概率论与数理统计 期中考核

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Problem 1

(1)

首先，对所求项进行适当转换：

$$
\begin{aligned}
\int_{-\infty}^{\infty} f\left(x;\mu,\sigma^2\right)dx &= \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}d(x-\mu) \\
&= \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-\left(\frac{x}{\sqrt{2}\sigma}\right)^2}d\frac{x}{\sqrt{2}\sigma} \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}dx
\end{aligned}
$$

接着，求上一步最终等价项的值：

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}dx\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-y^2}dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi}e^{-\left(x^2+y^2\right)}dxdy \\
&= \int_{0}^{2\pi}d\theta\int_{0}^{\infty}\frac{1}{\pi}\cdot e^{-r^2}\cdot rdr \\
&= \int_{0}^{\infty}e^{-r^2}dr^2 \\
&= 1
\end{aligned}
$$

又因为：

1. $\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}dx > 0$
2. $\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-y^2}dy$

$\therefore \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}dx = 1$

$\therefore \int_{-\infty}^{\infty} f\left(x;\mu,\sigma^2\right)dx = 1$

因此: $f\left(x;\mu,\sigma^2\right)$ is a legitimate pdf.

(2)

Because X is a continuous rv that has normal distribution

$\therefore E(x)=\mu \quad V(x)=\sigma^2$

(3)

$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$

$F\left(\frac{1}{2}\right)=\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx=\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$

Let : $\sqrt{x}=t, x=t^2$

$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty \frac{e^{-t^2}}{t}dt^2=\int_0^\infty 2e^{-t^2}dt=2\times\frac{\sqrt{\pi}}{2}=\sqrt{\pi}$

$\therefore F\left(\frac{1}{2}\right)=\sqrt{\pi}$

(4)

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x;\alpha,\beta)dx &= \int_0^\infty f(x;\alpha,\beta)dx \\
&= \frac{\int_0^\infty \frac{1}{\beta^\alpha}\cdot x^{\alpha-1}\cdot e^{\frac{-x}{\beta}}dx}{\Gamma(\alpha)} \\
&= \frac{\int_0^\infty (\frac{x}{\beta^2})^{\alpha-1}\cdot e^{-\left(\frac{x}{R}\right)}d\left(\frac{x}{\beta}\right)}{\Gamma(\alpha)} \\
&= \frac{\int_0^\infty x^{\alpha-1}e^{-x}dx}{\sqrt{(\alpha)}} \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \\
&= 1
\end{aligned}
$$

$\therefore f(x;\alpha,\beta)$ is a legitimate pdf.

(5)

$f(x;0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

Let:
1. $Y=x^2$
2. the distribution function of X and Y : $F_X(x), F_Y(y)$

$\because Y=x^2 \geqslant 0$

$\therefore y \leqslant 0$ 时 $F_Y(y)=0$

当$y>0$时:

$$
\begin{aligned}
F_Y(y)=P(Y \leqslant y) \\
& =P\left(x^2 \leq y\right)=P(-\sqrt{y} \leq x \leq \sqrt{y}) \\
& =F_x\left(\sqrt{y}\right)-F_x(-\sqrt{y})
\end{aligned}
$$

将$F_Y(y)$关于y求导得:

$$
f_{Y(y)}=\begin{cases}\frac{1}{2\sqrt{y}}\left[f_x(\sqrt{y})+f_x(-\sqrt{y})\right] & y>0 \\ 0 & y \leq 0\end{cases}
$$

代入$f_x(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ 得:

$$
f_Y(y)=\begin{cases}\frac{1}{\sqrt{2\pi}}y^{-\frac{1}{2}}e^{-\frac{y}{2}} & y>0 \\ 0 & y \leq 0\end{cases}
$$

(6)

$f(x,y)=f_X(x)\cdot f_Y(y) \quad f(x;0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \quad f(y;0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$

$\therefore f(x,y)=\frac{1}{2\pi}\cdot e^{-\frac{x^2+y^2}{2}}$

Let: $Z=X^2+Y^2 \quad W=\frac{X}{Y}$

可假设 $W>0$, 而 $W \leq 0$ 的情况同理亦可证得

$$
\begin{aligned}
F_Z(z)=P\left(X^2+Y^2 \leq z\right) & =\frac{1}{2\pi}\iint_{X^2+Y^2 \leq z} e^{-\frac{x^2+y^2}{2}} dxdy \\
& =\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^z e^{-\frac{r^2}{2}} d\frac{r^2}{2} \\
& =1-e^{-\frac{z^2}{2}}
\end{aligned}
$$

Let: $\alpha=\tan^{-1} w$ which means ($\alpha=\arctan w$)

$$
\begin{aligned}
F_W(w) = P\left(\frac{X}{Y} \le w\right) &= \frac{1}{2\pi} \iint_{\frac{X}{Y}\le w} e^{-\frac{x^2+y^2}{2}} dxdy \\
&= \frac{1}{2\pi} \int d\theta \int_0^{\infty} e^{-\frac{r^2}{2}} d\frac{r^2}{2} \\
&= \frac{1}{2\pi} \int d\theta \\
&= \frac{1}{2\pi} \left(\int_{\frac{\pi}{2}-\alpha}^{\pi} d\theta + \int_{\frac{3\pi}{2}-\alpha}^{2\pi} d\theta \right) \\
&= \frac{1}{2\pi} \times (2\alpha + \pi) = \frac{\alpha}{\pi} + \frac{1}{2} \\
&= \frac{1}{\pi} \tan^{-1} w + \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
F_{ZW}(z,w) &= \iint_{D_{ZW}} f_{XY}(x,y) dxdy \\
&= \iint_{D_{ZW}} f\left(\mathrm{rcos}\theta,\ \mathrm{rsin}\ \theta\right) r d\theta dr \\
&= \int_{\frac{\pi}{2}-\alpha}^{\pi} d\theta \int_0^z \frac{1}{2\pi} \cdot e^{-\frac{r^2}{2}} r dr + \int_{\frac{3\pi}{2}-\alpha}^{2\pi} d\theta \int_0^z \frac{1}{2\pi} \cdot e^{-\frac{r^2}{2}} r dr \\
&= \frac{1}{2\pi} \times (\alpha + \alpha + \pi) \times \left(1 - e^{-\frac{z^2}{2}}\right) \\
&= (\frac{1}{\pi} \times \tan^{-1} w + \frac{1}{2}) \times \left(1 - e^{-\frac{z^2}{2}}\right)
\end{aligned}
$$

$\therefore f_Z(z) = F_Z'(z) = ze^{-\frac{z^2}{2}} \quad f_W(w) = F_w'(w) = \frac{1}{\pi} \cdot \frac{1}{1+w^2}$

$f(z,w) = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial w} F_{ZW}(z,w) = \frac{1}{\pi} \cdot \frac{1}{1+w^2} \cdot z \cdot e^{-\frac{z^2}{2}}$

$\therefore f(z,w) = f_Z(z) \cdot f_W(w)$

$\therefore$ The rv $x^2 + y^2$ and $\frac{X}{Y}$ are also independent.