测度：Lebesgue测度,测度空间

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首先规定记号 ∞ 的运算

$$\infty + a = \infty. \quad \infty + \infty = \infty \qquad (a \in \mathbb{R})$$
$$-\infty + a = -\infty. \quad (-\infty) + (-\infty) = -\infty$$
$$a \times (\pm\infty) = \pm(\operatorname{sgn} a)\infty \qquad (a \neq 0)$$
$$0 \times \infty = 0$$

Def. 设有集 X. 并设 $B(\subset \mathcal{P}(X))$ 是一 Borel 集合体.

$\forall A \in B$. 有一值 $m(A)$ 与之对应 $(0 \le m(A) \le \infty)$. 并满足

MI $\quad m(\phi) = 0$

MII 设 $A_1, A_2 \cdots \in B$. $A_i \cap A_j = \phi\ (i \neq j)$. 则

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

这时就说在 B 上定义了 Lebesgue 测度 m.

称 (X, B, m) 为测度空间

Def. (X, B, m). $\quad m(X) < \infty$. 有界测度

$m(X) = \infty$. $X = \bigcup_{n=1}^{\infty} E_n$, $m(E_n) < \infty$. 准有界测度

以下 $A, B \cdots$ 除特别注明外. 都是属于 B 的集合

THM $\quad A \subset B. \quad m(A) \le m(B)$.

一般地. $m(A) + m(B) = m(A \cup B) + m(A \cap B)$

Pf.
$$A = (A \cap B) \cup (A - A \cap B)$$
$$B = (A \cap B) \cup (B - A \cap B)$$
$$\begin{aligned} m(A) + m(B) &= m(A \cap B) + m(A - A \cap B) \\ &\quad + m(A \cap B) + m(B - A \cap B) \\ &= m(A \cap B) + m(A \cup B) \end{aligned}$$

THM
$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m(A_n)$$

Pf. 令 $B_n = A_n - (A_1 \cup \cdots A_{n-1})$. 则

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n, \quad B_n \subset A_n. \quad B_i \cap B_j = \phi$$

$$\therefore m\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} m(B_n) \le \sum_{n=1}^{\infty} m(A_n) \quad \#$$

THM (i) 若 $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$. 则 $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty} m(A_n)$

(ii) 若 $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, $m(A_1) < \infty$. 则

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

(iii) $\quad m(\underline{\lim}\, A_n) \le \underline{\lim}\, m(A_n)$

(iv) 若 $m\left(\bigcup_{n=1}^{\infty} A_n\right) < +\infty$, 则 $m(\overline{\lim}\, A_n) \ge \overline{\lim}\, m(A_n)$

(v) 特别地. 若 $A = \lim A_n$. 且 $m\left(\bigcup_{n=1}^{\infty} A_n\right) < \infty$. 则

$$m(A) = \lim m(A_n)$$

Pf. (i) $\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 - A_1) \cup \cdots \cup (A_n - A_{n-1}) \cup \cdots$

$$\therefore m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n - A_{n-1})$$

此外. $m(A_n) = \sum_{i=1}^{n} m(A_i - A_{i-1})$

$$\therefore m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

(ii) 令 $B_n = A_1 - A_n$. 则 $B_1 \subset B_2 \subset \cdots$.

由(i). $m\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim\limits_{n\to\infty} m(B_n) = m(A_1) - \lim\limits_{n\to\infty} m(A_n)$

$$\bigcup_{n=1}^{\infty} B_n = A_1 - \bigcap_{n=1}^{\infty} A_n$$

$$\therefore m\left(\bigcup_{n=1}^{\infty} B_n\right) = m(A_1) - m\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$\therefore m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

(iii) 令 $B_n = \bigcap_{k=n}^{\infty} A_k$. 那么 $B_n \subset A_n$. 且 $B_1 \subset B_2 \subset \cdots$

$$\therefore m(\underline{\lim}\, A_n) = m\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim m(B_n) \le \underline{\lim}\, m(A_n)$$

(iv) 令 $B_n = \bigcup_{k=n}^{\infty} A_k$. $A_n \subset B_n$. 且 $B_1 \supset B_2 \supset \cdots$

$$\therefore m(\overline{\lim}\, A_n) = m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim m(B_n) \le \overline{\lim}\, m(A_n)$$

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(v) 由(iii). (iv)立得