## 概率论与数理统计 期中考核

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## **Problem 1**

(1)

首先,对所求项进行适当转换:

$$\int_{-\infty}^{\infty} f\left(x;\mu,\sigma^2\right) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-rac{(x-\mu)^2}{2\sigma^2}} dx \ = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-rac{(x-\mu)^2}{2\sigma^2}} d(x-\mu) \ = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-rac{x^2}{2\sigma^2}} dx \ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\left(rac{x}{\sqrt{2}\sigma}
ight)^2} drac{x}{\sqrt{2}\sigma} \ = \int_{-\infty}^{\infty} rac{1}{\sqrt{\pi}} e^{-x^2} dx$$

接着, 求上一步最终等价项的值:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2 + y^2)} dx dy$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{\infty} \frac{1}{\pi} \cdot e^{-r^2} \cdot r dr$$

$$= \int_{0}^{\infty} e^{-r^2} dr^2$$

$$= 1$$

又因为:

1. 
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx > 0$$
2.  $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$ 

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 1$$

$$\therefore \int_{-\infty}^{\infty} f\left(x;\mu,\sigma^2
ight) dx = 1$$

因此:  $f\left(x;\mu,\sigma^2\right)$  is a legitimate pdf.

**(2)** 

Because X is a continuous rv that has normal distribution

$$\therefore E(x) = \mu \quad V(x) = \sigma^2$$

(3)

$$\Gamma(lpha)=\int_0^\infty x^{lpha-1}e^{-x}dx$$
  $F\left(rac{1}{2}
ight)=\int_0^\infty x^{-rac{1}{2}}e^{-x}dx=\int_0^\infty rac{e^{-x}}{\sqrt{x}}dx$  Let  $:\sqrt{x}=t,x=t^2$ 

$$egin{aligned} \Gamma\left(rac{1}{2}
ight) &= \int_0^\infty rac{e^{-t^2}}{t} dt^2 = \int_0^\infty 2e^{-t^2} dt = 2 imes rac{\sqrt{\pi}}{2} = \sqrt{\pi} \ dots F\left(rac{1}{2}
ight) &= \sqrt{\pi} \end{aligned}$$

**(4)** 

$$\begin{split} \int_{-\infty}^{\infty} f(x; \alpha, \beta) dx &= \int_{0}^{\infty} f(x; \alpha, \beta) dx \\ &= \frac{\int_{0}^{\infty} \frac{1}{\beta^{\alpha}} \cdot x^{\alpha - 1} \cdot e^{\frac{-x}{\beta}} dx}{\Gamma(\alpha)} \\ &= \frac{\int_{0}^{\infty} (\frac{x}{\beta^{2}})^{\alpha - 1} \cdot e^{-\left(\frac{x}{R}\right)} d\left(\frac{x}{\beta}\right)}{\Gamma(\alpha)} \\ &= \frac{\int_{0}^{\infty} x^{\alpha - 1} e^{-x} dx}{\sqrt{(\alpha)}} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \\ &= 1 \end{split}$$

 $\therefore f(x; \alpha, \beta)$  is a legitimate pdf.

(5)

$$f(x;0,1)=rac{1}{\sqrt{2\pi}}e^{-rac{x^2}{2}}$$

Let:

1. 
$$Y = x^2$$

2. the distribution function of X and Y :  $F_X(x), F_Y(y)$ 

$$\therefore Y = x^2 \geqslant 0$$

$$\therefore y \leqslant 0$$
 时  $F_Y(y) = 0$ 

当y > 0时:

$$egin{aligned} F_Y(y) &= P(Y \leqslant y) \ &= P\left(x^2 \leq y
ight) = P(-\sqrt{y} \leq x \leq \sqrt{y}) \ &= F_x\left(\sqrt{y}
ight) - F_x(-\sqrt{y}) \end{aligned}$$

将 $F_Y(y)$ 关于y求导得:

$$f_{Y(y)} \, = egin{cases} rac{1}{2\sqrt{y}}igl[f_x(\sqrt{y}) + f_x(-\sqrt{y})igr] & y>0 \ 0 & y\leq 0 \end{cases}$$

代入
$$f_x(x)=rac{1}{\sqrt{2\pi}}e^{-rac{x^2}{2}}$$
得:

$$f_Y(y) = \left\{ egin{array}{ll} rac{1}{\sqrt{2\pi}}y^{-rac{1}{2}}e^{-rac{y}{2}} & y>0 \ 0 & y\leq 0 \end{array} 
ight.$$

(6)

$$f(x,y) = f_X(x) \cdot f_Y(y) \quad f(x;0,1) = rac{1}{\sqrt{2\pi}} e^{-rac{x^2}{2}} \quad f(y;0,1) = rac{1}{\sqrt{2\pi}} e^{-rac{y^2}{2}}$$

$$\therefore f(x,y) = rac{1}{2\pi} \cdot e^{-rac{x^2+y^2}{2}}$$

Let: 
$$Z=X^2+Y^2$$
  $W=rac{X}{Y}$ 

可假设 W>0, 而  $W\leq0$  的情况同理亦可证得

$$egin{align} F_Z(z) &= P\left(X^2 + Y^2 \leq z
ight) = rac{1}{2\pi} \iint_{X^2 + Y^2 \leq z} e^{-rac{x^2 + y^2}{2}} dx dy \ &= rac{1}{2\pi} \int_0^{2\pi} d heta \int_0^z e^{-rac{r^2}{2}} drac{r^2}{2} \ &= 1 - e^{-rac{z^2}{2}} \end{aligned}$$

Let:  $\alpha = an^{-1} w$  which means  $(\alpha = \arctan w)$ 

$$egin{aligned} F_W(w) &= P\left(rac{X}{Y} \leq w
ight) = rac{1}{2\pi} \iint_{rac{X}{Y} \leq w} e^{-rac{x^2+y^2}{2}} dx dy \ &= rac{1}{2\pi} \int d heta \int_0^\infty e^{-rac{r^2}{2}} drac{r^2}{2} \ &= rac{1}{2\pi} \int d heta \ &= rac{1}{2\pi} \left(\int_{rac{\pi}{2} - lpha}^\pi d heta + \int_{rac{3\pi}{2} - lpha}^{2\pi} d heta 
ight) \ &= rac{1}{2\pi} imes (2lpha + \pi) = rac{lpha}{\pi} + rac{1}{2} \ &= rac{1}{\pi} an^{-1} w + rac{1}{2} \end{aligned}$$

$$egin{aligned} F_{ZW}(z,w) &= \iint_{D_{ZW}} f_{XY}(x,y) dx dy \ &= \iint_{D_{ZW}} f\left( \mathrm{rcos} heta, \ \mathrm{rsin} \ heta 
ight) r d heta dr \ &= \int_{rac{\pi}{2} - lpha}^{\pi} d heta \int_{0}^{z} rac{1}{2\pi} \cdot e^{-rac{r^2}{2}} r dr + \int_{rac{3\pi}{2} - lpha}^{2\pi} d heta \int_{0}^{z} rac{1}{2\pi} \cdot e^{-rac{r^2}{2}} r dr \ &= rac{1}{2\pi} imes (lpha + lpha + \pi) imes \left( 1 - e^{-rac{z^2}{2}} 
ight) \ &= (rac{1}{\pi} imes an^{-1} w + rac{1}{2}) imes \left( 1 - e^{-rac{z^2}{2}} 
ight) \end{aligned}$$

$$f_{Z}(z)=F_{Z}'(z)=ze^{-rac{z^2}{2}} \quad f_{W}(w)=F_{w}'(w)=rac{1}{\pi}\cdotrac{1}{1+w^2} \ f(z,w)=rac{\partial}{\partial z}\cdotrac{\partial}{\partial w}F_{ZW}(z,w)=rac{1}{\pi}\cdotrac{1}{1+w^2}\cdot z\cdot e^{-rac{z^2}{2}}$$

$$\therefore f(z,w) = f_Z(z) \cdot f_W(w)$$

 $\therefore$  The rv  $x^2+y^2$  and  $rac{X}{Y}$  are also independent.