/60$; $P_3(4) = 1.8$

5. (a) $f^{(4)}(c) = 0$ for all c, thus $E_3(x) = 0$ for all x.

   (b) $f^{(4)}(c) = 24$ for all c, thus $E_3(x) = (x+1)(x-0)(x-3)(x-4)$.

   (c) $E_3(x) = \frac{(x+1)x(x-3)(x-4)(120c-120)}{4!}$

6. (a) $P_2(x) = (1)\frac{(x-1.25)(x-1.5)}{(-0.25)(-0.5)} + (1.25^{1.25})\frac{(x-1)(x-1.5)}{(0.25)(-0.25)} + (1.5^{1.5})\frac{(x-1)(x-1.25)}{(0.5)(0.25)}$

   (b) $\frac{1}{1.5-1}\int_1^{1.5} x^x\,dx \approx 0.676833$

   (c) The fourth derivative of $f(x) = x^x$:

$$f^{(4)}(x) = (1+\ln(x))^4 x^x + 2x^{x-2} + \left(\frac{1-x}{x^2} + \frac{2}{x} + 3(1+\ln(x))^2 + 2(1+\ln(x))\left(\frac{x-1}{x} + \ln(x)\right) + \left(\frac{x-1}{x} + \ln(x)\right)^2\right) x^{x-1}$$

is positive on the interval $[1, 1.5]$. Thus the third derivative of f:

$$f'''(x) = 2x^{x-1}(1+\ln(x)) + x^x(1+\ln(x))^2 + x^{x-1}\left(\frac{x-1}{x} + \ln(x)\right)$$

is strictly increasing on the interval $[1, 1.5]$ and the maximum value of $|f'''(x)|$ occurs at 1.5:

$$\left|f^{(2+1)}(x)\right| \le f^{(3)}(1.5) \approx 7.97643 = M_3.$$

Using $h = 0.25$ in (27):

$$|E_2(x)| \le \frac{(0.25)^3(7.97643)}{9\sqrt{3}} \approx 0.0239854.$$

7. (a)

$$\begin{aligned} g(x) &= L_{2,0}(x) + L_{2,1}(x) + L_{2,2}(x) - 1 \\ &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ &\quad + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} - 1 \end{aligned}$$

The function $g(x)$ is a sum of polynomials of degree less than or equal to two. Therefore, g is a polynomial of degree less than or equal to two.

(b) $g(x_0) = L_{2,0}(x_0) + L_{2,1}(x_0) + L_{2,2}(x_0) - 1 = 1 + 0 + 0 - 1 = 0$. Similarly $g(x_1) = 0 = g(x_2)$.

(c) In part (b) it was shown that the polynomial $g(x)$, of degree less than or equal to two, has at least three distinct real zeros. But, the Fundamental Theorem of Algebra says a nonconstant polynomial of degree less than or equal to two can have at most two distinct real zeros. Therefore, $g(x) \equiv 0$, a polynomial of degree zero, for all x.

8. For $x = x_k$

$$L_{N,k} = \frac{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_N)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_N)} = 1$$

For $x = x_j$, $j \ne k$

$$L_{N,j} = \frac{(x_j-x_0)\cdots(x_j-x_j)\cdots(x_j-x_{k-1})(x_j-x_{k+1})\cdots(x_j-x_N)}{(x_k-x_0)\cdots(x_k-x_j)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_N)} = 0$$

Thus $\sum_{k=0}^{N} L_{N,k} = 1$ for any real number x.

9. Let $f(x) = c_N x^N + c_{N-1}x^{N-1} + \cdots + c_1 x + c_0$. Thus $f^{(N+1)} \equiv 0$. It follows from Theorem 4.3 that

$$\begin{aligned} f(x) &= P_N(x) + \tfrac{(x-x_0)(x-x_1)\cdots(x-x_N)}{(N+1)!} f^{(N+1)}(c) \\ &= P_N(x) + \tfrac{(x-x_0)(x-x_1)\cdots(x-x_N)}{(N+1)!}(0) \\ &= P_N(x) \end{aligned}$$

10. $|f''(c)| \le |-\sin(1)| = 0.84147098 = M_2$

$\left|f^{(3)}(c)\right| \le |-\cos(0)| = 1 = M_3$

$\left|f^{(4)}(c)\right| \le |\sin(1)| = 0.84147098 = M_4$

(a) If $h^2M_2/8 = h^2(0.84147098)/8 < 5\times 10^{-7}$, then $h^2 < 4.753580\times 10^{-6}$ and $h < 0.00218027$.

(b) If $\frac{h^3M_3}{9\sqrt{3}} = \frac{h^3(1)}{9\sqrt{3}} < 5 \times 10^{-7}$, then $h^3 < 7.794228 \times 10^{-6}$ and $h < 0.01982703$.

(c) If $\frac{h^4M_4}{24} = \frac{h^4(0.84147098)}{24} < 5 \times 10^{-7}$ then $h^4 < 1.426074 \times 10^{-7}$ and $h < 0.06145193$.

11. $|x-x_0||x-x_1||x-x_2| = |t+h||t||t-h| = |v(t)|$ where $v(t) = (t+h)(t)(t-h) = t^3 - th^2$ for $-h \le t \le h$. The critical points of $v(t)$ occur where $v'(t) = 3t^2 - h^2 = 0$; they are $t = \pm h/\sqrt{3}$. The values at the endpoints of the interval are $v(\pm h) = 0$, and

$$v\left(\frac{h}{\sqrt{3}}\right) = \frac{h^3}{3\sqrt{3}} - \frac{h^3}{\sqrt{3}} = \frac{-2h^3}{3\sqrt{3}}$$

$$v\left(-\frac{h}{\sqrt{3}}\right) = \frac{-h^3}{3\sqrt{3}} + \frac{h^3}{\sqrt{3}} = \frac{2h^3}{3\sqrt{3}}$$

Hence $|v(t)| \le max\left\{0, \frac{2h^3}{3\sqrt{3}}\right\}$ for $-h \le t \le h$. Therefore, $|x-x_0||x-x_1||x-x_2| \le \frac{2h^3}{3\sqrt{3}}$ for $x_0 \le x \le x_2$, where $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$.

12. (a) $z = 3 - 3x + 4y$

(b) $z = 3.5 - 2.5x + 1.5y$

(c) $z = \frac{22}{3} - \frac{4}{3}x + \frac{1}{3}y$

(d) No. The linear system is inconsistent. The two points $(1, 2, 5)$ and $(1, 2, 0)$ have the same x and y coordinates.

13. Let

$$g(t) = f(t) - P_N(t) - E_N(x)\left(\frac{\prod_{k=0}^N (t-x_k)}{\prod_{k=0}^N (x-x_k)}\right)$$

Note:

(a) By hypothesis $x, x_0, \ldots, x_N \in [a, b]$ and are constants with respect to t.

(b) $g(x) = 0$ and $g(x_k) = 0$ for $k = 0, 1, \ldots, N$.

(c) $g(t)$ satisfies the hypotheses of the Generalized Rolle's Theorem (Theorem 1.7).

(d)

$$g^{(N+1)}(t) = f^{(N+1)}(t) - 0 - E_N(x)\left(\frac{(N+1)!}{\prod_{k=0}^{N}(x - x_k)}\right)$$

By the Generalized Rolle's Theorem there is a $c \in (a, b)$ such that

$$\begin{aligned} 0 &= g^{(N+1)}(c) \\ &= f^{(N+1)}(c) - E_N(x)\left(\frac{(N+1)!}{\prod_{k=0}^{N}(x - x_k)}\right) \end{aligned}$$

or

$$E_N(x) = \frac{\left(\prod_{k=0}^{N}(x - x_k)\right) f^{(N+1)}(c)}{(N+1)!}$$

4.4 Newton Polynomials

1. $P_1(x) = 4 - (x - 1)$
   $P_2(x) = 4 - (x - 1) + 0.4(x - 1)(x - 3)$
   $P_3(x) = 4 - (x - 1) + 0.4(x - 1)(x - 3) + 0.01(x - 1)(x - 3)(x - 4)$
   $P_4(x) = P_3(x) - 0.002(x - 1)(x - 3)(x - 4)(x - 4.5)$
   $P_1(2.5) = 2.5$, $P_2(2.5) = 2.2$, $P_3(2.5) = 2.21$, $P_4(2.5) = 2.21575$

2. $P_1(x) = 5 - 2(x - 0)$
   $P_2(x) = 5 - 2(x - 0) + 0.5(x - 0)(x - 1)$
   $P_3(x) = 5 - 2(x - 0) + 0.5(x - 0)(x - 1) - 0.1(x - 0)(x - 1)(x - 2)$
   $P_4(x) = P_3(x) + 0.003(x - 0)(x - 1)(x - 2)(x - 3)$
   $P_1(2.5) = 0.0$, $P_2(2.5) = 1.875$, $P_3(2.5) = 1.6875$, $P_4(2.5) = 1.6846875$

3. $P_1(x) = 7 + 3(x + 1)$
   $P_2(x) = 7 + 3(x + 1) + 0.1(x + 1)(x - 0)$
   $P_3(x) = P_2(x) + 0.05(x + 1)(x - 0)(x - 1)$
   $P_4(x) = P_3(x) - 0.004(x + 1)(x - 0)(x - 1)(x - 4)$
   $P_1(3.0) = 19$, $P_2(3.0) = 20.2$, $P_3(3.0) = 21.4$, $P_4(3.0) = 21.496$

4. $P_1(x) = -2 + 4(x+3)$
   $P_2(x) = -2 + 4(x+3) - 0.04(x+3)(x+1)$
   $P_3(x) = P_2(x) + 0.06(x+3)(x+1)(x-1)$
   $P_4(x) = P_3(x) + 0.005(x+3)(x+1)(x-1)(x-4)$
   $P_1(2.0) = 18$, $P_2(2.0) = 17.4$, $P_3(2.0) = 18.3$, $P_4(2.0) = 18.15$

5. $f(x) = x^{1/2}$

$$\begin{aligned} P_4(x) &= 2.0 + 2.3607(x-4) - 0.01132(x-4)(x-5) \\ &\quad +0.00091(x-4)(x-5)(x-6) \\ &\quad -0.00008(x-4)(x-5)(x-6)(x-7) \end{aligned}$$

   $P_1(4.5) = 2.11804$, $P_2(4.5) = 2.12086$, $P_3(4.5) = 2.12121$,
   $P_4(4.5) = 2.12128$

6. $f(x) = \frac{3.6}{x}$

$$\begin{aligned} P_4(x) &= 3.6 - 1.8(x-1) + 0.6(x-1)(x-2) \\ &\quad -0.15(x-1)(x-2)(x-3) \\ &\quad +0.03(x-1)(x-2)(x-3)(x-4) \end{aligned}$$

   $P_1(2.5) = 0.9$, $P_2(2.5) = 1.35$, $P_3(2.5) = 1.40625$, $P_4(2.5) = 1.423125$
   $P_1(3.5) = -0.9$, $P_2(3.5) = 1.35$, $P_3(3.5) = 1.06875$, $P_4(3.5) = 1.040625$

7. $f(x) = 3\sin^2(\pi x/6)$

$$\begin{aligned} P_4(x) &= 0.0 + 0.75(x-0) + 0.375(x-0)(x-1) \\ &\quad -0.25(x-0)(x-1)(x-2) \\ &\quad -0.03125(x-0)(x-1)(x-2)(x-3) \end{aligned}$$

   $P_1(1.5) = 1.125$, $P_2(1.5) = 1.40625$, $P_3(1.5) = 1.5$, $P_4(1.5) = 1.51758$

8. $f(x) = e^{-x}$

$$\begin{aligned} P_4(x) &= 1.0 - 0.63212(x-0) + 0.19979(x-0)(x-1) \\ &\quad -0.04210(x-0)(x-1)(x-2) \\ &\quad +0.0065(x-0)(x-1)(x-2)(x-3) \end{aligned}$$

   $P_1(1.5) = 0.05182$, $P_2(1.5) = 0.20166$, $P_3(1.5) = 0.21745$, $P_4(1.5) = 0.22119$

9. (a) We are given that all $(N+1)st$ divided differences are zero. If we assume that all $(N+i)th$ divided differences are zero, then using the recursive definition (14) all $(N+(i+1))st$ divided differences are zero; i.e.,

$$\begin{aligned} & f\left[x_{k-(N+(i+1))}, x_{k-(N+(i+1))+1}, \ldots, x_k\right] \\ &= \frac{f\left[x_{k-(N+(i+1))+1}, \ldots, x_k\right] - f\left[x_{k-(N+(i+1))}, \ldots, x_{k-1}\right]}{x_k - x_{k-(N+(i+1))}} \\ &= \frac{0-0}{x_k - x_{k-(N+(i+1))}} = 0 \end{aligned}$$

Therefore, by the principle of mathematical induction; if all $(N+1)st$ divided differences are zero, then all $(N+k)th$ divided differences are zero for $k > 1$.

(b) The Newton interpolatory polynomial for the set of points $\{(x_k, y_k)\}_{k=0}^{M}$ is

$$\begin{aligned}
P_M(x) &= a_0 + \sum_{k=1}^{M}\left(a_k \prod_{j=0}^{k-1}(x - x_j)\right) \\
&= f[x_0] + \sum_{k=1}^{M}\left(f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1}(x - x_j)\right) \\
&= f[x_0] + \sum_{k=1}^{N}\left(f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1}(x - x_j)\right) \\
&\quad + \sum_{k=N+1}^{M}\left(f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1}(x - x_j)\right) \\
&= f[x_0] + \sum_{k=1}^{N}\left(f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1}(x - x_j)\right) \\
&= P_N(x),
\end{aligned}$$

a polynomial of degree N, assuming that not all Nth divided differences equal zero.

10. $P_2(x) = -2 + 4x - x(x-1) = -x^2 + 5x - 2$

11. $P_2(x) = 8 + 9(x-1) - (x-1)(x-2)$

12. $P_3(x) = 5 - x(x-1) + x(x-1)(x-2)$

13.

$$\begin{aligned}
|E_2(x)| &= \left|\frac{x(x-\pi/2)(x-\pi)f'''(c)}{3!}\right| \\
&= \left|\frac{x(x-\pi/2)(x-\pi)\pi^3\sin(\pi c)}{6}\right| \\
&\le \frac{\pi^3}{6}|x(x-\pi/2)(x-\pi)|
\end{aligned}$$

The function $g(x) = |x(x-\pi/2)(x-\pi)|$ is continuous on the closed interval $[0, \pi]$. The maximum value of g occurs at an endpoint or at a critical number in the open interval $(0,\pi)$. In particular, the maximum value occurs at the critical number $x = \frac{\sqrt{3}-1}{2\sqrt{3}}\pi$. Therefore,

$$|E_2(x)| \le \frac{\pi^3}{6} g\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\pi\right) \approx 7.709142$$

4.5 Chebyshev Polynomials (Optional)

1.

$$\begin{aligned} T_4(x) &= 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1) \\ &= 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1 \end{aligned}$$

$$\begin{aligned} T_5(x) &= 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) \\ &= 16x^5 - 16x^3 + 2x - 4x^3 + 3x = 16x^5 - 20x^3 + 5x \end{aligned}$$

2.

$$\begin{aligned} T_6(x) &= 2xT_5(x) - T_4(x) = 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) \\ &= 32x^6 - 40x^4 + 10x^2 - 8x^4 + 8x^2 - 1 = 32x^6 - 48x^4 + 18x^2 - 1 \end{aligned}$$

$$\begin{aligned} T_7(x) &= 2xT_6(x) - T_5(x) \\ &= 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x) \\ &= 64x^7 - 96x^5 + 36x^3 - 2x - 16x^5 + 20x^3 - 5x \\ &= 64x^7 - 112x^5 + 56x^3 - 7x \end{aligned}$$

3. When $N = 1 : T_1(x) = x$ and the leading coefficient is $1 = 2^{1-1}$. Thus the statement is true when $N = 1$. For the inductive hypothesis assume the statement is true for $N = j$, i.e.; the leading coefficient of $T_j(x)$ is 2^{j-1}. From Property 1: $T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$, where $T_j(x)$ and $T_{j-1}(x)$ are polynomials of degree j and $j-1$, respectively. Thus the leading coefficient of $T_{j+1}(x)$ is $2(2^{j-1}) = 2^{(j+1)-1}$. Therefore, by the principle of mathematical induction: "The coefficient of x^N in $T_N(x)$ is 2^{N-1} when $N \ge 1$."

4. $T_0(x) + 1$ is even and $T_1(x)$ is odd. Hence $xT_0(x)$ is odd and $xT_1(x)$ is even. The sum of two even functions is even and the sum of two odd functions is odd. It follows that $T_2(x) = 2xT_1(x) - T_0(x)$ is even and $T_3(x) = 2xT_2(x) - T_1(x)$ is odd. Assume that $T_{2k}(x)$ is even and $T_{2k+1}(x)$ is odd for $k = 1, \ldots, n$. Then $T_{2k+2}(x) = 2xT_{2k+1}(x) - T_{2k}(x)$ is even and $T_{2k+3} = 2xT_{2k+2}(x) - T_{2k+1}(x)$ is odd. Therefore, it follows that $T_{2M}(x)$ is even and $T_{2M+1}(x)$ is odd for all M.

5. $T_2(x) = 2x^2 - 1$, $T_2'(x) = 4x$. The critical value in $(-1, 1)$ is $x = 0$. At the endpoints $T_2(-1) = 1 = T_2(1)$ and at the critical value $T_2(0) = -1$. The minimum value of $T_2(x)$ over $[-1, 1]$ is -1 and the maximum value is 1.

6. $T_3(x) = -3x + 4x^3$, $T_3'(x) = -3 + 12x^2 = 3(-1 + 2x)(1 + 2x)$. At the endpoints $T_3(-1) = -1$ and $T_3(1) = 1$. The critical values are $x_1 = 1/2$ and $x_2 = -1/2$. At the critical values $T_3(1/2) = -1$ and $T_3(-1/2) = 1$. The minimum of $T_3(x)$ over $[-1, 1]$ is -1 and the maxmimum is 1.

7. $T_4(x) = 1 - 8x^2 + 8x^4$, $T_4' = -16x + 32x^3 = 16x(2x^2 - 1) = 16x(x - 1/\sqrt{2})(x + 1/\sqrt{2})$. The critical values are $x_1 = 0$, $\pm 1/\sqrt{2}$. At the endpoints $T_4(-1) = 1$ and $T_4(1) = 1$. At the critical values $T_4(-1/\sqrt{2}) = -1$, $T_4(0) = 1$, and $T_4(1/\sqrt{2}) = -1$. The minimum of $T_4(x)$ over $[-1, 1]$ is -1 and the maximum is 1.

8. (a) $\sin(x) \approx 0.99898284x - 0.15850489x^3$

   (b) $\frac{|f^{(4)}(x)|}{2^3(4!)} \leq \frac{|\sin(1)|}{2^3(4!)} = 0.00438266$

9. (a) $\ln(x+2) \approx 0.69549038 + 0.49905042x - 0.14334605x^2 + 0.04909073x^3$

   (b) $\frac{|f^{(4)}(x)|}{2^3(4!)} \leq \frac{|-6|}{2^3(4!)} = 0.03125000$

10. $x_0 = \cos(5\pi/6) = -\sqrt{3}/2$, $x_1 = 0$, $x_2 = \cos(\pi/6) = \sqrt{3}/2$.

$$
\begin{aligned}
L_{2,0}(x) &= \frac{(x-0)\left(x - \frac{\sqrt{3}}{2}\right)}{\left(-\frac{\sqrt{3}}{2} - 0\right)\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)} = \frac{x^2 - \frac{\sqrt{3}}{2}x}{\frac{3}{2}} = \frac{2}{3}x^2 - \frac{1}{\sqrt{3}}x \\
L_{2,1}(x) &= \frac{\left(x + \frac{\sqrt{3}}{2}\right)\left(x - \frac{\sqrt{3}}{2}\right)}{\left(0 + \frac{\sqrt{3}}{2}\right)\left(0 - \frac{\sqrt{3}}{2}\right)} = \frac{x^2 - \frac{3}{4}}{-\frac{3}{4}} = -\frac{4}{3}x^2 + 1 \\
L_{2,2}(x) &= \frac{\left(x + \frac{\sqrt{3}}{2}\right)(x-0)}{\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2} - 0\right)} = \frac{x^2 + \frac{\sqrt{3}}{2}x}{\frac{3}{2}} = \frac{2}{3}x^2 + \frac{1}{\sqrt{3}}x
\end{aligned}
$$

11. (a) $\cos(x) \approx 1 - 0.46952087x^2$

    (b) $\frac{|f^{(3)}(x)|}{2^2(3!)} \leq \frac{|sin(1)|}{2^2(3!)} = 0.03506129$

12. (a) $e^x \approx 1 + 1.12977208x + 0.53204180x^2$

    (b) $\frac{|f^{(3)}(x)|}{2^2(3!)} \leq \frac{|e^1|}{2^2(3!)} = 0.11326174$

13. The error bound for Taylor's polynomial is:

$$\frac{|f^{(8)}(x)|}{8!} \leq \frac{|\sin(1)|}{8!} = 0.00002087.$$

The error bound for the minimax approximation is:

$$\frac{|f^{(8)}(x)|}{2^7(8!)} \leq \frac{|\sin(1)|}{2^7(8!)} = 0.00000261.$$

14. The error bound for Taylor's polynomial is:

$$\frac{|f^{(7)}(x)|}{7!} \leq \frac{|\sin(1)|}{7} = 0.00016696$$

The error bound for the minimax approximation is:

$$\frac{|f^{(7)}(x)|}{2^6(7!)} \leq \frac{|\sin(1)|}{2^6(7!)} = 0.00000261.$$

15. The error bound for Taylor's polynomial is:

$$\frac{|f^{(8)}(x)|}{8!} \leq \frac{|e^1|}{8!} = 0.00006742$$

The error bound for the minimax approximation is:

$$\frac{|f^{(8)}(x)|}{2^7(8!)} \leq \frac{|e^1|}{2^7(8!)} = 0.00000053.$$

16.

$$\begin{aligned}\sum_{k=0}^{N} T_i(x_k)T_j(x_k) &= \sum_{k=0}^{N} \cos(ix_k)\cos(jx_k) \\ &= \sum_{k=0}^{N} \frac{1}{2}(\cos((i-j)x_k) + \cos((i+j)x_k))\end{aligned}$$

Assume $i \neq j$. Thus:

Case 1: If N is odd, then

$$\cos\left((i+j)\frac{2k+1}{2N+2}\pi\right) = -\cos\left((i+j)\frac{2(N-k)+1}{2N+2}\pi\right)$$

for $k = 0, 1, \ldots, N$. A similar result occurs for the term $\cos((i-j)x_k)$. Therefore, in this case

$$\begin{aligned}\sum_{k=0}^{N} T_i(x_k)T_j(x_k) &= \sum_{k=0}^{N} \frac{1}{2}(\cos((i-j)x_k) + \cos((i+j)x_k)) \\ &= \frac{1}{2}(0+0) = 0\end{aligned}$$

Case 2: The case when N is even is similar.

17. When $i = j \neq 0$

$$\begin{aligned}
\sum_{k=0}^{N} T_i(x_k)T_j(x_k) &= \sum_{k=0}^{N} \cos(ix_k)\cos(jx_k) \\
&= \sum_{k=0}^{N} \frac{1}{2}(\cos((i+j)x_k) + \cos((i-j)x_k)) \\
&= \sum_{k=0}^{N} \frac{1}{2}(\cos(2jx_k) + \cos(0)) \\
&= \frac{1}{2}\sum_{k=0}^{N} \cos\left(\frac{2k+1}{N+1}j\pi\right) + \frac{1}{2}\sum_{k=0}^{N} 1 \\
&= 0 + \frac{N+1}{2} = \frac{N+1}{2}
\end{aligned}$$

4.6 Padé Approximations

1. $1 = p_0$, $1 + q_1 = p_1$, $1/2 + q_1 = 0$, $q_1 = -1/2$, $p_1 = 1/2$
   $e^x \approx R_{1,1}(x) = \frac{2+x}{2-x}$

2. (a) $1 = p_0$, $-1/2 + q_1 = p_1$, $\frac{1}{3} - \frac{q_1}{2} = 0$, $q_1 = 2/3$, $p_1 = 1/6$
   $\frac{\ln(x+1)}{x} \approx R_{1,1}(x) = \frac{1+\frac{1}{6}x}{1+\frac{2}{3}x}$

   (b) $\ln(x+1) \approx xR_{1,1}(x) = R_{2,1}(x) = \frac{6x+x^2}{6+4x}$

3. (a) $1 = p_0$, $\frac{1}{3} + \frac{2q_1}{15} = p_1$, $\frac{2}{15} + \frac{q_1}{3} = 0$, $q_1 = -\frac{2}{5}$, $p_1 = -\frac{1}{15}$
   $\frac{\tan(x^{1/2})}{x^{1/2}} \approx R_{1,1} = \frac{15-x}{15-6x}$

   (b) If $\frac{\tan(x^{1/2})}{x^{1/2}} \approx \frac{15-x}{15-6x}$, then

   $$\frac{\tan(x)}{x} \approx \frac{15-x^2}{15-6x^2}$$

   or

   $$\tan(x) \approx \frac{15x - x^3}{16-6x^2} = R_{3,2}(x)$$

4. (a) $1 = p_0$, $-\frac{1}{3} + q_1 = p_1$, $\frac{1}{5} - \frac{q_1}{3} = 0$, $q_1 = \frac{3}{5}$, $p_1 = \frac{4}{15}$
   $\frac{\arctan(x^{1/2})}{x^{1/2}} \approx R_{1,1} = \frac{x + \frac{4}{15}x^3}{1+ \frac{3}{5}x^2}$

   (b)

   $$\frac{\arctan(x)}{x} \approx \frac{15+4x^2}{15+9x^2}$$

   or

   $$\arctan(x) \approx \frac{15x + 4x^3}{15+9x^2} = R_{3,2}(x)$$

(c) $\frac{4x^3+15x}{9x^2+15} = \frac{4}{9}x + \frac{25}{3}\left(\frac{1}{9x+\frac{15}{x}}\right)$

5. (a) $1 = p_0$, $1 + q_1 = p_1$, $1/2 + q_1 + q_2 = p_2$, $1/6 + q_1/2 + q_2 = 0$, $1/24 + q_1/6 + q_2/2 = 0$ First solve this system:

$$\begin{aligned} \tfrac{1}{6} + \tfrac{q_1}{2} + q_2 &= 0 \\ \tfrac{1}{24} + \tfrac{q_1}{6} + \tfrac{q_2}{2} &= 0 \end{aligned}$$

$q_1 = -1/2$, $q_2 = 1/12$, $p_1 = 1/2$, $p_2 = 1/12$.

$e^x \approx R_{2,2}(x) = \frac{1+\frac{1}{2}x + \frac{1}{12}x^2}{1-\frac{1}{2}x + \frac{1}{12}x^2}$

(b)

$$\begin{aligned} \frac{x^2+6x+12}{x^2-6x+12} &= 1 + 12\left(\frac{x}{x^2-6x+12}\right) \\ &= 1 + 12\left(\frac{1}{\frac{x^2-6x+12}{x}}\right) \\ &= 1 + 12\left(\frac{1}{x-6+\frac{12}{x}}\right) \end{aligned}$$

6. (a) $1 = p_0$, $-1/2 + q_1 = p_1$, $1/3 - q_1/2 + q_2 = p_2$, $-1/4 + q_1/3 - q_1/2 = 0$, $1/5 + q_1/4 + q_2/3 = 0$. Solving the last two equations for q_1 and q_2 yields $q_1 = 6/5$, $q_2 = 3/10$, $p_1 = 7/10$, and $p_2 = 1/30$

$\frac{\ln(1+x)}{x} \approx R_{2,2}(x) = \frac{1+\frac{7}{10}x + \frac{1}{30}x^2}{1+\frac{6}{5}x + \frac{3}{10}x^2}$

(b) If

$$\frac{\ln(1+x)}{x} \approx R_{2,2}(x) = \frac{30 + 21x + x^2}{30 + 36x + 9x^2},$$

then

$$\ln(1+x) \approx \frac{30x + 21^2 + x^3}{30 + 36x + 9x^2} = R_{3,2}(x)$$

(c) $\frac{x^3+21x^2+30x}{9x^2+36x+30} = \frac{1}{9}x + \frac{17}{9} - \left(\frac{1}{\frac{27}{124}x + \frac{4401}{7688} - \frac{9375}{3844}\left(\frac{1}{\frac{124}{3}x + \frac{170}{3}}\right)}\right)$

7. (a) $1 = p_0$, $1/3 + q_0 = p_1$, $2/15 + q_1/3 + q_2 = p_2$, $17/315 + 2q_1/15 + q_2/3 = 0$, $62/2835 + 17q_1/315 + 2q_2/15 = 0$. Solving for q_1 and q_2 yields $q_1 = -4/9$, $q_2 = 1/63$, $p_1 = -1/9$, $p_2 = 1/945$.

$\frac{\tan(x^{1/2})}{x^{1/2}} \approx R_{2,2}(x) = \frac{1-\frac{1}{9}x + \frac{1}{945}x^2}{1-\frac{4}{9}x + \frac{1}{63}x^2}$

(b)

$$\frac{\tan(x)}{x} \approx \frac{945 - 105x + x^2}{945 - 420^2 + 15x^4}$$

$$\tan(x) \approx \frac{945x - 105x^3 + x^5}{945 - 420x^2 + 15x^4} = R_{5,4}(x)$$

(c) Recursively dividing yields the quotients:

$$q_1 = \frac{x}{15},\ q_2 = \frac{15x}{28},\ q_3 = \frac{63}{484} - \frac{14x}{165}$$

$$q_4 = \frac{16798369500}{28591087} + \frac{212960x}{14147},\ q_5 = -\frac{408441\left(\frac{397845}{484} - \frac{42441x}{1936}\right)}{1968733861380}$$

The continued fraction form is:

$$R_{5,4}(x) = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \frac{1}{q_5}}}}$$

8. (a) $1 = p_0$, $-1/3 + q_1 = p_1$, $1/5 - q_1/3 + q_2 = p_2$, $-1/7 + q_1/5 - q_2/3 = 0$, $1/9 - q_1/7 + q_2/5 = 0$. Solving for q_1 and q_2 yields $q_1 = 10/9$, $q_2 = 5/21$, $p_1 = 7/9$, $p_2 = 64/945$

$\frac{\arctan(x^{1/2})}{x^{1/2}} \approx R_{2,2}(x) = \frac{1+\frac{7}{9}x + \frac{64}{945}x^2}{1+\frac{10}{9}x + \frac{5}{21}x^2}$

(b) Substitute x for $x^{1/2}$ in the result from part (a):

$$\begin{aligned}\frac{\arctan(x)}{x} &\approx \frac{945 + 735x^2 + 64x^4}{945 + 1050x^2 + 225x^4}\\ \arctan(x) &\approx R_{5,4}(x)\\ &= \frac{945x735x^3 64x^5}{945 + 1050x^2 + 225x^4}\end{aligned}$$

(c) Recursively dividing yields the quotients:

$$q_1 = \frac{64x}{225},\ q_2 = \frac{675x}{1309},\ q_3 = \frac{34969x}{56205}$$

$$q_4 = \frac{39000025x}{4907628},\ q_5 = \frac{2916x}{31225}$$

The continued fraction form for $R_{5,4}(x)$ is

$$R_{5,4}(x) = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \frac{1}{q_5}}}}$$

9. Note: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. From formula (5) we consider:

$$(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots)(1+q_1x+q_2x^2+q_3x^3)-(p_0+p_1x+p_2x^2+p_3x^3).$$

Matching common terms yields the linear system:

$$\begin{aligned}
1-p_0 &= 0\\
1+q_1-p_1 &= 0\\
\frac{1}{2}+q_1+q_2-p_2 &= 0\\
\frac{1}{6}+\frac{1}{2}q_1+q_2+q_3-p_3 &= 0\\
\frac{1}{24}+\frac{1}{6}q_1+\frac{1}{2}q_2+q_3 &= 0\\
\frac{1}{120}+\frac{1}{24}q_1+\frac{1}{6}q_2+\frac{1}{2}q_3 &= 0\\
\frac{1}{720}+\frac{1}{120}q_1+\frac{1}{24}q_2+\frac{1}{6}q_3 &= 0
\end{aligned}$$

Solving the last three equations first, yields $q_1 = -\frac{1}{2}$, $q_2 = \frac{1}{10}$, and $q_3 = -\frac{1}{120}$. Substituting these values into the first four equations yields $p_0 = 1$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{10}$, and $p_3 = \frac{1}{20}$. Therefore,

$$\begin{aligned}
R_{3,3}(x) &= \frac{1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{120}x^2}{1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3}\\
&= \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3}
\end{aligned}$$

10. Note: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. From formula (5) we consider:

$$\begin{aligned}
&(1+x+\tfrac{x^2}{2}+\tfrac{x^3}{6}+\tfrac{x^4}{24}+\cdots)(1+q_1x+q_2x^2+q_3x^3q_4x^4)\\
&-(p_0+p_1x+p_2x^2+p_3x^3+p_4x^4).
\end{aligned}$$

Matching common terms yields the linear system:

$$\begin{aligned}
1-p_0 &= 0\\
1+q_1-p_1 &= 0\\
\frac{1}{2}+q_1+q_2-p_2 &= 0\\
\frac{1}{6}+\frac{1}{2}q_1+q_2+q_3-p_3 &= 0
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{24} + \frac{1}{6}q_1 + \frac{1}{2}q_2 + q_3 + q_4 - p_4 &= 0 \\
\frac{1}{120} + \frac{1}{24}q_1 + \frac{1}{6}q_2 + \frac{1}{2}q_3 + q_4 &= 0 \\
\frac{1}{6!} + \frac{1}{5!}q_1 + \frac{1}{24}q_2 + \frac{1}{6}q_3 + \frac{1}{2}q_4 &= 0 \\
\frac{1}{7!} + \frac{1}{6!}q_1 + \frac{1}{5!}q_2 + \frac{1}{4!}q_3 + \frac{1}{3!}q_4 &= 0 \\
\frac{1}{8!} + \frac{1}{7!}q_1 + \frac{1}{6!}q_2 + \frac{1}{5!}q_3 + \frac{1}{4!}q_4 &= 0
\end{aligned}
$$

Solving the last four equations first, yields $q_1 = -\frac{1}{2}$, $q_2 = \frac{3}{28}$, $q_3 = -\frac{1}{84}$, and $q_4 = \frac{1}{1680}$. Substituting these values into the first five equations yields $p_0 = 1$, $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{28}$, $p_3 = \frac{1}{84}$, and $p_4 = \frac{1}{1680}$. Therefore,

$$
\begin{aligned}
R_{4,4}(x) &= \frac{1 + \frac{1}{2}x + \frac{3}{28}x^2 + \frac{1}{84}x^3 + \frac{1}{1680}x^4}{1 - \frac{1}{2}x + \frac{3}{28}x^2 - \frac{1}{84}x^3 + \frac{1}{1680}x^4} \\
&= \frac{1680 + 840x + 180x^2 + 20x^3 + x^4}{1680 - 840x + 180x^2 - 20x^3 + x^4}
\end{aligned}
$$

Chapter 5

Curve Fitting

5.1 Least-Squares Line

1. (a) Solve:
$$\begin{cases} 10A + 0B &= 7 \\ 0A + 5B &= 13 \end{cases}$$
$y = 0.7x + 2.60$, $E_2(f) \approx 0.2449$

   (b) Solve:
$$\begin{cases} 80A + 0B &= -48 \\ 0A + 5B &= 17 \end{cases}$$
$y = -0.6x + 3.40$, $E_2(f) \approx 0.2828$

   (c) Solve:
$$\begin{cases} 30A + 0B &= 21 \\ 0A + 5B &= -1 \end{cases}$$
$y = 0.7 - x$, $E_2(f) \approx 0.1414$

2. (a) Solve:
$$\begin{cases} 40A + 0B &= 58 \\ 0A + 5B &= 31.2 \end{cases}$$
$y = 1.45x + 6.24$, $E_2(f) \approx 0.8958$

   (b) Solve:
$$\begin{cases} 80A + 0B &= 63.2 \\ 0A + 5B &= -6.3 \end{cases}$$
$y = 0.79x - 1.26$, $E_2(f) \approx 0.5299$

   (c) Solve:
$$\begin{cases} 120A + 0B &= -15 \\ 0A + 5B &= 13.2 \end{cases}$$
$y = 0.585x + 2.64$, $E_2(f) \approx 0.6488$

3. (a) $\sum_{k=1}^{N} x_k y_k \Big/ \sum_{k=1}^{N} x_k^2 = \frac{21}{30} = 0.7$, $y = 0.7x$, $E_2(f) \approx 0.2449$

   (b) $\sum_{k=1}^{N} x_k y_k \Big/ \sum_{k=1}^{N} x_k^2 = \frac{86.1}{150} = 0.574$, $y = 0.574x$, $E_2(f) \approx 0.0756$

   (c) $\sum_{k=1}^{N} x_k y_k \Big/ \sum_{k=1}^{N} x_k^2 = \frac{86.9}{55} = 1.58$, $y = 1.58x$, $E_2(f) \approx 0.1720$

4. Use the second normal equation $\left(\sum_{k=1}^{N} x_k\right) A + NB = \sum_{k=1}^{N} y_k$ and divide through by N, and obtain $A\left(\frac{1}{N}\sum_{k=1}^{N} x_k\right) + B = \frac{1}{N}\sum_{k=1}^{N} y_k$. Therefore, $A\overline{x} + B = \overline{y}$.

5. In (10) the sum of N times the first equation and $-\sum_{k=1}^{N} x_k$ times the second equation is:

$$N\left(\sum_{k=1}^{N} x_k^2\right) A - \left(\sum_{k=1}^{N} x_k\right)^2 A = N\sum_{k=1}^{N} x_k y_k - \sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k$$

Thus

$$\begin{aligned} A &= \frac{N\sum_{k=1}^{N} x_k y_k - \sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k}{N\left(\sum_{k=1}^{N} x_k^2\right) - \left(\sum_{k=1}^{N} x_k\right)^2} \\ &= \frac{1}{D}\left(N\sum_{k=1}^{N} x_k y_k - \sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k\right) \end{aligned}$$

Substituting the expression for A into the first equation in (10) and solving for B yields

$$\begin{aligned} B &= \frac{\sum_{k=1}^{N} y_k - \dfrac{\sum_{k=1}^{N} x_k \left(N\sum_{k=1}^{N} x_k y_k - \sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k\right)}{D}}{N} \\ &= \frac{D\sum_{k=1}^{N} y_k - N\sum_{k=1}^{N} x_k \sum_{k=1}^{N} x_k y_k + \left(\sum_{k=1}^{N} x_k\right)^2 \sum_{k=1}^{N} y_k}{ND} \\ &= \frac{N\sum_{k=1}^{N} y_k \sum_{k=1}^{N} x_k^2 - N\sum_{k=1}^{N} x_k \sum_{k=1}^{N} x_k y_k}{ND} \\ &= \frac{1}{D}\left(\sum_{k=1}^{N} y_k \sum_{k=1}^{N} x_k^2 - \sum_{k=1}^{N} x_k \sum_{k=1}^{N} x_k y_k\right) \end{aligned}$$

6.

$$\begin{aligned} N\sum_{k=1}^{N} (x_k - \overline{x})^2 &= N\sum_{k=1}^{N} \left(x_k^2 - 2x_k\overline{x} + \overline{x}^2\right) \\ &= N\left(\sum_{k=1}^{N} x_k^2 - 2\overline{x}\sum_{k=1}^{N} x_k + \sum_{k=1}^{N} \overline{x}^2\right) \\ &= N\left(\sum_{k=1}^{N} x_k^2 - \frac{2}{N}\left(\sum_{k=1}^{N} x_k\right)^2 + \frac{N}{N^2}\left(\sum_{k=1}^{N} x_k\right)^2\right) \\ &= N\sum_{k=1}^{N} x_k^2 - \left(\sum_{k=1}^{N} x_k\right)^2 \\ &= D \end{aligned}$$

Clearly $D = N\sum_{k=1}^{N}(x_k - \overline{x})^2 > 0$ since $N \geq 2$ and the $x_k's$ are all distinct.

7. To find the least-squares line $Y = AX$ for the data $\{X_k, Y_k\}_{k=1}^{N}$ first minimize $E(A) = \sum_{k=1}^{N}(AX_k - Y_k)^2$. Solving

$$0 = \frac{\partial E}{\partial A} = \sum_{k=1}^{N}(AX_k - Y_k)\,X_k,$$

yields

$$A = \frac{\sum_{k=1}^{N} X_k Y_k}{\sum_{k=1}^{N} X_k^2}.$$

Making the substitutions $X_k = x_k - \overline{x}$ and $Y_k = y_k - \overline{y}$ yields

$$A = \frac{\sum_{k=1}^{N}(x_k - \overline{x})(y_k - \overline{y})}{\sum_{k=1}^{N}(x_k - \overline{x})^2} = \frac{1}{C}\sum_{k=1}^{N}(x_k - \overline{x})(y_k - \overline{y})$$

To see that this is equivalent to the result obtained for A in Exercise 5 make the substitutions indicated in Exercise 4 to obtain

$$\begin{aligned} A &= \tfrac{1}{C}\sum_{k=1}^{N}(x_k - \overline{x})(y_k - \overline{y}) \\ &= \frac{N\sum_{k=1}^{N}(x_k y_k - x_k\overline{y} - \overline{x}y_k + \overline{x}\,\overline{y})}{D} \\ &= \frac{N\left(\sum_{k=1}^{N} x_k y_k - \frac{1}{N}\sum_{k=1}^{N} x_k y_k - \frac{1}{N}\sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k + \frac{N}{N^2}\sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k\right)}{D} \\ &= \frac{N\sum_{k=1}^{N} x_k y_k - \sum_{k=1}^{N} x_k \sum_{k=1}^{N} y_k}{D} \end{aligned}$$

In Exercise 4 it was shown that $(\overline{x}, \overline{y})$ lies on the least-squares line $y = Ax + B$ for the data points $\{x_k, y_k\}_{k=1}^{N}$. Thus $\overline{y} = A\overline{x} + B$ or $B = \overline{y} - A\overline{x}$. Therefore, the expressions given for A and B in Exercise 7 are equivalent to those given in Exercise 5.

8. (a) $y = 1.6866x^2$, $E_2(f) \approx 1.3$
   $y = 0.5902x^3$, $E_2(f) \approx 0.29$ The best fit.

   (b) $y = 1.6177x^2$, $E_2(f) \approx 0.359$ The best fit.
   $y = 0.5606x^3$, $E_2(f) \approx 1.165$

9. (a) $y = \frac{3.5339}{x}$, $E_2(f) \approx 0.034$. The best fit.
   $y = \frac{2.0060}{x^2}$, $E_2(f) \approx 1.189$

   (b) $y = \frac{4.6013}{x}$, $E_2(f) \approx 1.082$
   $y\frac{3.9755}{x^2}$, $E_2(f) \approx 0.035$. The best fit.

10. (a) Given $f(x) = Ax$ and $\{x_k, y_k\}_{k=1}^{N}$ minimize
$E(A) = \sum_{k=1}^{N} (Ax_k - y_k)^2$.

$$\begin{aligned}\frac{\partial E}{\partial A} &= \sum_{k=1}^{N} 2x_k (Ax_k - y_k) \\ &= 2\left(\sum_{k=1}^{N} Ax_k^2 - \sum_{k=1}^{N} x_k y_k\right)\end{aligned}$$

Setting $\partial E/\partial A = 0$ yields the single normal equation:

$$\left(\sum_{k=1}^{N} x_k^2\right) A = \sum_{k=1}^{N} x_k y_k$$

(b) Given $f(x) = Ax^2$ and $\{x_k, y_k\}_{k=1}^{N}$ minimize
$E(A) = \sum_{k=1}^{N} (Ax_k^2 - y_k)^2$.

$$\begin{aligned}\frac{\partial E}{\partial A} &= \sum_{k=1}^{N} 2x_k^2 (Ax_k^2 - y_k) \\ &= 2\left(\sum_{k=1}^{N} Ax_k^4 - \sum_{k=1}^{N} x_k^2 y_k\right)\end{aligned}$$

Setting $\partial E/\partial A = 0$ yields the single normal equation:

$$\left(\sum_{k=1}^{N} x_k^4\right) A = \sum_{k=1}^{N} x_k^2 y_k$$

(c) Given $f(x) = Ax^2 + B$ and $\{x_k, y_k\}_{k=1}^{N}$ minimize
$E(A, B) = \sum_{k=1}^{N} (Ax_k^2 + B - y_k)^2$.

$$\begin{aligned}\frac{\partial E}{\partial A} &= \sum_{k=1}^{N} 2x_k^2 (Ax_k^2 + B - y_k) \\ &= 2\left(\sum_{k=1}^{N} Ax_k^4 + \sum_{k=1}^{N} Bx_k^2 - \sum_{k=1}^{N} x_k^2 y_k\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial E}{\partial B} &= \sum_{k=1}^{N} 2 (Ax_k^2 + B - y_k) \\ &= 2\left(\sum_{k=1}^{N} Ax_k^2 + \sum_{k=1}^{N} B - \sum_{k=1}^{N} y_k\right)\end{aligned}$$

Setting $\partial E/\partial A = 0$ and $\partial E/\partial B = 0$ yields the normal equations:

$$\begin{aligned}\left(\sum_{k=1}^{N} x_k^4\right) A + \left(\sum_{k=1}^{N} x_k^2\right) B &= \sum_{k=1}^{N} x_k^2 y_k \\ \left(\sum_{k=1}^{N} x_k^2\right) A + NB &= \sum_{k=1}^{N} y_k\end{aligned}$$

11. (a)

$$\begin{aligned}lim_{n\to\infty} \overline{x}_N &= lim_{n\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{k}{N} \\ &= lim_{n\to\infty} \left(\frac{1}{N^2}\right) \left(\frac{N(N+1)}{2}\right) \\ &= \frac{1}{2} = \hat{x}\end{aligned}$$

(b)

$$\begin{aligned} lim_{n\to\infty}\overline{y}_N &= lim_{n\to\infty}\tfrac{1}{N}\textstyle\sum_{k=1}^{N}\tfrac{k^2}{N^2} \\ &= lim_{n\to\infty}\left(\tfrac{1}{N^3}\right)\left(\tfrac{N(N+1)(2N+1)}{6}\right) \\ &= \tfrac{2}{6}=\tfrac{1}{3} \end{aligned}$$

To show that $\hat{y}=1/3$ use the Mean Value Theorem for Integrals:

$$\hat{y}=\frac{1}{1-0}\int_0^1 x^2\,dx=\left.\frac{x^3}{3}\right|_0^1=\frac{1}{3}$$

12. (a)

$$\begin{aligned} lim_{n\to\infty}\overline{x}_N &= lim_{n\to\infty}\tfrac{1}{N}\textstyle\sum_{k=1}^{N}\left((b-a)\tfrac{k}{N}+a\right) \\ &= lim_{n\to\infty}\tfrac{1}{N}\left(\left(\tfrac{b-a}{N}\right)\left(\tfrac{N(N+1)}{2}\right)+Na\right) \\ &= \tfrac{b-a}{2}+a \\ &= \tfrac{b}{2}-\tfrac{a}{2}+a=\tfrac{b}{2}+\tfrac{a}{2}=\tfrac{b+a}{2}=\hat{x} \end{aligned}$$

(b)

$$\begin{aligned} lim_{n\to\infty}\overline{y}_N &= lim_{n\to\infty}\tfrac{1}{N}\textstyle\sum_{k=1}^{N}f\left((b-a)\tfrac{k}{N}+a\right) \\ &= lim_{n\to\infty}\tfrac{1}{b-a}\textstyle\sum_{k=1}^{N}f\left((b-a)\tfrac{k}{N}+a\right)\left(\tfrac{b-a}{N}\right) \\ &= \tfrac{1}{b-a}lim_{n\to\infty}\textstyle\sum_{k=1}^{N}f\left((b-a)\tfrac{k}{N}+a\right)\left(\tfrac{b-a}{N}\right) \\ &= \tfrac{1}{b-a}\textstyle\int_a^b f(x)\,dx \\ &= \hat{y} \end{aligned}$$

5.2 Curve Fitting

1. (a)

$$\left\{\begin{aligned} 164A+20C &= 186 \\ 20B &= -34 \\ 20A+4C &= 26 \end{aligned}\right. ,\qquad y=\frac{7}{8}x^2-\frac{17}{10}x+\frac{17}{8}$$

(b)

$$\left\{\begin{aligned} 164A+20C &= 50 \\ 20B &= 6 \\ 20A+4C &= 50 \end{aligned}\right. ,\qquad y=-\frac{25}{8}x^2+\frac{3}{10}x+\frac{225}{8}$$

2. (a)

$$\left\{\begin{aligned} 34A+10C &= -24.8 \\ 10B &= 10.8 \\ 10A+5C &= 0.9 \end{aligned}\right. ,\qquad y=-1.9x^2+1.08x+3.98$$

(b)

$$\begin{cases} 34A + 10C &= 65.3 \\ 10B &= 21.3 \\ 10A + 5C &= 25.65 \end{cases}, \qquad y = x^2 + 2.13x + 3.13$$

(c)

$$\begin{cases} 34A + 10C &= 79 \\ 10B &= -1 \\ 10A + 5C &= 22 \end{cases}, \qquad y = 2.5x^2 - 0.1x - 0.6$$

3. (a) $\begin{cases} 55A + 15B = 25.9297 \\ 15A + 5B = 6.1515 \end{cases}$; $A = 0.7475$, $B = -1.0123$,
$C = e^B = 0.3634$; $y = 0.3634e^{0.7475x}$
   (b) $\begin{cases} 6.1993A + 4.7874B = 8.9366 \\ 4.7874A + 5B = 6.1515 \end{cases}$; $A = 1.8859$, $B = -0.5755$,
$C = e^B = 0.5625$; $y = 0.5625x^{1.8859}$
   (c) In part (a) $E_2(f) \approx 1.2723$ and in part (b) $E_2(f) \approx 0.4761$.

4. (a) $\begin{cases} 15A + 5B = -0.8647 \\ 5A + 5B = 4.2196 \end{cases}$; $A = -0.5084$, $B = 1.3524$,
$C = e^B = 3.8665$; $y = 3.8665e^{-0.5084x}$
   (b) $\begin{cases} 15A + 5B = 5.1619 \\ 5A + 5B = 2.73 \end{cases}$; $A = 0.2432$, $B = 0.3028$;
$y = \dfrac{1}{0.2432x + 0.3028}$
   (c) In part (a) $E_2(f) \approx 0.1192$ and in part (b) $E_2(f) \approx 2.1663$.

5. (a) (i) $y = 3.0053e^{-1.4991x}$ (ii) $y = 2.3995e^{-1.0586x}$
   (b) (i) $y = (0.7573x + 0.7845)^{-2}$ (ii) $y = (0.5777x + 0.8499)^{-2}$
   (c) In part (a i) $E_2(f) \approx 0.0041$ and in part (a ii) $E_2(f) \approx 3.4097$. In part (b i) $E_2(f) \approx 0.8195$ and in part (b ii) $E_2(f) \approx 0.0767$

6. (a) Using linearization: $y = \dfrac{1000}{1 + 4.3018e^{-1.0802t}}$
Minimizing least square: $y = \dfrac{1000}{1 + 4.2131e^{-1.0456t}}$
   (b) Using linearization: $y = \dfrac{5000}{1 + 8.9991e^{-0.81138t}}$
Minimizing least square: $y = \dfrac{5000}{1 + 8.9987e^{-0.81157t}}$

7. (a) Using linearization: $y = \dfrac{800}{1 + 11.682e^{-0.23893t}}$, $P(10) = 386.3$ (millioins)
Minimizing least squares: $y = \dfrac{800}{1 + 9.3560e^{-0.16401t}}$, $P(10) = 303.6$ (millions)

(b) Using linearization: $y = \dfrac{800}{1 + 9.3784e^{-0.164457t}}$, $P(10) = 284.6$ (millioins)

Minimizing least squares: $y = \dfrac{800}{1 + 9.3560e^{-0.164011t}}$, $P(10) = 284.2$ (millions)

8. $Y = \frac{A}{x} + B$. Use the change of variables $X = 1/x$ and $Y = x$ to get $Y = AX + B$.

9.

$$\begin{aligned} y &= \frac{D}{x + C} \\ xy + Cy &= D \\ y &= -\frac{1}{C}xy + \frac{D}{C} \end{aligned}$$

Use the change of variables and constants $X = xy$, $Y = y$, $A = -1/C$ and $B = D/C$ to get $Y = AX + B$.

10.

$$\begin{aligned} y &= \frac{1}{Ax + B} \\ \frac{1}{y} &= Ax + B \end{aligned}$$

Use the change of variables $X = x$ and $Y = 1/y$ to get $Y = AX + B$.

11. If $y = \frac{x}{A+Bx}$, then $\frac{1}{y} = \frac{A+Bx}{x} = A\left(\frac{1}{x}\right) + B$. Use the change of variables $X = 1/x$ and $Y = 1/y$ to get $Y = AX + B$.

12. Substituting $X = \ln(x)$ and $Y = y$ into $y = A\ln(x) + B$ yields $Y = AX + B$.

13.

$$\begin{aligned} y &= Cx^A \\ \ln(y) &= \ln(C) + \ln(x^A) \\ \ln(y) &= \ln(C) + A\ln(x) \end{aligned}$$

Use the change of variables and constants $X = \ln(x)$, $Y = \ln(y)$, and $B = \ln(C)$ to get $Y = AX + B$.

14. If $y = (Ax + B)^{-2}$, then $y^{-1/2} = Ax + B$. Use the change of variables $X = x$ and $Y = y^{-1/2}$ to get $Y = Ax + B$.

15.

$$\begin{aligned} y &= Cxe^{-Dx} \\ \ln\left(\frac{y}{x}\right) &= \ln(C) - Dx \\ Y &= B + Ax \\ Y &= Ax + B \end{aligned}$$

16. (a) To minimize $E(A,B) = \sum_{k=1}^{N}\left(A\cos(x_k) + B\sin(x_k) - y_k\right)^2$, first find the partials with respect to A and B:

$$\begin{aligned}
\frac{\partial E}{\partial A} &= \sum_{k=1}^{N} 2\left(A\cos(x_k) + B\sin(x_k) - y_k\right)\cos(x_k)\\
\frac{\partial E}{\partial B} &= \sum_{k=1}^{N} 2\left(A\cos(x_k) + B\sin(x_k) - y_k\right)\sin(x_k)
\end{aligned}$$

Setting the partial derivatives equal to zero yields the normal equations:

$$\begin{aligned}
\left(\sum_{k=1}^{N}\cos^2(x_k)\right)A + \left(\sum_{k=1}^{N}\cos(x_k)\sin(x_k)\right)B &= \sum_{k=1}^{N} y_k\cos(x_k)\\
\left(\sum_{k=1}^{N}\cos(x_k)\sin(x_k)\right)A + \left(\sum_{k=1}^{N}\sin^2(x_k)\right)B &= \sum_{k=1}^{N} y_k\sin(x_k)
\end{aligned}$$

(b) $f(x) = 0.4998\cos(x) + 2.1541\sin(x)$

17. First take the partial derivatives of $E(A,B,C)$ with respect to A, B and C:

$$\begin{aligned}
\frac{\partial E}{\partial A} &= \sum_{k=1}^{N} 2(Ax_k + By_k + C - z_k)x_k\\
\frac{\partial E}{\partial B} &= \sum_{k=1}^{N} 2(Ax_k + By_k + C - z_k)y_k\\
\frac{\partial E}{\partial C} &= \sum_{k=1}^{N} 2(Ax_k + By_k + C - z_k)
\end{aligned}$$

Setting each partial derivative equal to zero and using the properties of finite summations yields:

$$\begin{aligned}
\left(\sum_{k=1}^{N}x_k^2\right)A + \left(\sum_{k=1}^{N}x_ky_k\right)B + \left(\sum_{k=1}^{N}x_k\right)C - \sum_{k=1}^{N}x_kz_k &= 0\\
\left(\sum_{k=1}^{N}x_ky_k\right)A + \left(\sum_{k=1}^{N}y_k^2\right)B + \left(\sum_{k=1}^{N}y_k\right)C - \sum_{k=1}^{N}y_kz_k &= 0\\
\left(\sum_{k=1}^{N}x_k\right)A + \left(\sum_{k=1}^{N}y_k\right)B + \sum_{k=1}^{N}C - \sum_{k=1}^{N}z_k &= 0
\end{aligned}$$

or

$$\begin{aligned}
\left(\sum_{k=1}^{N}x_k^2\right)A + \left(\sum_{k=1}^{N}x_ky_k\right)B + \left(\sum_{k=1}^{N}x_k\right)C &= \sum_{k=1}^{N}x_kz_k\\
\left(\sum_{k=1}^{N}x_ky_k\right)A + \left(\sum_{k=1}^{N}y_k^2\right)B + \left(\sum_{k=1}^{N}y_k\right)C &= \sum_{k=1}^{N}y_kz_k\\
\left(\sum_{k=1}^{N}x_k\right)A + \left(\sum_{k=1}^{N}y_k\right)B + NC &= \sum_{k=1}^{N}z_k
\end{aligned}$$

18. (a)

$$\left\{\begin{aligned}
14A + 15B + 8C &= 82\\
15A + 19B + 9C &= 93\\
8A + 9B + 5C &= 49
\end{aligned}\right.$$

$A = 2.4,\ B = 1.2,\ C = 3.8:\ z = 2.4x + 1.2y + 3.8$

(b)

$$\begin{cases} 14A + 15B + 8C &= 62 \\ 15A + 19B + 9C &= 66 \\ 8A + 9B + 5C &= 38 \end{cases}$$

$A = 1.6$, $B = -1.2$, $C = 7.2 : z = 1.6x - 1.2y + 7.2$

(c)

$$\begin{cases} 19A + 12B + 9C &= -7 \\ 12A + 11B + 7C &= 3 \\ 9A + 7B + 5C &= 0 \end{cases}$$

$A = -2.2$, $B = -1.4$, $C = 2.0 : z = -2.2x - 1.4y + 2.0$

19. For the first function $E_2(f) \approx 5.689905$ and for the second function $E_2(f) \approx 25.151736$. The second function has a vertical asymptote between 4 and 5.

5.3 Interpolation by Spline Functions

1. (a) Making the substitutions $S(1) = 1$, $S'(1) = 0$, $S(2) = 2$, and $S'(2) = 0$ into $S(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $S'(x) = a_1 + 2a_2x + 3a_3x^2$ produces the system

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 1 \\ a_1 + 2a_2 + 3a_3 &= 0 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 2 \\ a_1 + 4a_2 + 12a_3 &= 0 \end{aligned}$$

   (b) $a_0 = 6\, a_1 = -12\, a_2 = 9\, a_3 = -2 : S(x) = 6 - 12x + 9x^2 - 2x^3$

2. (a) Making the substitutions $S(1) = 3$, $S'(1) = -4$, $S(2) = 1$, and $S'(2) = 2$ into $S(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $S'(x) = a_1 + 2a_2x + 3a_3x^2$ produces the system

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 3 \\ a_1 + 2a_2 + 3a_3 &= -4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 1 \\ a_1 + 4a_2 + 12a_3 &= 2 \end{aligned}$$

   (b) $a_0 = 5\, a_1 = 2\, a_2 = -6\, a_3 = 2 : S(x) = 5 + 2x - 6x^2 + 2x^3$

3. (a) Cubic spline
   (b) Part III not satisfied: $S_0(2) \neq S_1(2)$.
   (c) Cubic spline
   (d) Part IV not satisfied: $S_0'(2) \neq S_1'(2)$.

4. $h_0 = 1$, $h_1 = 3$, $h_2 = 3$; $d_0 = -2$, $d_1 = 1$, $d_2 = -2/3$; $u_1 = 18$, $u_2 = -10$
   Solve $\begin{cases} \frac{15}{2}m_1 + m_2 &= 21 \\ 3m_1 + \frac{21}{2}m_2 &= -15 \end{cases}$, to get $m_1 = 314/101$ and $m_2 = -234/101$.
   Then $m_0 = -460/101$ and $m_3 = 856/303$. The cubic spline is

$$\begin{cases} S_0(x) = \frac{129}{101}(x+3)^3 - \frac{230}{101}(x+3)^2 - (x+3) + 2 & x \in [-3,-2] \\ S_1(x) = -\frac{274}{909}(x+2)^3 + \frac{157}{101}(x+2)^2 - \frac{96}{101}(x+2) & x \in [-2,1] \\ S_2(x) = \frac{779}{2727}(x-1)^3 - \frac{117}{101}(x-1)^2 + \frac{72}{303}(x-1) + 3 & x \in [1,4] \end{cases}$$

5. $h_0 = 1$, $h_1 = 3$, $h_2 = 3$; $d_0 = -2$, $d_1 = 1$, $d_2 = -2/3$; $u_1 = 18$, $u_2 = -10$
   Solve $\begin{cases} 8m_1 + 3m_2 &= 18 \\ 3m_1 + \frac{21}{2}m_2 &= -10 \end{cases}$, to get $m_1 = 82/29$ and $m_2 = -134/87$.
   Set $m_0 = 0 = m_3$. The cubic spline is

$$\begin{cases} S_0(x) = \frac{41}{87}(x+3)^3 - \frac{215}{87}(x+3) + 2 & x \in [-3,-2] \\ S_1(x) = -\frac{190}{783}(x+2)^3 + \frac{41}{29}(x+2)^2 - \frac{92}{87}(x+2) & x \in [-2,1] \\ S_2(x) = \frac{67}{783}(x-1)^3 - \frac{67}{87}(x-1)^2 + \frac{76}{87}(x-1) + 3 & x \in [1,4] \end{cases}$$

6. $h_0 = 1$, $h_1 = 3$, $h_2 = 3$; $d_0 = -2$, $d_1 = 1$, $d_2 = -2/3$; $u_1 = 18$, $u_2 = -10$
   Solve $\begin{cases} \frac{28}{3}m_1 + \frac{8}{3}m_2 &= 18 \\ 0m_1 + 18m_2 &= -10 \end{cases}$, to get $m_1 = 263/126$ and $m_2 = -5/9$.
   Then $m_0 = 187/63$ and $m_3 = -403/126$. The cubic spline is

$$\begin{cases} S_0(x) = -\frac{37}{252}(x+3)^3 + \frac{187}{126}(x+3)^2 - \frac{841}{252}(x+3) + 2 & x \in [-3,-2] \\ S_1(x) = -\frac{37}{252}(x+2)^3 + \frac{263}{252}(x+2)^2 - \frac{17}{21}(x+2) & x \in [-2,1] \\ S_2(x) = -\frac{37}{252}(x-1)^3 - \frac{5}{18}(x-1)^2 + \frac{125}{84}(x-1) + 3 & x \in [1,4] \end{cases}$$

7. $h_0 = 1$, $h_1 = 3$, $h_2 = 3$; $d_0 = -2$, $d_1 = 1$, $d_2 = -2/3$; $u_1 = 18$, $u_2 = -10$
   Solve $\begin{cases} 9m_1 + 3m_2 &= 18 \\ m_1 + 15m_2 &= -10 \end{cases}$, to get $m_1 = 25/11$ and $m_2 = -9/11$.
   Then $m_0 = 25/11$ and $m_3 = -9/11$. The cubic spline is

$$\begin{cases} S_0(x) = \left(\frac{25}{22}(x+3) - \frac{69}{22}\right)(x+3) + 2 & x \in [-3,-2] \\ S_1(x) = \left(\left(-\frac{17}{99}(x+2) + \frac{25}{22}\right)(x+2) - \frac{19}{22}\right)(x+2) & x \in [-2,1] \\ S_2(x) = \left(-\frac{9}{22}(x-1) - \frac{2}{3}\right)(x-1) + 3 & x \in [1,4] \end{cases}$$

8. $h_0 = 1$, $h_1 = 3$, $h_2 = 3$; $d_0 = -2$, $d_1 = 1$, $d_2 = -2/3$; $u_1 = 18$, $u_2 = -10$
   Solve $\begin{cases} 8m_1 + 3m_2 &= 19 \\ 3m_1 + 12m_2 &= -16 \end{cases}$, to get $m_1 = 92/29$ and $m_2 = -185/87$.
   Then $m_0 = S''(-3) = -1$ and $m_3 = S''(4) = 2$. The cubic spline is

$$\begin{cases} S_0(x) = \left(\left(\frac{121}{174}(x+3) - \frac{1}{2}\right)(x+3) - \frac{191}{87}\right)(x+3) + 2 & x \in [-3,-2] \\ S_1(x) = \left(\left(-\frac{461}{1566}(x+2) + \frac{46}{29}\right)(x+2) - \frac{193}{174}\right)(x+2) & x \in [-2,1] \\ S_2(x) = \left(\left(\frac{359}{1566}(x-1) - \frac{185}{174}\right)(x-1) + \frac{40}{87}\right)(x-1) + 3 & x \in [1,4] \end{cases}$$

9. (a) $h_0 = 1/2$, $h_1 = 1/2$, $h_2 = 1/2$; $d_0 = -3$, $d_1 = -1/3$, $d_2 = 1/3$; $u_1 = 16$, $u_2 = 4$ Solve $\begin{cases} \frac{7}{4}m_1 + \frac{1}{2}m_2 &= 4 \\ \frac{1}{2}m_1 + \frac{7}{4}m_2 &= \frac{11}{4} \end{cases}$, to get $m_1 = 2$ and $m_2 = 1$. Then $m_0 = 23$ and $m_3 = 2$. The cubic spline is

$$\begin{cases} S_0(x) = \left(\left(-7\left(x-\frac{1}{2}\right) + \frac{23}{2}\right)\left(x-\frac{1}{2}\right) - \frac{13}{4}\right)\left(x-\frac{1}{2}\right) + \frac{9}{2} & x \in \left[\frac{1}{2},1\right] \\ S_1(x) = \left(\left(-\frac{1}{3}(x-1) + 1\right)(x-1) - \frac{3}{4}\right)(x-1) + 3 & x \in \left[1,\frac{3}{2}\right] \\ S_2(x) = \left(\left(\frac{1}{6}\left(x-\frac{3}{2}\right) + \frac{1}{2}\right)\left(x-\frac{3}{2}\right) + 0\right)\left(x-\frac{3}{2}\right) + \frac{17}{6} & x \in \left[\frac{3}{2},2\right] \end{cases}$$

(b) Solve $\begin{cases} 2m_1 + \frac{1}{2}m_2 &= 16 \\ \frac{1}{2}m_1 + 2m_2 &= 4 \end{cases}$, to get $m_1 = 8$ and $m_2 = 0$. Then $m_0 = S''(1/2) = 0$ and $m_3 = S''(1/2) = 0$. The cubic spline is

$$\begin{cases} S_0(x) = \left(\left(-7\left(x-\frac{1}{2}\right) + 0\right)\left(x-\frac{1}{2}\right) - 7\right)\left(x-\frac{1}{2}\right) + \frac{9}{2} & x \in \left[\frac{1}{2},1\right] \\ S_1(x) = \left(\left(-\frac{1}{3}(x-1) + 4\right)(x-1) - \frac{3}{4}\right)(x-1) + 3 & x \in \left[1,\frac{3}{2}\right] \\ S_2(x) = \left(\left(\frac{1}{3}\left(x-\frac{3}{2}\right) + 0\right)\left(x-\frac{3}{2}\right) + 0\right)\left(x-\frac{3}{2}\right) + \frac{17}{6} & x \in \left[\frac{3}{2},2\right] \end{cases}$$

10. (a) $h_0 = \sqrt{\pi/2}$, $h_1 = \sqrt{3\pi/2} - \sqrt{\pi/2}$, $h_2 = \sqrt{5\pi/2} - \sqrt{3\pi/2}$; $d_0 = -\sqrt{2/\pi}$, $d_1 = 0$, $d_2 = 0$; $u_1 = 6\sqrt{2/\pi}$, $u_2 = 0$ Solve

$$\begin{cases} 3.714959m_1 + 0.917490m_2 &= 7.181061 \\ 0.917490m_1 + 2.782252m_2 &= 16.849736 \end{cases}$$

to get $m_1 = 0.479571$ and $m_2 = 5.884950$. Then $m_0 = -2.149645$ and $m_3 = -29.561425$. The cubic spline is

$$\begin{cases} S_0(x) = ((0.349635x - 1.074822)\,x + 0)\,x + 1 \\ \qquad x \in \left[0, \sqrt{\frac{\pi}{2}}\right] \\ S_1(x) = \left(\left(0.981915\left(x - \sqrt{\frac{\pi}{2}}\right) + 0.239785\right)\left(x - \sqrt{\frac{\pi}{2}}\right) \right. \\ \qquad \left. -1.046564\right)\left(x - \sqrt{\frac{\pi}{2}}\right) \\ \qquad x \in \left[\sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{2}}\right] \\ S_2(x) = \left(\left(-9.352233\left(x - \sqrt{\frac{3\pi}{2}}\right) + 2.942474\right)\left(x - \sqrt{\frac{3\pi}{2}}\right) \right. \\ \qquad \left. +1.873127\right)\left(x - \sqrt{\frac{3\pi}{2}}\right) \\ \qquad x \in \left[\sqrt{\frac{3\pi}{2}}, \sqrt{\frac{5\pi}{2}}\right] \end{cases}$$

(b) Let $m_0 = S''(0) = 0$ and $m_3 = S''(\sqrt{5\pi/2}) = 0$.

$$\begin{cases} S_0(x) = ((0.239785x - 0.898060)\,x + 1 \\ \qquad x \in \left[0, \sqrt{\frac{\pi}{2}}\right] \\ S_1(x) = ((2.702690\left(x - \sqrt{\frac{\pi}{2}}\right) + 0.239785) \\ \qquad \left(x - \sqrt{\frac{\pi}{2}}\right) - 1.046564)\left(x - \sqrt{\frac{\pi}{2}}\right) \\ \qquad x \in \left[\sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{2}}\right] \\ S_2(x) = \left(\left(-2.942475\left(x - \sqrt{\frac{3\pi}{2}}\right) + 2.942474\right)\left(x - \sqrt{\frac{3\pi}{2}}\right)\right. \\ \qquad -1.239158)\left(x - \sqrt{\frac{3\pi}{2}}\right) \\ \qquad x \in \left[\sqrt{\frac{3\pi}{2}}, \sqrt{\frac{5\pi}{2}}\right] \end{cases}$$

11.

$$\begin{aligned} S_k(x) &= \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(x_{k+1} - x) \\ &\quad + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)(x - x_k) \\ &= \frac{m_k}{6h_k}\left(h_k^3 - 3h_k^2(x - x_k) + 3h_k(x - x_k)^2 - (x - x_k)^3\right) \\ &\quad + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(h - (x - x_k)) \\ &\quad + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)(x - x_k) \end{aligned}$$

The spline has the form:

$$S_k(x) = s_{k0} + s_{k1}(x - x_k) + s_{k2}(x - x_k)^2 + s_{k3}(x - x_k)^3$$

The coefficients s_{kj} are now determined:

First, $\frac{m_k}{6h_k}h_k^3 + y_k - \frac{m_k h_k^2}{6} = y_k = s_{k0}$

Second,

$$\begin{aligned} & -\frac{m_k}{6h_k}(3h_k^2) - \frac{y_k}{h_k} + \frac{m_k h_k}{6} + \frac{y_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6} \\ =\ & \frac{y_{k+1} - y_k}{h_k} - \frac{h_k(2m_k + m_{k+1})}{6} \\ =\ & d_k - \frac{h_k(2m_k + m_{k+1})}{6} \\ =\ & s_{k1} \end{aligned}$$

Third, $\frac{m_k}{6h_k}(3h_k) = \frac{m_k}{2} = s_{k2}$

Last, $-\frac{m_k}{6h_k} + \frac{m_{k+1}}{6h_k} = \frac{(m_{k+1} - m_k)}{6h_k} = s_{k3}$

12. (a)

$$\begin{aligned}
&\int_{x_k}^{x_{k+1}} S_k(x)\,dx \\
&= \int_{x_k}^{x_{k+1}} s_{k,3}(x-x_k)^3 + s_{k,2}(x-x_k)^2 + s_{k,1}(x-x_k) + y_k\,dx \\
&= \tfrac{s_{k,3}}{4}(x-x_k)^4 + \tfrac{s_{k,2}}{3}(x-x_k)^3 + \tfrac{s_{k,1}}{2}(x-x_k)^2 + y_k(x-x_k)\Big|_{x_k}^{x_{k+1}} \\
&= .\tfrac{s_{k,3}}{4}(x_{k+1}-x_k)^4 + \tfrac{s_{k,2}}{3}(x_{k+1}-x_k)^3 + \tfrac{s_{k,1}}{2}(x_{k+1}-x_k)^2 \\
&\qquad + y_k(x_{k+1}-x_k) \\
&= \left(\left(\left(\tfrac{s_{k,3}}{4}h_k + \tfrac{s_{k,2}}{3}\right)h_k + \tfrac{s_{k,1}}{2}\right)h_k + y_k\right)h_k
\end{aligned}$$

(b) $\int_{1/2}^{2} S(x)\,dx \approx 5.054688$

(c) $\int_{0}^{\sqrt{5\pi/2}} S(x)\,dx \approx 0.80749$

13. When $k = 1$ (12) becomes

$$h_0m_0 + 2(h_0+h_1)m_1 + h_1m_2 = u_1$$

Substituting the end-point constraint $m_0 = \frac{3}{h_0}(d_0 - S'(x_0)) - \frac{m_1}{2}$ into this equation yields

$$\begin{aligned}
&h_0\left(\tfrac{3}{h_0}(d_0 - S'(x_0)) - \tfrac{m_1}{2}\right) + 2(h_0+h_1)m_1 + h_1m_2 = u_1 \\
&\left(-\tfrac{1}{2}h_0 + 2(h_0+h_1)\right)m_1 + h_1m_2 = u_1 - 3(d_0 - S'(x_0)) \\
&\left(\tfrac{3}{2}h_0 + 2h_1\right)m_1 + h_1m_2 = u_1 - 3(d_0 - S'(x_0))
\end{aligned}$$

Similarly, letting $k = N-1$ in (12) and making the end-point constraint substitution $m_N = \frac{3}{h_{N-1}}(S'(x_N) - d_{N-1}) - \frac{m_{N-1}}{2}$ yields

$$\begin{aligned}
&h_{N-2}m_{N-2} + 2(h_{N-2}+h_{N-1})m_{N-1} \\
&\quad + h_{N-1}\left(\tfrac{3}{h_{N-1}}(S'(x_N) - d_{N-1}) - \tfrac{m_{N-1}}{2}\right) \\
&= u_{N-1},
\end{aligned}$$

or

$$h_{N-2}m_{N-2} + (2h_{N-2} + \frac{3}{2}h_{N-1})m_{N-1} = u_{N-1} - 3(S'(x_N) - d_{N-1})$$

Thus solving the linear system in Lemma 5.1 yields $m_1, m_2, \ldots, m_{N-1}$. Then m_1 and m_{N-1} are used in the corresponding formulas in Table 5.8 (i) to solve for m_0 and m_N, respectively.

14. When $k = 1$ (12) becomes

$$h_0m_0 + 2(h_0+h_1)m_1 + h_1m_2 = u_1$$

Substituting the end-point constraint $m_0 = m_1 - \frac{h_0(m_2-m_1)}{h_1}$ into this equation yields

$$h_0\left(m_1 - \frac{h_0(m_2 - m_1)}{h_1}\right) + 2(h_0 + h_1)m_1 + h_1m_2 = u_1$$

or

$$\left(3h_0 + 2h_1 + \frac{h_0^2}{h_1}\right) m_1 + \left(h_1 - \frac{h_0^2}{h_1}\right) m_2 = u_1$$

Similarly, letting $k = N - 1$ in (12) and making the end-point constraint substitution $m_N = m_{N-1} + \frac{h_{N-1}(m_{N-1}-m_{N-2})}{h_{N-2}}$ yields

$$\begin{aligned}&h_{N-2}m_{N-2} + 2(h_{N-2} + h_{N-1})m_{N-1}\\ &\quad+h_{N-1}\left(m_{N-1} + \tfrac{h_{N-1}(m_{N-1}-m_{N-2})}{h_{N-2}}\right)\\ &= u_1\end{aligned}$$

or

$$\left(h_{N-2} - \frac{h_{N-1}^2}{h_{N-2}}\right) m_{N-2} + \left(2h_{N-2} + 3h_{N-1} + \frac{h_{N-1}^2}{h_{N-2}}\right) m_{N-1} = u_{N-1}$$

Thus solving the linear system in Lemma 5.3 yields $m_1, m_2, \ldots, m_{N-1}$. Then m_1 and m_2, and m_{N-2} and m_{N-1} are used in the corresponding formulas in Table 5.8 (iii) to solve for m_0 and m_N, respectively.

15. (a) Given $f(x) = x^3 - x$ we note that $f(-2) = -6$, $f(0) = 0$, $f'(-2) = 11$, and $f'(0) = -1$. Thus we seek the clamped cubic spline $S(x) = ax^3 + bx^2 + cx + d$ with derivative $S'(x) = 3ax^2 + 2bx + c$ that satisfies the constraints $S(-2) = -6$, $S(0) = 0$, $S'(-2) = 11$, and $S'(0) = -1$. Substituting the constraints into $S(x)$ and $S'(x)$ yields the linear system

$$\begin{aligned}-8a + 4b - 2c + d &= -6\\ d &= 0\\ 12a - 4b + c &= 11\\ c &= = -1\end{aligned}$$

with solution $a = 1$, $b = 0$, $c = -1$, $d = 0$. Therefore, $S(x) = f(x)$ on $[-2, 0]$.

(b) Given $f(x) = x^3 - x$ on the interval $[-2, 2]$ consider the clamped cubic spline $S(x)$:

$$\begin{cases} S_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0 & [-2, 0]\\ S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_0 & [0, 2]\end{cases}$$

It is sufficient to show $S_0(x) = f(x)$ and $S_1(x) = f(x)$ on $[-2,0]$ and $[0,2]$, respectively. For $S_0(x)$ the constraints $S_0(-2) - 6$, $S_0(0) = 0$, $S_0'(-2) = 11$, and $S_0''(0) = 0$ yields the linear system:

$$\begin{aligned} -8a_0 + 4b_0 - 2c_0 + d_0 &= 0 \\ d_0 &= 0 \\ 12a_0 - 4b_0 + c_0 &= 11 \\ 2b_0 &= 0 \end{aligned}$$

with solution $a_0 = 1$, $b_0 = 0$, $c_0 = -1$, and $d_0 = 0$. Thus $S_0(x) = f(x)$ on the interval $[-2,0]$. A similar argument shows that $S_1(x) = f(x)$ on the interval $[0,2]$. Therefore, $f(x) = x^3 - x$ is its own clamped cubic spline for the given nodes over the interval $[-2,2]$.

(c) First genralize the results in parts (a) and (b) to any third-degree polynomial on an interval $[a,b]$. As an inductive hypothesis assume that every third-degree polynomial is its own clamped cubic spline on an interval $[a,b]$ with k nodes. Then using the generalized results from parts (a) and (b), and the inductive hypothesis show that every third-degree polynomial is its own clamped cubic spline on an interval $[a,b]$ with $k+1$ nodes. Therefore, the statement is true for any interval $[a,b]$ with N nodes.

(d) In general the statement in part (c) is false for the other four type of cubic splines.

5.4 Fourier Series and Trigonometric Polynomials

1. $f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots\right)$

2. $f(x) = \frac{4}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \frac{1}{7^2}\cos(7x) + \cdots\right)$

3. $f(x) = \frac{\pi}{4} + \sum_{j=1}^{\infty}\left(\frac{(-1)^j - 1}{\pi j^2}\right)\cos(jx) - \sum_{j=1}^{\infty}\left(\frac{(-1)^j}{j}\right)\sin(jx)$

4. $f(x) = \frac{4}{\pi}\left(\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \cdots\right)$

5. $f(x) = \frac{4}{\pi}\left(\sin(x) - \frac{1}{3^2}\sin(3x) + \frac{1}{5^2}\sin(5x) - \frac{1}{7^2}\sin(7x) + \cdots\right)$

6. From Exercise 1:

$$f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots\right).$$

Letting $x = \pi/2$ yields $f(\pi/2) = 1$. Thus

$$\begin{aligned} 1 &= \tfrac{4}{\pi}\left(\sin\left(\tfrac{\pi}{2}\right) + \tfrac{1}{3}\sin\left(\tfrac{3\pi}{2}\right) + \tfrac{1}{5}\sin\left(\tfrac{5\pi}{2}\right) + \tfrac{1}{7}\sin\left(\tfrac{7\pi}{2}\right) + \cdots\right) \\ \tfrac{\pi}{4} &= 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \end{aligned}$$

7. From Exercise 2:

$$f(x) = \frac{4}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \frac{1}{7^2}\cos(7x) + \cdots\right).$$

Letting $x = 0$ yields $f(0) = \pi/2$. Thus

$$\begin{aligned} \tfrac{\pi}{2} &= \tfrac{4}{\pi}\left(\cos(0) + \frac{1}{3^2}\cos(3(0)) + \frac{1}{5^2}\cos(5(0)) + \frac{1}{7^2}\cos(7(0)) + \cdots\right) \\ \tfrac{\pi^2}{8} &= 1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \cdots \end{aligned}$$

8. $f(x) = \frac{\pi^2}{12} - \cos(x) + \frac{1}{2^2}\cos(2x) - \frac{1}{3^2}\cos(3x) + \frac{1}{4^2}\cos(4x) + \cdots$

9. If $g(x)$ has period 2π, then $a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(jx)dx$ for $j = 0, 1, 2, \ldots$. Let $x = \frac{\pi}{P}t$. Thus

$$\begin{aligned} a_j &= \tfrac{1}{\pi}\int_{-P}^{P} g\left(\tfrac{\pi}{P}t\right)\cos\left(\tfrac{j\pi t}{P}\right)\left(\tfrac{\pi}{P}\right)dt \\ &= \tfrac{1}{P}\int_{-P}^{P} g\left(\tfrac{\pi}{P}t\right)\cos\left(\tfrac{j\pi t}{P}\right)dt \end{aligned}$$

Let $f(t) = g\left(\frac{\pi}{P}t\right)$ a function of period $2P$.

$$a_j = \tfrac{1}{P}\int_{-P}^{P} f(t)\cos\left(\tfrac{j\pi t}{P}\right)dt$$

Let $t = x$ (dummy variable)

$$a_j = \tfrac{1}{P}\int_{-P}^{P} f(x)\cos\left(\tfrac{j\pi x}{P}\right)dx$$

Similary $b_j = \frac{1}{P}\int_{-P}^{P} f(x)\sin(j\pi x/P)dx$ for $j = 1, 2, \ldots$.

10. $f(x) = \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{2}{(2j-1)\pi}\right)\sin\left(\frac{(2j-1)\pi x}{2}\right)$

11.

$$\begin{aligned} a_0 &= \tfrac{1}{3} \\ a_j &= \tfrac{2}{3}\left(\tfrac{3}{j\pi}\sin\left(\tfrac{j\pi}{3}\right) + \left(\tfrac{3}{j\pi}\right)^2\left(\cos\left(\tfrac{j\pi}{3}\right) - 1\right)\right) \\ b_j &= \tfrac{2}{j\pi}\left(\cos\left(\tfrac{j\pi}{3}\right) - \cos(j\pi)\right) \end{aligned}$$

12. $f(x) = 6 + \frac{36}{\pi^2}\sum_{j=1}^{\infty}\left(\frac{(-1)^{j+1}}{j^2}\right)\cos\left(\frac{j\pi x}{3}\right)$

13. The function f satisfies the hypotheses of Theorem 5.5. Thus f has a Fourier series expansion:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_j \cos(jx) + b_j \sin(jx))$$

In addition, $f(-x) = f(x)$ for all $x \in D_f$. Thus $f(x)\sin(jx)$ is an odd function and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)dx = 0$$

since the integral of an odd function over an interval symmetric about the origin is zero. Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_j \cos(jx)$$

14. The proof of Theorem 5.7 is similar to the proof of Theorem 5.6.

5.5 Bézier Curves

1. $B_{2,4}(t) = 6t^2 - 12t^3 + 6t^4$
   $B_{3,5}(t) = 10t^3 - 20t^4 + 10t^5$
   $B_{5,7}(t) = 21t^5 - 42t^6 + 21t^7$

2. $(1-t)B_{i,N-1}(t) + tB_{i-1,N-1}(t)$
   $= (1-t)\begin{pmatrix} N-1 \\ i \end{pmatrix} t^i(1-t)^{N-1-i} + t\begin{pmatrix} N-1 \\ i-1 \end{pmatrix} t^{i-1}(1-t)^{N-1-(i-1)}$
   $= \begin{pmatrix} N-1 \\ i \end{pmatrix} t^i(1-t)^{N-i} + \begin{pmatrix} N-1 \\ i-1 \end{pmatrix} t^i(1-t)^{N-i}$
   $= \left(\begin{pmatrix} N-1 \\ i \end{pmatrix} + \begin{pmatrix} N-1 \\ i-1 \end{pmatrix}\right) t^i(1-t)^{N-i}$
   $= \begin{pmatrix} N \\ i \end{pmatrix} t^i(1-t)^{N-i} = B_{i,N}(t)$

3. Note: The binomial coefficients are non-negative, and t^i and $(1-t)^{N-i}$ are non-negative for $t \in [0,1]$. Can also be established using formula (4) and mathematical induction.

4. $\sum_{i=0}^{3} B_{i,3}(t) = B_{0,3}(t) + B_{1,3}(t) + B_{2,3}(t) + B_{3,3}(t)$
   $= (1-t)^3 + 3t(1-t)^2 + 3t^2(1-t) + t^3$
   $= 1 + (-3+3)t + (3-6-3)t^2 + (-1+3-3+1)t^3 = 1$

5. $\frac{d}{dt}B_{3,5}(t) = 5(B_{2,4}(t) - B_{3,4}(t)) = 5(6t^2(1-t)^2 - 4t^3(1-t))$
$\frac{d}{dt}B_{3,5}(1/3) = 80/81$ and $\frac{d}{dt}B_{3,5}(2/3) = -40/81$

6.

$$\begin{aligned}
\frac{d}{dt}B_{i,N}(t) &= \binom{N}{i}(t^i(1-t)^{N-i})' \\
&= \binom{N}{i}\left(it^{i-1}(1-t)^{N-i} + (N-i)t^i(1-t)^{N-i-1}(-1)\right) \\
&= \binom{N}{i}t^{i-1}(1-t)^{N-i-1}\left(i(1-t) - (N-i)t\right) \\
&= \binom{N}{i}t^{i-1}(1-t)^{N-i-1}(i - Nt)
\end{aligned}$$

On the open interval $(0,1)$, $B'_{i,N}(t) = 0$ when $t = i/N$. Since $B_{i,N}(0) = 0 = B_{i,N}(1)$ and $B_{i,N}(i/N) > 0$, the maximum value of the Bernstein polynomial $B_{i,N}(t)$, on the interval $[0,1]$, occurs at $t = i/N$.

7.

$$\begin{aligned}
tB_{i,N}(t) &= \binom{N}{i}t^{i+1}(1-t)^{N-i} \\
&= \binom{N}{i}t^{i+1}(1-t)^{(N+1)-(i+1)} \\
&= \frac{\binom{N}{i}}{\binom{N+1}{i+1}}B_{i+1,N+1}(t) \\
&= \frac{i+1}{N+1}B_{i+1,N+1}(t)
\end{aligned}$$

8. (a) $P(t) = (1 + 6t - 9t^2 + 5t^3,\ 3 - 12t + 27t^2 - 18t^3)$
(b) $P(t) = (-2 + 4t + 18t^2 - 28t^3 + 10t^4,\ 3 + 12t^2 - 20t^3 + 8t^4)$
(c) $P(t) = (1 + 5t,\ 1 + 5t + 10t^2 - 30t^3 + 15t^4)$

9. $P(t) = (1 + 3t,\ 1 + 6t)$.

10. By Definition 5.5

$$B_{i-1,N-1}(t) = \begin{cases} 0 & i \neq N \\ 1 & i = N \end{cases} \quad \text{and} \quad B_{i,N-1}(t) = \begin{cases} 0 & i \neq N-1 \\ 1 & i = N-1 \end{cases}.$$

Thus

$$\begin{aligned}
P'(1) &= \sum_{i=0}^{N}\mathbf{P}_i N(B_{i-1,N-1}(1) - B_{i,N-1}(1)) \\
&= N(\mathbf{P}_N - \mathbf{P}_{N-1}).
\end{aligned}$$

11. Note: by Definition 5.5

$$B_{i-2,N-2}(0) = \begin{cases} 0 & i \neq 2 \\ 1 & i = 2 \end{cases}, \quad B_{i-1,N-2}(0) = \begin{cases} 0 & i \neq 1 \\ 1 & i = 1 \end{cases}$$

and $B_{i-,N-2}(0) = \begin{cases} 0 & i \neq 0 \\ 1 & i = 0 \end{cases}$.

Taking the derivative of $P'(t)$, evaluated at $t = 1$, shown in Exercise 10 above, yields

$$\begin{aligned}
P''(1) &= \sum_{i=0}^{N} \mathbf{P}_i N \left(\frac{d}{dt} B_{i-1,N-1}(1) - \frac{d}{dt} B_{i,N-1}(1) \right) \\
&= \sum_{i=0}^{N} \mathbf{P}_i N((N-1)(B_{i-2,N-2}(1) - B_{i-1,N-2}(1)) \\
&\qquad -(N-1)(B_{i-1,N-2}(1) - B_{i,N-1}(1))) \\
&= N(N-1) \sum_{i=0}^{N} \mathbf{P}_i((B_{i-2,N-2}(1) - B_{i-1,N-2}(1)) \\
&\qquad -(B_{i-1,N-2}(1) - B_{i,N-1}(1))) \\
&= N(N-1)((\mathbf{P}_1 - \mathbf{P}_1) - (\mathbf{P}_1 - \mathbf{P}_0)) \\
&= N(N-1)(\mathbf{P}_2 - 2\mathbf{P}_1 + \mathbf{P}_0)
\end{aligned}$$

12. (a) The line segment (including endpoints) with endpoints $(1, 1)$ and $(7, -3)$.
    (b) The parallelogram and its interior with vertices $(-4, 2)$, $(-3, 5)$, $(2, 5)$, and $(1, 2)$.
    (c) The triangle and its interior with vertices $(0, 0)$, $(0, 1)$, and $(1/2, 0)$.

Chapter 6

Numerical Differentiation

6.1 Approximating the Derivative

1. $f(x) = \sin(x)$: Find approximations to $f'(0.8)$.

h	$f'(x)$	Error	Truncation Error
0.1	0.695546112	0.001160597	0.001274737
0.01	0.696695100	0.000011609	0.000012747
0.001	0.696706600	0.000000109	0.000000127

2. $f(x) = e^x$: Find approximations to $f'(2.3)$.

h	$f'(x)$	Error	Truncation Error
0.1	9.990814405	−0.016631050	0.018371961
0.01	9.974348950	−0.000166495	0.000183720
0.001	9.974184000	−0.000001545	0.000001837

3. $f(x) = \sin(x)$: Find approximations to $f'(0.8)$.

h	$f'(x)$	Error	Truncation Error
0.1	0.696704390	0.000002320	0.000002322
0.01	0.696706710	−0.000000001	0.000000000

4. $f(x) = e^x$: Find approximations to $f'(2.3)$.

h	$f'(x)$	Error	Truncation Error
0.1	9.974149167	0.000033288	0.000040608
0.01	9.974182750	−0.000000295	0.000000004

5. (a) $f'(2) \approx 12.0025000$

   (b) $f'(2) \approx 12.0000000$

   (c) For part (a): $\mathbf{O}(h^2) = -(0.05)^2 f^{(3)}(c)/6 = -0.0025000$.
   For part (b): $\mathbf{O}(h^4) = -(0.05)^4 f^{(3)}(c)/30 = -0.0000000$.

6. (a) Taylor's Theorem applied to $f(x+h)$ expanded about $x_0 = x$ yields

$$f(x+h) = \sum_{k=0}^{n} \frac{f^k(h)(x+h-x)^k}{k!} + \frac{f^{(n+1)}(c)(x+h-x)^{n+1}}{(n+1)!}.$$

When $n = 1$ this yields

$$\begin{aligned} f(x+h) &= \sum_{k=0}^{1} \frac{f^k(x)h^k}{k!} + \frac{f''(c)h^2}{2!} \\ &= f(x) + f'(x)h + \tfrac{f''(c)h^2}{2} \end{aligned},$$

where $|x-c| < h$.

(b)

$$\begin{aligned} f(x+h_k) - f(x) &= h_k f'(x) + \frac{h_k^2 f''(c)}{2} \\ \frac{f(x+h_k) - f(x)}{h_k} &= f'(x) + \frac{h_k f''(c)}{2} \\ f'(x) &= \frac{f(x+h_k) - f(x)}{h_k} - \frac{h_k f''(c)}{2} \\ f'(x) &= D_k - \frac{h_k f''(c)}{2} = D_k + \mathbf{O}(h_k) \end{aligned}$$

(c) In (3) larger values of h can be used, thus, in most cases, avoiding the subtractive cancellation difficulties that can occur for the smaller values of h used in (2).

7. (a) $f_x(x,y) = (y/(x+y))^2$, $f_x(2,3) = 0.36$

h	$\approx f_x(2,3)$	Error
0.1	0.360144060	−0.000144060
0.01	0.360001400	−0.000001400
0.001	0.360000000	0.000000000

$f_y(x,y) = (x/(x+y))^2$, $f_y(2,3) = 0.16$

h	$\approx f_y(2,3)$	Error
0.1	0.160064030	−0.000000600
0.01	0.160000600	−0.000000600
0.001	0.160000000	0.000000000

(b) $f_x(x,y) = \arctan(y/x)$, $f_x(3,4) = -0.16$

h	$\approx f_x(3,4)$	Error
0.1	−0.160009380	0.000009380
0.01	−0.160000100	0.000000100
0.001	−0.160000000	0.000000000

$f_y(x,y) = \dfrac{x}{x^2+y^2}$, $f_y(3,4) = 0.12$

h	$\approx f_y(3,4)$	Error
0.1	0.120024960	−0.000024960
0.01	0.120000255	−0.000000255
0.001	0.120000050	−0.000000050

8. Under the assumption that $f^{(5)}(c_1) \approx f^{(5)}(c_2)$ equations (31) and (32) can be expressed as

$$\begin{aligned} f'(x_0) &\approx 16D_1(h) + Ch^4 \\ f'(x_0) &\approx D_1(2h) + 16Ch^4 \end{aligned}$$

respectively. Multiplying both sides of the first expression by 16 and taking the difference in the two resulting expressions yields

$$\begin{aligned} 15f'(x_0) &\approx 16D_1(h) - D_1(2h) \\ f'(x_0) &\approx \frac{16D_1(h) - D_1(2h)}{15} \end{aligned}$$

9. (a) Let $g(h) = \frac{\epsilon}{h} + \frac{Mh^2}{6}$. Then

$$g'(h) = -\frac{\epsilon}{h^2} + \frac{Mh}{3} = \frac{-3\epsilon + Mh^3}{3h^2}$$

Solving $g'(h) = 0$ yields

$$\begin{aligned} \frac{-3\epsilon + Mh^3}{3h^2} &= 0 \\ -3\epsilon + Mh^3 &= 0 \\ h &= \left(\frac{3\epsilon}{M}\right)^{1/3} \end{aligned}$$

Substituting $h = \left(\frac{3\epsilon}{M}\right)^{1/3}$ into $g''(h) = \frac{2\epsilon}{h^3} + \frac{M}{3}$ yields $g''\left(\left(\frac{3\epsilon}{M}\right)^{1/3}\right) > 0$. Therefore, by the second derivative test, $g(h)$ has a minimum at $h = \left(\frac{3\epsilon}{M}\right)^{1/3}$

(b) Let $g(h) = \frac{3\epsilon}{2h} + \frac{Mh^4}{30}$. Then

$$g'(h) = -\frac{3\epsilon}{2h^2} + \frac{2Mh^3}{15} = \frac{-45\epsilon + 4Mh^5}{30h^2}$$

Solving $g'(h) = 0$ yields

$$\begin{aligned} \frac{-45\epsilon + 4Mh^5}{30h^2} &= 0 \\ -45\epsilon + 4Mh^5 &= 0 \\ h &= \left(\frac{45\epsilon}{4M}\right)^{1/5} \end{aligned}$$

Substituting $h = \left(\frac{45\epsilon}{4M}\right)^{1/5}$ into $g''(h) = \frac{3\epsilon}{h^3} + \frac{2Mh^2}{5}$ yields $g''\left(\left(\frac{45\epsilon}{4M}\right)^{1/5}\right) > 0$. Therefore, by the second derivative test, $g(h)$ has a minimum at $h = \left(\frac{45\epsilon}{4M}\right)^{1/5}$

10. (a) Formula (3) gives $I'(1.2) \approx -13.5840$ and $E(1.2) \approx 11.3024$. Formula (10) given $I'(1.2) \approx -13.6824$ and $E(1.2) \approx 11.2975$.

    (b) Using differentiation rules from calculus, obtain $I'(1.2) \approx -13.6793$ and $E(1.2) \approx 11.2976$.

11. Numerical differentiation formulas (3) and (10) yield $D'(10.0) \approx 4.4205$ and $D'(10.0) \approx 4.4248$, respectively. Differentiation rules from calculus yield $D'(10.0) \approx 4.4248$.

12.

h	$\approx f'(1.2)$	Error	$\lvert$round-off$\rvert$+$\lvert$truncation$\rvert$
0.1	-0.93050	-0.00154	$0.00005 + 0.00161 = 0.00166$
0.01	-0.93200	-0.00004	$0.00050 + 0.00002 = 0.00052$
0.001	-0.93000	-0.00204	$0.00500 + 0.00000 = 0.00500$

13.

h	$\approx f'(3.0)$	Error	$\lvert$round-off$\rvert$+$\lvert$truncation$\rvert$
0.1	0.33345	-0.00012	$0.00005 + 0.00014 = 0.00019$
0.01	0.33350	-0.00017	$0.00050 + 0.00000 = 0.00050$
0.001	0.33500	-0.00167	$0.00500 + 0.00000 = 0.00500$

14. (a) Maximize $E(h) = \frac{0.0005}{h} + 2.5h^2$. Solve $E'(h) = 0$ and obtain $h = 0.1$.

    (b) Maximize $E(h) = \frac{0.00075}{h} + 0.05h^4$. Solve $E'(h) = 0$ and obtain $h = 0.327195$.

15.

h	$\approx f'(1.2)$	Error	$\lvert$round-off$\rvert$+$\lvert$truncation$\rvert$
0.1	-0.93206	0.00002	$0.00008 + 0.00000 = 0.00008$
0.01	-0.93208	0.00004	$0.00075 + 0.00000 = 0.00075$
0.001	-0.92917	-0.00287	$0.00750 + 0.00000 = 0.00750$

16.

h	$\approx f'(4.0)$	Error	|round-off|+|truncation|
0.1	0.33333	0.00000	$0.00008 + 0.00000 = 0.00008$
0.01	0.33350	-0.00017	$0.00075 + 0.00000 = 0.00075$
0.001	0.33583	-0.00250	$0.00750 + 0.00000 = 0.00750$

6.2 Numerical Differentiation Formulas

1.

(a) $f''(5) \approx -0.040001600$ (b) $f''(5) \approx -0.040007900$
(c) $f''(5) \approx -0.039999833$ (d) $f''(5) \approx -0.040000000 = -1/5^2$

2.

(a) $f''(1) \approx -0.540190000$ (b) $f''(1) \approx -0.539852200$
(c) $f''(1) \approx -0.540301667$ (d) $f''(1) \approx -0.540302306 \approx -\cos(1)$

3.

(a) $f''(5) \approx 0.0000$ (b) $f''(5) \approx -0.0400$
(c) $f''(5) \approx -0.0133$ (d) $f''(5) \approx -0.0400 = -1/5^2$

4.

(a) $f''(1) \approx -0.5200$ (b) $f''(1) \approx -0.5400$
(c) $f''(1) \approx -0.5133$ (d) $f''(1) \approx -0.5403 \approx -\cos(1)$

5. (a) $f(x) = x^2$, $f''(1) \approx 2.0000$
   (b) $f(x) = x^4$, $f''(1) \approx 12.0002$

6. (a) $f(x) = x^4$, $f''(1) \approx 12.0000$
   (b) $f(x) = x^6$, $f''(1) \approx 29.9992$

7. $f(x+h) = f(x) + hf'(x) + \dfrac{h^2 f''(x)}{2!} + \dfrac{h^3 f^{(3)}(x)}{3!} + \cdots$

$f(x-h) = f(x) - hf'(x) + \dfrac{h^2 f''(x)}{2!} - \dfrac{h^3 f^{(3)}(x)}{3!} + \cdots$

$f(x+h) - f(x-h) = 2hf'(x) + \dfrac{2h^3 f^{(3)}(x)}{3!} + \dfrac{2h^5 f^{(5)}(x)}{5!} + \cdots$

Similarly, (or substitute $2h$ for h in the previous equation)

$$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16h^3 f^{(3)}(x)}{3!} + \frac{64h^5 f^{(5)}(x)}{5!} + \cdots$$

Then

$$f(x+2h) - 2f(x+h) + f(x-h) - f(x-2h) = \frac{12h^3 f^{(3)}(x)}{3!} + \frac{60h^5 f^{(5)}(x)}{5!} + \cdots$$

Now solve for $f^{(3)}(x)$ and get

$$f^{(3)} = \frac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3} - \frac{h^2 f^{(5)}(x)}{4} + \cdots$$

8.

$$f(x+h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)h^k}{k!}, \; f(x-h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)(-h)^k}{k!}$$

Substituting $2h$ for h in the two previous expressions yields:

$$f(x+2h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)(2h)^k}{k!}, \; f(x-2h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)(-2h)^k}{k!}$$

Then

$$f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h) = h^4 f^{(4)}(x) + \mathbf{O}(h^5)$$

Thus

$$f^{(4)}(x) \approx \frac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4}$$

9.

(a)

x	$f'(x)$
0.0	0.141345
0.1	0.041515
0.2	−0.058275
0.3	−0.158025

(b)

x	$f'(x)$
0.0	−0.993310
0.1	−0.997470
0.2	−0.996635
0.3	−0.990805

10. We are given, as approximations, that the points $(x + h/2,\, f'(x + h/2))$ and $(x - h/2,\, f'(x - h/2))$ lie on the graph of $f'(x)$. The slope of the tangent line at x can be approximated with the slope of the secant line passing through the two given points. Thus

$$\begin{aligned} f''(x) &\approx m_{sec} \\ &= \frac{f'(x+h/2) - f'(x-h/2)}{(x+h/2) - (x-h/2)} \\ &= \frac{f'(x+h/2) - f'(x-h/2)}{h} \\ &\approx \frac{\frac{f_1 - f_0}{h} - \frac{f_0 - f_{-1}}{h}}{h} \\ &= \frac{f_1 - 2f_0 + f_{-1}}{h^2} \end{aligned}$$

11. Let $t_0 = x$, $t_1 = x + h$, and $t_2 = x + 3h$. Then

$$a_1 = \frac{f_1 - f_0}{x + h - x} = \frac{f_1 - f_0}{h}$$

and

$$\begin{aligned} a_2 &= \frac{\left(\frac{f_2 - f_1}{x + 3h - x - h} - \frac{f_1 - f_0}{x + h - x}\right)}{x + 3h - x} \\ &= \frac{\left(\frac{f_2 - f_1}{2h} - \frac{f_1 - f_0}{h}\right)}{3h} \\ &= \frac{\left(\frac{f_2 - f_1}{2h} - \frac{2f_1 - 2f_0}{2h}\right)}{3h} \\ &= \frac{(2f_0 - 3f_1 + f_2)}{6h^2} \end{aligned}$$

Thus

$$\begin{aligned} f'(x) &= a_1 + a_2(x - (x + h)) \\ &= \frac{f_1 - f_0}{h} + \frac{2f_0 - 3f_1 + f_2}{6h^2}(-h) \\ &= \frac{-8f_0 + 9f_1 - f_2}{6h} \\ &= \frac{-8f(x) + 9f(x + h) - f(x + 3h)}{6h} \end{aligned}$$

12. Let $t_0 = x$, $t_1 = x - h$, and $t_2 = x + 2h$. Then

$$a_1 = \frac{f_1 - f_0}{x - h - x} = \frac{f_1 - f_0}{-h} = \frac{f_0 - f_1}{h}$$

and

$$\begin{aligned} a_2 &= \frac{\left(\frac{f_2 - f_1}{x + 2h - x + h} - \frac{f_1 - f_0}{x - h - x}\right)}{x + 2h - x} \\ &= \frac{\left(\frac{f_2 - f_1}{3h} - \frac{f_1 - f_0}{h}\right)}{2h} \\ &= \frac{\left(\frac{f_2 - f_1}{3h} - \frac{3f_1 - 3f_0}{3h}\right)}{2h} \\ &= \frac{(-3f_0 + 2f_1 + f_2)}{6h^2} \end{aligned}$$

Thus

$$\begin{aligned}
f'(x) &= a_1 + a_2(x - (x - h)) \\
&= \frac{f_0 - f_1}{h} + \frac{-3f_0 + 2f_1 + f_2}{6h^2}(h) \\
&= \frac{6(f_0 - f_1)}{6h} + \frac{-3f_0 + 2f_1 + f_2}{6h} \\
&= \frac{3f_0 - 4f_1 + f_2}{6h} \\
&= \frac{3f(x) - 4f(x-h) + f(x+2h)}{6h}
\end{aligned}$$

13. (a)

$$\begin{aligned}
f''(x) + f'(x) &\approx \frac{f_1 - 2f_0 + f_{-1}}{h^2} + \frac{f_1 - f_{-1}}{2h} \\
&= \frac{2f_1 - 4f_0 + 2f_{-1} + hf_1 - hf_{-1}}{2h^2} \\
&= \frac{(2+h)f_1 - 4f_0 + (2-h)f_{-1}}{2h^2}
\end{aligned}$$

(b)

$$\begin{aligned}
f''(x) + f'(x) &\approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{-3f_0 + 4f_1 - f_2}{2h} \\
&= \frac{4f_0 - 10f_1 + 8f_2 - 2f_3 - 3hf_0 + 4hf_1 - hf_2}{2h^2} \\
&= \frac{(4-3h)f_0 + (-10+4h)f_1 + (8-h)f_2 - 2f_3}{2h^2}
\end{aligned}$$

(c) $\mathbf{O}(h^2)$

14. In the two representations the value of c lies between $x + h$ and x, and $x - h$ and x, respectively. Thus, in general, each representation will have a different value of c, say c_1 and c_2, respectively. As given the formula for $f''(x)$ is inappropriate, but under the appropriate assumptions about the size of h and the behavior of $f^{(3)}(x)$ the formula for $f''(x)$ could be a good approximation.

Chapter 7

Numerical Integration

7.1 Introduction to Quadrature

1. (a)

trapezoidal rule	0.0
Simpson's rule	0.666667
Simpson's 3/8 rule	0.649519
Boole's rule	0.636165

(b)

trapezoidal rule	1.379769
Simpson's rule	0.958319
Simpson's 3/8 rule	0.986927
Boole's rule	1.008763

(c)

trapezoidal rule	0.420735
Simpson's rule	0.573336
Simpson's 3/8 rule	0.583143
Boole's rule	0.593376

2. (a)

trapezoidal rule	0.603553
Simpson's rule	0.638071
Boole's rule	0.636165

(b)

trapezoidal rule	1.020128
Simpson's rule	1.005610
Boole's rule	1.008763

(c)

trapezoidal rule	0.577889
Simpson's rule	0.592124
Boole's rule	0.593376

3. (a)

$$\begin{aligned}
\textstyle\int_a^b x^2 dx &= \frac{\frac{b-a}{2}(f(a) + 4f(\frac{a+b}{2}) + f(b)}{3} \\
&= \frac{(b-a)(a^2 + (a+b)^2 + b^2)}{6} \\
&= \frac{(b-a)(2a^2 + 2ab + 2b^2)}{6} \\
&= \tfrac{b^2}{3} - \tfrac{a^2}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
\textstyle\int_a^b x^3 dx &= \frac{\frac{b-a}{2}(f(a) + 4f(\frac{a+b}{2}) + f(b)}{3} \\
&= \frac{(b-a)(a^3 + \frac{(a+b)^3}{2} + b^3)}{6} \\
&= \frac{(b-a)(3a^3 + 3a^2b + 3ab^2 + 3b^3)}{6} \\
&= \tfrac{b^4}{4} - \tfrac{a^4}{2}
\end{aligned}$$

4.

$$\begin{aligned}
\int_{x_0}^{x_1} P_1(x)dx &= f_0 \int_{x_0}^{x_1} \frac{x - x_1}{x_0 - x_1} dx + f_1 \int_{x_0}^{x_1} \frac{x - x_0}{x_1 - x_0} dx \\
&= f_0 \left. \frac{(x - x_1)^2}{(x_0 - x_1)} \right|_{x=x_0}^{x=x_1} + f_1 \left. \frac{(x - x_0)^2}{(x_1 - x_0)} \right|_{x=x_0}^{x=x_1} \\
&= -f_0 \frac{x_1 - x_0}{2} + f_1 \frac{x_1 - x_0}{2} \\
&= \frac{x_1 - x_0}{2}(f_0 + f_1)
\end{aligned}$$

5. The degree of precision is $n = 1$.

6. The degree of precision is $n = 3$.

7. The degree of precision is $n = 5$.

8. (a) $m = (b-a)/(d-c)$

$$x - a = \frac{b-a}{d-c}(t - c)$$

$$\begin{aligned} x &= \frac{b-a}{d-c}t - \frac{bc-ac}{d-c} + a \\ &= \frac{b-a}{d-c}t - \frac{bc-ac}{d-c} + \frac{ad-ac}{d-c} \\ &= \frac{b-a}{d-c}t + \frac{ad-bc}{d-c} \end{aligned}$$

(b) The precision is 1.

(c) The precision is 2.

(d) The precision is 5.

9.

$$\begin{aligned} \int_{x_0}^{x_3} f(x)\,dx &\approx \int_{x_0}^{x_3} P_3(x)\,dx \\ &= \int_{x_0}^{x_3} f_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx \\ &\quad + \int_{x_0}^{x_3} f_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,dx \\ &\quad + \int_{x_0}^{x_3} f_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\,dx \\ &\quad + \int_{x_0}^{x_3} f_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,dx \end{aligned}$$

Now make the substitution $x = x_0 + ht$ where $h = (x_3 - x_0)/3$. Then $dx = hdt$, $x_k - x_j = (k-j)h$, $x - x_i = h(t-i)$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$. This yields the first line of the expression in Exercise 9. Starting at the last line of the expression in Exercise 9:

$$\begin{aligned} \cdots &= \frac{f_0 h}{6}\left(-\frac{81}{4} + 54 - \frac{99}{2} + 18\right) + \frac{f_1 h}{2}\left(\frac{81}{4} - 45 + 27\right) \\ &\quad + \frac{f_2 h}{2}\left(-\frac{81}{4} + 36 - \frac{27}{2}\right) + \frac{f_3 h}{6}\left(\frac{81}{4} - 27 + 9\right) \\ &= \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \end{aligned}$$

Which is Simpson's $\frac{3}{8}$ rule.

10.

$$\begin{aligned} \int_{x_0}^{x_5} f(x)\,dx &\approx \int_{x_0}^{x_5} P_5(x)\,dx \\ &= \int_{x_0}^{x_5} \left(\sum_{j=0}^{5} f_j \prod_{\substack{i=0 \\ i\neq j}}^{5} \frac{(x-x_i)}{(x_j-x_i)} \right) dx \end{aligned}$$

Now make the substitution $x = x_0 + ht$ where $h = (x_3 - x_0)/3$. Then $dx = hdt$, $x_k - x_j = (k-j)h$, $x - x_i = h(t-i)$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$ and $x_5 = 5$.

$$\begin{aligned}
&= \frac{f_0 h^6}{-120h^5}\int_0^5 (t-1)(t-2)(t-3)(t-4)(t-5)\,dt \\
&\quad + \frac{f_1 h^6}{24h^5}\int_0^5 (t-0)(t-2)(t-3)(t-4)(t-5)\,dt \\
&\quad + \frac{f_2 h^6}{-12h^5}\int_0^5 (t-0)(t-1)(t-3)(t-4)(t-5)\,dt \\
&\quad + \frac{f_3 h^6}{12h^5}\int_0^5 (t-0)(t-1)(t-2)(t-4)(t-5)\,dt \\
&\quad + \frac{f_4 h^6}{-24h^5}\int_0^5 (t-0)(t-1)(t-2)(t-3)(t-5)\,dt \\
&\quad + \frac{f_5 h^6}{120h^5}\int_0^5 (t-0)(t-1)(t-2)(t-3)(t-4)\,dt \\
&= \frac{hf_0}{-120}\left(-\frac{475}{12}\right) + \frac{hf_1}{24}\left(\frac{125}{4}\right) + \frac{hf_2}{-12}\left(-\frac{125}{12}\right) \\
&\quad + \frac{hf_3}{12}\left(\frac{125}{12}\right) + \frac{hf_4}{-24}\left(-\frac{125}{4}\right) + \frac{hf_5}{120}\left(\frac{475}{12}\right) \\
&= \frac{5h}{288}(19f_0 + 75f_1 + 50f_2 + 50f_3 + 75f_4 + 19f_5)
\end{aligned}$$

11. Constructing a divided-difference table for the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ produces the second-degree Newton polynomial

$$P_2(x) = f_0 + \left(\frac{f_1 - f_0}{h}\right)(x - x_0) + \left(\frac{f_2 - 2f_1 + f_0}{2h^2}\right)(x - x_0)(x - x_1).$$

Thus

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &\approx \int_{x_0}^{x_2} P_2(x)\,dx \\
&= f_0\int_{x_0}^{x_2} dx + \frac{f_1 - f_0}{h}\int_{x_0}^{x_2}(x - x_0)\,dx \\
&\quad + \frac{f_2 - 2f_1 + f_0}{2h^2}\int_{x_0}^{x_2}(x - x_0)(x - x_1)\,dx
\end{aligned}$$

Now make the substitution $x = x_0 + ht$ where $h = (x_3 - x_0)/3$. Then $dx = hdt$, $x_k - x_j = (k-j)h$, $x - x_i = h(t-i)$, $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.

$$\begin{aligned}
&= hf_0 \int_0^2 dt + h(f_1 - f_0)\int_0^2 t\,dt + \frac{h(f_2 - 2f_1 + f_0)}{2}\int_0^2 t(t-1)\,dt \\
&= hf_0\,(t)_0^2 + h(f_1 - f_0)\left(\frac{t^2}{2}\right)_0^2 + \frac{h(f_2 - 2f_1 + f_0)}{2}\left(\frac{t^3}{3} - \frac{t^2}{2}\right)_0^2 \\
&= 2hf_0 + 2h(f_1 - f_0) + \frac{2}{3}\left(\frac{h(f_2 - 2f_1 + f_0)}{2}\right) \\
&= \frac{h}{3}(f_0 + 4f_1 + f_2)
\end{aligned}$$

Simpson's rule.

7.2 Composite Trapezoidal and Simpson's Rule

1. (a) $F(x) = \arctan(x)$, $F(1) - F(-1) = \pi/2 \approx 1.57079632679$
   - i. $M = 10$, $h = 0.2$, $T(f,h) = 1.56746305691$, $E_T(f,h) = 0.00333326989$
   - ii. $M = 5$, $h = 0.2$, $S(f,h) = 1.57079538808$, $E_S(f,h) = 0.00000093870$

   (b) $F(x) = 2x - \sqrt{x}\cos(2\sqrt{x}) + \sin(2\sqrt{x})/2$
   $F(1) - F(0) = 2\cos(2) + \sin(2)/2 \approx 2.87079554995998$
   - i. $M = 10$, $h = 0.1$, $T(f,h) = 2.85740884711$, $E_T(f,h) = 0.1338670225$
   - ii. $S(f,h) = 2.865601$

   (c) $F(x) = 2\sqrt{x}$, $F(4) - F(1/2) = 3$
   - i. $M = 10$, $h = 0.375$, $T(f,h) = 3.04191993765$, $E_T(f,h) = -0.04191993765$
   - ii. $M = 5$, $h = 0.375$, $S(f,h) = 3.00762208163$, $E_S(f,h) = -0.00762208163$

   (d) $F(x) = -(x^2 + 2x + 2)e^{-x}$
   $F(4) - F(0) = 2 - 26e^{-x} \approx 1.52379338889291$
   - i. $M = 10$, $h = 0.4$, $T(f,h) = 1.52162778130$, $E_T(f,h) = 0.00216560759$
   - ii. $S(f,h) = 1.524599$

   (e) $F(x) = 2\cos(x) + 2x\sin(x)$
   $F(2) - F(0) = 2\cos(2) + 4\sin(2) - 2 \approx 0.804896034208442$
   - i. $M = 10$, $h = 0.2$, $T(f,h) = 0.78330408729$, $E_T(f,h) = 0.02159194692$
   - ii. $S(f,h) = 0.805005$

(f) $F(x) = -\frac{2}{5}\cos(2x)e^{-x} - \frac{1}{5}\sin(2x)e^{-x}$
$F(\pi) - F(0) = \frac{2}{5} - \frac{2}{5}e^{-\pi} \approx 0.382714432694491$

i. $M = 10$, $h = \pi/10$, $T(f,h) = 0.36695122058$,
$E_T(f,h) = 0.01576321211$

ii. $S(f,h) = 0.382793$

2. (a) $\displaystyle\int_0^1 \sqrt{1+9x^4}dx = 1.54786565469019$

i. $M = 10$, $T(f, 1/10) = 1.55260945$

ii. $M = 5$, $S(f, 1/10) = 1.54786419$

(b) $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos^2(x)}dx = 1.31144249821555$

i. $M = 10$, $T(f, \pi/40) = 1.3110891$

ii. $M = 5$, $S(f, \pi/40) = 1.31144352$

(c) $\displaystyle\int_0^1 \sqrt{1+e^{-2x}}dx = 1.19270140197215$

i. $M = 10$, $T(f, 1/10) = 1.19318470$

ii. $M = 5$, $S(f, 1/10) = 1.19270186$

3. (a) $\displaystyle 2\pi\int_0^1 x^3\sqrt{1+9x^4}dx = 3.5631218520124$

i. $M = 10$, $T(f, 1/10) = 3.64244664$

ii. $M = 5$, $S(f, 1/10) = 3.56372816$

(b) $\displaystyle 2\pi\int_0^{\pi/4} \sin(x)\sqrt{1+\cos^2(x)}dx = 3.65837776892123$

i. $M = 10$, $T(f, \pi/40) = 3.65242104$

ii. $M = 5$, $S(f, \pi/40) = 3.65869833$

(c) $\displaystyle 2\pi\int_0^1 e^{-x}\sqrt{1+e^{-2x}}dx = 4.8492184914728$

i. $M = 10$, $T(f, 1/10) = 4.85802252$

ii. $M = 5$, $S(f, 1/10) = 4.84924283$

4. (a) The trapezoidal rule ($M = 1$, $h = 1$) is exact for $f(x) = c_1x + c_0$ over $[0, 1]$ since:

$$T(f,h) = \frac{1}{2}(f(0) + f(1)) = \frac{1}{2}(c_0 + (c_1 + c_0)) = \frac{1}{2}c_1 + c_0$$

and

$$\int_0^1 (c_1x + c_0)dx = \left(\frac{c_1x^2}{2} + c_0x\right)_0^1 = \frac{1}{2}c_1 + c_0.$$

(b) It is sufficient to show $E_T(f,1) = \int_0^1 c_0x^2dx - T(c_2x^2,1)$. Thus

$$E_T(f,1) = \frac{-(1-0)(2c_2)(1^2)}{12} = -\frac{c_2}{6}$$

and

$$\begin{aligned}\int_0^1 c_2x^2dx - \frac{1}{2}(0+c_2) &= \left.\frac{c_2x^3}{3}\right|_0^1 - \frac{c_2}{2}\\ &= \frac{c_2}{3} - \frac{c_2}{2} = -\frac{c_2}{6}\end{aligned}$$

5. (a) By the summation properties of definite integrals, it is sufficient to show Simpson's rule is exact for each of the polynomials c_3x^2, c_2x^2, c_1x_1, and c_0. Thus

$$\begin{array}{ll}\int_0^2 c_0\,dx = 2c_0 & \frac{1}{3}(c_0+4c_0+c_0) = 2c_0\\ \int_0^2 c_1x\,dx = \left.\frac{c_1x^2}{2}\right|_0^2 = 2c_1 & \frac{1}{3}(0+4c_1+2c_1) = 2c_1\\ \int_0^2 c_2x^2dx = \left.\frac{c_2x^2}{3}\right|_0^2 = \frac{8}{3}c_2 & \frac{1}{3}(0+4c_2+4c_2) = \frac{8}{3}c_2\\ \int_0^2 c_3x^2dx = \left.\frac{c_3x^4}{4}\right|_0^2 = 4c_3 & \frac{1}{3}(0+4c_3+8c_3) = 4c_3\end{array}$$

(b) It is sufficient to show $\int_0^2 c_4x^4dx - S(c_4x^4,1) = E_S(c_4x^4,1)$.

$$E_X(c_4x^4,1) = \frac{-(2-0)(24c_4)(1)^2}{180} = -\frac{4}{15}c_4$$

$$\begin{aligned}\int_0^2 c_4x^2dx - \frac{1}{3}(0+4c_4+16c_4) &= \left.\frac{c_4x^5}{5}\right|_0^2 - \frac{20}{3}c_4\\ &= \left(\frac{32}{5}-\frac{20}{3}\right)c_4 = -\frac{4}{15}c_4\end{aligned}$$

6. (a) We want a quadrature formula:

$$\int_0^1 g(t)dt \approx \omega_0 g(0) + \omega_1 g(1),$$

that is exact for polynomials of degree less than or equal to one. Thus we find values of ω_0 and ω_1 satisfying the equations:

$$\begin{aligned}\int_0^1 1\,dt &= \omega_0(1) + \omega_1(1)\\ \int_0^1 t\,dt &= \omega_0(0) + \omega_1(1)\end{aligned}$$

or $1 = \omega_0 + \omega_1$ and $\frac{1}{2} = \omega_1$. The solution is $\omega_0 = \omega_1 = \frac{1}{2}$. Thus the quadrature scheme

$$\int_0^1 g(t)dt \approx \frac{1}{2}g(0) + \frac{1}{2}g(1)$$

is exact over the interval $[0, 1]$ for all polynomials of degree less than or equal to one.

(b) Make the substitution $x = x_0 + ht$ into the approimation scheme (note: $f(x) = g(t)$, $dt = \frac{1}{h}dx$, $x_0 = 0$, and $x_1 = 1$):

$$\frac{1}{h}\int_{x_0}^{x_1} f(x)dx \approx \frac{1}{2}f(x_0) + \frac{1}{2}f(x_1)$$

or

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0 + f_1),$$

the trapezoidal rule for $M = 1$ and $h = x_1 - x_0$ on the interval $[x_0, x_1]$.

7. (a) We want a quadrature formula

$$\int_0^2 g(t)dt \approx \omega_0 g(0) + \omega_1 g(1) + \omega_2 g(2),$$

that is exact for polynomials of degree less than or equal to two. Thus we find values ω_0, ω_1, and ω_2 such that

$$\begin{aligned} \int_0^2 1\, dt &= \omega_0(1) + \omega_1(1) + \omega_2(1) \\ \int_0^2 t\, dt &= \omega_0(0) + \omega_1(1) + \omega_2(2) \\ \int_0^2 t^2 dt &= \omega_0(0) + \omega_1(1) + \omega_2(4) \end{aligned}$$

or

$$\begin{aligned} \omega_0 + \omega_1 + \omega_2 &= 2 \\ \omega_1 + 2\omega_2 &= 2 \\ \omega_1 + 4\omega_2 &= \frac{8}{3} \end{aligned}$$

The solution of the linear system is $\omega_0 = 1/3$, $\omega_1 = 4/3$ and $\omega_2 = 1/3$. Thus the quadrature scheme

$$\int_0^2 g(t)dt \approx \frac{1}{3}g(0) + \frac{4}{3}g(1) + \frac{1}{3}g(2)$$

is exact for polynomials of degree less than two on the interval $[0, 2]$.

(b) Make the substitution $x = x_0 + ht$ into the quadrature scheme (note: $f(x) = g(t)$, $dt = \frac{1}{h}dx$, $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$):

$$\frac{1}{h}\int_{x_0}^{x_2} f(x)dx \approx \frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{1}{3}f(x_2)$$

or

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2),$$

Simpson's rule for $M = 1$ and $h = \frac{x_2 - x_0}{2}$ on the interval $[x_0, x_2]$.

8. (a) Use the bound $|f^{(2)}(x)| = |-\cos(x)| \leq |\cos(0)| = 1$, and obtain $((\pi/3 - 0)\, h^2)\,/12 \leq 5 \times 10^{-9}$, then substitute $h = \pi/(3M)$ and get $\pi^3/162 \times 10^8 \leq M^2$. Solve and get $4374.89 \leq M$; since M must be an integer, $M = 4375$ and $h = 0.000239359$.

   (b) Use the bound $|f^{(2)}(x)| = \frac{2}{|5-x|^3} \leq |f^{(2)}(3)| = 1/4$, and obtain $\frac{(3-2)\frac{1}{4}h^2}{12} \leq 5 \times 10^{-9}$, then substitute $h = 1/M$ and get $\frac{1}{24} \times 10^8 \leq M^2$. Solve and get $2041.24145 \leq M$; since M must be an integer, $M = 2042$ and $h = 0.000489716$.

   (c) Use the bound $|f^{(2)}(x)| = |(x-2)e^{-x}| \leq |f^{(2)}(0)| = 2$, and obtain $\frac{(2-0)2h^2}{12} \leq 5 \times 10^{-9}$, then substitute $h = 2/M$ and get $\frac{4}{15} \times 10^9 \leq M^2$. Solve and get $16329.9316 \leq M$; since M must be an integer, $M = 16330$ and $h = 0.000122474$.

9. (a) Use the bound $|f^{(4)}(x)| = |\cos(x)| \leq |\cos(0)| = 1$, and obtain $((\pi/3 - 0)\, h^2)\,/12 \leq 5 \times 10^{-9}$, then substitute $h = \pi/(6M)$ and get $\pi^5/34.992 \times 10^7 \leq M^4$; since M must be an integer, $M = 18$ and $h = 0.029088821$.

   (b) Use the bound $|f^{(4)}(x)| = \frac{24}{|5-x|^5} \leq |f^{(4)}(3)| = 3/4$, and obtain $\frac{(3-2)\frac{3}{4}h^4}{180} \leq 5 \times 10^{-9}$, then substitute $h = 1/(2M)$ and get $\frac{1}{192} \times 10^7 \leq M^4$. Solve and get $15.106877 \leq M$; since M must be an integer, $M = 16$ and $h = 0.03125$.

   (c) Use the bound $|f^{(4)}(x)| = |(x-4)e^{-x}| \leq |f^{(4)}(0)| = 4$, and obtain $\frac{(2-0)4h^4}{180} \leq 5 \times 10^{-9}$, then substitute $h = 2/(2M)$ and get $\frac{1}{1125} \times 10^{10} \leq M^4$. Solve and get $54.602417 \leq M$; since M must be an integer, $M = 55$ and $h = 0.0181818$.

10.

M	h	$T(f,h)$	$E_T(f,h) = \mathbf{O}(h^2)$
1	0.2	0.199008	0.0006660
2	0.1	0.1995004	0.0001664
4	0.05	0.1996252	0.0000416
8	0.025	0.1996564	0.0000104
16	0.0125	0.1996642	0.0000026

11.

M	h	$S(f,h)$	$E_S(f,h) = \mathbf{O}(h^4)$
1	0.75	1.3658444	-0.0025669
2	0.375	1.3634298	-0.0001523
4	0.1875	1.3632869	-0.0000094
8	0.09375	1.3632781	-0.0000006

12. (a) Let $h = \frac{x_1 - x_0}{2}$, $x^* = x_0 + h$, and $F'(x) = f(x)$. Then

$$\begin{aligned} F(x) &= \sum_{k=0}^{2} \frac{F^{(k)}(x^*)(x-x^*)^k}{k!} + \frac{F^{(3)}(c)(x-x^*)^3}{3!} \\ &= P_2(x) + E_M(f,h), \end{aligned}$$

where c is between x and x^*. By the Fundamental Theorem of Calculus:

$$\begin{aligned} \int_{x_0}^{x_1} f(x)dx &= F(x_1) - F(x_0) \\ &\approx P_2(x_1) - P_2(x_0) \\ &= (F(x^*) + F'(x^*)(x_1 - x^*) + \frac{1}{2}F''(x^*)(x_1 - x^*)^2) \\ &\quad -(F(x^*) + F'(x^*)(x_0 - x^*) + \frac{1}{2}F''(x^*)(x_0 - x^*)^2) \\ &= 2hF'(x^*) + \frac{1}{2}(h^2 - (-h)^2)F''(x^*) \\ &= 2hF'(x^*) = 2hF'(x_0 + h) = 2hf(x_0 + h), \end{aligned}$$

the midpoint rule. The error term is:

$$\begin{aligned} E_M(f,h) &= \frac{1}{3!}F^{(3)}(c_1)(x_1 - x^*)^3 - \frac{1}{3!}F^{(3)}(c_0)(x_0 - x^*)^3 \\ &= \frac{1}{6}F^{(3)}(c_1)h^3 - \frac{1}{6}F^{(3)}(c_0)(-h)^3 \\ &= \frac{h^3}{6}(F^{(3)}(c_1) + F^{(3)}(c_0)) = \frac{h^3}{6}(f''(c_1) + f''(c_0)). \end{aligned}$$

If $f''(x)$ is continuous on $[x_0, x_1]$, then, by the Intermediate Value Theorem, there exists $c \in [x_0, x_1]$ such that $f''(c_1)+f''(c_0) = 2f''(c)$. Thus $E_M(f, h) = \frac{h^3}{3} f''(c)$.

(b) Note: We now let $h = \dfrac{b-a}{N}$, the length of the ith subinterval. Then

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} f(x)dx \\
&\approx \sum_{k=1}^{N} hf\left(a + \left(k - \frac{1}{2}\right)h\right) \\
&= h\sum_{k=1}^{N} f\left(a + \left(k - \frac{1}{2}\right)h\right),
\end{aligned}$$

the composite midpoint formula.

(c) From part(a) over the interval $[x_{i-1}, x_i]$ we have $E_M(f, h) = \frac{h^3}{3} f''(c_i)$. Thus over the interval $[x_0, x_N]$

$$\begin{aligned}
E_M(f, h) &= \sum_{k=1}^{N} \frac{h^3}{3} f''(c_k) = \frac{h^3}{3} \sum_{k=1}^{N} f''(c_k) \\
&= \frac{(b-a)h^2}{3} \left(\frac{1}{N} \sum_{k=1}^{N} f''(c_k)\right).
\end{aligned}$$

The term in parentheses can be recognized as an average of values for the second derivative and hence is replaced by $f''(c)$. Therefore, we have established that for the composite midpoint formula:

$$E_M(f, h) = \frac{(b-a)f''(c)h^2}{3} = \mathbf{O}(h^2)$$

13. (a) $\int_{-1}^{1} (1 + x^2)^{-1} dx \approx 1.1074$
(b) $\int_0^1 (2 + \sin(2\sqrt{x})dx \approx 5.5111$
(c) $\int_{0.25}^4 \frac{1}{\sqrt{x}} dx \approx 2.5582$
(d) $\int_0^4 x^2 e^{-x} dx \approx 0.6467$
(e) $\int_0^2 2x \cos(x) dx \approx 0.8157$
(f) $\int_0^{\pi} \sin(2x) e^{-x} dx \approx 0.4593$

14. We first determine the error when the rule is applied over the interval $[x_0, x_2]$. In the proof of Corollary 7.2 we were able to use the Second

Mean Value Theorem for Integrals. But, in this case the expression $(x - x_0)(x - x_1)(x - x_2)$ changes sign on $[x_0, x_2]$. Hence, we proceed in a different manner.

Let $P_2(x)$ be the Newton interpolatory polynomial of degree two for $f(x)$ at the nodes x_0, x_1, and x_2. Let x be the value at which we want to calculate the error. Adding x to the divided-difference table for $P_2(x)$ yields $P_3(x)$ an interpolatory polynomial agreeing with $f(x)$ at x_0, x_1, x_2, and x. Thus

$$P_3(x) = P_2(x) + f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)$$

But $P_3(x) = f(x)$, at x, thus

$$f(x) = P_2(x) + f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)$$

Integrating:

$$\begin{aligned}
&\textstyle\int_{x_0}^{x_2} f(x)dx \\
&= \textstyle\int_{x_0}^{x_2} P_2(x)dx \\
&\quad + \textstyle\int_{x_0}^{x_2} f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)dx \\
&= \tfrac{h}{3}(f_0 + 4f_1 + f_2) + \textstyle\int_{x_0}^{x_2} f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)dx
\end{aligned}$$

We now consider the error terms.

$$E_S(f, h) = \int_{x_0}^{x_2} f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)dx.$$

Define

$$g(x) = \int_{x_0}^{x} (t - x_0)(t - x_1)(t - x_2)dt.$$

Note: By the Second Fundamental Theorem of Calculus; $g'(x) = (x - x_0)(x - x_1)(x - x_2)$, and by the Fundamental Theorem of Calculus; $g(x) = \frac{h^2}{4}(x - x_0)^2(x - x_1)^2$. Thus $g(x_0) = 0 = g(x_2)$, $g(x) > 0$ for all x in (x_0, x_2), and

$$E_S(f, h) = \int_{x_0}^{x_2} f[x_0, x_1, x_2, x]g'(x)dx.$$

Integration by parts yields:

$$
\begin{aligned}
E_S(f,h) &= f[x_0,x_1,x_2,x]g(x)|_{x_0}^{x_2} - \int_{x_0}^{x_2} g(x)(f[x_0,x_1,x_2,x])'dx \\
&= -\int_{x_0}^{x_2} g(x)(f[x_0,x_1,x_2,x])'dx \\
&= -\int_{x_0}^{x_2} g(x)\frac{f^{(4)}(d(x))}{4!}dx,
\end{aligned}
$$

where $d = d(x) \in (x_0, x_2)$, by Theorem 1.7 Generalized Rolle's Theorem. Applying Theorem 1.11 Weighted Integral Mean Value Theorem yields

$$
\begin{aligned}
E_x(f,h) &= \frac{f^{(4)}(c)}{4!}\int_{x_0}^{x_2} g(x)dx \\
&= \frac{f^{(4)}(c)}{4!}\int_{x_0}^{x_2}\left(\int_{x_0}^{x}(t-x_0)(t-x_1)(t-x_2)dt\right)dx \\
&= \frac{f^{(4)}(c)}{4!}\int_{x_0}^{x_2}\left(\int_{t}^{x_2}(t-x_0)(t-x_1)(t-x_2)dx\right)dt \\
&= \frac{f^{(4)}(c)}{4!}\int_{x_0}^{x_2}(t-x_2)(t-x_1)(t-x_2)(x_2-t)dt \\
&= \frac{h^5 f^{(4)}(c)}{4!}\left(\frac{32}{5} - 10 + \frac{64}{3} - 8\right) \\
&= -\frac{h^5}{90}f^{(4)}(c).
\end{aligned}
$$

where $c = c(x) \in (x_0, x_2)$. Now sum the error over each subinterval:

$$
E_S(f,h) = -\frac{h^5}{90}\sum_{k=1}^{2M} f^{(4)}(c_k)
$$

Assuming $f^{(4)}$ is bounded and continuous over $[a,b]$, then by Theorem 1.2 Intermediate Value Theorem there exists $c \in (a,b)$ such that

$$
f^{(4)}(c) = \frac{1}{M}\sum_{k=1}^{M} f^{(4)}(c_k)
$$

Therefore,

$$
\begin{aligned}
E_S(f,h) &= -\frac{h^5}{90}\left(Mf^{(4)}(c)\right) = -\frac{(b-a)Mf^{(4)}(c)}{90(2M)} \\
&= -\frac{(b-a)f^{(4)}(c)}{180} = \mathbf{O}(h^4)
\end{aligned}
$$

7.3 Recursive Rules and Romberg Integration

1. (a)

J	$R(J,0)$	$R(J,1)$	$R(J,2)$
0	-0.00171772		
1	0.02377300	0.03226990	
2	0.60402717	0.79744521	0.84845691

(b)

J	$R(J,0)$	$R(J,1)$	$R(J,2)$
0	-0.00199505		
1	-0.02186444	-0.2848757	
2	0.01611754	0.02877821	0.03259592

(c)

J	$R(J,0)$	$R(J,1)$	$R(J,2)$
0	2.88		
1	2.10564024	1.84752031	
2	1.78167637	1.67368841	1.66209962

(d)

J	$R(J,0)$	$R(J,1)$	$R(J,2)$
0	10.24390244		
1	6.03104213	4.62675536	
2	4.65685845	4.19879722	4.17026668

(e)

J	$R(J,0)$	$R(J,1)$	$R(J,2)$
0	0.44127407		
1	0.95641862	1.12813347	
2	1.21628836	1.30291160	1.31456348

(f)

J	$R(J,0)$	$R(J,1)$	$R(J,2)$
0	2.0		
1	2.73205081	2.97606774	
2	2.99570907	3.08359515	3.0976365

2. (a)

$$\lim_{J\to\infty} S(J) = \lim_{J\to\infty} \frac{4T(J) - T(J-1)}{3} = \frac{4L - L}{3} = L$$

(b)

$$\lim_{J\to\infty} B(J) = \lim_{J\to\infty} \frac{16S(J) - S(J-1)}{15} = \frac{16L - L}{15} = L$$

3. (a)

$$\begin{array}{lll} \int_0^4 1\,dx = 4 & \int_0^4 x\,dx = 8 & \int_0^4 x^2dx = \frac{64}{3} \\ \int_0^4 x^3dx = 64 & \int_0^4 x^4dx = \frac{1024}{5} & \int_0^4 x^5dx = \frac{2048}{3} \end{array}$$

$$\begin{aligned} B(1,1) &= \tfrac{2}{45}(7 + 32 + 12 + 32 + 7) = 4 \\ B(x,1) &= \tfrac{2}{45}(0 + 32 + 24 + 96 + 28) = 8 \\ B(x^2,1) &= \tfrac{2}{45}(0 + 32 + 12(4) + 32(9) + 7(10)) = \tfrac{64}{3} \\ B(x^3,1) &= \tfrac{2}{45}(7(0) + 32(1) + 12(8) + 32(27) + 7(64)) = 64 \\ B(x^4,1) &= \tfrac{2}{45}(7(0) + 32(1) + 12(16) + 32(81) + 7(256)) = \tfrac{1024}{5} \\ B(x^5,1) &= \tfrac{2}{45}(7(0) + 32(1) + 12(32) + 32(243) + 7(1024)) = \tfrac{2048}{3} \end{aligned}$$

(b)

$$\int_0^4 x^6dx - B(x^6,1) = \frac{16384}{7} - \frac{7040}{3} = -\frac{128}{21}$$

$$E_B(x^6,1) = \frac{-2(4-0)(6!)(1)^6}{945} = -\frac{128}{21}$$

4. We want a quadrature scheme

$$\int_0^4 g(t)dt = \omega_0 g(0) + \omega_1 g(1) + \omega_2 g(2) + \omega_3 g(3) + \omega_4 g(4)$$

that is exact for polynomials of degree less than or equal to four. Thus we seek ω_0, ω_1, ω_2, ω_3, and ω_4 such that

$$\begin{aligned} \int_0^4 1\,dt &= \omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 \\ \int_0^4 t\,dt &= \omega_1 + 2\omega_2 + 3\omega_3 + 4\omega_4 \\ \int_0^4 t^2dt &= \omega_1 + 4\omega_2 + 9\omega_3 + 16\omega_4 \\ \int_0^4 t^3dt &= \omega_1 + 8\omega_2 + 27\omega_3 + 64\omega_4 \\ \int_0^4 t^4dt &= \omega_1 + 16\omega_2 + 81\omega_3 + 256\omega_4 \end{aligned}$$

or

$$\begin{aligned}
4 &= \omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 \\
8 &= \omega_1 + 2\omega_2 + 3\omega_3 + 4\omega_4 \\
\frac{64}{3} &= \omega_1 + 4\omega_2 + 9\omega_3 + 16\omega_4 \\
64 &= \omega_1 + 8\omega_2 + 27\omega_3 + 64\omega_4 \\
\frac{1024}{5} &= \omega_1 + 16\omega_2 + 81\omega_3 + 256\omega_4
\end{aligned}$$

The solution of the linear system is: $\omega_0 = \frac{14}{45}$, $\omega_1 = \frac{64}{45}$, $\omega_2 = \frac{8}{15}$, $\omega_3 = \frac{64}{45}$, and $\omega_4 = \frac{14}{45}$. Therefore,

$$\int_0^4 g(t)dt = \frac{2}{45}(7g(0) + 32g(1) + 12g(2) + 32g(3) + 7g(4))$$

or

$$\int_0^4 f(x)dx \approx \frac{2}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4),$$

Boole's composite rule for $h = 1$ and $M = 1$ over the interval $[0, 4]$.

5.

$$\begin{aligned}
&\frac{16(\frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)) - \frac{2h}{3}(f_0 + 4f_2 + f_4)}{15} \\
&= \tfrac{h}{45}\left(16f_0 + 64f_1 + 32f_2 + 64f_3 + 16f_4 - 2f_0 - 8f_2 - 2f_4\right) \\
&= \tfrac{h}{45}\left(14f_0 + 64f_1 + 24f_2 + 64f_3 + 14f_4\right) \\
&= \tfrac{2h}{45}\left(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\right)
\end{aligned}$$

Boole's Composite rule for $J = 2$.

6.

$$\begin{aligned}
&\frac{9T(f,h) - T(f,3h)}{8} \\
&= \frac{9\left(\frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + f_3\right)\right) - \frac{3h}{2}\left(f_0 + f_3\right)}{8} \\
&= \tfrac{3h}{16}\left(3f_0 + 6f_1 + 6f_2 + 3f_3 - f_0 - f_3\right) \\
&= \tfrac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right)
\end{aligned}$$

Simpson's $\frac{3}{8}$ rule (6) from Section 7.1.

7. Multiply both sides of (25) by 16 and subtract (26) from the result:

$$\begin{aligned}
&16\int_a^b f(x)dx - \int_a^b f(x)dx \\
&= 16S(f,h) - S(f,2h) + b_2(16-64)h^6 + b_3(16-256)h^8 + \cdots
\end{aligned}$$

or

$$\int_a^b f(x)dx = \frac{16S(f,h) - S(f,2h)}{15} - \frac{48b_2h^6}{15} - \frac{240b_3h^8}{15}$$

8. Multiply both sides of (28) by 4^K and subtract (29) to obtain:

$$\begin{aligned} 4^KQ - Q &= 4^KR(h, K-1) - R(2h, K-1) \\ &\quad + c_1(4^K - 4^K)h^{2K} + c_2(4^K - 4^{K+1})h^{2k+2} + \cdots \end{aligned}$$

or

$$\begin{aligned} Q &= \frac{4^KR(h,K-1) - R(2h,K-1)}{4^K - 1} + \frac{c_2 4^K(-3)h^{2K+2}}{4^K - 1} + \cdots \\ &= \frac{4^KR(h,K-1) - R(2h,K-1)}{4^K - 1} + \mathbf{O}(h^{2k+2}) \end{aligned}$$

9. (a) $f^{(8)}(x) = 0$, thus by Theorem 7.7 $R(3,3) = 256$. The following Romberg tableau shows the result.

$$\begin{array}{cccc} 1024 & 0 & 0 & 0 \\ 520 & 352 & 0 & 0 \\ \frac{2627}{8} & \frac{529}{2} & \frac{776}{3} & 0 \\ \frac{6314}{23} & \frac{32841}{128} & \frac{6145}{24} & 256 \end{array}$$

(b) $f^{(11)}(x) = 0$, thus by Theorem 7.7 $R(4,4) = 2048$. The following Romberg tableau shows the result.

$$\begin{array}{ccccc} 11264 & 0 & 0 & 0 & 0 \\ 5643 & \frac{11308}{3} & 0 & 0 & 0 \\ \frac{288757}{2} & \frac{5990}{26} & \frac{35299}{16} & 0 & 0 \\ \frac{28031}{12} & \frac{184082}{89} & \frac{84158}{41} & \frac{125062}{61} & 0 \\ \frac{233309}{110} & \frac{47135}{23} & \frac{24577}{12} & \frac{200705}{48} & 2048 \end{array}$$

10. (a) $x = 0 \Rightarrow t = 0$, $x = 1 \Rightarrow t = 1$, and $\sqrt{t^2} = t$ on the interval $(0,1)$.

$$\int_0^1 \sqrt{x}\,dx = \int_0^1 \sqrt{t^2}\,(2t)dt = \int_0^1 t(2t)dt = \int_0^1 2t^2dt$$

(b) For $\int_0^1 \sqrt{x}\,dx$, Romberg integration converges slowly because the higher-order derivatives of the integrand $f(x) = \sqrt{x}$ are not bounded near $x = 0$.

11. (a) Substitute h_J for h in the formula for $M(f,h)$ to get the sequential midpoint rule:

$$M(J) = M(f, h_J) = h_J \sum_{k=1}^{2^J} f\left(a + \left(k - \frac{1}{2}\right) h_J\right)$$

(b) Since the given error term $E_M(f,h)$ has the same form as the error term for the trapezoidal rule, we follow the work outlined in formulas (20) through (33) and see that $R(J,0) = M(J)$ for $J \geq 0$, and for $J \geq K$

$$R(J,K) = \frac{4^K R(J,K-1) - R(J-1,K-1)}{4^K - 1}$$

7.4 Adaptive Quadrature

There are no Exercises for this section–just Algorithms and Programs.

7.5 Gauss-Legendre Integration

1. $\int_0^2 6t^5\,dt = 64$, $G(f,2) = 58.6666667$
2. $\int_0^2 \sin(t)\,dt \approx 1.41647$, $G(f,2) = 1.410157$
3. $\int_0^1 \frac{\sin(t)}{t}\,dt \approx 0.9460831$, $G(f,2) = 0.9460411$
4. $\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-t^2/2} dt \approx 0.3413447$, $G(f,2) = 0.3412211$
5. $\frac{1}{\pi} \int_0^\pi \cos(0.6\sin(t))\,dt \approx 0.91200486$, $G(f,2) = 0.91200411$
6. (a) $N = 4$ (b) $N = 6$
7. The roots of the Legendre polynomials are (a) $\pm\frac{1}{\sqrt{3}}$, (b) $0, \pm\sqrt{0.6}$, and (c) ± 0.8611363116, ± 0.3399810436.
8. If the fourth derivative does not change too much, then

$$\left|\frac{f^{(4)}(c_1)}{135}\right| < \left|\frac{-f^{(4)}(c_2)}{90}\right|.$$

The truncation error term for the Gauss-Legendre rule will be less than the truncation error term for Simpson's rule.

9.

$$\int_{-1}^{1} 1\,dt = 2 \quad \int_{-1}^{1} t\,dt = 0 \quad \int_{-1}^{1} t^2 dt = \tfrac{2}{3}$$
$$\int_{-1}^{1} t^3 dt = 0 \quad \int_{-1}^{1} t^4 dt = \tfrac{2}{5} \quad \int_{-1}^{1} t^5 dt = 0$$

$$\begin{aligned}
G_3(1) &= \frac{1}{9}(5+8+5) = 2 \\
G_3(t) &= \frac{1}{9}(-5\sqrt{0.6} + 8(0) + 5\sqrt{0.6}) = 0 \\
G_3(t^2) &= \frac{1}{9}(5(0.6) + 5(0.6)) = \frac{2}{3} \\
G_3(t^3) &= \frac{1}{9}(5(-\sqrt{0.6})^3 + 8(0) + 5\sqrt{0.6}^3) = 0 \\
G_3(t^4) &= \frac{1}{9}(5(0.6)^2 + 8(0) + 5(0.6)^2) = \frac{2}{5} \\
G_3(t^5) &= \frac{1}{9}(-5(0.6)^{5/2} + 8(0) + 5(0.6)^{5/2}) = 0
\end{aligned}$$

10. If the sixth derivative does not change too much, then

$$\left|\frac{f^{(6)}(c_1)}{15750}\right| < \left|\frac{-f^{(6)}(c_2)}{15120}\right|.$$

Since the magnitude of the truncation error terms are about the same, both methods have about the same precision. But the Gauss-Legendre rule requires three calculations compared to five calculations required by Boole's rule.

11. We want a quadrature scheme

$$\int_{-1}^{1} f(x)dx \approx \omega_1 f(-\sqrt{0.6}) + \omega_2 f(0) + \omega_3 f(\sqrt{0.6})$$

that is exact for polynomials of degree less than or equal to two. Setting $f(x)$ equal to 1, x, and x^2 in the previous scheme yields the linear system

$$\begin{aligned}
2 &= \omega_1 + \omega_2 + \omega_3 \\
0 &= -\sqrt{0.6}\omega_1 + \sqrt{0.6}\omega_3 \\
\frac{2}{3} &= 0.6\omega_1 + 0.6\omega_3
\end{aligned}$$

The solution of the linear system is $\omega_1 = \frac{5}{9}$, $\omega_2 = \frac{8}{9}$, and $\omega_3 = \frac{5}{9}$. Therefore,

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{9}(5f(-\sqrt{0.6}) + 8f(0) + 5f(\sqrt{0.6})),$$

the three-point Gauss-Legendre rule.

12. Truncation Error for Gauss-Legendre: $\frac{f^{(34)}(c)2^{35}(17!)^4}{(34!)^3(35!)}$. Truncation error for Romberg: $\mathbf{O}\left(\left(\frac{2}{2^4}\right)^{2(4)+2}\right) = \mathbf{O}\left(\left(\frac{1}{8}\right)^{10}\right)$.

Chapter 8

Numerical Optimization

8.1 Minimization of a function

1. (a) $f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$
   On $(-\infty, 1) : f'(x) > 0$, thus f is increasing.
   On $(1, 2) : f'(x) < 0$, thus f is decreasing.
   On $(2, \infty) : f'(x) > 0$: thus f is increasing.

   (b) $f'(x) = \dfrac{1}{x^2+1} > 0$ for all x in the domain of f, thus f is increasing for all x in the domain of f.

   (c) $f'(x) = -1/x^2 < 0$ for all x in the domain of f, thus f is decreasing for all x in the domain of f.

   (d) $f'(x) = x^x(a + \ln(x))$
   On $(0, e^{-1}) : f'(x) < 0$, thus f is decreasing.
   On $(e^{-1}, \infty) : f'(x) > 0$, thus f is increasing.

2. (a) $f(x)$ is unimodal on $[0, 4]$, since $f'(x) = 2x - 2 = 0$ when $x = 1$, and $f'(x) \leq 0$ on $[0, 1]$ and $f'(x) \geq 0$ on $[1, 4]$.

   (b) $f(x)$ is unimodal on $[0, 4]$, since $f'(x) = -\sin(x) = 0$ when $x = \pi$, and $f'(x) \leq 0$ on $[0, \pi]$ and $f'(x) \geq 0$ on $[\pi, 4]$.

   (c) $f(x)$ is unimodal on $[0.1, 10]$, since $f'(x) = x^x(1 + \ln(x)) = 0$ when $x = e^{-1}$, and $f'(x) \leq 0$ on $[0.1, e^{-1}]$ and $f'(x) \geq 0$ on $[e^{-1}, 10]$.

   (d) $f(x)$ is unimodal on $[0, 3]$, since $f'(x) = (3-x)^{2/3}(\frac{8}{3}x - 3) = 0$ when $x = 9/8$, and $f'(x) \leq 0$ on $[0, 9/8]$ and $f'(x) \geq 0$ on $[9/8, 3]$.

3. (a) $f'(x) = 12x^3 - 16x - 11$; local minimum at $x = 11/6$

   (b) $f'(x) = 1 - 6/x^3$; local minimum at $x = \sqrt[3]{6}$

   (c) $f'(x) = (x^2 + 5x + 4)/((4 - x^2)^2$; local minimum at $x = -1$

   (d) $f'(x) = e^x(x-2)/x^3$; local minimum at $x = 2$

   (e) $f'(x) = -\cos(x) - \cos(3x)$; local minimum at $x = 0.785398163$

(f) $f'(x) = -2(\cos(x) - \cos(2x) + \cos(3x))$; local minimum at $x = 2.356194490$

4. Minimize the distance squared: $d(x) = (x-3)^2 + (x^2-1)^2$; $d'(x) = 2(2x^3 - x - 3)$. The minimum occurs at $x = 1.28962390$.

5. Minimize the distance squared: $d(x) = (x-2)^2 + (\sin(x)-1)^2$; $d'(x) = 2(x - 2 + \sin(x)\cos(x) - \cos(x))$. The minimum occurs at $x = 1.96954061$.

6. $x = -1.09638$, $y = -4.87831$

7. (a) $[a_0, b_0] = [-2.4000, -1.6000]$, $[a_1, b_1] = [-2.4000, -1.9056]$, $[a_2, b_2] = [-2.2111, -1.9056]$
   (b) $[a_0, b_0] = [0.8000, 1.6000]$, $[a_1, b_1] = [1.1056, 1.6000]$, $[a_2, b_2] = [1.1056, 1.4111]$
   (c) $[a_0, b_0] = [0.5000, 2.5000]$, $[a_1, b_1] = [1.2639, 2.5000]$, $[a_2, b_2] = [1.7361, 2.5000]$
   (d) $[a_0, b_0] = [1.000, 5.0000]$, $[a_1, b_1] = [2.5279, 5.0000]$, $[a_2, b_2] = [2.5279, 4.0557]$

8. (a) $[a_0, b_0] = [-2.4000, -1.6000]$, $[a_1, b_1] = [-2.4000, -1.9056]$, $[a_2, b_2] = [-2.2112, -1.9056]$
   (b) $[a_0, b_0] = [0.8000, 1.6000]$, $[a_1, b_1] = [1.1056, 1.6000]$, $[a_2, b_2] = [1.1056, 1.4112]$
   (c) $[a_0, b_0] = [0.5000, 2.5000]$, $[a_1, b_1] = [1.2640, 2.5000]$, $[a_2, b_2] = [1.7360, 2.5000]$
   (d) $[a_0, b_0] = [1.0000, 5.0000]$, $[a_1, b_1] = [2.5281, 5.0000]$, $[a_2, b_2] = [2.5281, 4.0562]$

9. (a) $p_0 = -2.4000$, $p_{min_1} = -2.1220$, $p_{min_2} = -2.1200$
   (b) $p_0 = 0.8000$, $p_{min_1} = 1.2776$, $p_{min_2} = 1.2834$
   (c) $p_0 = 0.5000$, $p_{min_1} = 1.9608$, $p_{min_2} = 1.8920$
   (d) $p_0 = 1.0000$, $p_{min_1} = 2.8750$, $p_{min_2} = 3.3095$

10. (a) $p_0 = -2.4000$, $p_1 = -2.1180$, $p_2 = -2.1200$
    (b) $p_0 = 0.8000$, $p_1 = 1.2809$, $p_2 = 1.2834$
    (c) $p_0 = 0.5000$, $p_1 = 1.8566$, $p_2 = 1.8906$
    (d) $p_0 = 1.0000$, $p_1 = 3.3333$. Note; p_1 is the abscissa of the minimum, since f is a cubic.

11. (a) $b_k - a_k = \left(\frac{-1+\sqrt{5}}{2}\right)^4 (1-0) = 0.14590$
    (b) $b_k - a_k = \left(\frac{-1+\sqrt{5}}{2}\right)^5 (-1.6 - (-2.3)) = 0.063119$

(c) $b_k - a_k = \left(\frac{-1+\sqrt{5}}{2}\right)^{10} (3.5 - (-4.6)) = 0.065858$

12. (a) $\frac{b_0 - a_0}{\epsilon} = 35,000 < 46,368 = F_{24}$

(b) $\frac{b_0 - a_0}{\epsilon} = 7,800,000 < 9,227,465 = F_{35}$

(c) $\frac{b_0 - a_0}{\epsilon} = 66,000,000 < 102,334,155 = F_{40}$

13. $1 - \frac{F_{n-k-1}}{F_{n-k}} = \frac{F_{n-k} - F_{n-k-1}}{F_{n-k}} = \frac{F_{n-2}}{F_{n-k}}$

14. **(19)**

$$\begin{aligned}
F &= f(p_1) - f(p_0) = P(p_1) - P(p_0) \\
&= \frac{\alpha(p_1 - p_2)^3}{h^3} + \frac{\beta(p_1 - p_2)^2}{h^2} - \frac{\alpha(p_0 - p_2)^3}{h^3} - \frac{\beta(p_0 - p_2)^2}{h^2} \\
&= -\frac{\alpha(h\gamma - h)^3}{h^3} + \frac{\beta(h\gamma - h)^2}{h^2} + \frac{\alpha(h\gamma)^3}{h^3} - \frac{\beta(h\gamma)^2}{h^2} \\
&= -\alpha(\gamma - 1)^3 + \beta(\gamma - 1)^2 + \alpha\gamma^3 - \beta\gamma^2 \\
&= \alpha(\gamma^3 - (\gamma^3 - 3\gamma^2 + 3\gamma - 1)) + \beta(\gamma^2 - 2\gamma + 1 - \gamma^2) \\
&= \alpha(3\gamma^2 - 3\gamma + 1) + \beta(1 - 2\gamma)
\end{aligned}$$

(20)

$$\begin{aligned}
G &= h(f'(p_1) - f'(p_0)) = h(P'(p_1) - P'(p_0)) \\
&= h\left(\frac{3\alpha(p_1 - p_2)^2}{h^3} + \frac{2\beta(p_1 - p_2)}{h^2} - \frac{3\alpha(p_0 - p_2)^2}{h^3} - \frac{2\beta(p_0 - p_2)}{h^2}\right) \\
&= h\left(\frac{3\alpha(h\gamma - h)^2}{h^3} - \frac{2\beta(h\gamma - h)}{h^2} - \frac{3\alpha(h\gamma)^2}{h^3} + \frac{2\beta(h\gamma)}{h^2}\right) \\
&= 3\alpha(\gamma - 1)^2 - 2\beta(\gamma - 1) - 3\alpha\gamma^2 + 2\beta\gamma \\
&= 3\alpha(\gamma^2 - 2\gamma + 1 - \gamma^2) + 2\beta(\gamma - \gamma + 1) \\
&= 3\alpha(1 - 2\gamma) + 2\beta
\end{aligned}$$

(21)

$$\begin{aligned}
hf'(p_0) &= hP'(p_0) \\
&= h\left(\frac{3\alpha(p_0 - p_2)^2}{h^3} + \frac{2\beta(p_0 - p_2)}{h^2}\right) \\
&= h\left(\frac{3\alpha(h\gamma)^2}{h^3} - \frac{2\beta h\gamma}{h^2}\right) \\
&= 3\alpha\gamma^2 - 2\beta\gamma
\end{aligned}$$

15. The result follows immediately by adding twice the difference of equations (21) and (19) to equation (20).

16. The sum of equation (21) and γ times equation (20) yields

$$\begin{aligned} hf'(p_0) + \gamma G &= 3\alpha\gamma^2 - 2\beta\gamma + \gamma(3\alpha(1-2\gamma) + 2\beta\gamma) \\ hf'(p_0) + \gamma G &= 3\alpha\gamma - 3\alpha\gamma^2 \\ 3\alpha\gamma^2 + (G - 3\alpha)\gamma + hf'(p_0) &= 0 \end{aligned}$$

17. (a) $[a_1, b_1] = [-2.4000, -1.9000]$, $[a_2, b_2] = [-2.2500, -1.9000]$

(b) (Principle of Mathemaical Induction) When $k = 1$:

$$b_1 - a_1 = \left(\frac{a_0 + b_0}{2} + \epsilon\right) - a_0,$$

or

$$b_1 - a_1 = b_0 - \left(\frac{a_0 + b_0}{2} - \epsilon\right).$$

In either case:

$$b_1 - a_1 = \frac{b_0 - a_0}{2} + \epsilon.$$

Thus the statement is true for $k = 1$. It follows,

$$\begin{aligned} b_{k+1} - a_{k+1} &= \frac{b_k - a_k}{2} + \epsilon \\ &= \frac{\frac{b_0 - a_0}{2^k} + 2\epsilon\left(1 - \frac{1}{2^k}\right)}{2} + \epsilon \\ &= \frac{b_0 - a_0}{2^{k+1}} + \epsilon\left(1 - \frac{1}{2^k}\right) + \epsilon \\ &= \frac{b_0 - a_0}{2^{k+1}} + 2\epsilon\left(1 - \frac{1}{2^{k+1}}\right) \end{aligned}$$

Therefore, by the principle of mathematical induction:

$$b_k - a_k = \frac{b_0 - a_0}{2^k} + 2\epsilon\left(1 - \frac{1}{2^k}\right),$$

for all $k \geq 1$.

(c) $k = 13$

18. (a) $P'(x) = 3\alpha(x - a_0)^2 + \beta(x - a_0) + \gamma$. Solving $P'(x) = 0$ yields:

$$\begin{aligned} x - a_0 &= \frac{-2\beta \pm \sqrt{4\beta^2 - 12\alpha\gamma}}{6\alpha} \\ &= \frac{-\beta \pm \sqrt{\beta^2 - 3\alpha\gamma}}{3\alpha} \end{aligned}$$

Thus

$$p_{min} = a_0 + \frac{-\beta + \sqrt{\beta^2 - 3\alpha\gamma}}{3\alpha}.$$

(b) Substituting a_0 into $P(x)$ and $P'(x)$ yields: $P(a_0) = f(a_0) = \rho$ and $P'(a_0) = f'(a_0) = \gamma$.

(c) Substituting b_0 into P and P', and letting $h = b_0 - a_0$ yields:

$$\begin{aligned} \alpha h^3 + \beta h^2 + \gamma h + \rho &= f(b_0) \\ 3\alpha h^2 + 2\beta h + \gamma &= f'(b_0) \end{aligned}$$

or

$$\begin{aligned} \alpha h^3 + \beta h^2 &= f(b_0) - f(a_0) - hf'(a_0) \\ 3\alpha h^2 + 2\beta h &= f'(b_0) - f'(a_0) \end{aligned}$$

Eliminating β $(-2E_1 + hE_2)$ yields:

$$\begin{aligned} \alpha h^3 &= -2(f(b_0) - f(a_0)) + 2hf'(a_0) + h(f'(b_0) - f'(a_0)) \\ \alpha &= \frac{-2(f(b_0) - f(a_0)) + 2hf'(a_0) + h(f'(b_0) - f'(a_0))}{h^3} \\ &= \frac{\frac{f'(b_0)-f'(a_0)}{h} - \frac{2\left(\frac{f(b_0)-f(a_0)}{h} - f'(a_0)\right)}{h}}{h} \\ &= \frac{G - 2\left(\frac{F-\gamma}{h}\right)}{h} = \frac{G-2D}{b_0 - a_0} \end{aligned}$$

Similarly, eliminating α yields:

$$\begin{aligned} \beta h^2 &= 3(f(b_0) - f(a_0) - hf'(a_0)) - h(f'(b_0) - f'(a_0)) \\ \beta &= 3\left(\frac{\frac{f(b_0)-f(a_0)}{h} - f'(a_0)}{h}\right) - \frac{f'(b_0) - f'(a_0)}{h} \\ &= 3\left(\frac{F-\gamma}{h}\right) - G = 3D - G \end{aligned}$$

(d) $[a_1, b_1] = [-2.4000, -2.1197]$, $[a_2, b_2] = [-2.1200, -2.1197]$

8.2 Nelder-Mead and Powell's Methods

1. (a) $f_x(x,y) = 3x^2 - 3$, $f_y(x,y) = 3y^2 - 3$
   Critical points: $(1,1), (1,-1), (-1,1), (-1,-1)$
   Local minimum at $(1,1)$

   (b) $f_x(x,y) = 2x - y + 1$, $f_y(x,y) = 2y - x - 2$
   Critical point: $(0,1)$. Local minimum at $(0,1)$.

   (c) $f_x(x,y) = 2xy + y^2 - 3y$, $f_y(x,y) = x^2 + 2xy - 3x$
   Critical point: $(0,0), (0,3), (3,0), (1,1)$
   Local minimum at $(1,1)$.

(d) $f_x(x,y) = \dfrac{y^2+2xy+2-x^2}{(x^2+y^2+2)^2}$,
$f_y(x,y) = \dfrac{y^2-2xy-2-x^2}{(x^2+y^2+2)^2}$
Critical point: $(1,-1),(-1,1)$. Local minimum at $(-1,1)$.

(e) $f_x(x,y) = -400x(-x^2+y)-2(1-x)$, $f_y(x,y) = 200(-x^2+y)$
Critical point: $(1,1)$. Local minimum at $(1,1)$.

2. $\mathbf{M} = \frac{1}{2}(\mathbf{B}+\mathbf{G}) = (3/2,1)$
$\mathbf{R} = 2\mathbf{M}-\mathbf{W} = (-2,0)$
$\mathbf{E} = 2\mathbf{R}-\mathbf{M} = (-11/2,-1)$

3. $\mathbf{M} = \frac{1}{2}(\mathbf{B}+\mathbf{G}) = (-3/2,-3/2)$
$\mathbf{R} = 2\mathbf{M}-\mathbf{W} = (-6,-4)$
$\mathbf{E} = 2\mathbf{R}-\mathbf{M} = (-21/2,-13/2)$

4. $\mathbf{M} = \frac{1}{3}(\mathbf{B}+\mathbf{G}+\mathbf{P}) = \frac{1}{3}(1,1,1)$
$\mathbf{R} = 2\mathbf{M}-\mathbf{W} = (-1/3,1,1)$
$\mathbf{E} = 2\mathbf{R}-\mathbf{M} = (-1,5/3,5/3)$

5. $\mathbf{M} = \frac{1}{3}(\mathbf{B}+\mathbf{G}+\mathbf{P}) = \frac{1}{3}(0,3,1)$
$\mathbf{R} = 2\mathbf{M}-\mathbf{W} = (-2,1,2/3)$
$\mathbf{E} = 2\mathbf{R}-\mathbf{M} = (-4,1,1)$

6. Let $\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{P}_0 = \mathbf{X}_0 = (1/2,1/3)$. When $i=1$ the function

$$\begin{aligned} f(\mathbf{P}_0+\gamma_1\mathbf{U}_1) &= f((1/2,1/3)+\gamma_1(1,0)) \\ &= f(1/2+\gamma_1,1/3) \\ &= \frac{109}{27} - 3\left(\frac{1}{2}+\gamma_1\right) + \left(\frac{1}{2}+\gamma_1\right)^3 \end{aligned}$$

has a minimum at $\gamma_1 = 0.5$. Thus $\mathbf{P}_1 = (1,1/3)$. When $i=2$ the function

$$\begin{aligned} f(\mathbf{P}_1+\gamma_2\mathbf{U}_2) &= f((1/1/3)+\gamma_2(0,1)) \\ &= f(1,1/3+\gamma_2) \\ &= 3 - 3\left(\frac{1}{3}+\gamma_2\right) + \left(\frac{1}{3}+\gamma_2\right)^3 \end{aligned}$$

has a minimum at $\gamma_2 = 0.666667$. Thus $\mathbf{P}_2 = (1,1)$. Set $\mathbf{U}_2' = (\mathbf{P}_2-\mathbf{P}_0)'$ and

$$\mathbf{U} = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.666667 \end{bmatrix}.$$

The function

$$\begin{aligned} f(\mathbf{P}_0+\gamma\mathbf{U}_2) &= f((1/2,1/3)+\gamma(0.5,0.666667)) \\ &= f(1/2+0.5\gamma, 1/3+0.666667\gamma) \\ &= 5-3\left(\frac{1}{2}+0.5\gamma\right)+\left(\frac{1}{2}+0.5\gamma\right)^3 \\ &\quad -3\left(\frac{1}{3}+0.666667\gamma\right)+\left(\frac{1}{3}+0.666667\gamma\right)^3 \end{aligned}$$

has a minimum at $\gamma = 1$. Thus $\mathbf{X}_1 = (1,1)$.

Set $\mathbf{P}_0 = \mathbf{X}_1$. When $i = 1$ the function

$$\begin{aligned} f(\mathbf{P}_0+\gamma_1\mathbf{U}_1) &= f((1,1)+\gamma_1(0,1)) \\ &= f(1+\gamma_1, 1) \\ &= 3-3(1+\gamma_1)+(1+\gamma_1)^3 \end{aligned}$$

has a minimum at $\gamma_1 = 0$. Thus $\mathbf{P}_1 = (1,1)$. When $i = 2$ the function

$$\begin{aligned} f(\mathbf{P}_1+\gamma_2\mathbf{U}_2) &= f((1,1)+\gamma_2(0.5,0.666667)) \\ &= f(1+0.5\gamma_2, 1+0.666667\gamma_2) \\ &= 5-3(1+0.5\gamma_2)+(1+0.5\gamma_2)^3 \\ &\quad -3(1+0.666667\gamma_2)+(1+0.666667\gamma_2)^3 \end{aligned}$$

has a minimum at $\gamma_2 = 0$. Thus $\mathbf{P}_2 = (1,1)$. Set $\mathbf{U}_2' = (\mathbf{P}_2 - \mathbf{P}_0)'$ and

$$\mathbf{U} = \begin{bmatrix} 0.5 & 0 \\ 0.666667 & 0 \end{bmatrix}.$$

The function

$$\begin{aligned} f(\mathbf{P}_0+\gamma\mathbf{U}_2) &= f((1/2,1/3)+\gamma(0,0)) \\ &= f(1,1) \\ &= 1 \end{aligned}$$

is a constant function. Thus the minimum occurs at $\mathbf{X}_2 = (1,1)$.

7. Let $\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{P}_0 = \mathbf{X}_0 = (1/2,1/3)$. When $i = 1$ the function

$$\begin{aligned} f(\mathbf{P}_0+\gamma_1\mathbf{U}_1) &= f((1/2,1/3)+\gamma_1(1,0)) \\ &= f(1/2+\gamma_1, 1/3) \\ &= -\frac{1}{2}-\gamma_1+\frac{1}{9}\left(\frac{1}{2}+\gamma_1\right)+\frac{1}{3}\left(\frac{1}{2}+\gamma_1\right) \end{aligned}$$

has a minimum at $\gamma_1 = 0.833333$. Thus $\mathbf{P}_1 = (1.33333, 0.33333)$. When $i = 2$ the function

$$\begin{aligned} f(\mathbf{P}_1 + \gamma_2 \mathbf{U}_2) &= f((1.33333, 0.33333) + \gamma_2(0,1)) \\ &= f(1.33333, 0.33333 + \gamma_2) \\ &= -2.22222(0.33333 + \gamma_2) + 1.33333(0.33333 + \gamma - 2)^2 \end{aligned}$$

has a minimum at $\gamma_2 = 0.500002$. Thus $\mathbf{P}_2 = (1.33333, 0.833335)$. Set $\mathbf{U}_2' = (\mathbf{P}_2 - \mathbf{P}_0)'$ and

$$\mathbf{U} = \begin{bmatrix} 0 & 0.833333 \\ 1 & 0.500002 \end{bmatrix}.$$

The function

$$\begin{aligned} f(\mathbf{P}_0 + \gamma \mathbf{U}_2) &= f((1/2, 1/3) + \gamma(0.833333, 0.500002)) \\ &= f(1/2 + 0.833333\gamma, 1/3 + 0.500002\gamma) \\ &= -3(0.33333 + 0.500002\gamma)(0.5 + 0.833333\gamma \\ &\quad +(0.33333 + 0.500002\gamma)^2(0.5 + 0.833333\gamma) \\ &\quad +(0.33333 + 0.500002\gamma)(0.5 + 0.83333\gamma)^2 \end{aligned}$$

has a minimum at $\gamma = 0.872467$. Thus $\mathbf{X}_1 = (1.22706, 0.769569)$.
Set $\mathbf{P}_0 = \mathbf{X}_1$. When $i = 1$ the function

$$\begin{aligned} f(\mathbf{P}_0 + \gamma_1 \mathbf{U}_1) &= f((1.22706, 0.769569) + \gamma_1(0,1)) \\ &= f(1.22706, 0.76969 + \gamma_1) \\ &= -2.1755(0.769569 + \gamma_1) + 1.22706(0.769569 + \gamma_1)^2 \end{aligned}$$

has a minimum at $\gamma_1 = 0.116901$. Thus $\mathbf{P}_1 = (1.22706, 0.88647)$. When $i = 2$ the function

$$\begin{aligned} f(\mathbf{P}_1 + \gamma_2 \mathbf{U}_2) &= f((1.22706, 0.88647) + \gamma_2(0.833333, 0.500002)) \\ &= f(1.22706 + 0.833333\gamma_2, 0.88647 + 0.833333 + 0.500002\gamma_2) \\ &= -3(0.88647 + 0.500002\gamma_2)(1.22706 + 0.833333\gamma_2) \\ &\quad +(0.88647 + 0.500002\gamma_2)^2(1.22706 + 0.833333\gamma_2) \\ &\quad (0.88646 + 0.500002\gamma_2)(1.22706 + 0.833333\gamma_2)^2 \end{aligned}$$

has a minimum at $\gamma_2 = 0.0927493$. Thus $\mathbf{P}_2 = (1.14977, 0.840095)$. Set $\mathbf{U}_2' = (\mathbf{P}_2 - \mathbf{P}_0)'$ and

$$\mathbf{U} = \begin{bmatrix} 0.833333 & -.0772911 \\ 0.500002 & 0.0705262 \end{bmatrix}.$$

The function

$$f(\mathbf{P}_0 + \gamma \mathbf{U}_2) = f((1.22706, 0.769569) + \gamma(-0.0772922, 0.0705262))$$

$$\begin{aligned}
&= f(1.22706 - 0.0772011\gamma, 0.760569 + 0.0705262\gamma) \\
&= -3(1.22706 - 0.0772911\gamma)(0.769569 + 0.0705262\gamma) \\
&\quad +(1.22706 - 0.0772911\gamma)^2(0.769569 + 0.0705262\gamma) \\
&\quad +(1.22706 - 0.0772911\gamma)(0.769569 + 0.0705262\gamma)^2
\end{aligned}$$

has a minimum at $\gamma = 3.07333$. Thus $\mathbf{X}_2 = (1.4646, 0.552819)$.

8. The midpoint of the segment joining the points $\mathbf{B}$ and $\mathbf{G}$ can represented by the vector $\mathbf{B} + \frac{1}{2}(\mathbf{G} - \mathbf{B})$. Thus

$$\mathbf{B} + \frac{1}{2}(\mathbf{G} - \mathbf{B}) = \mathbf{B} + \frac{1}{2}\mathbf{G} - \frac{1}{2}\mathbf{B} = \frac{1}{2}(\mathbf{B} - \mathbf{G})$$

9. Reflecting the triangle through the side $\mathbf{BG}$ implies that the terminal points of the vectors $\mathbf{W}$, $\mathbf{M}$, and $\mathbf{R}$ all lie on the same line segment. Thus, by the definition of scalar multiplication and vector addition, we have $\mathbf{R} - \mathbf{W} = 2(\mathbf{M} - \mathbf{W})$ or $\mathbf{R} = 2\mathbf{M} - \mathbf{W}$.

10. The expansion requires moving from R a distance equal to the length of the vector $\mathbf{R} - \mathbf{M}$ in the direction of the vector $\mathbf{R} - \mathbf{M}$. Thus $\mathbf{E} = \mathbf{R} + (\mathbf{R} - \mathbf{M}) = 2\mathbf{R} - \mathbf{M}$.

11. Given a triangle with vertices A, B, and C. Let D and E be the midpoints of the sides BC and AC, respectively, and let F be the point of intersection of segments AD and BE. It follows that $F - A = k_1(D - F)$ and $F - B = k_2(E - F)$. Thus

$$\begin{aligned}
F - A &= k_1((E - F) + (C - E) + (D - C)) \\
&= k_1\left(\frac{1}{k_2}(F - B) + \frac{1}{2}(C - A) + \frac{1}{2}(B - C)\right) \\
&= k_1\left(\frac{1}{k_2}(F - B) + \frac{1}{2}(B - A)\right) \\
&= k_1\left(\frac{1}{k_2}(F - B) + \frac{1}{2}((F - A) + (B - F))\right)
\end{aligned}$$

or

$$\begin{aligned}
\left(1 - \frac{k_1}{2}\right)(F - A) &= k_1\left(\frac{1}{k_2}(F - B) - \frac{1}{2}(F - B)\right) \\
\left(1 - \frac{k_1}{2}\right)(F - A) &= k_1\left(\frac{1}{k_2} - \frac{1}{2}\right)(F - B)
\end{aligned}$$

Since $F - A$ and $F - B$ are not parallel we must have $1 - \frac{k_1}{2} = 0$ and $k_1\left(\frac{1}{k_2} - \frac{1}{2}\right) = 0$. Therefore $k_1 = 2 = k_2$ and $F - A = \frac{2}{3}(D - A)$.

8.3 Gradient and Newton's Methods

1. (a) $\nabla f(x,y) = (2x-3, 3y^2-3)$
   $\nabla f(-1,2) = (2(-1)-3, 3(2)^2-3) = (-5,9)$

   (b) $\nabla f(x,y) = (200(y-x^2)(-2x) - 2(1-x), 200(y-x^2))$
   $\nabla f(1/2,4/3) = \left(200\left(\frac{4}{3}-\left(\frac{1}{2}\right)^2\right)\left(-2\left(\frac{1}{2}\right)\right) - 2\left(1-\frac{1}{2}\right), 200\left(\frac{4}{3}-\left(\frac{1}{2}\right)^2\right)\right) = \left(-\frac{653}{3}, \frac{650}{3}\right)$

   (c) $\nabla f(x,y,z) = (-y\sin(xy) - z\cos(xz), -x\sin(xy), -x\cos(xz))$
   $\nabla f(0,\pi,\pi/2) = (-\pi\sin(0) - \frac{\pi}{2}\cos(0), -0\sin(0), -0\cos(0)) = (-\pi/2, 0, 0)$

2. (a) When $\mathbf{P}_0 = (-1,2)$

$$\begin{aligned} S_0 &= \frac{1}{\|-\nabla f(\mathbf{P}_0)\|}(-\nabla f(\mathbf{P}_0)) \\ &= \frac{1}{\|-\nabla f(-1,2)\|}(-\nabla f(-1,2)) \\ &= \left(\frac{5}{\sqrt{106}}, -\frac{9}{\sqrt{106}}\right). \end{aligned}$$

The function

$$\begin{aligned} f(\mathbf{P}_0) + \gamma\mathbf{S}_0) &= f((-1,2) + \gamma(5/\sqrt{106}, -9/\sqrt{106})) \\ &= 5 - 3\left(2 - \frac{9\gamma}{\sqrt{106}}\right) + \left(2 - \frac{9\gamma}{\sqrt{106}}\right)^3 \\ &\quad -3\left(-1 + \frac{5\gamma}{\sqrt{106}}\right) + \left(-1 + \frac{5\gamma}{\sqrt{106}}\right)^2 \end{aligned}$$

has a minimum at $\gamma = h_{min} = 1.5998$. Thus $\mathbf{P}_1 = \mathbf{P}_0 + h_{min}\mathbf{S}_0 = (-0.223068, 0.601523)$. Similary,

$$\mathbf{S}_1 = \frac{1}{\|-\nabla f(\mathbf{P}_1)\|}(-\nabla f(\mathbf{P}_1)) = (0.874158, 0.485641).$$

The function

$$\begin{aligned} f(\mathbf{P}_1 + \gamma\mathbf{S}_1) &= f(-.223068 + 0.874158\gamma, 0.6 - 1523 + 0.485642\gamma) \\ &= 5 - 3(0.601523 + 0.485641\gamma) + (0.601523 + 0.485641\gamma)^3 \\ &\quad -3(-0.223068 + 0.874158\gamma) + (-0.223068 + 0.874158\gamma)^2 \end{aligned}$$

has a minimum at $\gamma = h_{min} = 1.38124$. Thus

$$\mathbf{P}_2 = \mathbf{P}_1 + h_{min}\mathbf{S}_1 = (0.984354, 1.27231).$$

(b) When $\mathbf{P}_0 = (1/2, 4/3)$

$$\begin{aligned} S_0 &= \frac{1}{\| - \nabla f(\mathbf{P}_0)\|}(-\nabla f(\mathbf{P}_0)) \\ &= \frac{1}{\| - \nabla f(1/2, 4/3)\|}(-\nabla f(1/2, 4/3)) \\ &= (0.708735, -0.705479). \end{aligned}$$

The function

$$\begin{aligned} f(\mathbf{P}_0) + \gamma \mathbf{S}_0) &= f((0.5, 1.33333) + \gamma(0.708735, -0.705479)) \\ &= (0.5 - 0.708735\gamma)^2 \\ &\quad +100(1.33333 - (0.5 + 0.708735\gamma)^2 - 0.705479\gamma)^2 \end{aligned}$$

has a minimum at $\gamma = h_{min} = 0.626677$. Thus $\mathbf{P}_1 = \mathbf{P}_0 + h_{min}\mathbf{S}_0 =$ $(0.944148, 0.891226)$. Similary,

$$\mathbf{S}_1 = \frac{1}{\| - \nabla f(\mathbf{P}_1)\|}(-\nabla f(\mathbf{P}_1)) = (0.727355, 0.686261).$$

The function

$$\begin{aligned} f(\mathbf{P}_1 + \gamma \mathbf{S}_1) &= f(0.944148 + 0.727353\gamma, 0.891226 + 0.686261\gamma) \\ &= (0.055852 - 0.727355\gamma)^2 \\ &\quad +100(0.891226 - (0.944148 + 0.727355\gamma)^2 + 0.686261\gamma)^2 \end{aligned}$$

has a minimum at $\gamma = h_{min} = 0.00057708$. Thus

$$\mathbf{P}_2 = \mathbf{P}_1 + h_{min}\mathbf{S}_1 = (0.944568, 0.891622).$$

(c) When $\mathbf{P}_0 = (0, \pi, \pi/2)$

$$\begin{aligned} S_0 &= \frac{1}{\| - \nabla f(\mathbf{P}_0)\|}(-\nabla f(\mathbf{P}_0)) \\ &= \frac{1}{\| - \nabla f(0, \pi, \pi/2)\|}(-\nabla f(0, \pi, \pi/2)) \\ &= (1, 0, 0). \end{aligned}$$

The function

$$\begin{aligned} f(\mathbf{P}_0) + \gamma \mathbf{S}_0) &= f((0, \pi, \pi/2) + \gamma(1, 0, 0)) \\ &= \cos(\pi\gamma) - \sin\left(\frac{\pi\gamma}{2}\right) \end{aligned}$$

has a minimum at $\gamma = h_{min} = -1.0$. Thus $\mathbf{P}_1 = \mathbf{P}_0 + h_{min}\mathbf{S}_0 =$ $(-1, \pi, \pi/2)$. The process ends at this point since $\nabla f(-1, \pi, \pi/2) =$ $\mathbf{0}$. Specifically, a critical value of the function f has been located–at which there may be a local extremum.

3. (a) $\begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}$

   (b) $\begin{bmatrix} -\frac{694}{3} & -200 \\ -200 & 200 \end{bmatrix}$

   (c) $\begin{bmatrix} -\pi^2 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

4. (a)

$$\begin{aligned} Q(\mathbf{X}) &= f(\mathbf{P}_0) + \nabla f(\mathbf{P}_0)\cdot(\mathbf{X}-\mathbf{P}_0) + \frac{1}{2}(\mathbf{X}-\mathbf{P}_0)\mathbf{H}f(\mathbf{P}_0)(\mathbf{X}-\mathbf{P}_0)' \\ &= 11 + \begin{bmatrix} -5 & 9 \end{bmatrix}\cdot\begin{bmatrix} x+1 & y-2 \end{bmatrix} \\ &\quad + \frac{1}{2}\begin{bmatrix} x+1 & y-2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}\begin{bmatrix} x+1 \\ y-2 \end{bmatrix} \\ &= 11 - 5(x+1) + 9(y-2) + (x+1)^2 + 6(y-2)^2 \end{aligned}$$

   (b)

$$\begin{aligned} Q(\mathbf{X}) &= f(\mathbf{P}_0) + \nabla f(\mathbf{P}_0)\cdot(\mathbf{X}-\mathbf{P}_0) + \frac{1}{2}(\mathbf{X}-\mathbf{P}_0)\mathbf{H}f(\mathbf{P}_0)(\mathbf{X}-\mathbf{P}_0)' \\ &= \frac{2117}{18} + \begin{bmatrix} \frac{-653}{3} & \frac{650}{3} \end{bmatrix}\cdot\begin{bmatrix} x-\frac{1}{2} & y-\frac{4}{3} \end{bmatrix} \\ &\quad + \frac{1}{2}\begin{bmatrix} x-\frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{694}{3} & -200 \\ -200 & 200 \end{bmatrix}\begin{bmatrix} x-\frac{1}{2} \\ y-\frac{4}{3} \end{bmatrix} \\ &= -\frac{563}{12} + \frac{493}{3}x + 50y - 200xy + 100y^2 - \frac{347}{3}x^2 \end{aligned}$$

   (c)

$$\begin{aligned} Q(\mathbf{X}) &= f(\mathbf{P}_0) + \nabla f(\mathbf{P}_0)\cdot(\mathbf{X}-\mathbf{P}_0) + \frac{1}{2}(\mathbf{X}-\mathbf{P}_0)\mathbf{H}f(\mathbf{P}_0)(\mathbf{X}-\mathbf{P}_0)' \\ &= 1 + \begin{bmatrix} -\frac{\pi}{2} & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} x & y-\pi & z-\frac{\pi}{2} \end{bmatrix} \\ &\quad + \frac{1}{2}\begin{bmatrix} x & y-\pi & z-\frac{\pi}{2} \end{bmatrix}\begin{bmatrix} -\pi^2 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-\pi \\ z-\frac{\pi}{2} \end{bmatrix} \\ &= 1 - xz - \frac{\pi^2}{2}x^2 \end{aligned}$$

5. (a) If $\mathbf{P}_0 = (-1, 2)$, then

$$\mathbf{P}_1 = \mathbf{P}_0 - \nabla f(\mathbf{P}_0)((\mathbf{H}f(\mathbf{P}_0))^{-1})') = \left(\frac{3}{2}, \frac{5}{4}\right)$$

$$\mathbf{P}_2 = \mathbf{P}_1 - \nabla f(\mathbf{P}_1)((\mathbf{H}f(\mathbf{P}_1))^{-1})') = \left(\frac{3}{2}, \frac{41}{40}\right)$$

(b) If $\mathbf{P}_0 = (0.5, 1.33333)$, then

$$\mathbf{P}_1 = \mathbf{P}_0 - \nabla f(\mathbf{P}_0)((\mathbf{H}f(\mathbf{P}_0))^{-1})') = (0.498424, 0.248424)$$
$$\mathbf{P}_2 = \mathbf{P}_1 - \nabla f(\mathbf{P}_1)((\mathbf{H}f(\mathbf{P}_1))^{-1})') = (0.493401, 0.24342)$$

(c) The matrix $\mathbf{H}f(\mathbf{P}_0)$ is not invertible.

6. (a) When $\mathbf{P}_0 = (-1, 2)$

$$\mathbf{S}_0 = -\nabla f(\mathbf{P}_0)((\mathbf{H}f(\mathbf{P}_0))^{-1})') = (2.5, -0.75).$$

The function

$$\begin{aligned} f(\mathbf{P}_0 + \gamma\mathbf{S}_0) &= 5 - 3(2 - 0.75\gamma) + (2 - 0.75\gamma)^3 \\ &\quad -3(-1 + 2.5\gamma) + (-1 + 2.5\gamma)^2 \end{aligned}$$

has a minimum at $\gamma = h_{min} = 1.07614$. Thus

$$\mathbf{P}_1 = \mathbf{P}_0 + h_{min}\mathbf{S}_0 = (1.69033, 1.1929)$$

(b) When $\mathbf{P}_0 = (0.5, 1.33333)$

$$\mathbf{S}_0 = -\nabla f(\mathbf{P}_0)((\mathbf{H}f(\mathbf{P}_0))^{-1})') = (-0.0023184, -1.08563).$$

The function

$$\begin{aligned} f(\mathbf{P}_0 + \gamma\mathbf{S}_0) &= 100(1.33333 - (0.5 - 0.0023184\gamma)^2 - 1.08565\gamma)^2 \\ &\quad +(0.5 + 0.0023184\gamma)^2 \end{aligned}$$

has a minimum at $\gamma = h_{min} = 0.999984$. Thus

$$\mathbf{P}_1 = \mathbf{P}_0 + h_{min}\mathbf{S}_0 = (0.5, 1.33324)$$

(c) The matrix $\mathbf{H}f(\mathbf{P}_0)$ is not invertible.

7. By definition $\nabla f(x, y) = [f_x(x, y)\ f_y(x, y)]$. It follows from Definition 3.8 of Section 3.3 that

$$\begin{aligned} \mathbf{J}\nabla f(x, y) &= \begin{bmatrix} \frac{\partial f_x(x,y)}{\partial x} & \frac{\partial f_x(x,y)}{\partial y} \\ \frac{\partial f_y(x,y)}{\partial x} & \frac{\partial f_y(x,y)}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix} = \mathbf{H}f(x, y) \end{aligned}$$

8. Show left-hand sides of formulas (6) and (7) are equal. Substituting $\mathbf{X} = (x, y)$ and $\mathbf{P}_0 = (a, b)$ into formula (5) yields

$$\begin{aligned} Q(x, y) &= f(a, b) + \begin{bmatrix} f_x(a, b) & f_y(a, b) \end{bmatrix} \cdot \begin{bmatrix} x - a & y - b \end{bmatrix} \\ &\quad + \frac{1}{2} \begin{bmatrix} x - a & y - b \end{bmatrix} \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} \\ &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + f_{xy}(a, b)(x - a)(y - b) \\ &\quad + \frac{1}{2} f_{xx}(a, b)(x - a)^2 + \frac{1}{2} f_{yy}(a, b)(y - b)^2. \end{aligned}$$

Taking partial derivatives with respect to x and y:

$$\begin{aligned}
Q_x(x,y) &= f_x(a,b) + f_{xy}(a,b)(y-b) + f_{xx}(a,b)(x-a) \\
Q_y(x,y) &= f_y(a,b) + f_{yx}(a,b)(x-a) + f_{yy}(a,b)(y-b)
\end{aligned}$$

Thus

$$\begin{aligned}
\nabla Q(x,y) &= \begin{bmatrix} Q_x(x,y) & Q_y(x,y) \end{bmatrix} \\
&= \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} + \begin{bmatrix} x-a & y-b \end{bmatrix} \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}' \\
&= \nabla f(a,b) + (\begin{bmatrix} x & y \end{bmatrix} - \begin{bmatrix} a & b \end{bmatrix})(\mathbf{H}f(a,b))'.
\end{aligned}$$

9. Solve formula (7)

$$\nabla f(\mathbf{P}_0) + (\mathbf{X} - \mathbf{P}_0)(\mathbf{H}f(\mathbf{P}_0))' = \mathbf{0}$$

for $\mathbf{X}$:

$$(\mathbf{X} - \mathbf{P}_0)(\mathbf{H}f(\mathbf{P}_0))' = -\nabla f(\mathbf{P}_0),$$

assume $(\mathbf{H}f(\mathbf{P}_0))'$ is invertible,

$$\begin{aligned}
\mathbf{X} - \mathbf{P}_0 &= -\nabla f(\mathbf{P}_0((\mathbf{H}f(\mathbf{P}_0))')^{-1} \\
\mathbf{X} &= \mathbf{P}_0 - \nabla f(\mathbf{P}_0((\mathbf{H}f(\mathbf{P}_0))^{-1})'.
\end{aligned}$$

Note: If a matrix A is invertible, then $(A')^{-1} = (A^{-1})'$.

Chapter 9

Solution of Differential Equations

9.1 Introduction to Differential Equations

1. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = -Ce^{-t} + 2t - 2$. Substituting $y(t)$ into the right-side of the differential equation yields

$$y' = t^2 - (Ce^{-t} + t2 - 2t + 2) = -Ce^{-t} + 2t - 2$$

   (b) $L = 1$

2. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = 3Ce^{3t} - 1$. Substituting $y(t)$ into the right-side of the differential equation yields

$$y' = 3(Ce^{3t} - t - \frac{1}{3}) + 3t = 3Ce^{3t} - 1$$

   (b) $L = 3$

3. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = -Cte^{-t^2/2}$. Substituting $y(t)$ into the right-side of the differential equation yields

$$y' = -t(Ce^{-t^2/2}) = -Cte^{-t^2/2}$$

   (b) $L = 3$

4. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = -2Ce^{-2t} + e^{-2t} - 2te^{-2t}$. Substituting $y(t)$ into the right-side of the differential equation yields

$$\begin{aligned} y' &= e^{-2t} - 2(Ce^{-2t} + te^{-2t}) \\ &= 2Ce^{-2t} + e^{-2t} - 2te^{-2t} \end{aligned}$$

(b) $L = 2$

5. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = 2t/(C - t^2)$. Substituting $y(t)$ into the right-side of the differential equation yields

$$y' = 2t\left(\frac{1}{C - t^2}\right)^2 = \frac{2t}{C - t^2}$$

(b) $L = 60$

6. Use the MATLAB M-file given prior to the beginning of the exercises. The differential equation and given general solution should be used to avoid points at which y' is undefined.

7. Use the MATLAB M-file given prior to the beginning of the exercises. The differential equation and given general solution should be used to avoid points at which y' is undefined.

8. Use the MATLAB M-file given prior to the beginning of the exercises. The differential equation and given general solution should be used to avoid points at which y' is undefined.

9. Use the MATLAB M-file given prior to the beginning of the exercises. The differential equation and given general solution should be used to avoid points at which y' is undefined.

10. (a) Differentiating $y(t) = 0$ with respect to t yields $y'(t) = 0$. Substituting $y(t) = 0$ into the right-side of the differential equation yields $y' = \frac{3}{2}(0) = 0$.

    (b) Differentiating $y(t) = t^{3/2}$ with respect to t yields $y'(t) = \frac{3}{2}t^{1/2}$. Substituting $y(t)$ into the right-side of the differential equation yields $y' = \frac{3}{2}(t^{3/2})^{1/3} = \frac{3}{2}t^{1/2}$.

    (c) No, because $f_y(t, y) = \frac{1}{2}y^{-2/3}$ is not continuous when $t = 0$, and $\lim_{y \to 0} f_y(t, y) = \infty$.

11. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = \cos(t)$. Substituting $y(t)$ into the right-side of the differential equation yields $y' = (1 - \sin^2(t))^{1/2} = (\cos^2(t))^{1/2} = \cos(t)$, since $\cos(t) \geq 0$ on $[0, \pi/4]$..

    (b) $-\pi/2 \leq t \leq \pi/2$

12. Given $y' = f(t)$ and $y(a) = 0$. Let $F(t)$ be an antiderivative of $f(t)$. Thus $y = \int y' dt = \int f(t)dt$ or $y = F(t) + C$. Making the substitution $y(a) = 0$ yields $C = -F(a)$. Thus $y = F(t) - F(a)$ and by the Fundamental Theorem of Calculus

$$\begin{aligned}\int_a^b f(t)dt &= (F(t) - F(a))_a^b \\ &= (F(b) - F(a)) - (F(a) - F(a)) = F(b) - F(a) = y\end{aligned}$$

13. $y(t) = t^3 - \cos(t) + 3$

14. $y(t) = \arctan(t)$

15. $y(t) = \int_0^t e^{-s^2/2}\,ds$

16. Differentiate the first integral: $F'(t) = p(t)F(t)$. By the product rule

$$\begin{aligned}(F(t)y(t))' &= F'(t)y(t) + F(t)y'(t) \\ &= p(t)F(t)y(t) + F(t)y'(t)\end{aligned}$$

Multiply bogth sides of the second integral by $F(t)$ and differentiate: $(F(t)y(t))' = F(t)q(t)$. Equate the righthand sides of the equations for $(F(t)y(t))'$ and divide both sides by $F(t)$ to obtain $y'(t) + p(t)y(t) = q(t)$.

17. (a) Differentiating $y(t)$ with respect to t yields $y'(t) = -ky_0e^{-kt}$. Substituting $y(t)$ into the differential equation yields $y' = -k(y_0e^{-kt}) = -ky_0e^{-kt}$.
    (b) $y(t) = y_0e^{-0.000120968t}$
    (c) 2808 years
    (d) 6.9237 seconds

18. Solve the initial value problem: $y' = k(300 - y)$, $y(0) = 0$, and $y(3) = 40$. Using the technique of separable variables:

$$\int \frac{y'}{300 - y}dt = \int k\,dt$$

$$-ln(300 - y) = kt + C$$

Substituting $y(0) = 0$ yields

$$-\ln(300 - y) = kt - \ln(300)$$

$$y = 300 - 300e^{-kt}$$

Substituting $y(3) = 40$ yields

$$y = 300 - 300e^{-\frac{\ln(15/13)t}{3}}$$

Setting $y = 220$ and solving for t yields $t = -\frac{3\ln(4/13)}{\ln(15/13)} \approx 27.71$ years.

9.2 Euler's Method

1. **(a) and (b)**

t_k	$y_k\ (h=0.1)$	$y_k\ (h=0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.90000		0.90516258
0.2	0.81000	0.80000	0.82126925
0.3	0.73390		0.74918180
0.4	0.66951	0.64800	0.68967959

(c) $E(y(0.4), 0.1) \approx 0.02017$ and $E(y(0.4), 0.2) \approx 0.04167$. As expected $E(y(0.4), 0.1) \approx \frac{1}{2}E(y(0.4), 0.2)$.

2. **(a) and (b)**

t_k	$y_k\ (h=0.1)$	$y_k\ (h=0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.3000		1.36647810
0.2	1.7200	1.6000	1.89615801
0.3	2.2960		2.64613748
0.4	3.0748	2.6800	3.69348923

(c) $E(y(0.4), 0.1) \approx 0.618689$ and $E(y(0.4), 0.2) \approx 1.013489$. As expected $E(y(0.4), 0.1) \approx \frac{1}{2}E(y(0.4), 0.2)$.

3. **(a) and (b)**

t_k	$y_k\ (h=0.1)$	$y_k\ (h=0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.00000		0.99501248
0.2	0.99000	1.00000	0.98019867
0.3	0.97020		0.95599748
0.4	0.94109	0.96000	0.92311635

(c) $E(y(0.4), 0.1) \approx 0.01797$ and $E(y(0.4), 0.2) \approx 0.03688$. As expected $E(y(0.4), 0.1) \approx \frac{1}{2}E(y(0.4), 0.2)$.

4. **(a) and (b)**

t_k	$y_k\ (h=0.1)$	$y_k\ (h=0.2)$	$y(t_k)$
0.0	0.1	0.1	0.1
0.1	0.1800		0.16374615
0.2	0.2259	0.2600	0.20109601
0.3	0.2477		0.21952465
0.4	0.2531	0.2901	0.22466448

(c) $E(y(0.4), 0.1) \approx 0.02844$ and $E(y(0.4), 0.2) \approx 0.06543$. As expected $E(y(0.4), 0.1) \approx \frac{1}{2}E(y(0.4), 0.2)$.

5. **(a) and (b)**

t_k	$y_k\ (h = 0.1)$	$y_k\ (h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.0000		1.01010101
0.2	1.0200	1.00000	1.04166667
0.3	1.0616		1.09890110
0.4	1.1292	1.0800	1.19047615

(c) $E(y(0.4), 0.1) \approx 0.06128$ and $E(y(0.4), 0.2) \approx 0.11048$. As expected $E(y(0.4), 0.1) \approx \frac{1}{2}E(y(0.4), 0.2)$.

6. Euler's approximation to $P(t)$:

$$P_0 = 76.1,\ P_{k+1} = P_k + 10(0.02P_k - 0,00004P_k^2)$$

$P(20.0) = 103.6$, $P(30.0) = 120.0$, $P(50.0) = 158.2$, $P(60.0) = 179.8$, $P(80.0) = 226.3$

7.

$$\begin{aligned}
y_0 &= 0 \\
y_1 &= 0 + hf(t_0) \\
y_2 &= y_1 + hf(t_1) \\
&= hf(t_0) + hf(t_1) \\
y_3 &= y_2 + hf(t_2) \\
&= hf(t_0) + hf(t_1) + hf(t_2) \\
&\vdots \\
y_{m-1} &= y_{m-2} + hf(t_{m-2}) \\
&= \sum_{k=0}^{m-2} hf(t_k) \\
y_m &= y_{m-1} + hf(t_{m-1}) \\
&= \sum_{k=0}^{m-1} f(t_k)
\end{aligned}$$

The result follows, since $y(b) \approx y_m$.

8. The given I.V.P. does not have a unique soloution. Euler's method for the given initial condition approaches teh solution $y(t) = 0$.

9. No. Note: the solution $y(t) = \tan(t)$ is not continuous on $[0, 3]$.

9.3 Heun's Method

1. **(a) and (b)**

t_k	y_k $(h=0.1)$	y_k $(h=0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.90550		0.90516258
0.2	0.82193	0.82400	0.82126925
0.3	0.75014		0.74918180
0.4	0.69092	0.69488	0.68967959

(c) $E(y(0.4), 0.1) \approx 0.020170$ and $E(y(0.4), 0.2) \approx 0.005200$. As expected $E(y(0.4), 0.1) \approx \frac{1}{4}E(y(0.4), 0.2)$.

2. **(a) and (b)**

t_k	y_k $(h=0.1)$	y_k $(h=0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.3600		1.36647810
0.2	1.8787	1.8400	1.89615801
0.3	2.6109		2.64613748
0.4	3.6301	3.4912	3.69348923

(c) $E(y(0.4), 0.1) \approx 0.07339$ and $E(y(0.4), 0.2) \approx 0.20229$. As expected $E(y(0.4), 0.1) \approx \frac{1}{4}E(y(0.4), 0.2)$.

3. **(a) and (b)**

t_k	y_k $(h=0.1)$	y_k $(h=0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.9950		0.99501248
0.2	0.98107	0.98000	0.98019867
0.3	0.95596		0.95599748
0.4	0.92308	0.92277	0.92311635

(c) $E(y(0.4), 0.1) \approx 0.00003635$ and $E(y(0.4), 0.2) \approx 0.00034635$. As expected $E(y(0.4), 0.1) \approx \frac{1}{4}E(y(0.4), 0.2)$.

4. **(a) and (b)**

t_k	y_k $(h=0.1)$	y_k $(h=0.2)$	$y(t_k)$
0.0	0.1	0.1	0.1
0.1	0.1629		0.16374615
0.2	0.1999	0.1950	0.20109601
0.3	0.2181		0.21952465
0.4	0.2233	0.2178	0.22466448

(c) $E(y(0.4), 0.1) \approx 0.0013645$ and $E(y(0.4), 0.2) \approx 0.0068645$. As expected $E(y(0.4), 0.1) \approx \frac{1}{4}E(y(0.4), 0.2)$.

5. **(a) and (b)**

t_k	$y_k\ (h = 0.1)$	$y_k\ (h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.0100		1.01010101
0.2	1.0414	1.0400	1.04166667
0.3	1.0984		1.09890110
0.4	1.1895	1.1848	1.19047615

(c) $E(y(0.4), 0.1) \approx 0.00097615$ and $E(y(0.4), 0.2) \approx 0.0056762$. As expected $E(y(0.4), 0.1) \approx \frac{1}{4}E(y(0.4), 0.2)$.

6.

$$
\begin{aligned}
y_0 &= 0 \\
y_1 &= \frac{h}{2}(f(t_0) + f(t_1)) \\
y_2 &= y_1 + \frac{h}{2}(f(t_1) + f(t_2)) \\
&= \frac{h}{2}(f(t_0) + 2f(t_1) + f(t_2)) \\
y_3 &= y_2 + \frac{h}{2}(f(t_2) + f(t_3)) \\
&= \frac{h}{2}(f(t_0) + 2f(t_1) + 2f(t_2) + f(t_3)) \\
&\vdots \\
y_{m-2} &= y_{m-3} + \frac{h}{2}\left(f(t_0) + 2\sum_{k=1}^{m-3} f(t_k) + f(t_{m-2})\right) \\
y_{m-1} &= y_{m-2} + \frac{h}{2}(f(t_{m-2}) + f(t_{m-1})) \\
&= \frac{h}{2}\left(f(t_0) + 2\sum_{k=1}^{m-2} f(t_k) + f(t_{m-2})\right) \\
&= \frac{h}{2}\sum_{k=0}^{m-1}(f(t_k) + f(t_{k+1}))
\end{aligned}
$$

The results follows since $y(b) \approx y_m$.

7. Richardson improvement for solving $y' = (t - y)/2$ with $y(0) = 1$. The table entries are approximations to $y(3)$.

h	y_k	$(4y_h - y_{2h})/3$
1	1.732422	
1/2	1.682121	1.665354
1/4	1.672269	1.668985
1/8	1.670076	1.669345
1/16	1.669558	1.669385
1/32	1.669432	1.669390
1/64	1.669401	1.669391

8. $y' = f(t, y) = 1.5y^{1/3}$, $f_y(t, y) = 0.5y^{-2/3}$. The partial derivative $f_y(0, 0)$ does not exist. The I.V.P. is not well-posed on any rectangle that contains $(0, 0)$.

9.4 Taylor Series Method

1. **(a) and (b)**

t_k	$y_k\,(h = 0.1)$	$y_k\,(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.90516		0.90516258
0.2	0.82127	0.82127	0.82126925
0.3	0.74918		0.74918180
0.4	0.68968	0.68968	0.68967959

(c) $E(y(0.4), 0.1) \approx 0.0000004$ and $E(y(0.4), 0.2) \approx 0.0000004$. Unexpectedly $E(y(0.4), 0.1) \not\approx \frac{1}{16}E(y(0.4), 0.2)$, why?

2. **(a) and (b)**

t_k	$y_k\,(h = 0.1)$	$y_k\,(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.3664		1.36647810
0.2	1.8961	1.8952	1.89615801
0.3	2.6460		2.64613748
0.4	3.6932	3.6900	3.69348923

(c) $E(y(0.4), 0.1) \approx 0.00028923$ and $E(y(0.4), 0.2) \approx 0..00348923$. As expected $E(y(0.4), 0.1) \approx \frac{1}{16}E(y(0.4), 0.2)$.

3. **(a) and (b)**

t_k	y_k $(h = 0.1)$	y_k $(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.99501		0.99501248
0.2	0.98020	0.98000	0.98019867
0.3	0.96000		0.95599748
0.4	0.92312	0.92313	0.92311635

(c) $E(y(0.4), 0.1) \approx 0.00000365$ and $E(y(0.4), 0.2) \approx 0.00001365$. Unexpectedly, $E(y(0.4), 0.1) \not\approx \frac{1}{16}y(0.4), 0.2)$.

4. **(a) and (b)**

t_k	y_k $(h = 0.1)$	y_k $(h = 0.2)$	$y(t_k)$
0.0	0.1	0.1	0.1
0.1	0.1637		0.16374615
0.2	0.2011	0.2009	0.20109601
0.3	0.2195		0.21952465
0.4	0.2247	0.2244	0.22466448

(c) $E(y(0.4), 0.1) \approx 0.00003552$ and $E(y(0.4), 0.2) \approx 0.00026448$. Unexpectedly $E(y(0.4), 0.1) \not\approx \frac{1}{16}y(0.4), 0.2)$.

5. **(a) and (b)**

t_k	y_k $(h = 0.1)$	y_k $(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.0101		1.0101
0.2	1.0417	1.0416	1.0417
0.3	1.0989		1.0989
0.4	1.1904	1.1896	1.1905

(c) $E(y(0.4), 0.1) \approx 0.0001$ and $E(y(0.4), 0.2) \approx 0.0009$. Unexpectedly $E(y(0.4), 0.1) \not\approx \frac{1}{16}y(0.4), 0.2)$.

6. Richardson improvement for the Taylor solution $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$. The table entries are approximations to $y(3)$.

h	y_k	$(16y_h - y_{2h})/15$
1	1.6701860	
1/2	1.6694308	1.6693805
1/4	1.6693928	1.6693903
1/8	1.6693906	1.6693905

7. $E(y(h), h/2) = \mathbf{O}((h/2)^N) \approx C(h/2)^N = \frac{1}{2^N}Ch^N \approx \frac{1}{2^N}\mathbf{O}(h^N) = \frac{1}{2^N}E(y(h), h)$

8.

$$\begin{aligned}
y' &= \frac{3}{2}y^{1/3} \\
y'' &= \frac{1}{2}y^{-2/3}y' = \frac{3}{4}y^{-1/3} \\
y^{(3)} &= -\frac{1}{4}y^{-4/3}y' = -\frac{3}{8}y^{-1} \\
y^{(4)} &= -\frac{3}{8}y^{-2}y' = \frac{9}{16}y^{-5/3}
\end{aligned}$$

Evaluating y'', $y^{(3)}$, and $y^{(4)}$ at $y(0) = 0$ results in division by zero.

9. (a) If $y = 1/(1-t)$, then $y' = 1/(1-t)^2$. Thus $y' = y^2$.

   (b) First verify $y(t)$ satisfies the initial condition: $y(0) = \tan(0 + \pi/4) = 1$. Substitute $y(t) = \tan(t + \pi/4)$ and $y'(t) = \sec^2(t + \pi/4)$ into the differential equation:

   $$\sec^2(t + \pi/4) = 1 + \tan^2(t + \pi/4).$$

   Therefore, $y(t)$ is a solution to the given initial value problem.

   (c) Over the interval $[0, 1]$ the graph of the solution of $y' = y^2 + t^2$ lies between the graphs of $y = \tan(t + \pi/4)$ and $y = \frac{1}{1-t}$. Thus the solution to the initial value problem has a vertical asymptote between $t = \pi/4$ and $t = 1$.

10. (a)

$$\begin{aligned}
y' &= 1 + y^2 \\
y'' &= 2yy' = 2y + 2y^3 \\
y''' &= 2y' + 6y^2y' \\
&= 2 + 2y^2 + 6y^2 + 6y^4 = 2 + 8y^2 + 6y^4 \\
y^{(4)} &= 16yy' + 24y^3y' \\
&= 16y + 16y^3 + 24y^3 + 24y^5 = 16y + 40y^3 + 24y^5
\end{aligned}$$

   (b) $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 2$, $y^{(4)}(0) = 0$ The Maclaurin expansion for $n = 4$ is

$$\begin{aligned}
\sum_{k=0}^{4} \frac{\tan^{(k)}(0)x^k}{k!} &= 0 + (1)x + \frac{(0)x^2}{2} + \frac{2x^3}{6} + \frac{(0)x^4}{24} \\
&= x + \frac{1}{3}x^3
\end{aligned}$$

9.5 Runge-Kutta Methods

1. **(a) and (b)**

t_k	$y_k\,(h = 0.1)$	$y_k\,(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.90516		0.90516258
0.2	0.82127	0.82127	0.82126925
0.3	0.74918		0.74918180
0.4	0.68968	0.68969	0.68967959

(c) $E(y(0.4), 0.1) \approx 0..0000004$ and $E(y(0.4), 0.2) \approx 0.00001041$. As expected $E(y(0.4), 0.1) \approx \frac{1}{16}E(y(0.4), 0.2)$.

2. **(a) and (b)**

t_k	$y_k\,(h = 0.1)$	$y_k\,(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.3664		1.36647810
0.2	1.8961	1.8952	1.89615801
0.3	2.6460		2.64613748
0.4	3.6932	3.6900	3.69348923

(c) $E(y(0.4), 0.1) \approx 0.00028923$ and $E(y(0.4), 0.2) \approx 0.00348923$. As expected $E(y(0.4), 0.1) \approx \frac{1}{16}E(y(0.4), 0.2)$.

3. **(a) and (b)**

t_k	$y_k\,(h = 0.1)$	$y_k\,(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	0.99501		0.99501248
0.2	0.98020	0.98020	0.98019867
0.3	0.95600		0.95599748
0.4	0.92312	0.92312	0.92311635

(c) $E(y(0.4), 0.1) \approx 0.00000365$ and $E(y(0.4), 0.2) \approx 0.00000365$. Unexpectedly, $E(y(0.4), 0.1) \not\approx \frac{1}{16}E(y(0.4), 0.2)$.

4. **(a) and (b)**

t_k	$y_k\,(h = 0.1)$	$y_k\,(h = 0.2)$	$y(t_k)$
0.0	0.1	0.1	0.1
0.1	0.1637		0.16374615
0.2	0.2011	0.2010	0.20109601
0.3	0.2195		0.21952465
0.4	0.2247	0.2246	0.22466448

(c) $E(y(0.4), 0.1) \approx 0.00003552$ and $E(y(0.4), 0.2) \approx 0.00006448$. Unexpectedly, $E(y(0.4), 0.1) \not\approx \frac{1}{16}E(y(0.4), 0.2)$.

5. **(a) and (b)**

t_k	y_k $(h = 0.1)$	y_k $(h = 0.2)$	$y(t_k)$
0.0	1	1	1
0.1	1.0101		1.01010101
0.2	1.0417	1.0417	1.04166667
0.3	1.0989		1.09890110
0.4	1.1905	1.1905	1.19047615

(c) $E(y(0.4), 0.1) \approx 0.00002385$ and $E(y(0.4), 0.2) \approx 0.00002385$. Unexpectedly, $E(y(0.4), 0.1) \not\approx \frac{1}{16}E(y(0.4), 0.2)$.

6.

$$
\begin{aligned}
y_0 &= 0 \\
y_1 &= y_0 + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4) \\
&= \frac{h}{6}(f(t_0) + 2f(t_0 + \frac{h}{2}) + 2f(t_0 + \frac{h}{2}) + f(t_0 + h)) \\
&= \frac{h}{6}(f(t_0) + 4f(t_0 + \frac{h}{2}) + f(t_0 + h)) \\
y_2 &= y_1 + \frac{h}{6}(f(t_1) + 4f(t_1 + \frac{h}{2}) + f(t_1 + h)) \\
&= \frac{h}{6}(f(t_0) + 4f(t_0 + \frac{h}{2}) + 4f(t_1 + \frac{h}{2}) + 2f(t_1) + f(t_2)) \\
&\vdots \\
y_{m-2} &= \frac{h}{6}\left(f(t_0) + 4\sum_{k=1}^{m-2} f(t_k + \frac{h}{2}) + 2\sum_{k=1}^{m-3} f(t_k) + f(t_{m-3})\right) \\
y_{m-1} &= \frac{h}{6}\left(f(t_0) + 4\sum_{k=1}^{m-1} f(t_k + \frac{h}{2}) + 2\sum_{k=1}^{m-2} f(t_k) + f(t_{m-2})\right) \\
&= \frac{h}{6}\sum_{k=0}^{m-1}(f(t_k) + 4f(t_{k+1/2}) + f(t_{k+1}))
\end{aligned}
$$

The result follows since $y(b) \approx y_{m-1}$.

7. $y = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$.

h	y_h	$(16y_h - y_{2h})/15$
1	1.6701860	
1/2	1.6694308	1.6693805
1/4	1.6693928	1.6693903
1/8	1.6693906	1.6693906

8. (a) $P_2(t,y) = 1 - (t-1) + (y-1) + (t-1)^2 - (t-1)(y-1)$
   (b) $P_2(1.05, 1.1) = 1-(1.05-1)+(1.1-1)+(1.05-1)^2-(1.05-1)(1.1-1) = 1.0475$; $f(1.05, 1.1) = 1.0476$

9. (a) $P_2(t,y) = 1 + \frac{1}{2}t - \frac{1}{2}y - \frac{1}{8}t^2 + \frac{1}{4}ty - \frac{1}{8}y^2$
   (b)

$$\begin{aligned} P_2(0.02, 0.08) &= 1 + \frac{1}{2}(0.04) - \frac{1}{2}(0.08) - \frac{1}{8}(0.04)^2 \\ &\quad + \frac{1}{4}(0.04)(0.08) - \frac{1}{8}(0.08)^2 \\ &= 0.9798 \\ f(0.04, 0.08) &= 0.9797 \end{aligned}$$

9.6 Predictor-Corrector Methods

1. $y_4 = 0.82126825$, $y_5 = 0.78369923$
2. $y_4 = 1.2828055$, $y_5 = 1.3805509$
3. $y_4 = 0.74832050$, $y_5 = 0.66139979$
4. $y_4 = 0.98247692$, $y_5 = 0.97350099$
5. $y_4 = 1.0416670$, $y_5 = 1.0666673$
6. $y_4 = 1.5084982$, $y_5 = 1.6858030$
7. $y_4 = 1.1542232$, $y_5 = 1.2225213$
8. $y_4 = 0.1986693$, $y_5 = 0.2474040$
9. $y_4 = 1.0203389$, $y_5 = 1.0320851$

9.7 Systems of Differential Equations

1. (a) $(x_1, y_1) = (-2.5500000, 2.6700000)$
   $(x_2, y_2) = (-2.4040735, 2.5485015)$
   (b) $(x_1, y_1) = (-2.5521092, 2.6742492)$
2. (a) $(x_1, y_1) = (2.0050, 0.5150)$, $(x_2, y_2) = (2.2800, 0.8902)$
   (b) $(x_1, y_1) = (1.4623255, 3.2430091)$
3. (a) $(x_1, y_1) = (1.500, 3.2500)$, $(x_2, y_2) = (0.9250, 3.4875)$
   (b) $(x_1, y_1) = (1.4623255, 3.2430091)$
4. (a) $(x_1, y_1) = (0.8500, 1.1000)$, $(x_2, y_2) = (0.7350, 1.1975)$

(b) $(x_1, y_1) = (0.8663753, 1.0990094)$

5. (a) Substitute x, $x' = -2e^{-t/2} + 21e^{3t} - 18e^{2t}$ and $x'' = e^{-t/2} + 63e^{3t} - 36e^{2t}$ into the differential equation:

$$\begin{aligned}
&2x'' - 5x' - 3x \\
&= 2(e^{-t/2} + 63e^{3t} - 36e^{2t}) - 5(-2e^{-t/2} + 21e^{3t} - 18e^{2t}) \\
&\quad -3(4e^{-t/2} + 7e^{3t} - 9e^{2t}) \\
&= (2 + 10 - 12)e^{-t/2} + (126 - 105 - 21)e^{3t} \\
&\quad +(-72 + 90 + 27)e^{2t} \\
&= 45e^{2t}
\end{aligned}$$

(b) $x' = y$, $y' = 1.5x + 2.5y + 22.5e^{2t}$

(c) $x_1 = 2.05$, $x_2 = 2.17$

(d) $x_1 = 2.0875384$

6. (a) Substitute x, $x' = -4e^{-3t} - 24te^{-3t}$ and $x'' = -12e^{-3t} + 72te^{-3t}$ into the differential equation:

$$\begin{aligned}
x'' + 6x' + 9x &= (-12e^{-3t} + 72te^{-3t}) + 6(-4e^{-3t} - 24te^{-3t}) \\
&\quad +9(4e^{-3t} + 8t3^{-3t}) \\
&= (-12 - 24 + 36)e^{-3t} + (72 - 144 + 72)t3^{-3t} = 0
\end{aligned}$$

(b) $x' = y$, $y' = -9x - 6y$

(c) $x_1 = 3.6$, $x_2 = 3.08$

(d) $x_1 = 3.7871094$

7. (a) Substitute x and $x'' = 4\cos(t) - 3\sin(t) - 3t\sin(t)$ into the differential equation:

$$\begin{aligned}
&x'' + x \\
&= (4\cos(t) - 3\sin(t) - 3t\sin(t)) + (2\cos(t) + 3\sin(t) + 3t\sin(t)) \\
&= (4 + 2)\cos(t) + (-3 + 3)\sin(t) + (-3 + 3)t\sin(t) = 6\cos(t)
\end{aligned}$$

(b) $x' = y$, $y' = -x + 6\cos(t)$

(c) $x_1 = 2.3$, $x_2 = 2.64$

(d) $x_1 = 2.4648$

8. (a) Substitute $x' = 4 - 3e^{-3t}$ and $x'' = 9e^{-3t}$ into the differential equation:

$$\begin{aligned}
x'' + 3x' &= (9e^{-3t}) + 3(4 - 3e^{-3t}) \\
&= (9 - 9)e^{-3t} + 12 = 12
\end{aligned}$$

(b) $x' = y$, $y' = 12 - 3y$

(c) $x_1 = 5.1$, $x_2 = 5.29$

(d) $x_1 = 5.0607086$

9.8 Boundary Value Problems

1. (a) Substitute x,

$$x' = -0.335951 - \frac{8.6719}{t^3} + \frac{\cos(\ln(t))}{t} + \frac{3\sin(\ln(t))}{t}$$

and

$$x'' = \frac{26.0157}{t^4} + \frac{2\cos(\ln(t))}{t^2} - \frac{4\sin(\ln(t))}{t^2}$$

into the differential equation:

$$\begin{aligned}
&x'' + \left(\tfrac{2}{t}\right)x' - \left(\tfrac{2}{t^2}\right)x = \\
&\frac{26.0157}{t^4} + \frac{2\cos(\ln(t))}{t^2} - \frac{4\sin(\ln(t))}{t^2} \\
&+ \left(\tfrac{2}{t}\right)\left(-0.335951 - \frac{8.6719}{t^3} + \frac{\cos(\ln(t))}{t} + \frac{3\sin(\ln(t))}{t}\right) \\
&- \left(\tfrac{2}{t^2}\right)\left(\frac{4.33595 - 0.33595t^3 - 3t^2\cos(\ln(t)) + t^2\sin(\ln(t))}{t^2}\right) \\
&= \frac{10t^2\cos(\ln(t))}{t^4} = \frac{10\cos(\ln(t))}{t^2}
\end{aligned}$$

(b) Substitute x,

$$\begin{aligned}
x' &= -e^{-t} + 3.87023e^{-t}\cos(t) - \frac{1}{5}\cos^2(t) - 3.47023e^{-t}\sin(t) \\
&\quad + \frac{4}{5}\cos(t)\sin(t) + \frac{1}{5}\sin^2(t)
\end{aligned}$$

and

$$\begin{aligned}
x'' &= e^{-t} - 7.34045e^{-t}\cos(t) + \frac{4}{5}\cos^2(t) \\
&\quad -0.4e^{-t}\sin(t) + \frac{4}{5}\cos(t)\sin(t) - \frac{4}{5}\sin^2(t)
\end{aligned}$$

into the differential equation:

$$\begin{aligned}
&x'' + 2x' + 2x = \\
&\left(e^{-t} - 7.34045e^{-t}\cos(t) + \tfrac{4}{5}\cos^2(t)\right. \\
&\left.-0.4e^{-t}\sin(t) + \tfrac{4}{5}\cos(t)\sin(t) - \tfrac{4}{5}\sin^2(t)\right) \\
&+2\left(-e^{-t} + 3.87023e^{-t}\cos(t) - \tfrac{1}{5}\cos^2(t)\right. \\
&\left.-3.47023e^{-t}\sin(t) + \tfrac{4}{5}\cos(t)\sin(t) + \tfrac{1}{5}\sin^2(t)\right) \\
&+2\left(\tfrac{1}{5} + e^{-t} - \tfrac{1}{5}e^{-t}\cos(t) - \tfrac{2}{5}\cos^2(t)\right. \\
&\left.+3.67023e^{-t}\sin(t) - \tfrac{1}{5}\cos(t)\sin(t)\right) \\
&= e^{-t} + 2\cos(t)\sin(t) = e^{-t} + \sin(2t)
\end{aligned}$$

(c) Substitute x,

$$\begin{aligned} x' &= -3.96573e^{-2t} + 3.83146t3^{-2t} - \frac{4}{5}\cos^2 t + \frac{8}{5}\cos^4(t) \\ &\quad -\frac{19}{10}\cos(t)\sin(t) + \frac{24}{5}\cos^3(t)\sin(t) \\ &\quad +\frac{4}{5}\sin^2(t) - \frac{24}{5}\cos^2(t)\sin^2(t) \end{aligned}$$

and

$$\begin{aligned} x'' &= 11.7629e^{-2t} - 7.66292te^{-2t} - \frac{19}{10}\cos^2(t) + \frac{24}{5}\cos^4(t) \\ &\quad +\frac{16}{5}\cos(t)\sin(t) - 16\cos^3(t)\sin(t) \\ &\quad +\frac{19}{10}\sin^2(t) - \frac{72}{5}\cos^2(t)\sin^2(t) + \frac{48}{5}\cos(t)\sin^3(t)) \end{aligned}$$

into the differential equation:

$$\begin{aligned} & x'' + 4x' + 4x = \\ & \left(11.7629e^{-2t} - 7.66292te^{-2t} - \tfrac{19}{10}\cos^2(t)\right. \\ & +\tfrac{24}{5}\cos^4(t) + \tfrac{16}{5}\cos(t)\sin(t) - 16\cos^3(t)\sin(t) \\ & \left.+\tfrac{19}{10}\sin^2(t) - \tfrac{72}{5}\cos^2(t)\sin^2(t) + \tfrac{48}{5}\cos(t)\sin^3(t))\right) \\ & +4\left(-3.96573e^{-2t} + 3.83146t3^{-2t} - \tfrac{4}{5}\cos^2(t)\right. \\ & +\tfrac{8}{5}\cos^4(t) - \tfrac{19}{10}\cos(t)\sin(t) + \tfrac{24}{5}\cos^3(t)\sin(t) \\ & \left.+\tfrac{4}{5}\sin^2(t) - \tfrac{24}{5}\cos^2(t)\sin^2(t)\right) \\ & +4\left(-\tfrac{1}{40} + 1.025e^{-2t} - 1.915710te^{-2t} + \tfrac{19}{20}\cos^2(t)\right. \\ & \left.-\tfrac{6}{5}\cos^4(t) - \tfrac{4}{5}\cos(t)\sin(t) + \tfrac{8}{5}\cos^3(t)\sin(t)\right) \\ & = \tfrac{1}{10}\left(-1 - 13\cos^2(t) + 64\cos^4(t) - 76\cos(t)\sin(t)\right. \\ & +96\cos^3(t)\sin(t) + 51\sin^2(t) - 336\cos^2(t)\sin^2(t) \\ & \left.+96\cos(t)\sin^3(t)\right) \\ & = 5\cos(4t) + \sin(2t) \end{aligned}$$

(d) Substitute x,

$$x' = \frac{1.0013\cos(t) - 0.291384\sin(t)}{t^{1/2}} - \frac{0.29138\cos(t) + 1.0013\sin(t)}{2t^{3/2}}$$

and

$$\begin{aligned} x'' &= \frac{-0.291384\cos(t) - 1.0013\sin(t)}{t^{1/2}} - \frac{1.0013\cos(t) - 0.291384\sin(t)}{t^{3/2}} \\ &\quad +\frac{3(0.291384\cos(t) + 1.0013\sin(t))}{4t^{5/2}} \end{aligned}$$

into the differential equation:

$$
\begin{aligned}
&x'' + \tfrac{1}{t}x' + \left(1 - \tfrac{1}{4t^2}\right)x = \\
&\left(\frac{-0.291384\cos(t) - 1.0013\sin(t)}{t^{1/2}} - \frac{1.0013\cos(t) - 0.291384\sin(t)}{t^{3/2}}\right. \\
&\left.+\frac{(0.291384\cos(t) + 1.0013\sin(t))}{4t^{5/2}}\right) \\
&+\frac{1}{t}\left(\frac{1.0013\cos(t) - 0.291384\sin(t)}{t^{1/2}} - \frac{0.29138\cos(t) + 1.0013\sin(t)}{2t^{3/2}}\right) \\
&+\left(1 - \frac{1}{4t^2}\right)\left(\frac{0.291384\cos(t) + 1.0013\sin(t)}{t^{1/2}}\right) \\
&= 0
\end{aligned}
$$

(e) Substitute x, $x' = -2.78102 + 2t - 2.52844\ln(t)$, and $x'' = 2 - \frac{2.52844}{t}$ into the differential equation:

$$
\begin{aligned}
&x'' - \left(\tfrac{1}{t}\right)x' + \left(\tfrac{1}{t^2}\right)x = \\
&\left(2 - \tfrac{2.52844}{t}\right) - \left(\tfrac{1}{t}\right)(-2.78102 + 2t - 2.52844\ln(t)) \\
&+ \left(\tfrac{1}{t^2}\right)\left(t^2 - 0.25258t - t\ln(t)\right) = 1
\end{aligned}
$$

2. No; $q(t) = -1/t^2 < 0$ for all $t \in [0.5, 4.5]$.

3. By substitution we verify that $v(t) = 0$ is a solution of the given boundary value problem. To show it is the unique solution, it is sufficient to show that the hypotheses of Corollary 9.1 are satisfied. We are given that $g(t) > 0$ for all $t \in [a, b]$, and $f_v = g(t)$ and $v''(t)$ are continuous. Furthermore the functions v and v' are continuous, since they are differentiable. Thus $p(t)$ is continuous for all $t \in [a, b]$. By the Extreme Value Theorem $|p(t)| \le M = \max|p(t)|$ for all $t \in [a, b]$. Therefore, the solution $v(t) = 0$ is unique.

9.9 Finite-Difference Method

1. (a) $h_1 = 0.5$, $x_1 = 7.2857149$
   $h_2 = 0.25$, $x_1 = 6.0771913$, $x_2 = 7.2827433$

   (b)
$$
z_{j,i} = \frac{4x_{j,2} - x_{j,1}}{3} = \frac{4(7.2827433) - 6.0771913}{3} = 7.6839687
$$

   (c) $x(0 + 0.5) = 7.25$

2. (a) $h_1 = 0.5$, $x_1 = 0.85414295$
   $h_2 = 0.25$, $x_1 = 0.93524622$, $x_2 = 0.83762911$

(b)

$$z_{j,i} = \frac{4x_{j,2} - x_{j,1}}{3} = \frac{4(0.83762911) - 0.93524622}{3} = 0.80514847$$

(c) $x(1.5) = 0.83233892$

3. (a) $h = 0.5$, $x_1 = 0.66721941$
   $h = 0.25$, $x_1 = 0.90452638$, $x_2 = 0.72525982$

   (b)

$$z_{j,i} = \frac{4x_{j,2} - x_{j,1}}{3} = \frac{4(0.66721941) - 0.72525982}{3} = 0.64787261$$

   (c) $x(1) = 0.74741735$

4. If $h < \frac{2}{M}$, then $M < \frac{2}{h}$ or $|p_j| \leq M < \frac{2}{h}$ for $j = 1, 2, \ldots, N-1$. For the second through $N-2nd$ rows ($j = 2, \ldots, N-2$)

$$\begin{aligned} \left|-\frac{h}{2}p_j - 1\right| + \left|\frac{h}{2}p_j - 1\right| &< \left|\frac{h}{2}\frac{2}{h} + 1\right| + \left|\frac{h}{2}\frac{2}{h} - 1\right| \\ &= 2 \\ &\leq 2 + h^2 q_j \\ &= |2 + h^2 q_j|, \end{aligned}$$

since $q(t) \geq 0$ on $[a, b]$. Similarly for the first and $N+1\,st$ rows:

$$|2 + h^2 q_1| > \left|\frac{h}{2}p_1 - 1\right| \text{ and } |2 + h^2 q_{N-1}| > \left|-\frac{h}{2}p_{N-1} - 1\right|$$

So, by definition, the coefficient matrix is strictly diagonally dominant and thus nonsingular. Therefore, the given linear system has a unique solution.

5. (a) The tridiagonal coefficient matrix is:

$$\begin{bmatrix} 2 + h^2 c_2 & \frac{h}{2}c_1 - 1 & & \\ -\frac{h}{2}c_1 - 1 & 2 + h^2 c_2 & \frac{h}{2}c_1 - 1 & \mathbf{O} \\ & \ddots & \ddots & \ddots \\ & -\frac{h}{2}c_1 - 1 & 2 + h^2 c_2 & \frac{h}{2}c_1 - 1 \\ \mathbf{O} & & -\frac{h}{2}c_1 - 1 & 2 + h^2 c_2 \end{bmatrix}$$

(b) For the first row ($j = 1$)

$$\begin{aligned} c_1 &\le hc_2 \\ hc_1 &\le h^2c_2 \\ \frac{h}{2}c_1 &< h^2c_2 \\ \frac{h}{2}c_1 - 1 &< 2 + h^2c_2 \\ \left|\frac{h}{2}c_1 - 1\right| &< |2 + h^2c_2| \end{aligned}$$

Similarly, for the last row ($j = N - 1$)

$$\left|\frac{h}{2}c_1 - 1\right| < |2 + h^2c_2|$$

For rows $j = 2, \ldots, N - 2$

$$\begin{aligned} \left|-\frac{h}{2}c_1 - 1\right| + \left|\frac{h}{2}c_1 - 1\right| &\le \frac{h}{2}c_1 + 1 + \frac{h}{2}c_1 + 1 \\ &= hc_1 + 2 \\ &< h^2c_2 + 2 \\ &= |2 + h^2c_2| \end{aligned}$$

Chapter 10

Solution of Partial Differential Equations

10.1 Hyperbolic Equations

1. (a)

$$\begin{aligned} u(x,t) &= \sin(n\pi x)\cos(2n\pi t) \\ u_t(x,t) &= -2n\pi \sin(n\pi x)\sin(2n\pi t) \\ u_{tt}(x,t) &= -4n^2\pi^2 \sin(n\pi x)\cos(2n\pi t) \\ u_x(x,t) &= n\pi \cos(n\pi x)\cos(2n\pi t) \\ u_{xx}(x,t) &= -n^2\pi^2 \sin(n\pi x)\cos(2n\pi t) \end{aligned}$$

Thus $u_{tt}(x,t) = 4u_{xx}(x,t)$

(b)

$$\begin{aligned} u(x,t) &= \sin(n\pi x)\cos(cn\pi t) \\ u_t(x,t) &= -cn\pi \sin(n\pi x)\sin(cn\pi t) \\ u_{tt}(x,t) &= -c^2n^2\pi^2 \sin(n\pi x)\cos(cn\pi t) \\ u_x(x,t) &= n\pi \cos(n\pi x)\cos(cn\pi t) \\ u_{xx}(x,t) &= -n^2\pi^2 \sin(n\pi x)\cos(cn\pi t) \end{aligned}$$

Thus $u_{tt}(x,t) = c^2u_{xx}(x,t)$

2. If $u(x,y) = \frac{1}{2}(f(x+ct) + f(x-ct))$, then

$$u(x,0) = \frac{1}{2}(f(x) + f(x)) = f(x)$$

and

$$u_t(x,0) = \frac{c}{2}(f'(x) - f'(x)) = 0$$

Furthermore;

$$\begin{aligned} u_x &= \frac{1}{2}(f'(x+ct)+f'(x-ct)) \\ u_{xx} &= \frac{1}{2}(f''(x+ct)+f''(x-ct)) \\ u_t &= \frac{c}{2}(f'(x+ct)-f'(x-ct)) \\ u_{tt} &= \frac{c^2}{2}(f''(x+ct)+f''(x-ct)) \end{aligned}$$

Thus $c^2 u_{xx} = u_{tt}$.

3. Substituting $h = 2ck$ into (5) yields

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} = \frac{c^2\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)}{4c^2k^2}$$

$$u_{i,j+1}-2j_{i,j}+u_{i,j-1} = \frac{1}{4}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)$$

$$u_{i,j+1} = \frac{3}{2}u_{i,j}+\frac{1}{4}\left(u_{i+1,j}+u_{i-1,j}\right)-u_{i,j-1}$$

4.

t_j	x_2	x_3	x_4	x_5
0.0	0.587785	0.951057	0.951057	0.587785
0.1	0.475528	0.769421	0.769421	0.475528
0.2	0.181636	0.293893	0.293893	0.181636

5.

t_j	x_2	x_3	x_4	x_5
0.0	0.500	1.000	1.500	0.750
0.1	0.500	1.000	0.875	0.800
0.2	0.500	0.375	0.300	0.125

6. If $u(x,t) = \frac{1}{2}(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$, then

$$u(x,0) = \frac{1}{2}(2f(x)) + \frac{1}{2c}\int_x^x g(s)ds = f(x)+0 = f(x).$$

Secondly,

$$u_t(x,t) = \frac{c}{2}(f'(x+ct)-f'(x-ct)) + \frac{c}{2c}(g(x+ct)+g(x-ct)),$$

thus $u_t(x,0) = 0 + \frac{1}{2}(2g(x)) = g(x)$. Finally,

$$\begin{aligned}
u_{tt}(x,t) &= \frac{c^2}{2}\left(f''(x+ct)+f''(x-ct)\right)+\frac{c}{2}\left(g'(x+ct)-g'(x-ct)\right)\\
u_x(x,t) &= \frac{1}{2}\left(f'(x+ct)+f'(x-ct)\right)+\frac{1}{2c}\left(g(x+ct-g(x-ct)\right)\\
u_{xx}(x,t) &= \frac{1}{2}\left(f''(x+ct)+f''(x-ct)\right)+\frac{1}{2c}\left(g'(x+ct)-g'(x-ct)\right)
\end{aligned}$$

Therefore, $u_{tt}(x,t) = c^2 u_{xx}(x,t)$.

7. In (5) let $k = h/3$, then (7) becomes $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$.
8. If $c = 2$, $k = 0.02$, and $h = 0.03$, then

$$r = \frac{2(0.02)}{0.03} = \frac{0.04}{0.03} = \frac{4}{3} > 1.$$

Since $r \not\leq 1$ there is no guarantee of stability in formula (7).

10.2 Parabolic Equations

1. (a) If $u(x,t) = \sin(n\pi x)e^{-4n^2\pi^2 t}$, then

$$\begin{aligned}
u_t(x,t) &= -4n^2\pi^2\sin(n\pi x)e^{-4n^2\pi^2 t}\\
u_x(x,t) &= n\pi\cos(n\pi x)e^{-4n^2\pi^2 t}\\
u_{xx}(x,t) &= -n^2\pi^2\sin(n\pi x)e^{-4n^2\pi^2 t}
\end{aligned}$$

Therefore, $4u_{xx}(x,t) = u_t(x,t)$.

(b)

$$\begin{aligned}
u(x,t) &= \sin(n\pi x)e^{-(cn\pi)^2 t}\\
u_t(x,t) &= -(cn\pi)^2\sin(n\pi x)e^{-(cn\pi)^2 t}\\
u_x(x,t) &= n\pi\cos(n\pi x)e^{-(cn\pi)^2 t}\\
u_{xx}(x,t) &= -n^2\pi^2\sin(n\pi x)e^{-(cn\pi)^2 t}
\end{aligned}$$

Thus $u_t(x,t) = c^2 u_{xx}(x,t)$.

2. If $k = h^2/c^2$, then $r = \dfrac{c^2\left(\frac{h^2}{c^2}\right)}{h^2} = 1$. In this case $r \not\leq 1/2$ and stability is not guaranteed in formula (7).

3.

$x_1 = 0.0$	$x_2 = 0.2$	$x_3 = 0.4$	$x_4 = 0.6$	$x_5 = 0.8$	$x_6 = 1.0$
0.0	0.587785	0.951057	0.951057	0.587785	0.0
0.0	0.475528	0.769421	0.769421	0.475528	0.0
0.0	0.384710	0.622475	0.622475	0.384710	0.0

4.

$x_1 = 0.0$	$x_2 = 0.2$	$x_3 = 0.4$	$x_4 = 0.6$	$x_5 = 0.8$	$x_6 = 1.0$
0.0	0.4000	0.800	0.800	0.400	0.0
0.0	0.400	0.7500	0.7500	0.400	0.0
0.0	0.3938	0.7063	0.7063	0.3937	0.0

5. (a) Substituting $k = h^2/(2c^2)$ into (15) yields

$\frac{2c^2}{h^2}(u_{i,j+1} - u_{i,j})$
$= \frac{c^2}{2h^2}(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$

Solving for $u_{i-1,j+1}$, $u_{i,j+1}$, and $u_{i+1,j+1}$ yields

$$-u_{i-1,j+1} + 6u_{i,j+1} - u_{i+1,j+1} = 2u_{i,j} + u_{i-1,j} + u_{i+1,j},$$

for $i = 2, 3, \ldots, n-1$.

(b) **A** has the tridiagonal form:

$$\mathbf{A} = \begin{bmatrix} 6 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 6 & -1 & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & -1 & 6 & -1 & 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \cdots & 0 & -1 & 6 & -1 & 0 \\ \vdots & & & \cdots & 0 & -1 & 6 & 1 \\ 0 & \cdots & & & \cdots & 0 & -1 & 6 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2c_1 + 2u_{2,j} + u_{3,j} \\ u_{2,j} + 2u_{3,j} + u_{4,j} \\ \vdots \\ u_{n-1,j} + 2u_{n-2,j} + u_{n-3,j} \\ 2c_2 + 2u_{n-1,j} + u_{n-2,j} \end{bmatrix}$$

(c) Clearly, the answer is yes.

6. If $u(x,t) = \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t} \sin(j\pi x)$, then

$$\begin{aligned} u_t &= -(j\pi)^2 \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t} \sin(j\pi x) \\ u_x &= j\pi \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t} \cos(j\pi x) \\ u_{xx} &= -(j\pi)^2 \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t} \sin(j\pi x) \end{aligned}$$

Thus $u(x,t)$ is a solution of $u_t(x,t) = u_{xx}(x,t)$. Further

$$\begin{aligned}
u(0,t) &= \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t}\sin(0) = \sum_{j=1}^{N} 0 = 0\\
u(1,t) &= \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t}\sin(j\pi) = \sum_{j=1}^{N} 0 = 0\\
u(x,0) &= \sum_{j=1}^{N} a_j e^{-(j\pi)^2 0}\sin(j\pi x) = \sum_{j=1}^{N} a_j \sin(j\pi x).
\end{aligned}$$

Thus the given boundary conditions are satisfied.

7. (a)

$$\lim_{t\to\infty}\left(\frac{\sin(\pi x)}{e^{\pi^2 t}} + \frac{\sin(3\pi x)}{e^{(3\pi)^2 t}}\right) = 0$$

   (b) Through time the temperature at all points in the rod approaches zero.

8. (a) We have

$$u_t(x,t) = \frac{u(x,t+k) - u(x,t)}{k} + \mathbf{O}(k)$$

   and

$$u_{xx}(x,t) = \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} + \mathbf{O}(h^2).$$

   Thus

$$\begin{aligned}
\frac{u_{i,j+1} - u_{i,j}}{k} - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} &= h_i\\
u_{i,j+1} = u_{i,j} + \frac{k}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) &+ kh_i\\
u_{i,j+1} = \frac{k}{h^2}u_{i-1,j} + \left(1 - \frac{2k}{h^2}\right)u_{i,j} &+ \frac{k}{h^2}u_{i+1,j} + kh_i
\end{aligned}$$

   the explicit forward-difference equation.

   (b) Solve the difference equation

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2h^2} = h_i$$

   for $u_{i-1,j+1}$, $u_{i,j+1}$, and $u_{i-1,j+1}$. Thus

$$\begin{aligned}
&-\tfrac{k}{2h^2}u_{i-1,j+1} + \left(1 + \tfrac{k}{h^2}\right)u_{i,j+1} - \tfrac{k}{2h^2}u_{i+1,j+1}\\
&= \tfrac{k}{2h^2}u_{i-1,j} + \left(1 - \tfrac{k}{h^2}\right)u_{i,j} + \tfrac{k}{2h^2}u_{i+1,j} + kh_i,
\end{aligned}$$

or

$$-ku_{i-1,j+1} + 2(h^2 + k)u_{i,j+1} - ku_{i+1,j+1}$$
$$= ku_{i-1,j} + 2(h^2 - k)u_{i,j} + ku_{i+1,j} + 2kh^2h_i,$$

the implicit difference formula.

9.

10.3 Elliptic Equations

1. (a)

$$\begin{cases} -4p_1 + p_2 + p_3 &= -80 \\ p_1 - 4p_2 + p_4 &= -10 \\ p_1 - 4p_3 + p_4 &= -160 \\ p_2 + p_3 - 4p_4 &= -90 \end{cases}$$

(b) $p_1 = 41.25$, $p_2 = 23.75$, $p_3 = 61.25$, $p_4 = 43.75$

2. (a)

$$\begin{cases} -4q_1 + q_2 + 2q_3 &= -70 \\ q_1 - 4q_2 + 2q_4 &= 0 \\ q_1 - 4q_3 + q_4 + q_5 &= -70 \\ q_2 + q_3 - 4q_4 + q_6 &= 0 \\ q_3 - 4q_5 + q_6 &= -160 \\ q_4 + q_5 - 4q_6 &= -90 \end{cases}$$

(b) $q_1 = 52.5657$, $q_2 = 29.6566$, $q_3 = 55.3030$
$q_4 = 33.03030$, $q_5 = 65.6162$, $q_6 = 47.1616$

3. (a)

$$\begin{aligned} u_x &= a_1\cos(x)\sinh(y) + b_1\cosh(x)\sin(y) \\ u_{xx} &= -a_1\sin(x)\sinh(y) + b_1\sinh(x)\sin(y) \\ u_y &= a_1\sin(x)\cosh(y) + b_1\sinh(x)\cos(y) \\ u_{yy} &= a_1\sin(x)\sinh(y) - b_1\sinh(x)\sin(y) \end{aligned}$$

Substituting u_{xx} and u_{yy} into Laplace's equation we see that $u_{xx} + u_{yy} \equiv 0$.

(b)

$$\begin{aligned} u_x &= a_n n\cos(nx)\sinh(ny) + b_n n\cosh(nx)\sin(ny) \\ u_{xx} &= -a_n n^2\sin(nx)\sinh(ny) + b_n n^2\sinh(nx)\sin(ny) \\ u_y &= a_n n\sin(nx)\cosh(ny) + b_n n\sinh(nx)\cos(ny) \\ u_{yy} &= a_n n^2\sin(nx)\sinh(yn) - b_n n^2\sinh(nx)\sin(ny) \end{aligned}$$

Substituting u_{xx} and u_{yy} into Laplace's equation we see that $u_{xx} + u_{yy} \equiv 0$.

4.

$$u(x+h,y) = (x+h)^2 - y^2, \qquad u(x-h,y) = (x-h)^2 - y^2$$
$$u(x,y+h) = x^2 - (y+h)^2 \qquad u(x,y-h) = x^2 - (y-h)^2$$

Substituting into (7):

$$\frac{(x+h)^2 - y^2 + (x-h)^2 - y^2 + (^2 - (y+h)^2 + x^2 - (y-h)^2 - 4(x^2 - y^2)}{h^2}$$
$$= \frac{2xh + h^2 - 2xh + h^2 - 2hy - h^2 + 2hy - h^2}{h^2}$$
$$= \frac{0}{h^2} = 0$$

5. (a) $u_{xx} + u_{yy} = 2a + 2c = 0$, if $a = -c$.
   (b) $2a + 2c = -1$, if $a + c = -\frac{1}{2}$.

6. Determine if $u(x,y) = \cos(2x)+\sin(2y)$ is a solution; since it is also defined on the interior of R. That is:

$$\begin{aligned} u_{xx} + u_{yy} &= -4\cos(2x) - 4\sin(2y) \\ &= -4(\cos(2x) + \sin(2y)) \\ &= -4u \end{aligned}$$

7.

$$\begin{aligned} -4p_1 + p_2 + p_3 &= -u_{1,2} - u_{2,1} \\ p_1 - 4p_2 + p_4 &= -u_{3,1} - u_{4,2} \\ p_1 - 4p_3 + p_4 &= -u_{1,3} - u_{2,4} \\ p_2 + p_3 - 4p_4 &= -u_{3,4} - u_{4,3} \end{aligned}$$

Chapter 11

Eigenvalues and Eigenvectors

11.1 Homogeneous Systems: The Eigenvalue Problem

1. (a) $p(\lambda) = \lambda^2 - 3\lambda - 4$ and $\lambda_1 = -1$, $\mathbf{V}_1 = [\ -1 \quad 1\ ]'$; $\lambda_2 = 4$, $\mathbf{V}_2 = [\ 2 \quad 3\ ]'$.

   (b) $p(\lambda) = \lambda^2 - 3\lambda - 52$ and

$$\begin{aligned} \lambda_1 &= \left(3 - \sqrt{217}\right)/18, & \mathbf{V}_1 &= \left[\ \left(-1 - \sqrt{217}\right)/18 \quad 1\ \right]' \\ \lambda_2 &= \left(3 + \sqrt{217}\right)/18, & \mathbf{V}_2 &= \left[\ \left(-1 + \sqrt{217}\right)/18 \quad 1\ \right]' \end{aligned}$$

   (c) $p(\lambda) = \lambda^2 - 4\lambda - 5$ and $\lambda_1 = -5$, $\mathbf{V}_1 = [\ -1 \quad 1\ ]'$; $\lambda_2 = 1$, $\mathbf{V}_2 = [\ 1 \quad 1\ ]'$.

   (d) $p(\lambda) = \lambda^3 - 4\lambda_2 + 7$

$$\lambda_1 = \frac{1889}{557},\ \lambda_2 = \frac{1163}{656},\ \lambda_3 = -\frac{1765}{1516}$$

$$\mathbf{V_1} = \begin{bmatrix} \frac{563}{865} \\ \frac{696}{1429} \\ \frac{1856}{3187} \end{bmatrix},\ \mathbf{V_2} = \begin{bmatrix} -\frac{222}{235} \\ -\frac{659}{2154} \\ -\frac{1501}{12696} \end{bmatrix},\ \mathbf{V_3} = \begin{bmatrix} -\frac{512}{1851} \\ \frac{812}{1245} \\ -\frac{1113}{1577} \end{bmatrix}$$

   (e) $p(\lambda) = \lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 25$ and

$$\begin{aligned} \lambda_1 &= 1, & \mathbf{V}_1 &= [\ 1 \quad 0 \quad 0 \quad 0\ ]' \\ \lambda_2 &= 2, & \mathbf{V}_2 &= [\ 0 \quad 2 \quad 0 \quad 0\ ]' \\ \lambda_3 &= 3, & \mathbf{V}_3 &= [\ 0 \quad 0 \quad 3 \quad 0\ ]' \\ \lambda_4 &= 4, & \mathbf{V}_4 &= [\ 0 \quad 0 \quad 0 \quad 4\ ]' \end{aligned}$$

2. (a) 4 (b) $(3+\sqrt{217})/2$ (c) 5 (d) 1889/557 (e) 4

3. (a) $\| \mathbf{A} \|_2 = 4.130649$, $\| \mathbf{A} \|_\infty = 5$
   (b) $\| \mathbf{A} \|_2 = 9.379685$, $\| \mathbf{A} \|_\infty = 11$
   (c) $\| \mathbf{A} \|_2 = 5.000000$, $\| \mathbf{A} \|_\infty = 5$
   (d) $\| \mathbf{A} \|_2 = 4.680464$, $\| \mathbf{A} \|_\infty = 6$
   (e) $\| \mathbf{A} \|_2 = 6.281336$, $\| \mathbf{A} \|_\infty = 7$

4. (a) The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ is diagonalizable since the eigenvectors $\mathbf{V}_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}'$ and $\mathbf{V}_2 = \begin{bmatrix} 2 & 3 \end{bmatrix}$ are linearly independent. Let $\mathbf{V} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$, then $\mathbf{V}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & -1 \end{bmatrix}$ and

$$\begin{aligned}
\mathbf{V}^{-1}\mathbf{A}\mathbf{V} &= -\tfrac{1}{5}\begin{bmatrix} 3 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \\
&= -\tfrac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & -20 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \\
&= diag(-1,4) = \mathbf{D}
\end{aligned}$$

(b)

$$\mathbf{V} = \begin{bmatrix} -0.6581 & -0.6065 \\ 0.7530 & -0.7951 \end{bmatrix}, \quad \mathbf{V}^{-1} = \begin{bmatrix} -0.8114 & 0.6190 \\ -0.7684 & -0.6716 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} -5.8655 & 0.0000 \\ 0.0000 & 8.8655 \end{bmatrix}$$

(c)

$$\mathbf{V} = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}, \quad \mathbf{V}^{-1} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} -5.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

(d)

$$\mathbf{V} = \begin{bmatrix} 0.6509 & -0.9447 & -0.2766 \\ 0.4871 & -0.3059 & 0.6522 \\ 0.5824 & -0.1182 & -0.7058 \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} -0.5569 & 1.2050 & 1.3318 \\ -1.3752 & 0.5669 & 1.0628 \\ -0.2292 & 0.8993 & -0.4960 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{V}^{-1}\mathbf{AV} = \begin{bmatrix} 3.3914 & 0.0000 & 0.0000 \\ 0.0000 & 1.7729 & 0.0000 \\ 0.0000 & 0.0000 & -1.1642 \end{bmatrix}$$

(e)

$$\mathbf{V} = \begin{bmatrix} 1.0000 & 0.7071 & 0.5571 & 0.4625 \\ 0.0000 & 0.7071 & 0.7428 & 0.7471 \\ 0.0000 & 0.0000 & 0.3714 & 0.4269 \\ 0.0000 & 0.0000 & 0.0000 & 0.2135 \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} 1.0000 & 1.0000 & 0.5000 & 0.3333 \\ 0.0000 & 1.4142 & -2.8284 & 0.7071 \\ 0.0000 & 0.0000 & 2.6926 & -5.3852 \\ 0.0000 & 0.0000 & 0.0000 & 4.6845 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{V}^{-1}\mathbf{AV} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

5. (a)

$$\begin{aligned} \mathbf{R'R} &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \end{aligned}$$

(b) The characteristic polynomial of R is

$$\begin{aligned} \begin{vmatrix} \cos(\theta) - \lambda & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) - \lambda \end{vmatrix} &= (\cos(\theta) - \lambda)^2 + \sin(\theta) \\ &= \lambda^2 - 2\cos(\theta)\lambda + 1 \end{aligned}$$

Real eigenvalues occur when the discriminant is nonnegative:

$$\begin{aligned} (-2\cos(\theta))^2 - 4(1)(1) &\geq 0 \\ \cos^2(\theta) &\geq 1; \end{aligned}$$

or $\theta = n\pi$, where n is an integer.

6. (a)

$$\mathbf{R}_x(\alpha)(\mathbf{R}_x(\alpha))' =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2(\alpha) + \sin^2(\alpha) & \cos(\alpha)\sin(\alpha) - \sin(\alpha)\cos(\alpha) \\ 0 & \sin(\alpha)\cos(\alpha) - \cos(\alpha)\sin(\alpha) & \sin^2(\alpha) + cos^2(\alpha) \end{bmatrix}$$
$$= \mathbf{I}$$

$$\mathbf{R}_y(\beta)(\mathbf{R}_y(\beta))' =$$

$$\begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2(\beta) + \sin^2(\beta) & 0 & -\cos(\beta)\sin(\beta) + \sin(\beta)\cos(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta)\cos(\beta) & 0 & \sin^2(\beta) + \cos^2(\beta) \end{bmatrix}$$
$$= \mathbf{I}$$

$$\mathbf{R}_z(\gamma)(\mathbf{R}_z(\gamma))' =$$

$$\begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2(\gamma) + \sin^2(\gamma) & \cos(\gamma)\sin(\gamma) - \sin(\gamma)\cos(\gamma) & 0 \\ \sin(\gamma)\cos(\gamma) - \cos(\gamma)\sin(\gamma) & \sin^2(\gamma) + cos^2(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \mathbf{I}$$

(b) The characteristic polynomial of $R_x(\alpha)$ is

$$\begin{aligned} \rho(\lambda) &= (\lambda - 1)((\lambda - \cos(\alpha))^2 + \sin^2(\alpha)) \\ &= (\lambda - 1)(\lambda 62 - 2\lambda\cos(\alpha) + 1) \end{aligned}$$

The eigenvalues are real when $\alpha = n\pi$.
The characteristic polynomial of $R_y(\beta)$ is

$$\rho(\lambda) = (\lambda - 1)((\lambda - \cos(\beta))^2 + \sin^2(\beta))$$

The eigenvalues are real when $\beta = n\pi$.

The characteristic polynomial of $R_z(\gamma)$ is

$$\rho(\lambda) = (\lambda - 1)((\lambda - \cos(\gamma))^2 + \sin^2(\gamma))$$

The eigenvalues are real when $\gamma = n\pi$.

7. (a)

$$\begin{aligned}
\rho(\lambda) &= (\lambda - (a+3))(\lambda - a) - 4 \\
&= \lambda 62 + (-a - (a+3))\lambda + a(a+3) - 4 \\
&= \lambda^2 - (3+2a)\lambda + a^2 + 3a - 4
\end{aligned}$$

(b) Solving $\rho(\lambda) = 0$ yields

$$\begin{aligned}
\lambda &= \frac{(3+21) \pm \sqrt{(3+2a)^2 - 4(1)(a^2+3a-4)}}{2} \\
&= \frac{(3+2a) \pm \sqrt{25}}{2} \\
&= a+4,\ a-1
\end{aligned}$$

(c) Substituting $\lambda = a+4$ and $\lambda = a-1$ into $(\lambda\mathbf{I} - \mathbf{A}) = \mathbf{0}$ yields the linear systems

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ -2x_1 + 4x_2 &= 0 \end{aligned}, \quad \begin{aligned} -4x_1 - 2x_2 &= 0 \\ 02x_1 - x_2 &= 0 \end{aligned}$$

with solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

where t is an arbitrary constant.

8. By induction: Assume for $k = n$; $\mathbf{A}^n\mathbf{X} = \lambda^n\mathbf{X}$ where $\mathbf{X} \neq 0$. Then

$$\mathbf{A}^{n+1}\mathbf{X} = \mathbf{A}(\mathbf{A}^n\mathbf{X}) = \mathbf{A}(\lambda^n\mathbf{X}) = \lambda^n(\lambda\mathbf{X}) = \lambda^{n+1}\mathbf{X}.$$

Thus λ^k is an eigenvalue of $\mathbf{A}^k$ provided λ is an eigenvalue of $\mathbf{A}$. It follows that $\mathbf{V}$ is an eigenvector associated with λ^k (let $\mathbf{X} = \mathbf{V}$).

9. From Exercise 8: if 3, $\mathbf{V}$ is an eigenpair of A, then 3^2, $\mathbf{V}$ is an eigenpair of $\mathbf{A}^2$.

10.

$$\begin{aligned}
\mathbf{AV} &= 2\mathbf{V} \\
\mathbf{A}^{-1}(\mathbf{AV}) &= \mathbf{A}^{-1}(2\mathbf{V}) \\
\mathbf{V} &= 2(\mathbf{A}^{-1}\mathbf{V}) \\
\mathbf{A}^{-1}\mathbf{V} &= \tfrac{1}{2}\mathbf{V}
\end{aligned}$$

11. Since $\mathbf{AV} = 5\mathbf{V}$, it follows that; $(\mathbf{A} - \mathbf{I})\mathbf{V} = \mathbf{AV} - \mathbf{V} = 5\mathbf{V} - \mathbf{V} = 4\mathbf{V}$. Thus 5, $\mathbf{V}$ is an eigenpair of the matrix $\mathbf{A} - \mathbf{I}$.

12. (a) $(-1)^n c_n = p(0) = det(\mathbf{A} - 0\mathbf{I}) = det(\mathbf{A})$. Thus $c_n = (-1)^n det(\mathbf{A})$.

    (b) In the characteristic polynomial, the coefficient c_k is $(-1)^k$ times the sum of the principal minors of order k. Thus

$$c_1 = (-1)^1 \sum_{j=1}^{n} |a_{jj}| = -\sum_{j=1}^{n} a_{jj},$$

    where $|a_{jj}|$ represents the determinant of a 1×1 matrix.

13. By induction: Assume for $k = m$; $\mathbf{A}^m = \mathbf{V}diag(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m)\mathbf{V}^{-1}$. Then

$$\begin{aligned} \mathbf{A}^{m+1} &= \mathbf{A}^m\mathbf{A} \\ &= (\mathbf{V}diag(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m)\mathbf{V}^{-1})(\mathbf{V}diag(\lambda_1, \lambda_2, \ldots, \lambda_n)\mathbf{V}^{-1}) \\ &= \mathbf{V}diag(\lambda_1^{m+1}, \lambda_2^{m+1}, \ldots, \lambda_n^{m+1})\mathbf{V}^{-1} \end{aligned}$$

    Therfore, if k is a positive integer, then $\mathbf{A}^k = \mathbf{V}diag(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)\mathbf{V}^{-1}$.

11.2 Power Method

1. $(\mathbf{A} - \alpha\mathbf{I})\mathbf{V} = \mathbf{AV} - \alpha\mathbf{V} = \lambda\mathbf{V} - \alpha\mathbf{V} = (\lambda - \alpha)\mathbf{V}$. Therefore, $(\lambda - \alpha)$, $\mathbf{V}$ is an eigenpair of the matrix $\mathbf{A} - \alpha\mathbf{I}$.

2.

$$\begin{aligned} \mathbf{AV} &= \lambda\mathbf{V} \\ \mathbf{A}^{-1}(\mathbf{AV}) &= \mathbf{A}^{-1}(\lambda\mathbf{V}) \\ \mathbf{V} &= \lambda(\mathbf{A}^{-1}\mathbf{V}) \\ \mathbf{A}^{-1}\mathbf{V} &= \tfrac{1}{\lambda}\mathbf{V} \end{aligned}$$

    Thus $1/\lambda$, $\mathbf{V}$ is an eigenpair of the matrix $\mathbf{A}^{-1}$.

3.

$$\begin{aligned} \mathbf{AV} &= \lambda\mathbf{V} \\ \mathbf{AV} - \alpha\mathbf{V} &= \lambda\mathbf{V} - \alpha\mathbf{V} \\ (\mathbf{A} - \alpha\mathbf{I})\mathbf{V} &= (\lambda - \alpha)\mathbf{V} \\ \tfrac{1}{\lambda-\alpha}\mathbf{V} &= (\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{V}; \end{aligned}$$

    provided $\lambda \neq \alpha$. Therefore, $\frac{1}{\lambda-\alpha}$, $\mathbf{V}$ is an eigenpair of the matrix $(\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{V}$.

4.

$$\begin{aligned} \mathbf{BV}_1 &= (\mathbf{A} - \lambda_1\mathbf{V}_1\mathbf{X}')\mathbf{V}_1 \\ &= \mathbf{AV}_1 - \lambda_1\mathbf{V}_1\mathbf{X}'\mathbf{V}_1 \\ &= \lambda_1\mathbf{V}_1 - \lambda_1\mathbf{V}_1 \\ &= (\lambda_1 - \lambda_1)\mathbf{V}_1 \\ &= 0\mathbf{V}_1 \end{aligned}$$

Thus 0, $\mathbf{V}_1$ is an eigenpair of the matrix $\mathbf{V}$.

5. (a)

$$\begin{aligned} 0 &= |\mathbf{A} - \lambda\mathbf{I}| \\ &= (0.8 - \lambda)(0.7 - \lambda) - 0.06 \\ &= \lambda^2 - 1.5\lambda + 0.56 - 0.06 \\ &= \lambda^2 - 1.5\lambda + 0.5 \\ &= (\lambda - 1)(\lambda - 0.5) \end{aligned}$$

(b) The augmented matrix for the system $(\mathbf{A} - (1)\mathbf{I})\mathbf{V} = \mathbf{O}$ is

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}.$$

The set of nonzero solutions (eigenvectors) of the linear system is $\{t \begin{bmatrix} 3/2 & 1 \end{bmatrix}' : t \in \Re,\, t \neq 0\}$.

(c) $\mathbf{A} \begin{bmatrix} 30,000 & 20,000 \end{bmatrix}' = \begin{bmatrix} 30,000 & 20,000 \end{bmatrix}'$

11.3 Jacobi's Method

1. (a) Let K and M be the given matrices of spring constants and masses, respectively. Let $\mathbf{X} = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}'$. Making the substitution $x_j(t) = v_j \sin(\omega t + \theta)$ yields: $\mathbf{X} = \sin(\omega t + \theta) \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}'$ and $\mathbf{X}'' - \omega^2 \sin(\omega t + \theta) \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}'$. If $\mathbf{V} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}'$, then the given system can be written as:

$$\mathbf{K}(\sin(\omega t + \theta)\mathbf{V}) + \mathbf{M}(-\omega^2 \sin(\omega t + \theta)\mathbf{V}) = \mathbf{O}$$

or

$$\begin{aligned} \mathbf{KV} &= \mathbf{M}(-\omega^2\mathbf{V}) \\ (\mathbf{M}^{-1}\mathbf{K}(\mathbf{V} &= -\omega^2\mathbf{V}, \end{aligned}$$

which is the desired system.

(b) Let $\lambda_k = \omega_k^2$. Then $x_j^{(k)} = v_j^{(k)} \sin(\sqrt{\lambda_k} t + \theta)$ for $k = 1, 2, 3$. Therefore,

$$\mathbf{X}_k(t) = \sin(\sqrt{\lambda_k} t + \theta) \left[v_1^{(k)}\ v_2^{(k)}\ v_3^{(k)} \right]'$$

for $k = 1, 2, 3$.

2. (a)

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\left(e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} \left(e^{2t}\right)' \\ \left(e^{2t}\right)' \end{bmatrix} = e^{2t}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\left(e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = e^{3t}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

and

$$\begin{bmatrix} \left(e^{3t}\right)' \\ \left(e^{3t}\right)' \end{bmatrix} = e^{3t}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\left(c_1e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = c_1e^{2t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

and

$$\begin{bmatrix} \left(c_1e^{2t} + c_2e^{3t}\right)' \\ \left(c_1e^{2t} + 2c_2e^{3t}\right)' \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} + 3c_2e^{3t} \\ 2c_1e^{2t} + 6c_2e^{3t} \end{bmatrix} = c_1e^{2t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

3. (a) $\mathbf{X}(t) = \frac{5}{7}e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{3}{7}e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $\mathbf{X}(t) = 6e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + -4e^{-t}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(c) $\mathbf{X}(t) = -\frac{1}{3}e^{t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{2}e^{2t}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \frac{1}{6}e^{4t}\begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}$

11.4 Eigenvalues for Symmetric Matrices

1. Note: $\mathbf{Z}$ and $\mathbf{W}$ are perpendicular if and only if $\mathbf{Z}\cdot\mathbf{W} = 0$. Thus

$$\begin{aligned} \mathbf{Z}\cdot\mathbf{W} &= \tfrac{1}{2}(\mathbf{X}+\mathbf{Y})\cdot\tfrac{1}{c}(\mathbf{X}-\mathbf{Y}) \\ &= \frac{1}{2c}(\mathbf{X}\cdot\mathbf{X} + \mathbf{Y}\cdot\mathbf{X} - \mathbf{X}\cdot\mathbf{Y} + \mathbf{Y}\cdot\mathbf{Y}) \\ &= \frac{1}{2c}\left(\|\mathbf{X}\|^2 - \|\mathbf{Y}\|^2\right) \\ &= \frac{1}{2c}(0) = 0, \end{aligned}$$

since by hypothesis $\|\mathbf{X}\| = \|\mathbf{Y}\|$.

2. $\mathbf{P}' = \left(\mathbf{I} + (-2)\mathbf{X}\mathbf{X}'\right)' = \mathbf{I} + (-2)\left(\mathbf{X}\mathbf{X}'\right)' = \mathbf{I} + (-2)\mathbf{X}\mathbf{X}' = \mathbf{P}$

3. (a) From (2): $\mathbf{P}'\mathbf{P} = \mathbf{P}^2$

   (b) $\mathbf{P}'\mathbf{P} = \mathbf{I}$