Chapter 2. Probability

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What is probability?

- Probability

The term probability refers to the study of **randomness** and **uncertainty**.

In any situation in which one of a number of possible outcomes may occur, the theory of probability provides methods for **quantifying** the changes, or likehoods, associated with the various outcomes.

Chapter two: Probability

- 2.1 Sample Spaces and Events
- 2.2 Axioms, Interpretations, and Properties of Probability
- 2.3 Counting Techniques
- 2.4 Conditional Probability
- 2.5 Independence

2.1 Sample Spaces and Events

- Experiment

  An experiment is any activity or process whose outcome is subject to uncertainty, ***e.g.***

  tossing a coin once or several times

  selecting a card or cards from a deck, *etc*.

- Sample Space

  The sample space of an experiment, denoted by ***S*** , is the set of all possible outcomes of that experiment, ***e.g.***

  Examining whether a single fuse is defective or not. The two possible outcomes: D (defective) & N(not defective)

  Two fuses in sequence: ***S*** ={DD DN ND NN}

2.1 Sample Spaces and Events

- Example 2.3

Two gas stations are located at a certain intersection. Each one has six gas pumps. Consider the experiment in which the number of pumps in use at a particular time of day is determined for each of the stations. The possible outcomes:

	0	1	2	3	4	5	6
0	(0 0)	(0 1)	(0 2)	(0 3)	(0 4)	(0 5)	(0 6)
1	(1 0)	(1 1)	(1 2)	(1 3)	(1 4)	(1 5)	(1 6)
2	(2 0)	(2 1)	(2 2)	(2 3)	(2 4)	(2 5)	(2 6)
3	(3 0)	(3 1)	(3 2)	(3 3)	(3 4)	(3 5)	(3 6)
4	(4 0)	(4 1)	(4 2)	(4 3)	(4 4)	(4 5)	(4 6)
5	(5 0)	(5 1)	(5 2)	(5 3)	(5 4)	(5 5)	(5 6)
6	(6 0)	(6 1)	(6 2)	(6 3)	(6 4)	(6 5)	(6 6)

2.1 Sample Spaces and Events

- Example 2.4

 If a new type-D flashlight battery has a voltage that is outside certain limits, that battery is characterized as a failure(F); otherwise, it is a success(S).

 Suppose an experiment consists of testing each battery as it comes off an assembly line until we first observe a success. The sample space is

$$\boldsymbol{S} = \{S, FS, FFS, FFFS, \ldots\}$$

which contains an infinite number of possible outcomes.

2.1 Sample Spaces and Events

- Event

An event is any collection (subset) of outcomes contained in the sample space ***S***.

- Simple Event

  An event consists of exactly one outcome

- Compound Event

  An event consists of more than one outcome

2.1 Sample Spaces and Events

- Example 2.5

  Consider an experiment in which each of three vehicles taking a particular freeway exit turns left(L) or right(R) at the end of the exit ramp.

  The 8 possible outcomes (simple events):

  {LLL, RLL, LRL, LLR, LRR, RLR, RRL RRR }

  Some compound events include

- the event that exactly one of the three vehicles turns right : {RLL, LRL, LLR}
- the event that all three vehicles turns in the same direction: {LLL, RRR}

2.1 Sample Spaces and Events

- Example 2.6 (Ex.2.3 continued)

➢ The event that the number of pumps in use is the same for both stations:

$$\{(0,0),\{1,1\},\{2,2\},\{3,3\}, \{4,4\},\{5,5\},\{6,6\}\}$$

➢ The event that the total number of pumps in use is four

$$\{(0,4), \{1,3\}, \{2,2\}, \{3,1\}, \{4,0\}\}$$

➢ The event that at most one pump is in use at each station

$$\{(0,0),\{0,1\},\{1,0\},\{1,1\}\}$$

2.1 Sample Spaces and Events

- Example 2.7 (Ex. 2.4 continued)

➢ the event that at most three batteries are examined:

{S, FS, FFS}

➢ the event that an even number of batteries are examined

{FS, FFFS, FFFFFS, ...}

2.1 Sample Spaces and Events

- An event is nothing but a set, so that relationships and results from elementary set theory can be used to study events. The following concepts from set theory will be used to construct new events from given events.
- **Union** of two events A and B, denoted by $A \cup B$, and read "A or B", that is, all outcomes in at least one of the events A and B.
- **Intersection** of two events A and B, denoted by $A \cap B$ and read A and B is the event consisting of all outcomes that are in both A and B.
- **Complement** of an event A, denoted by A', is the set of all outcomes in S that are no contained in A

2.1 Sample Spaces and Events

- Example 2.8 (Ex. 2.3 continued)

  For the experiment in which the number of pumps in use at a single six-pump gas station is observed.

  Let A={0,1,2,3,4}, B={3,4,5,6} and C={1,3,5} . Then

  $A \cup B = \{0,1,2,3,4,5,6\} = \boldsymbol{S}$,

  $A \cup C = \{0,1,2,3,4,5\}$

  $A \cap B = \{3,4\}$

  $A \cap C = \{1,3\}$

  $A' = \{5,6\}$

  $(A \cup C)' = \{6\}$

2.1 Sample Spaces and Events

- Mutually exclusive (disjoint) events

A and B have no outcomes in common, namely

$$A \cap B = \Phi$$

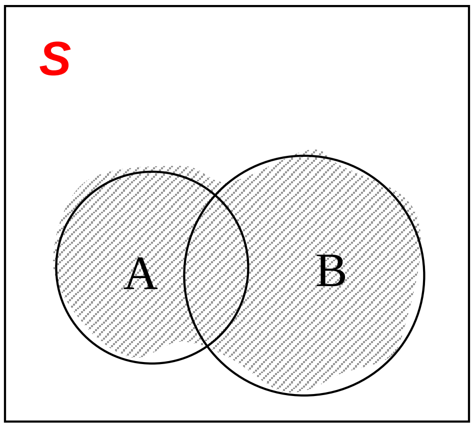
Example

A = {Chevrolet, Pontiac, Buick}

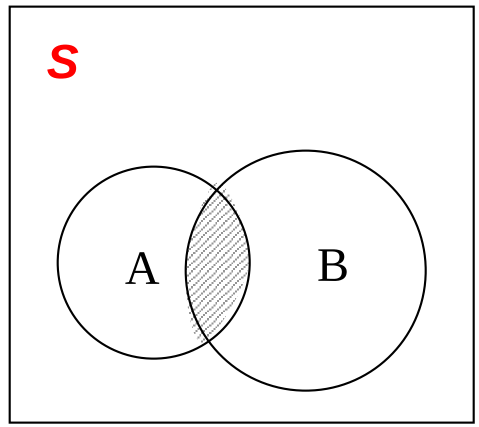
B = {Ford, Merrcury}

2.1 Sample Spaces and Events

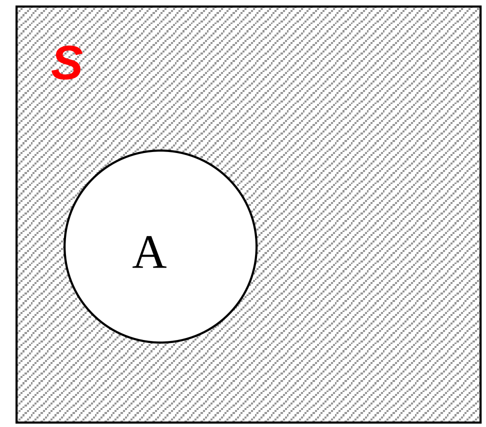
- Venn Diagrams



$A \cup B$

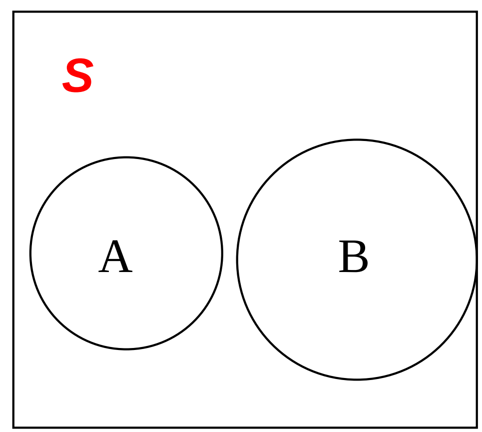


$A \cap B$



$\overline{A}$

Universal set: the sample space S
Event : Subset of S
Element (object): Individual Outcome



Disjoint events

Homework

- Ex. 2, Ex. 4, Ex. 9

2.2 Axioms, Interpretations, and Properties of Probability

- Given an experiment and a sample space ***S***, the objective of probability is to assign to each event A a number P(A), called the probability of the event A, which will give a precise measure of the chance that A will occur. All assignments should satisfy the three following axioms of probability.

- Axiom 1: for any event A, P(A)>=0
- Axiom 2: P(S)=1
- Axiom 3: if $A_1, A_2, ..A_k$ (/…)is a finite / infinite collection of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i) \qquad P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

- Example

In the experiment in which a single coin is tossed, the sample space is S={H,T}. Then

$$P(S) = P(H) + P(T) = 1$$

since $H \cup T = S$ & $H \cap T = \phi$

Let P(H) = p, where p is any fixed number between 0 and 1, then P(T) = 1- p is an assignment consistent with the axioms.

- Example 2.12 (Ex. 2.4 continued)

$E_1=\{S\}$, $E_2=\{FS\}$, $E_3=\{FFS\}$, $E_4=\{FFFS\}\ldots$

Support the probability of any particular battery being satisfactory is 0.99, then

$$P(E_1) = 0.99$$

$$P(E_2) = 0.01 \times 0.99$$

$$P(E_3) = (0.01)^2 \times 0.99 \ldots$$

Note: $\boldsymbol{S} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup \ldots$

and $E_i \cap E_j = \phi$ (i is not j)

$$P(S) = 1 = P(E_1) + P(E_2) + P(E_3) + \ldots$$

- Two Special Events
  - Impossible event

    The event contains no simple event
  - Certain event

    The event contains all simple events

Suppose A is an impossible event and B is a certain event, then $P(A)=0$, $P(B)=1$

Q: $P(A)=0 \rightarrow$ A is an impossible event ?

$P(B)=1 \rightarrow$ B is a certain event ?

2.2 Axioms, Interpretations, and Properties of Probability

- Interpreting Probability

- Axioms #1-3 serve only to rule out assignments inconsistent with our intuitive notions of probability.
- Methods for assigning appropriate/correct probability

  1. Based on repeatedly experiments **(objective)**, ***e.g.*** coin-tossing

  2. Based on some reasonable assumption or prior information **(subjective)**, ***e.g.*** a *fair* die

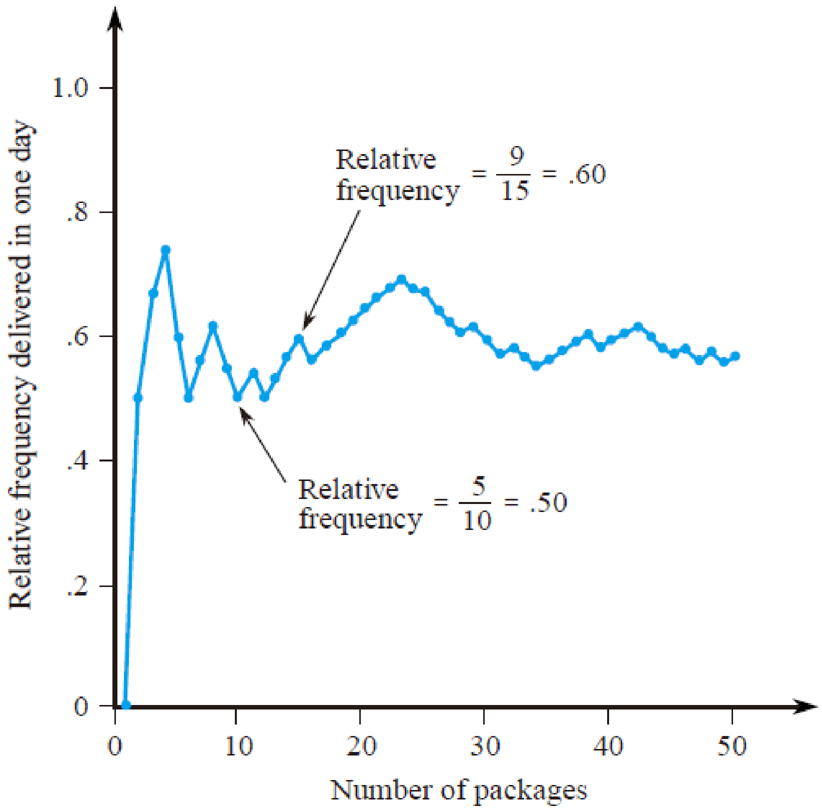
Note: May be different for different observers.

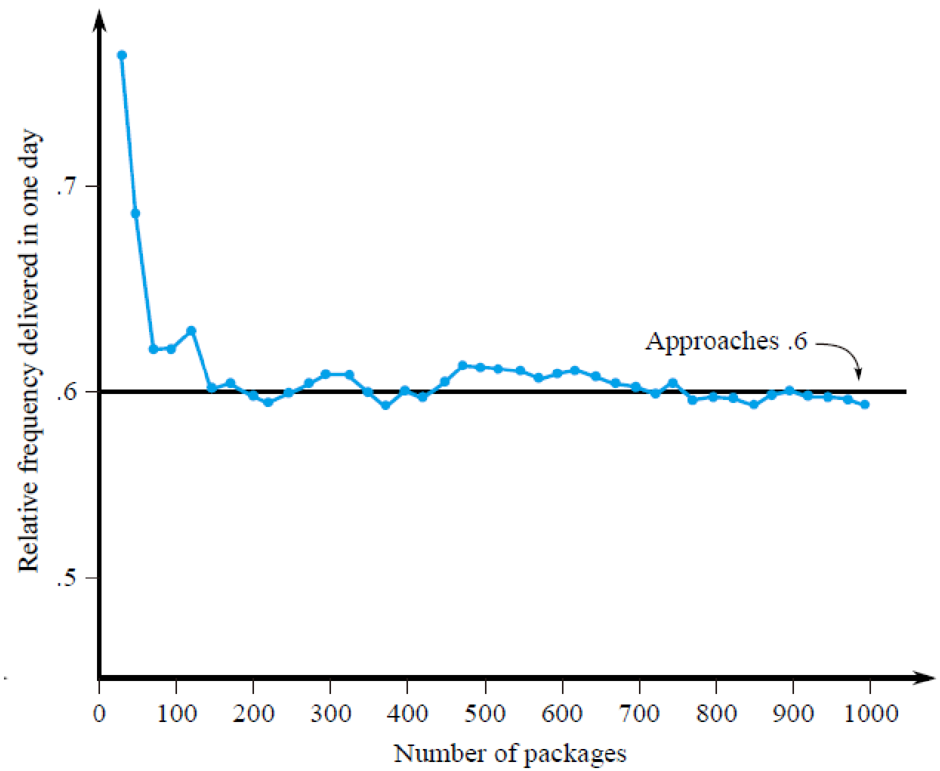
2.2 Axioms, Interpretations, and Properties of Probability

- Consider an experiment that can be repeatedly performed in an identical and independent fashion, and let *A* be an event consisting of a fixed set of outcomes of the experiment.
- If the experiment is performed *n* times, on some of the replications the event *A* will occur, and on others, *A* will not occur
- Let *n*(*A*) denote the number of replications on which *A* does occur. Then the ratio is called the *relative frequency* of occurrence of the event *A* in the sequence of *n* replications.

Package #	1	2	3	4	5	6	7	8	9	10
Did *A* occur?	N	Y	Y	Y	N	N	Y	Y	N	N
Relative frequency of *A*	0	.5	.667	.75	.6	.5	.571	.625	.556	.5

2.2 Axioms, Interpretations, and Properties of Probability





The *objective interpretation of probability* identifies this limiting relative frequency with *P*(*A*).

- **Property #1**

For any event A, $P(A)=1 - P(A')$

Proof:

By the definition of A', we have

$$\boldsymbol{S} = A \cup A' , A \cap A' = \phi$$

Since

$$1=P(S) = P(A \cup A') = P(A) + P(A')$$

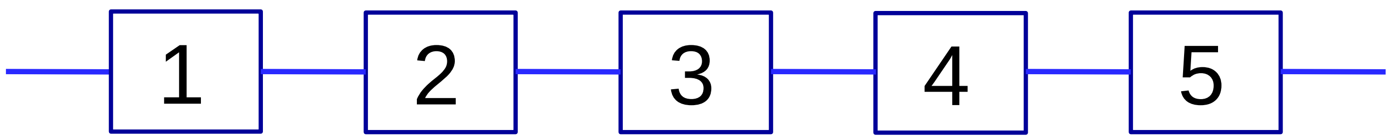
then

$$P(A) = 1- P(A')$$

2.2 Axioms, Interpretations, and Properties of Probability

- Example 2.13

Consider a system of five identical components connected in series, as illustrated in the following figure



Denote a component that fails by F and one that doesn't fail by S. Let A be the event that the system fails.

A={FSSSS, SFSSS, …} there are 31 different outcomes in A. However, A' the event that the system works, consists of the single outcome SSSSS.

$$P(A) = 1- P(A') = 1-0.9^5=0.41$$

- **Property #2:**

  If A and B are mutually exclusive, then $P(A \cap B) = 0$

Proof:

Because $A \cap B$ contains no outcomes, $(A \cap B)' = \boldsymbol{S}$.

Thus we have that

$$P(S) = P[(A \cap B)'] + P(A \cap B) = P(S) + P(A \cap B)$$

which implies

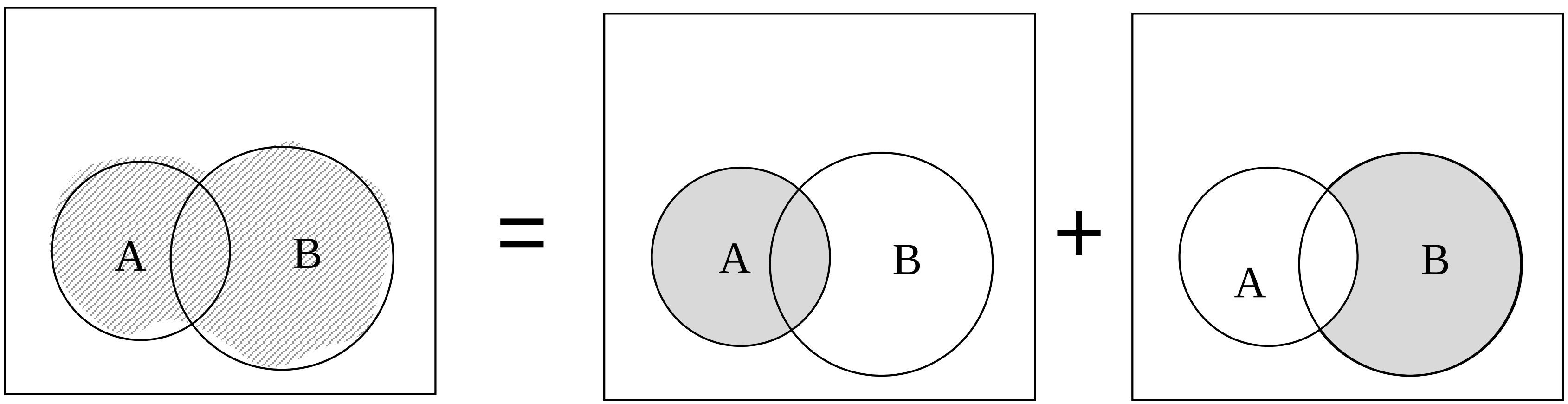
$$P(A \cap B) = 0$$

2.2 Axioms, Interpretations, and Properties of Probability

- **Property #3:**

  For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$A \cup B$ $\quad$ A $\quad$ $B \cap A'$

Proof:

$$P(A \cup B) = P(A) + P(B \cap A')$$
$$= P(A) + [P(B) - P(A \cap B)]$$

Note: $B = (B \cap A') \cup (A \cap B)$
& $(B \cap A') \cap (A \cap B) = \phi$

- Example 2.14

  A = {gets Internet service}

  B ={gets TV service}

  $P(A) = 0.6$, $P(B) = 0.8$, $P(A \cap B) = 0.5$

  P(subscribes to at least one of the two services)

$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6+0.8-0.5=0.9$

  P(exactly one)

$= P(A \cap B') + P(A' \cap B) =0.1+0.3 =0.4$

Note: $(A \cap B') + (A \cap B) = A$ & $(A \cap B') \cap (A \cap B) = \phi$

$(A' \cap B) + (A \cap B) = B$ & $(A' \cap B) \cap (A \cap B) = \phi$

- Determining Probabilities Systematically

  When the number of possible outcomes (simple events) is large, there will be many compound events. A simple way to determine probabilities for these events is that

- First determine probability $P(E_i)$ for all simple events.

  Note: $P(E_i) >= 0$ and $\sum_{\text{all } i} P(E_i) = 1$

- The probability of any compound event A is computed by adding together the $P(E_i)$'s for all E_i's in A

$$P(A) = \sum_{\text{all } E_i\text{'s in } A} P(E_i)$$

Note: $E_i \cap E_j = \phi$, i is not j

- Example 2.15

Denote the six elementary events {1}, {2}, …{6} associated with tossing a six-sided die once by $E_1, E_2, \ldots E_6$. If the die is constructed so that any of the three even outcomes is twice as likely to occur as any of the three odd outcomes (**unfair die**), then an appropriate assignment of probabilities to elementary events is $P(E_1)=P(E_3)=P(E_5) = 1/9$ and $P(E_2)=P(E_4)=P(E_6)=2/9$, then

the event A={outcome is even}= $E_2 \cup E_4 \cup E_6$

$$P(A) = P(E_2)+P(E_4)+P(E_6)=2/3$$

the event B={outcome <=3}= $E_1 \cup E_2 \cup E_3$

$$P(B) = P(E_1)+P(E_2)+P(E_3)=4/9$$

- Equally Likely Outcomes

In many experiments consisting of ***N*** outcomes, it is reasonable to assign equal probabilities to all ***N*** simple events. e.g. tossing a fair coin or fair die, selecting cards from a well-shuffled deck of 52.

With $p=P(E_i)$ for every i, then

$$1 = \sum_{i=1}^{N} P(E_i) = \sum_{i=1}^{N} p \Rightarrow p = 1/N$$

Consider an event A, with N(A) denoting the number of outcomes containing in A, then

$$P(A) = \sum_{Ei\text{'s in }A} P(E_i) = \sum_{Ei\text{'s in }A} 1/N = N(A)/N$$

- Example

When two dice are rolled separately, there are N=36 outcomes. If both the dice are fair, all 36 outcomes are equally likely, so $P(E_i) = 1/36$. Then the event A={sum of two number =7} consists of the six outcomes (1,6) (2,5) (3,4), (4,3), (5,2) and (6,1), so

$$P(A) = N(A)/N = 6/36 = 1/6$$

Homework

- Ex. 12, Ex. 17, Ex. 24, Ex. 27

2.3 Counting Techniques

- When the various outcomes of an experiment are equally likely, the task of computing probabilities reduces to counting. In particular, in ***N*** is the number of outcomes in a sample space and N(A) is the number of outcomes contained in an event A, then

$$P(A) = \frac{N(A)}{N}$$

- Product Rule
- Permutations
- Combinations

2.3 Counting Techniques

- Ordered Pair

By an ordered pair, we mean that, if O_1 and O_2 are ***different*** objects, then the pair (O_1,O_2) is different from the pair (O_2,O_1).

- Counting the number of ordered pair

If the first element of object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is n_1n_2

- Example 2.17

A homeowner doing some remodeling requires the services of both a plumbing contractor and an electrical contractor. If there are 12 plumbing contractor $P_1, P_2, \ldots P_{12}$ and 9 electrical contractors $Q_1, Q_2, \ldots, Q_9$ available in the area, in how many ways can the contractors be chosen?

Task: counting the number of pairs of the form (P_i, Q_j)

With $n_1 = 12$, $n_2 = 9$, the produce rule yields $N = 12 \times 9 = 108$

Note: In this example, the choice of the second element of the pair did not depend on which first element was chosen or occurred.

As long as there is the same number of choices of the second element for each first element, the product rule is valid even when the second elements depends on the first ones, see Ex. 2.18.

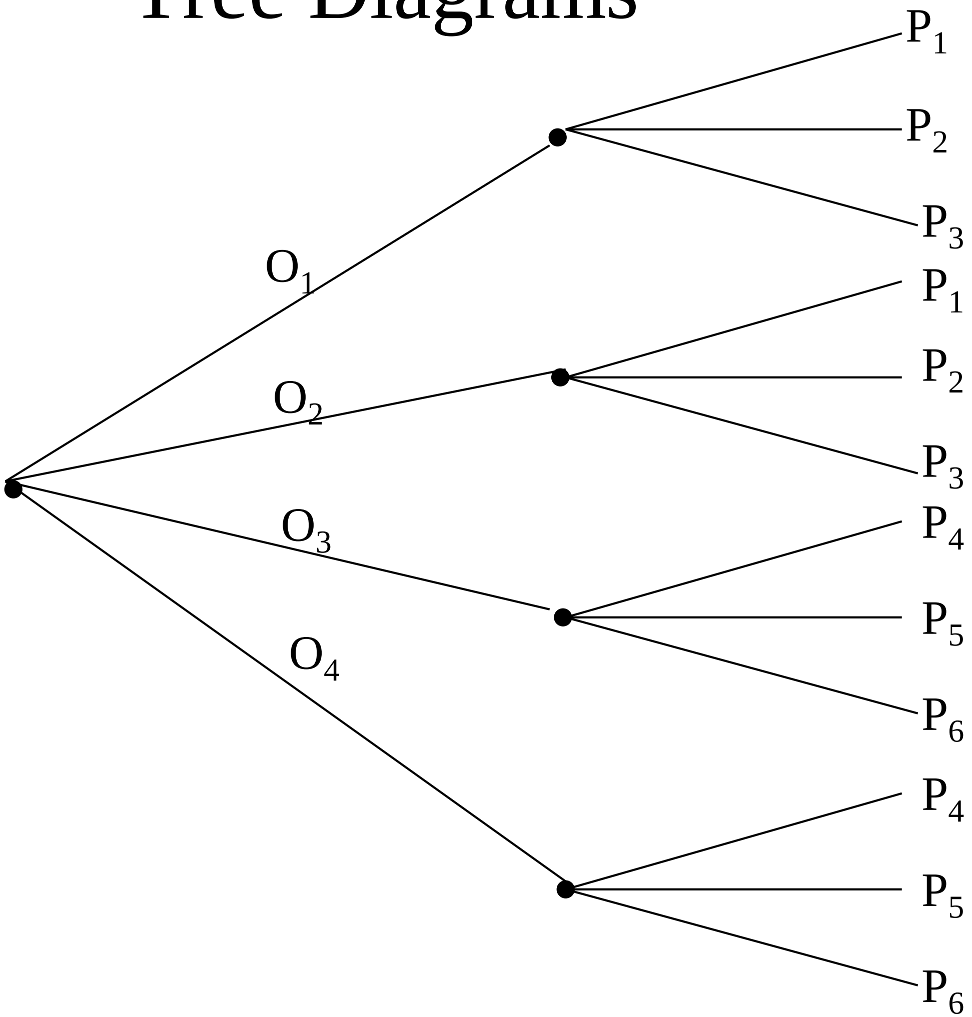
- Example 2.18

A family has just moved to a new city and requires the services of both an obstetrician and a pediatrician. There are two easily accessible medical clinics, each having 2 obstetricians and 3 pediatricians. The family will obtain maximum health insurance benefits by joining a clinic and selecting both doctors from that clinic. In how many ways can this be done?

Denote the obstetricians by O_1, O_2, O_3, and O_4 and the pediatricians by $P_1, \ldots, P_6$. Then we wish the number of pairs (O_i, P_j) for which O_i and P_j are associated with the same clinic. Because there are four obstetricians, $n_1=4$, and for each there are three choices of pediatrician, so $n_2=3$. Applying the product rule gives $N= n_1 n_2=12$ possible choices.

2.3 Counting Techniques

- Tree Diagrams



Note: The construction of a tree diagram does not depend on having the same number of Second-generation braches emanating from each First-generation branch.

Thus a tree diagram can be used to represent pictorially experiments other than those to which the product rule applies.

2.3 Counting Techniques

- K-tuple

  An ordered collection of k objects

- Product Rule for k-Tuple

  Support a set consists of ordered collections of k elements (k-tuples) and that there are n_1 possible choices for the first element; for each choice of the first element, there are n_2 possible choices of the second element; …, for each possible choice of the first k-1 elements, there are n_k choices of the k-th element. Then there are $n_1 n_2 \ldots n_k$ possible k-tuples.

2.3 Counting Techniques

- Example 2.19 (Ex.2.17 continued)

  Suppose the home remodeling job involves first purchasing several kitchen appliances. They will all be purchased from the same dealer, and there are five dealers in the area. With the dealers denoted by D_1, D_2, …D_5, there are $N=n_1n_2n_3=5\times 12\times 9=540$, 3-tuples of the form (D_i,P_j,Q_k), so there are 540 ways to choose first an appliance dealer, then a plumbing contractor, and finally an electrical contractor.

2.3 Counting Techniques

- Permutation

  Any **ordered** sequence of k objects taken from a set of n distinct objects is called a permutation of size k of the objects. The number of permutations of size k that can be constructed from the n objects is denoted by $P_{k,n}$.

$$P_{k,n} = n(n-1)(n-2)\ldots(n-k+1)$$

$$P_{k,n} = \frac{n!}{(n-k)!}$$

2.3 Counting Techniques

- Example 2.21

There are 10 teaching assistants available for grading papers in a particular course. The first exam consists of 4 questions, and the professor wishes to select a different assistant to grade each question (only one assistant per question). In how many ways can assistants be chosen to grade the exam?

Here

n = the number of assistants=10 &

k = the number of questions =4.

The number of different grading assignments is then

$$P_{k,n}=10\times 9\times 8\times 7=5040$$

2.3 Counting Techniques

- Birthday Paradox

In a set of n randomly chosen people (n is less than 366), what is the probability that some pair of them having the same birthday?

Let A={at least one pair of the n people having the same birthday}.

Then A' = {all the n people have different birthday}.

$$P(A') = \frac{P_{n,365}}{365^n} \qquad P(A) = 1 - P(A') = 1 - \frac{P_{n,365}}{365^n}$$

n	10	20	23	30	40	50
P(A)	0.12	0.41	0.51	0.71	0.89	0.97

- Combinations

Given a set of n distinct objects, any **unordered** subset of size k of the objects is called a combination. The number of combinations of size k that can be formed from the n distinct objects will be denoted by $C_{k,n}$

$$C_{k,n} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Note: the number of combinations of size k from a particular set is smaller than the number of permutations because, when order is disregarded, a number of permutations correspond to the same combination.

- Example 2.23

A university warehouse has received a shipment of 25 printers, of which 10 are laser printer and 15 are inkjet models. If 6 of these 25 are selected at random to be checked by a particular technician, what is the probability that exactly 3 of those selected are laser printers?

Let D_3 ={exactly 3 of the 6 selected are inkjet printers}

$$P(D_3) = \frac{N(D_3)}{N} = \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} = \frac{\frac{15!}{3!12!} \times \frac{10!}{3!7!}}{\frac{25!}{6!19!}} = 0.3083$$

Homework

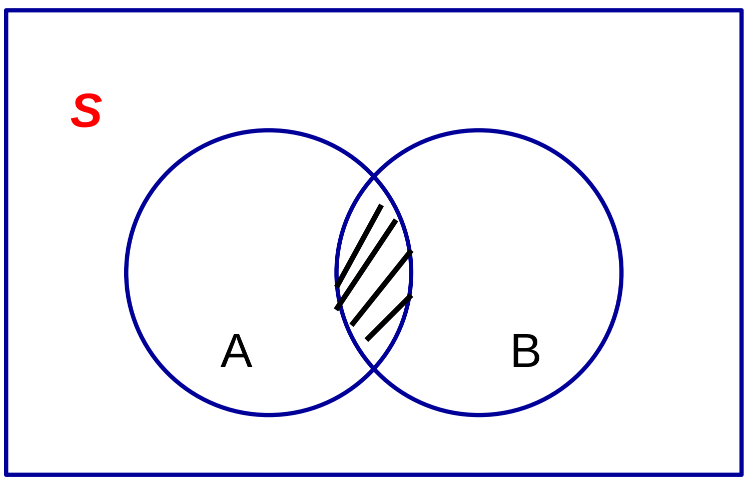
- Ex. 30, Ex. 39, Ex. 44

2.4 Conditional Probability

- Definition of Conditional Probability

For any two events A and B with P(B)>0, the conditional probability of A given that B has occurred is defined by

$$P(A|B)=\frac{P(A\bigcap B)}{P(B)}$$



Note:

1. Given that B has occurred, the relevant sample space is no longer ***S*** but consists of outcomes in B;
2. A has occurred if and only if one of the outcomes in the intersection occurred.

2.4 Conditional Probability

- Example 2.24

Complex components are assembled in a plant that uses two different assembly lines, A and A'. Line A uses older equipment than A', so it is somewhat slower and less reliable. Suppose on a given day line A has assembled 8 components, of which 2 have been identified as defective (B) and 6 as nondefective (B'), whereas A' has produced 1 defective and 9 nondefective components. This information is summarized in the accompanying table.

		Condition	
		B	B'
Line	A	2	6
	A'	1	9

2.4 Conditional Probability

		Condition	
		B	B’
Line	A	2	6
	A’	1	9

The sales manager randomly selects 1 of these 18 components for a demonstration. Prior to the demonstration

$$P(\text{line} \quad A \ \text{ component} \quad \text{selected}) = P(A) = \frac{N(A)}{N} = \frac{8}{18}$$

However, if the chosen component turns out to be defective, then the event B has occurred, so the component must have been 1 of the 3 in the B column of the table. Since these 3 components are equally likely among themselves after B has occurred,

$$P(A \mid B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A \cap B)}{P(B)}$$

2.4 Conditional Probability

- Example 2.25

Consider randomly selecting a buyer and let A={memory card purchased} and B={battery purchased}. Then P(A)=0.6, P(B)=0.4 and P(both purchased) = P(A∩B) = 0.3. Given that the selected individual purchased an extra battery, the probability that an optional card was also purchased is

$$P(A|B)=\frac{P(A\bigcap B)}{P(B)}=\frac{0.3}{0.4}=0.75$$

That is, of all those purchasing an extra battery, 75% purchased an optional memory card. Similarly,

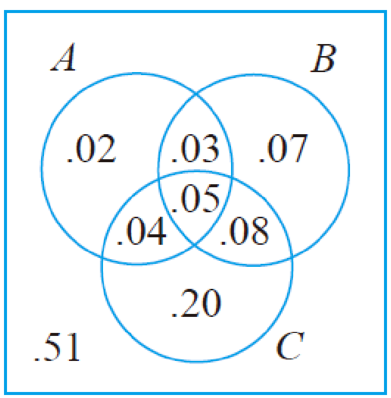
$$P(B|A)=\frac{P(A\bigcap B)}{P(A)}=\frac{0.3}{0.6}=0.5 \quad \neq P(A|B)$$

2.4 Conditional Probability

- Example 2.26

A news magazine publishes three columns entitled “Art”(A), “Books”(B), and “Cinema”(C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A\cap B$	$A\cap C$	$B\cap C$	$A\cap B\cap C$
Probability	0.14	0.23	0.37	0.08	0.09	0.13	0.05



$P(A|B) = 0.348$

$P(A|B\cup C) = 0.255$

$P(A|A\cup B\cup C) = 0.286$

$P(A\cup B|C) = 0.459$

2.4 Conditional Probability

- The Multiplication Rule

$$P(A \cap B) = P(A \mid B)\, P(B)$$

This rule is important because it is often the case that $P(A \cap B)$ is desired, whereas that both $P(B)$ and $P(A|B)$ can be specified from the problem description.

Note :

1. $P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$
2. $P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_1 \cap A_2)\, P(A_1 \cap A_2)$

$$= P(A_3 \mid A_1 \cap A_2)\, P(A_2 \mid A_1)\, P(A_1)$$

2.4 Conditional Probability

- Example 2.27

Four individuals have responded to request by a blood bank for blood donations. None of them has donated before, so their blood types are unknown. Suppose only type O+ is desired and only one of the four actually has this type. If the potential donors are selected in random order for typing, what is the probability that at least three individuals must be typed to obtain the desired type?

Let B={first type not O+}, A = {second type not O+}

P(at least three individuals must be typed) = $P(A \cap B)$

we know that $P(B) = 3/4$ and $P(A \mid B) = 2/3$ (why?)

Based on The Multiplication Rule, we have

$$P(A \cap B) = P(A \mid B)\, P(B) = 0.5$$

- Example 2.28

For the blood typing experiment of Example 2.27,

P(third type is O+)

= P(third is ∩ first isn't ∩ second isn't)

= P(third is | first isn't ∩ second isn't) P(first isn't ∩ second isn't)

= P(third is | first isn't ∩ second isn't) P(second isn't | first isn't) P(first isn't)

= 1/2 × 2/3 × 3/4 = 0.25

2.4 Conditional Probability

- Example 2.29

  A chain of video stores sells three different brands of VCRs. Of its VCR sales, 50% are brand 1, 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's VCRs require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

  1. What is the probability that a randomly selected purchaser has bought a brand 1 VCR that will need repair while under warranty?
  2. What is the probability that a randomly selected purchaser has a VCR that will need repair while under warranty?
  3. If a customer returns to the store with a VCR that needs warranty repair work, what is the probability that it is a brand 1 VCR? A brand 2 VCR? A brand 3 VCR?

2.4 Conditional Probability

- Example 2.29 (Cont')

First Stage:

a customer selecting one of the three brands of VCR

Let $P(A_i)$ = {brand i is purchased}, where i= 1, 2, 3

then $P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$

Second Stage:

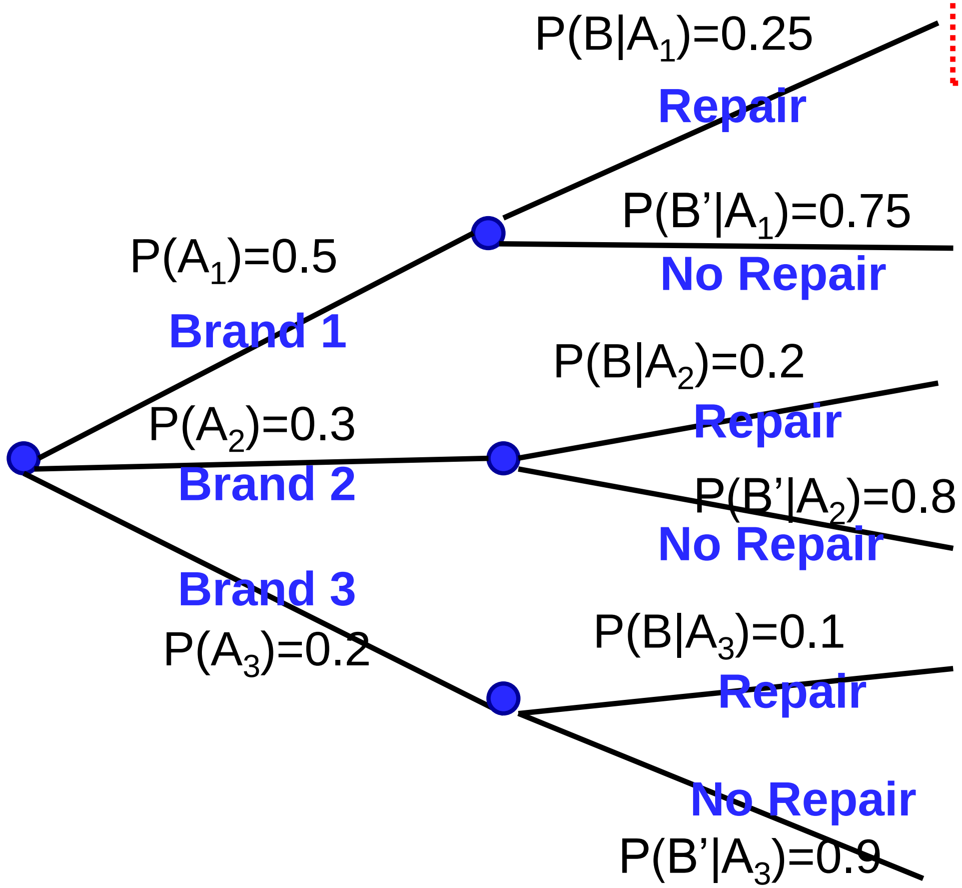
observing whether the selected VCR needs warranty repair

Let B = {needs repair} B'={doesn't need repair}

then $P(B|A_1) =0.25$, $P(B|A_2) =0.20$, $P(B|A_3) =0.10$

2.4 Conditional Probability

- Example 2.29 (Cont’)



Q1

$P(A_1 \cap B) = P(A_1)\ \ P(B|A_1) = 0.125$

$P(A_2 \cap B) = P(A_2)\ \ P(B|A_2) = 0.06$

$P(A_3 \cap B) = P(A_3)\ \ P(B|A_3) = 0.02$

Q2

$P(B) = 0.125+0.06+0.02=0.205$

2.4 Conditional Probability

- Example 2.29 (Cont')

Q3

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.125}{0.205} = 0.61$$

$$P(A_2 \mid B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.06}{0.205} = 0.29$$

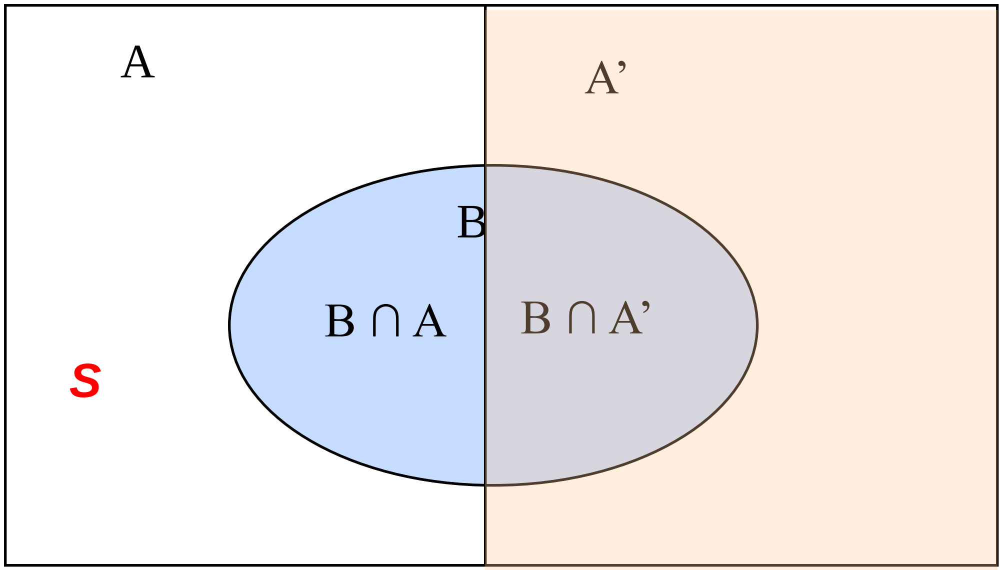
$$P(A_3 \mid B) = 1 - P(A_2 \mid B) - P(A_1 \mid B) = 0.1$$

2.4 Conditional Probability

- The Law of Total Probability (2-D case)

$$P(B) = P(B \cap A) + P(B \cap A')$$

$$= P(B \mid A)P(A) + P(B \mid A')P(A')$$



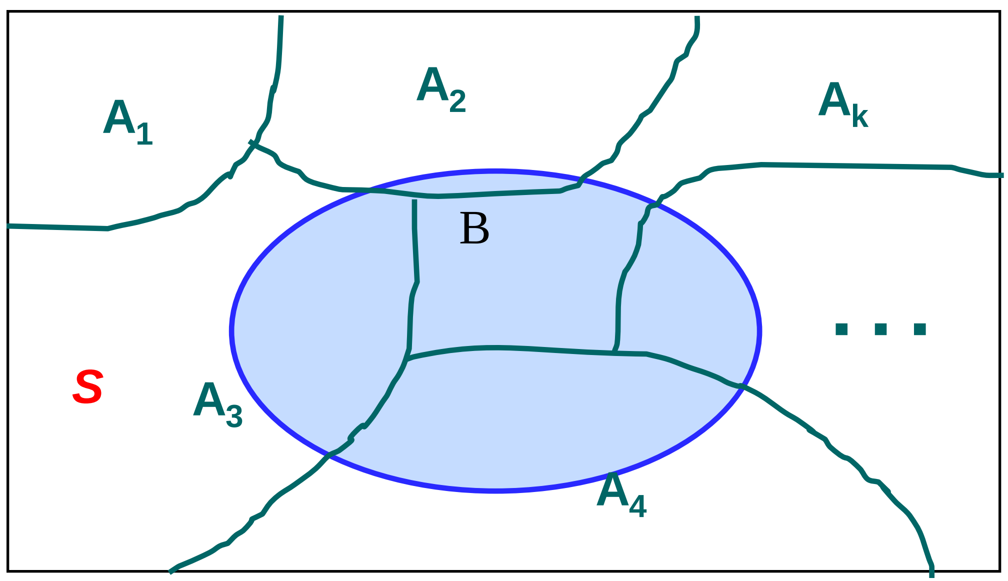
Note:
A U A' =S
A ∩ A' = ф

2.4 Conditional Probability

- The Law of Total Probability (general cases)

Let $A_1, \ldots A_k$ be mutually exclusive and exhaustive events (Partition of **S**). Then for any other event B

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(A_i) P(B \mid A_i)$$



2.4 Conditional Probability

- Bayes' Theorem

Let $A_1, A_2, \ldots A_k$ be a collection of k multually exclusive and exhaustive events with P(A)>0 for i=1,…k, then for any other event B for which P(B)>0.

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j)P(B \mid A_j)}{P(B)}$$

$$= \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)} \qquad j = 1, 2, \ldots, k$$

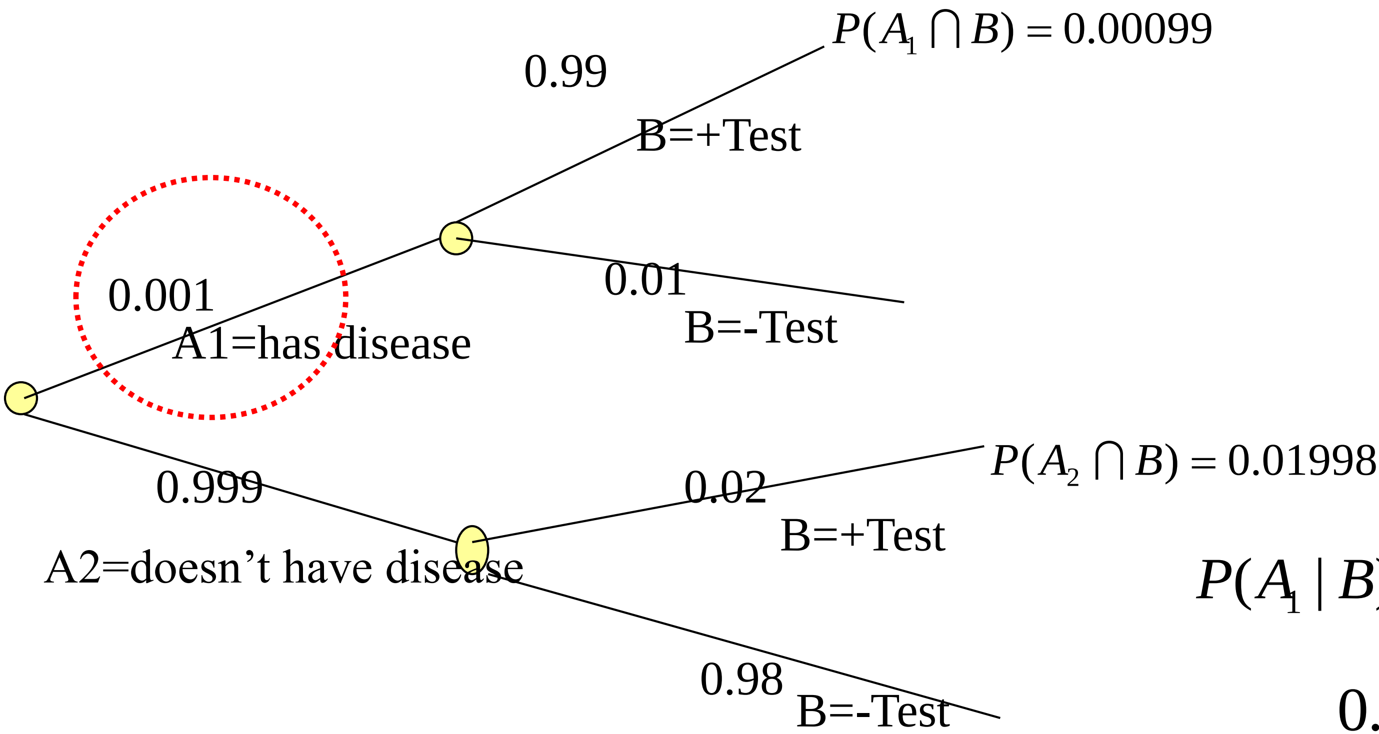
- Example 2.31

Incidence of a rare disease. Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, whereas an individual without the disease will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Let: A_1={individual has the disease} A_2={individual does not have the disease}, and B ={positive test result}. Then $P(A_1)=0.001$; $P(A_2)=0.999$, $P(B|A_1)=0.99$ and $P(B|A_2)=0.02$.

2.4 Conditional Probability

- Example 2.30 Cont’



$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)}$$

$$= \frac{0.00099}{0.02097} = 0.047$$

P(B) = $P(A_1 \cap B) + P(A_2 \cap B)$ =0.2097

2.4 Conditional Probability

- Homework

Ex. 45, Ex. 50, Ex. 58, Ex. 66

2.5 Independence

- Definition

  Two events A and B are independence if $P(A \mid B)=P(A)$ and are dependent otherwise.

Note:

1. Since $P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$

   if $P(A \mid B)=P(A)$, then we have

   $P(A)\,P(B) = P(B|A)P(A) \Rightarrow P(B|A) = P(B)$ (if $P(A) > 0$)

2. If A and B are independence, so are the following pairs of events:

   a. A’ and B   b. A and B’ c. A’ and B’

2.5 Independence

- Example

Consider tossing a fair six-sided die once and define events $A=\{2,4,6\}$, $B=\{1,2,3\}$, and $C=\{1,2,3,4\}$. We then have $P(A)=1/2$, $P(A \mid B)=1/3$ and $P(A \mid C)=1/2$. That is, events A and B are dependent, whereas events A and C are independent.

Note: Intuitively, if such a die is tossed and we are informed that the outcome was 1,2,3,or 4 (C has occurred), then the probability that A occurred is 1/2, as it originally was, since two of the four relevant outcomes are even and the outcomes are still equally likely.

2.5 Independence

- Example 2.33

  Let A and B be any two mutually exclusive events with P(A)>0. For example, for a randomly chosen automobile, let A={the car has four cylinders} and B={the car has six cylinders}.

  Since the events are mutually exclusive, if B occurs, then A cannot possibly have occurred, so $P(A|B) = 0 \neq P(A)$. The message here is that if ***two events are mutually exclusive, they cannot be independent.***

  (Here: P(A) & P(B) are not zero!)

2.5 Independence

- Proposition #1

  A and B are independent ***if and only if***

  $$P(A \cap B) = P(A)\,P(B)$$

  Proof:

  1. If A and B are independent, then

     $P(A|B) = P(A)$, and thus

     $$P(A \cap B) = P(A|B)P(B) = P(A)\,P(B)$$

  2. If $P(A \cap B) = P(A)\,P(B)$, then

     $$P(A \cap B) = P(A|B)P(B) = P(A)\,P(B)$$

  $P(A|B) = P(A)$ $(P(B)>0)$, A and B are independent

2.5 Independence

- Example 2.34

It is known that 30% of a certain company's washing machines require service while under warranty, whereas only 10% of its dryers need such service. If someone purchases both a washer and a dryer made by this company, what is the probability that both machines need warranty service?

Let A be the event that washer needs service while under warranty, and B be defined analogously for the dryer, then $P(A) = 0.3$, $P(B) = 0.1$. Assuming that the two machines function independently of one another, the desired probability is

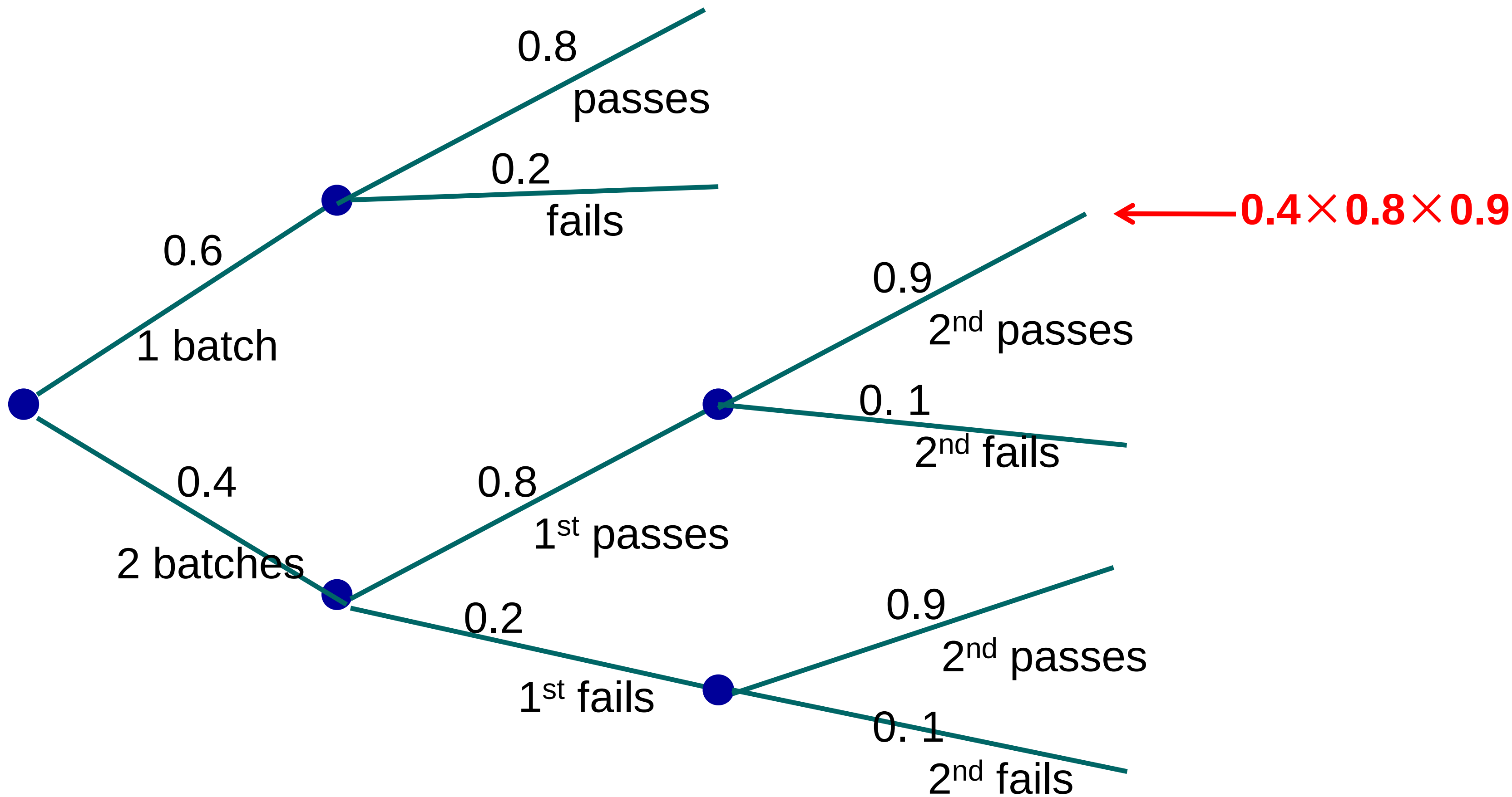
$$P(A \cap B) = P(A)\,P(B) = 0.3 \times 0.1 = 0.03.$$

The probability that neither machine needs service is

$$P(A' \cap B') = P(A')\,P(B') = (1-0.3)\,(1-0.1) = 0.63$$

2.5 Independence

- Example 2.35



2.5 Independence

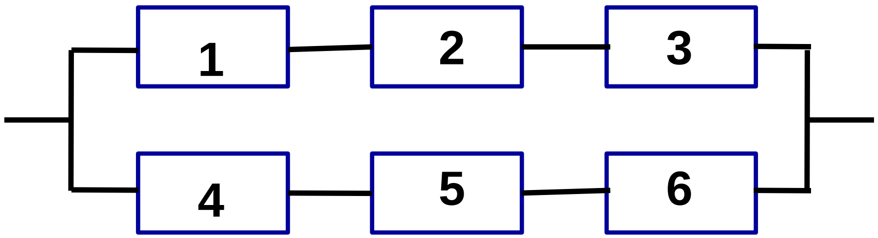
- Mutually Independent

Events A_1, A_2, …A_n are mutually independent if for every k (k=2,3…n) and every subset of indices i_1,i_2…i_k

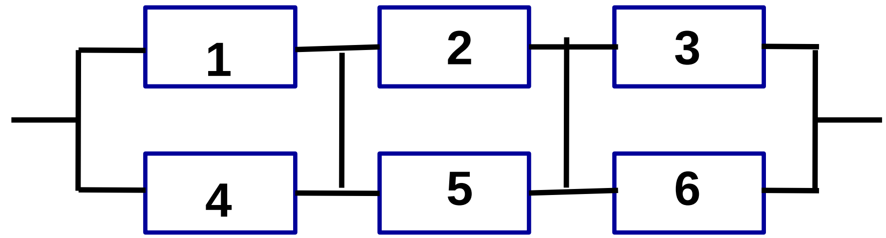
$$P(A_{i1} \cap A_{i2} \ldots \cap A i_k) = P(A_{i1})P(A_{i2})\ldots P(A_{ik})$$

2.5 Independence

- Example 2.36



System-1



System-2

P(system-1 lifetime exceeds t0)

$=P[(A_1 \cap A_2 \cap A_3) \cup (A_4 \cap A_5 \cap A_6)]$

$=P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6) - P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6)$

$=0.9^3+0.9^3-0.9^6=0.927$

P(system-2 lifetime exceeds t0)

$=P[(A_1 \cup A_4) \cap (A_2 \cup A_5) \cap (A_2 \cup A_5)]$

$= P(A_1 \cup A_4)^3$

$=[P(A_1)+P(A_4) - P(A_1 \cap A_4)]^3$

$=[P(A_1)+P(A_4) - P(A_1)P(A_4)]^3$

$=(0.9+0.9-0.9 \times 0.9)^3=0.97$

2.5 Independence

- Homework

  Ex. 72, Ex. 78, Ex. 82, Ex. 87