Lec 03

2019年3月6日 星期三 上午10:02

Chapter 1

algebra
(i) $S \in \Sigma$.
(ii) closed under taking complemented.
(iii) closed under finite union

σ-algebra (iii) → (iii)' closed under countable union

Measurable Space (S, Σ)
↑ set, ↑ σ-algebra

$\sigma(C)$ $\qquad \sigma(C) = \bigcap_{C \subset \Sigma} \Sigma$

Borel algebra
S topological space
C open sets of S
$B(S) = \sigma(C)$

$$\mathcal{B} := \mathcal{B}(\mathbb{R}) = \sigma(\pi(\mathbb{R}))$$
$$\pi(\mathbb{R}) := \{(-\infty, x] : x \in \mathbb{R}\}$$

Measure $\quad \mu: \Sigma \to [0, +\infty]$
可列可加性 $\quad \mu(+_i F_i) = \sum_i \mu(F_i)$

Measure Space $\quad (S, \Sigma, \mu)$

Moto convergence.
(a) $F_n \in \Sigma\ (n \in \mathbb{N})$. $F_n \uparrow F$. $\mu(F_n) \uparrow \mu(F)$
(b) $G_n \in \Sigma\ (G_n \downarrow G)$. $\exists k$. $\mu(G_k) < \infty$. $\mu(G_n) \downarrow \mu(G)$

e.g.

$$\bigcap_k \bigcup_n I_{n,k} \neq \bigcup_n \bigcap_k I_{n,k}$$

$\boxed{\text{Baire Category THM}}$ Baire纲定理

S是完备度量空间. $S = \bigcup_i F_i$. F_i是闭集. 则至少有一个F_i包含一个开球

$$V = \mathbb{Q} \cap [0,1]$$
$$V \subseteq G_k = \bigcup_{n \in \mathbb{N}} \left[\left(v_n - \varepsilon(k) 2^{-n}, v_n + \varepsilon(k) 2^{-n} \right) \cap S \right] := \bigcup_n I_{n,k}$$
$$H = \bigcap_k G_k = \bigcap_k \bigcup_n I_{n,k}. \qquad Leb(H) = 0$$
$$V = \bigcup_n \bigcap_k I_{n,k}$$

$\mu(H) = 0$. 但H是不可数集

Rota. 几何概率

☆ e.g. Suppose $P: \Sigma \to \mathbb{R}$ is a function, such that the following hold:

1. $P(C) \geq 0 \quad \forall C \in \Sigma$
2. $P(S) = 1$
3. $P(\sum_{i=1}^{n} C_i) = \sum_{i=1}^{n} P(C_i)$, + 为不交并

Then the following are equivalent:

a) P is σ-additive
b) For any sequence of events $C_i \in \Sigma$, $C_i \subseteq C_{i+1}$. we have $P(\lim C_i) = P(\bigcup C_i) = \lim P(C_i)$
c) For any sequence of events $C_i \in \Sigma$. $C_i \supseteq C_{i+1}$ we have $P(\lim C_i) = P(\bigcap C_i) = \lim P(C_i)$
d) For any sequence of events $C_i \in \Sigma$, $C_i \supseteq C_{i+1}$ $\quad \phi . S \to$ something else ?
$\bigcap_i C_i = \phi$, we have $\lim P(C_i) = 0$
e) For any sequence of events $C_i \in \Sigma$, $C_i \subseteq C_{i+1}$
$\bigcup_i C_i = S$, we have $\lim P(C_i) = 1$

☆ e.g. 2. Let (S, Σ, P) be a probability space.

For $A, B \in \Sigma$. define

$$d(A, B) = \begin{cases} \dfrac{P(A \triangle B)}{P(A \cup B)}, & \text{if } P(A \cup B) > 0 \\ 0 & \text{else} \end{cases}$$

Show that $d(A, C) \leq d(A, B) + d(B, C)$