Lec 04/05

2019年3月11日 星期一 上午11:00

Chapter 2 Event

$(\Omega, \mathcal{F}, P)$ probability space

样本空间 事件类

样本空间中的元素称为样本点

e.g. $\Omega = \{H, T\}^{\mathbb{N}}$

$\mathcal{F} = \sigma(\{\omega\in\Omega \mid \omega_n = W\}, n\in\mathbb{N}, W\in\{H,T\})$

$\boxed{\mathcal{F} \neq \mathcal{P}(\Omega)}$? why

$F = \{\omega\in\Omega \mid \frac{\#\{k\le n : \omega_k = H\}}{n} \to \frac{1}{2}\}$

$F \in \mathcal{F}$?

$F_\alpha = \{\omega\in\Omega \mid \frac{\#\{k\le n : \omega_{\alpha(k)} = H\}}{n} \to \frac{1}{2}\}$

$\alpha: \mathbb{N}\to\mathbb{N}$ strictly increasing

Chapter 3 Random Variables

$X: \Omega \to \mathbb{R}$

$(\Omega, \mathcal{F})$ $X^{-1}: \mathcal{B} \to \mathcal{F}$ Borel sets

Random event is random variable

$\Omega \supseteq A \longleftrightarrow 1_A: \Omega \to \mathbb{R}$

Σ-measurable function (S, Σ) measurable space

$h: S \xrightarrow{h} \mathbb{R}$, $\Sigma \xleftarrow{h^{-1}} \mathcal{B}$ h: Σ-measurable function

Borel Function $(S, \mathcal{O}(S))$ Topology Space

$h: S\to\mathbb{R}$ h是$\mathcal{B}(S)$可测的

Proposition (a) $h^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha h^{-1}(A_\alpha)$

$h^{-1}(A^c) = (h^{-1}(A))^c$

(b) $C \subseteq \mathcal{B}$. $\sigma(C) = \mathcal{B}$.

$h^{-1}: C\to\Sigma \implies h = m\Sigma$

(c) S topological space. $h: S\to\mathbb{R}$. continuous

$\implies h$ is Borel Function

(d) (S, Σ). $h: S\to\mathbb{R}$

$\{h\le c\} := \{s\in S \mid h(s)\le c\} \in \Sigma$ $\forall c\in\mathbb{R}$

$\implies h$ is Σ-measurable.

Proposition. (a). $h_1\in m\Sigma$. $h_2\in m\Sigma$

$\implies h_1 + h_2 \in m\Sigma$

Pf. $\{h_1+h_2 < c\} = \bigcup_{q\in\mathbb{Q}} (\{s : h_1(s) < q\} \cap \{s : q < c - h_2(s)\})$

(b) $c h_1$, $h_1 h_2 \in m\Sigma$

(c) $S \xrightarrow{h} R \xrightarrow{f} R$ $f\circ h\in mS$

$h\in mS$ $f\in mR$

☆Proposition. $(h_n : n\in\mathbb{N})$. $h_n\in m\Sigma$.

则 (i) $\inf h_n \in m\Sigma$

(ii) $\liminf h_n \in m\Sigma$

(iii) $\limsup h_n \in m\Sigma$

(iv) $\{s \mid \lim h_n(s)$ 存在且属于$\mathbb{R}\} \in \Sigma$

Pf. (i) $\{\inf h_n \ge c\} = \bigcap_n \{h_n \ge c\}$

(ii) $\liminf h_n$

$L_n(s) := \inf_{r\ge n}\{h(s)\}$. $L_n(s)\in m\Sigma$

$\liminf h_n = \uparrow\lim L_n = \sup L_n \in m\Sigma$

e.g. $\Omega = \{H, T\}^{\mathbb{N}}$

$\mathcal{F} = \sigma(\{\omega_n = W\}, n\in\mathbb{N}, W\in\{H,T\})$

则 $F = \{\omega \mid \frac{\#\{k\le n ; \omega_k = H\}}{n} \to \frac{1}{2}\}$

Pf. 设 $X_n(\omega) = \begin{cases} 1 & \omega_n = H \\ 0 & \omega_n = T \end{cases} \in m\Omega$

$\implies S_n := \sum_{i=1}^n X_i \in m\Omega$

$\implies \limsup \frac{S_n}{n} \in m\Omega$ $\liminf \frac{S_n}{n} \in m\Omega$

$\therefore \{\limsup \frac{S_n}{n} = p\} \in \mathcal{F}$, $\{\liminf \frac{S_n}{n} = p\}\in\mathcal{F}$

$\therefore \{\limsup \frac{S_n}{n} = p\} \cap \{\liminf \frac{S_n}{n} = p\} = \{\lim \frac{S_n}{n} = p\} \in \mathcal{F}$

$\therefore F = \{\lim \frac{S_n}{n} = \frac{1}{2}\} \in \mathcal{F}$ #

Def. Ω上的函数族生成的σ-代数

$\mathcal{F} = \sigma(X_n, n\in\mathbb{N})$

Def. Distribution Function

$\Omega \xrightarrow{X} \mathbb{R}$

$[0,1] \xleftarrow{P} \sigma(X) \xleftarrow{X^{-1}} \mathcal{B}$

$\mathcal{L}_X := P\circ X^{-1}$ $\mathcal{L}_X : \mathcal{B} \to [0,1]$

$F_X(c) := \mathcal{L}_X(-\infty, c]$

Def. A distribution function is a function F defined on $\mathbb{R}$.

such that (i) F is right continuous, monotonically increasing

(ii) $\lim_{x\to\infty} F = 1$. $\lim_{x\to\infty} F = 0$

Def. Vitali Set $\to$ 不可测集 $\mathbb{R}/\mathbb{Q}$

$\mathbb{R}/\mathbb{Q} = \{r + \mathbb{Q}\}$, $r_1 - r_2 \in\mathbb{Q} \iff r_1 + \mathbb{Q} = r_2 + \mathbb{Q}$

Def. Robin Tomas

$(\mathbb{R}, E)$ $xy\in E \iff |x-y| = 3^k$. $k\in\mathbb{Z}$.

$\forall k$. $A = B + 3^k$.

F closed. $\lambda F > 0$. $F\subseteq A$.

open set G $F\subseteq G$. $\frac{\lambda F}{\lambda G} > \frac{3}{4}$

e.g. Poincaré 1890

Poincaré's Recurrence THM

Let T be a measure-preserving transform on $(\Omega, \mathcal{F}, P)$.
then for any $E\in\mathcal{F}$ with $P(E) > 0$. almost all points of E return to E infinitely often under positive iterations by T

Pf. $A_n = \{x\in E \mid x\notin T^{-kn}E.\ \forall k\ge 1\}$.

$= E \setminus \bigcup_{k\ge 1} T^{-kn}E \in \mathcal{F}$

claim A_n, $T^{-n}A_n$, $T^{-2n}A_n$ ... are pairwise disjoint

$T^{-kn}A_n \cap T^{-(k+m)n}A_n \neq \phi$

$A_n \cap T^{-mn}A_n \supseteq T^{kn}(T^{-kn}(A_n) \cap T^{-(k+m)n}(A_n)) \neq \phi$

$P(A_n) = 0$

$P(\limsup A_n) \le P(\bigcup_n A_n) \le \sum_n P(A_n) = 0$

$\therefore P(E \setminus \limsup A_n) = P(E)$ #

e.g. Murphy's' Law. Whatever can go wrong will eventually go wrong

Pf. $E = \lim E_n$, Suppose E_n is an increasing sequence of events

Suppose that $P(E/\bar{E_n}) \ge \varepsilon$, for a constant $\varepsilon > 0$.

Then $P(E) = 1$

$P_n = P(E_n)$. $P(E) = (1-P_n)P(E|\bar{E_n}) + P_n P(E|E_n)$

$\ge \varepsilon(1-P_n) + P_n$

$P(E) \ge \varepsilon(1-P(E)) + P(E)$

$P(E) = 1$ #

☆ Zeta Distribution: $P_s(X = n) = \frac{\frac{1}{n^s}}{\zeta(s)}$

$\lim_{s\to 1^+} P_n = ?$

☆ Let $X, Y: \Omega\to\mathbb{R}$, Y is $\sigma(X)$-measurable $\iff \exists$ Borel Function f. $Y = f(X)$

Pf "$\Leftarrow$" $(f\circ X)^{-1} = X^{-1} f^{-1}(B) \xleftarrow{f^{-1}} B \in\mathcal{B}$, $\in\sigma(X)$, $\in\mathcal{B}$

$Y = f(X)$ is $\sigma(X)$-measurable

"$\Rightarrow$" $Y = Y_+ - Y_-$

$\therefore$ 不妨设 $Y\ge 0$ w.l.o.g.

Exercise

Fix n $\Omega \supseteq A_{kn} = Y^{-1}[\frac{k}{2^n}, \frac{k+1}{2^n}) \in \sigma(X)$ (Y is $\sigma(X)$-Measurable)

$B_{kn}\in\mathcal{B}$ $= X^{-1}(B_{kn})$, $= X^{-1}(B)$

$A_{0n} \xleftarrow{X^{-1}} B_{0n}$

$A_{1n} \xleftarrow{X^{-1}} B_{1n}$

$A_{2n} \xleftarrow{X^{-1}} B_{2n}$

$\vdots$

pairwise disjoint

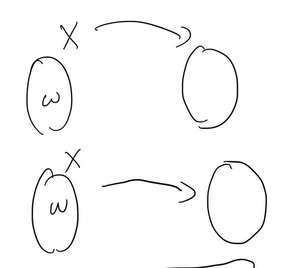
$\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{f_n} \mathbb{R}$.

$f_n(t) = \sum_k 1_{B_{kn}}(t)\frac{k}{2^n}$. f_n is $\mathcal{B}$-measurable

$f_n(X)|_{A_{kn}} = \frac{k}{2^n}$ $Y|_{A_{kn}} \in [\frac{k}{2^n}, \frac{k+1}{2^n})$

$A_{0n} = X^{-1}(B_{0n})$

$\tilde{A}_{1n} = X^{-1}(B_{1n})$



$Y(\omega_1) \neq Y(\omega_2)$

$X(\omega_1) = X(\omega_2)$

$Y - \sigma(X)$

Y is $\sigma(X)$-measurable

$\sigma(X) \supseteq \sigma(Y)$

$(-\infty$