Basic math & physics

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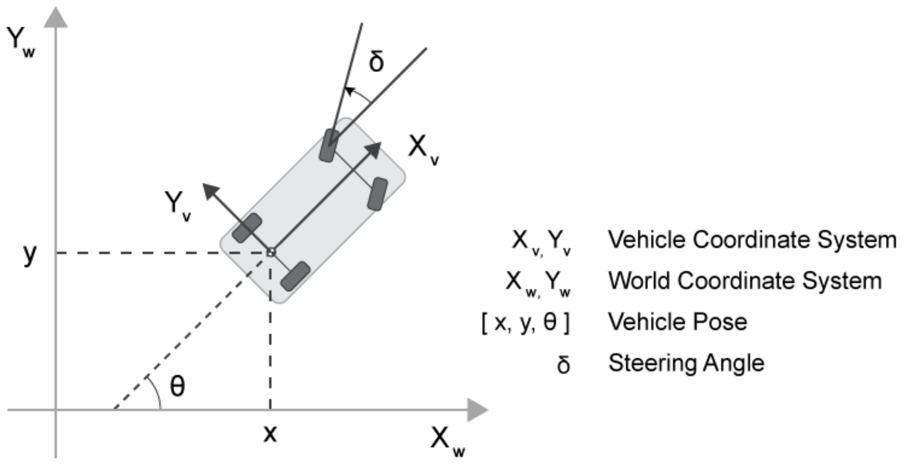
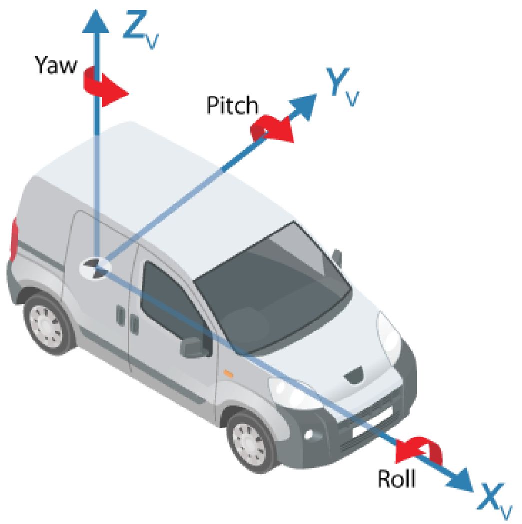
Contents

Coordinate system

- World(Global) & vehicle(Local) coordinate
  - 가장 먼저 해야 할 일 – 좌표계 약속
  - Right-handed cartesian coordinate
  - World coordinate 사용 시, 변환 필요 없도록 항상 동일한 좌표 사용(Planning, localization, mapping 등)
  - ISO vehicle coordinate : y가 왼쪽방향
  - World coordinate : 대문자 (X, Y, Z) 사용 / Vehicle coordinate : 소문자 (x, y, z) 사용



World coordinate



ISO vehicle coordinate

Linear algebra

- Rank of matrix
  - Number of orthogonal column/row vectors
  - Determined / underdetermined / overdetermined system
    - Determined : rank(A) = n
    - Underdetermined : rank(A) < n
    - Overdetermined : rank(A) > n

$$
\begin{gathered}
c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n = y_1 \\
c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n = y_2 \\
\vdots \\
c_{m1}x_1 + c_{m2}x_2 + \ldots + c_{mn}x_n = y_m
\end{gathered}
$$

$$\Downarrow$$

$$
\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}
$$

example

$$
\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
$$

Linear algebra

- Least square and pseudo inverse
  - Polynomial function(n-th order)

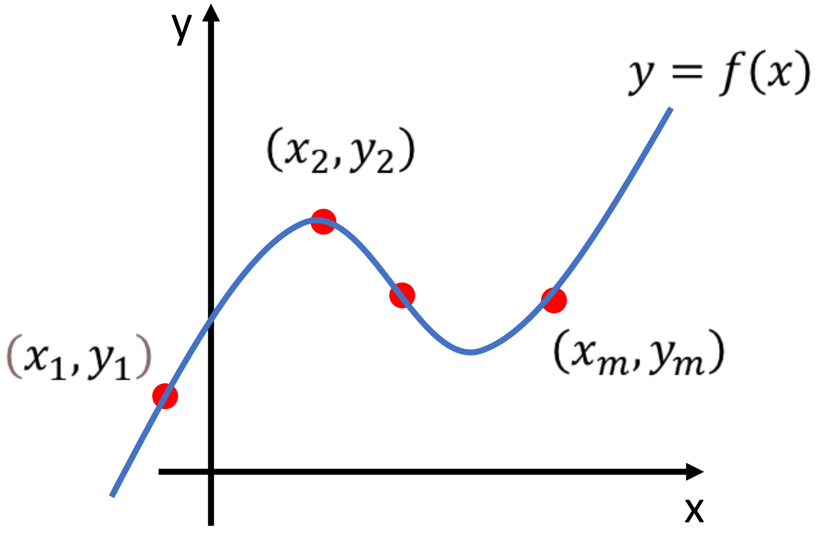
$$y = f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

  - Unknown coefficients

$$\begin{bmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \end{bmatrix}$$

  - Known data(m)

$$\begin{matrix} (x_1, y_1) \\ (x_2, y_2) \\ \vdots \\ (x_m, y_m) \end{matrix}$$



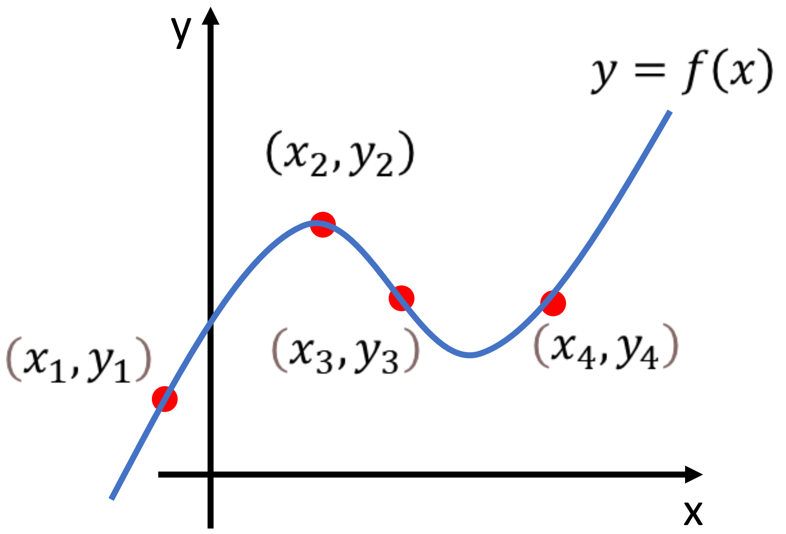
$$\begin{bmatrix} x_1^n & x_1^{n-1} & \cdots & x_1^0 \\ x_2^n & x_2^{n-1} & \cdots & x_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ x_m^n & x_m^{n-1} & \cdots & x_m^0 \end{bmatrix} \begin{bmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$Ax = b$$

Linear algebra

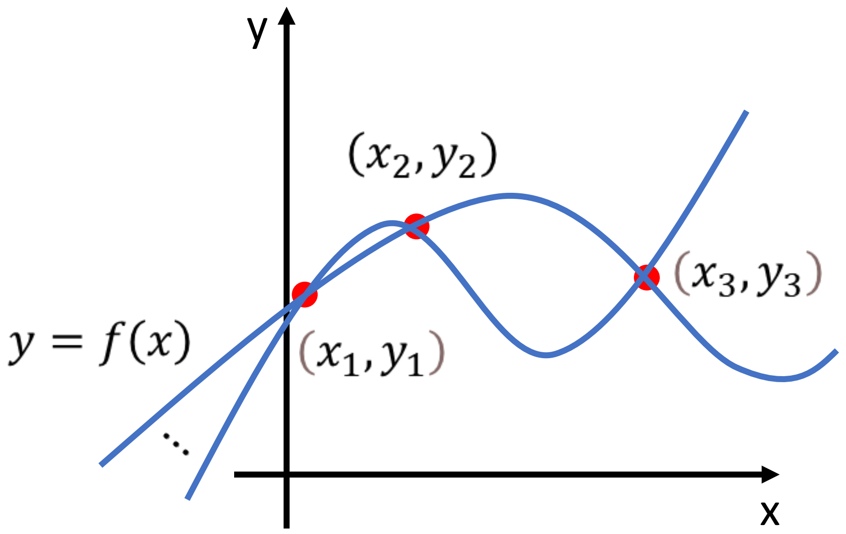
- Least square and pseudo inverse

Determined system : 1 solution



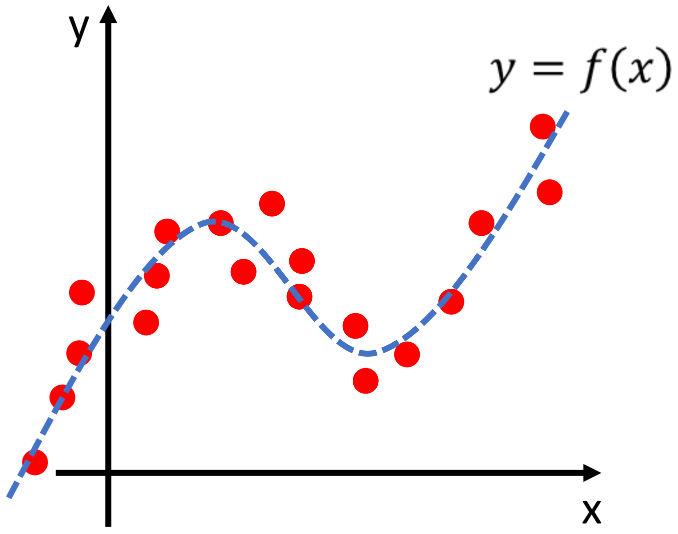
$rank(A) = n + 1$
$\det(A) \neq 0$

Underdetermined system : many solutions



$rank(A) < n + 1$
$\det(A) = 0$

Overdetermined system : no solution



$rank(A) > n + 1$
$\det(A) = 0$

Linear algebra

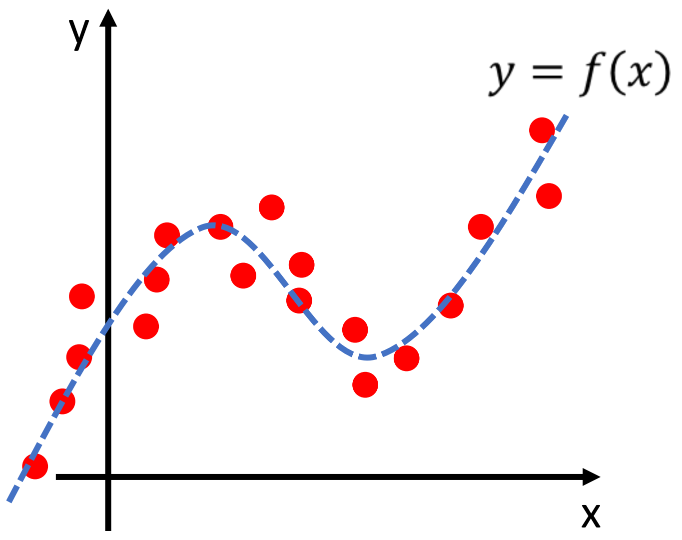
- Least square and pseudo inverse
  - Overdetermined system, fitting

Known

(x_1, y_1)
(x_2, y_2)
$\vdots$
(x_m, y_m)

Unknown

$$y \approx f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$



$$\begin{bmatrix} x_1^n & x_1^{n-1} & \cdots & x_1^0 \\ x_2^n & x_2^{n-1} & \cdots & x_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ x_m^n & x_m^{n-1} & \cdots & x_m^0 \end{bmatrix} \begin{bmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \end{bmatrix} \approx \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$Ax \approx b$$

$$rank(A) > n + 1$$
$$\det(A) = 0$$

Linear algebra

- Least square and pseudo inverse
  - Fitting error

$$
\begin{aligned}
e_1 &= (c_n x_1^n + c_{n-1} x_1^{n-1} + \cdots + c_0) - y_1 \\
e_2 &= (c_n x_2^n + c_{n-1} x_2^{n-1} + \cdots + c_0) - y_2 \\
&\vdots \\
e_m &= (c_n x_m^n + c_{n-1} x_m^{n-1} + \cdots + c_0) - y_m
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} = Ax - b \neq 0
$$

  - Minimize fitting error

$$Minimize(\|Ax - b\|^2)$$

$$Minimize\big((Ax - b)^T (Ax - b)\big)$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x}\big((Ax - b)^T (Ax - b)\big) = 0$$

$$\frac{\partial}{\partial x}\big((x^T A^T - b^T)(Ax - b)\big) = 0$$

$$\frac{\partial}{\partial x}(x^T A^T A x - x^T A^T b - b^T A x + b^T b) = 0$$

$$A^T A x + x^T A^T A - A^T b - b^T A = 0$$

$$\Rightarrow$$

$A^T A x = A^T b$ 일 때 error 최소화!

$$\therefore x = inv(A^T A) A^T b$$

$(A^T A)^{-1} A^T$: Pseudo inverse matrix of A

HL Klemove

Modeling

- What is modeling?
  - 동작/행동하는 방식을 수학적으로 나타냄
  - 법칙에 따라 움직이므로, 초기 상태를 안다면 미래의 상태를 예측 가능

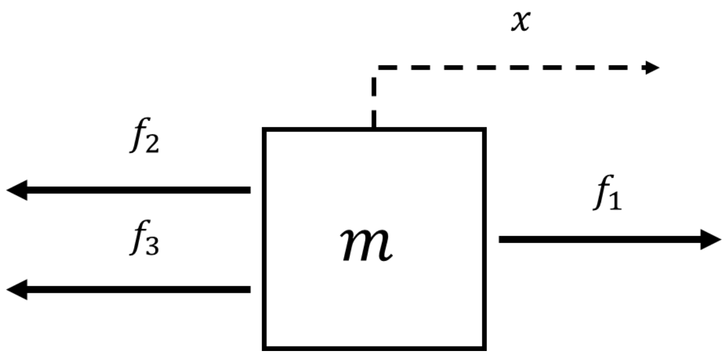
- Dynamics modeling : Newton’s second law
  - Translational system

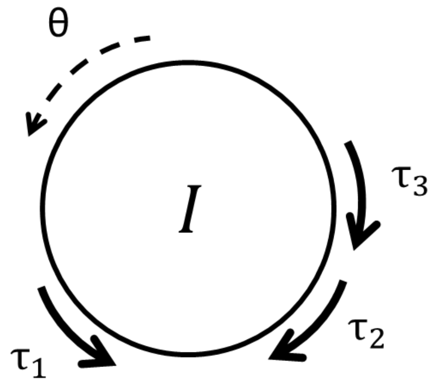
$$\sum F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

  - Rotational system

$$\sum T = I\alpha = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

  - D’Alemberts principle
    - 관성력 : 좌표계가 비 관성계일 때,
      좌표계가 움직이는 가속도의 반대 방향으로 힘(토크)이 작용

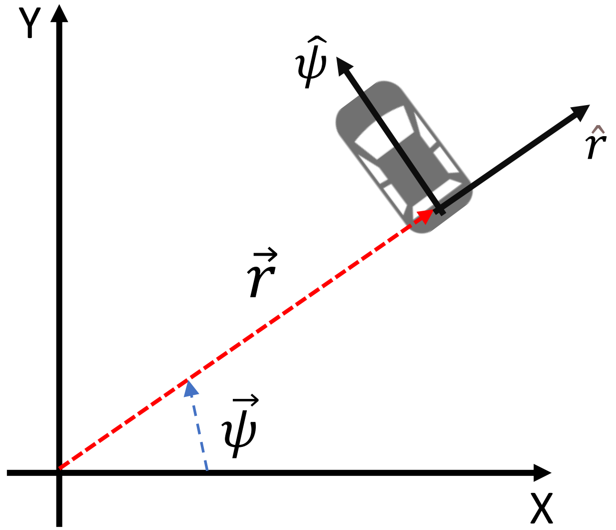


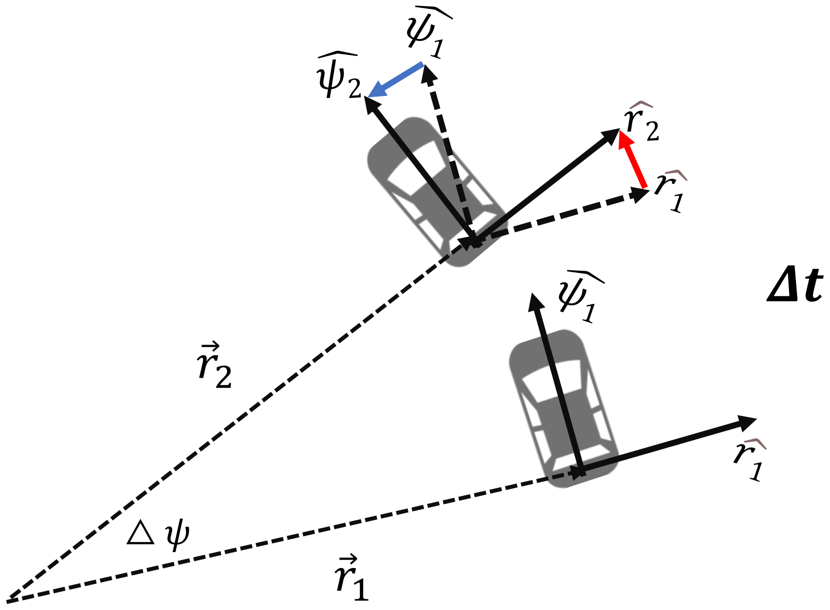


Newton’s law on moving coordinate

- Newton’s second law of motion

$$\sum \vec{F} = m\vec{a} = m\frac{d^2}{dt^2}\vec{r}$$





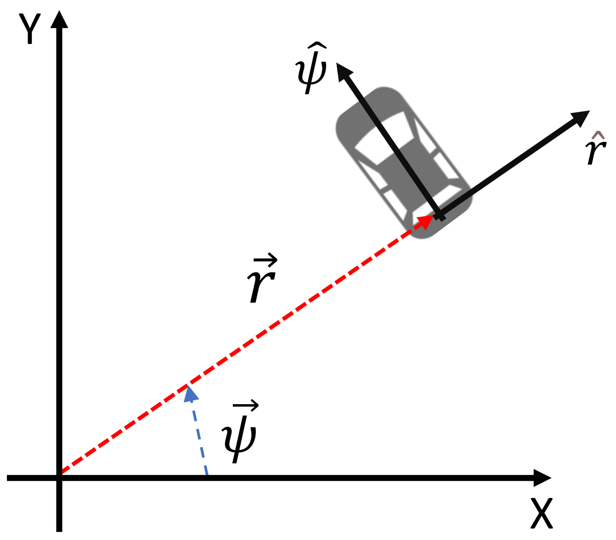
$$\frac{d\hat{r}}{dt} = \lim_{\Delta t \to 0}\frac{\hat{r}_2 - \hat{r}_1}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta\psi}{\Delta t}\hat{\psi} = \dot{\psi}\hat{\psi}$$

$$\frac{d\hat{\psi}}{dt} = \lim_{\Delta t \to 0}\frac{\hat{\psi}_2 - \hat{\psi}_1}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta\psi}{\Delta t}(-\hat{r}) = -\dot{\psi}\hat{r}$$

Newton’s law on moving coordinate

- Newton’s second law of motion

$$\sum \vec{F} = m\vec{a} = m\frac{d^2}{dt^2}\vec{r}$$



$$\frac{d\hat{r}}{dt} = \dot{\psi}\hat{\psi}$$

$$\frac{d\hat{\psi}}{dt} = -\dot{\psi}\hat{r}$$

$$\vec{r} = r\hat{r}$$

$$\frac{d}{dt}\vec{r} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + \dot{\psi}r\hat{\psi}$$

$$\frac{d^2}{dt^2}\vec{r} = \frac{d}{dt}\left(\frac{d}{dt}\vec{r}\right) = \frac{d}{dt}(\dot{r}\hat{r} + \dot{\psi}r\hat{\psi})$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\psi}\hat{\psi} + \ddot{\psi}r\hat{\psi} + \dot{\psi}\dot{r}\hat{\psi} - \dot{\psi}r\dot{\psi}\hat{r}$$

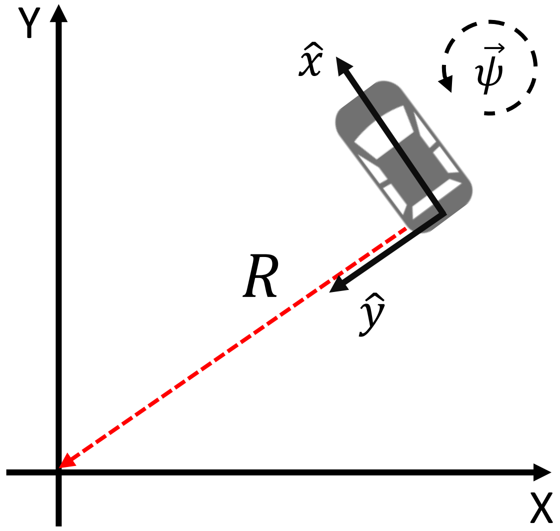
$$= \boldsymbol{(\ddot{r} - \dot{\psi}^2 r)\hat{r} + (\ddot{\psi}r + 2\dot{\psi}\dot{r})\hat{\psi}}$$

Newton's law on moving coordinate

- Newton's second law of motion

$$\hat{\psi} = \hat{x}$$

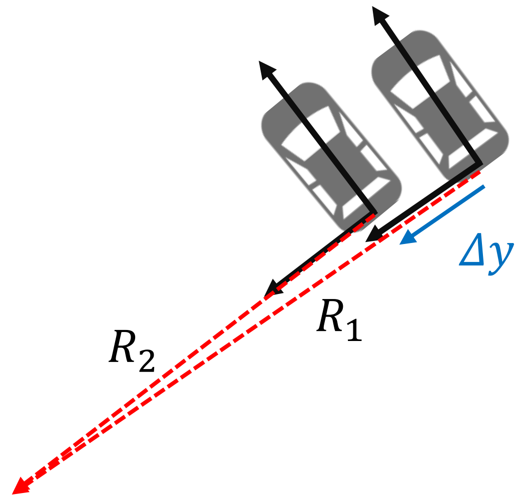
$$\hat{r} = -\hat{y}$$



$$\sum F = (\ddot{\psi}R + 2\dot{\psi}\dot{R})\hat{x} + (-\ddot{R} + \dot{\psi}^2 R)\hat{y}$$

$$\dot{R} = \lim_{\Delta t \to 0} \frac{R_2 - R_1}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_2 - (R_2 + \Delta y)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-\Delta y}{\Delta t} = -\dot{y}$$

$$\ddot{R} = \ddot{y}$$

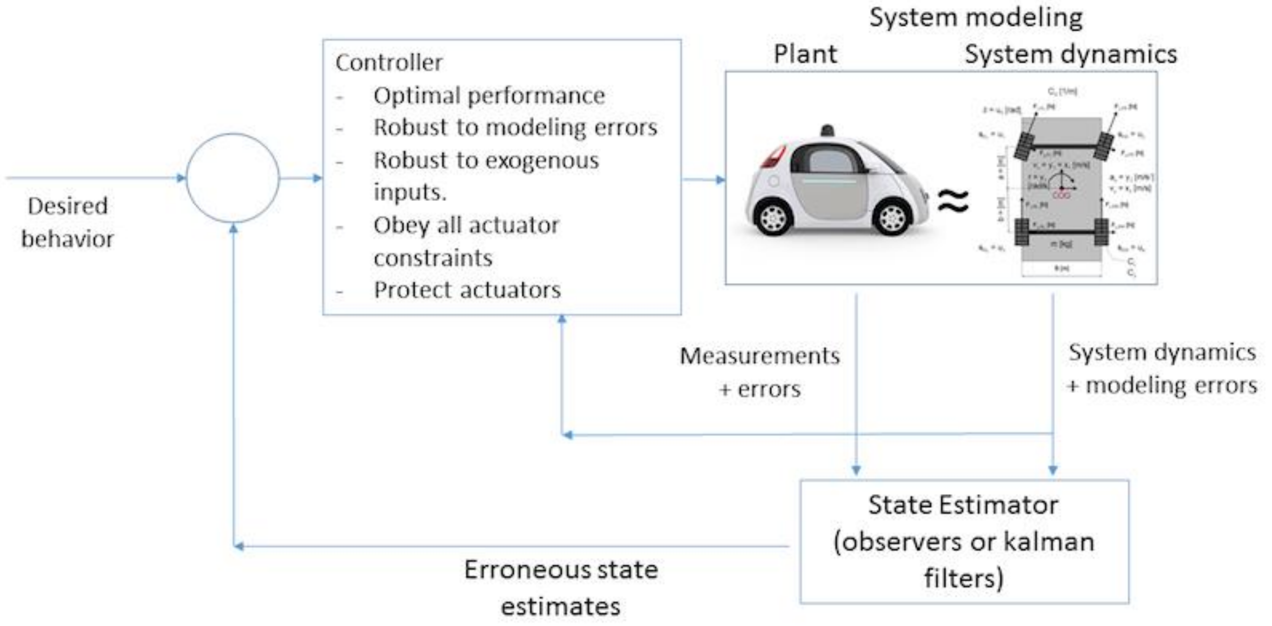


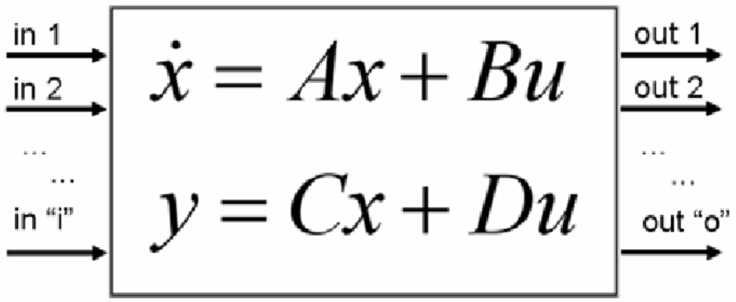
$$\sum \boldsymbol{F} = (\ddot{\boldsymbol{\psi}}\boldsymbol{R} - 2\dot{\boldsymbol{\psi}}\dot{\boldsymbol{y}})\hat{\boldsymbol{x}} + (\ddot{\boldsymbol{y}} + \dot{\boldsymbol{\psi}}^2\boldsymbol{R})\hat{\boldsymbol{y}}$$

$$\sum F = (\ddot{r} - \dot{\psi}^2 r)\hat{r} + (\ddot{\psi}r + 2\dot{\psi}\dot{r})\hat{\psi}$$

State space equation

- 상태 공간 방정식
  - Model 을 표현하는 방식
    - Linear model, Matrix form, 1st order differential equation form
    - State variables (x) : Set of system variables that can completely describe the system at any given time
    - Input variables (u) : Control signals or user applied inputs to the system.
    - Output variables (y) : measurements from the sensor of a system



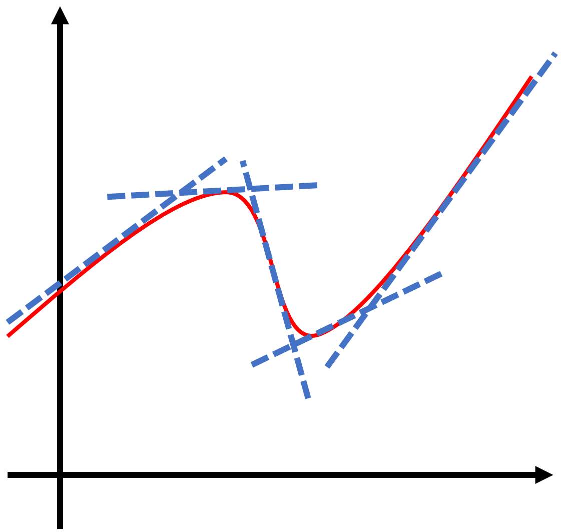


* Image credit: https://mec560sbu.github.io/2016/09/11/Systems_Dynamics/

State space equation

- 상태 공간 방정식
  - Nonlinear system vs linear system
    - Linearlize!

$$\dot{x} = f(x,u) \qquad \dot{x} = Ax + Bu$$
$$y = h(x,u) \qquad y = Cx + Du$$



  - 1st order differential equation form
    - Recurrence relation(점화식)

$$\dot{x} = Ax + Bu$$

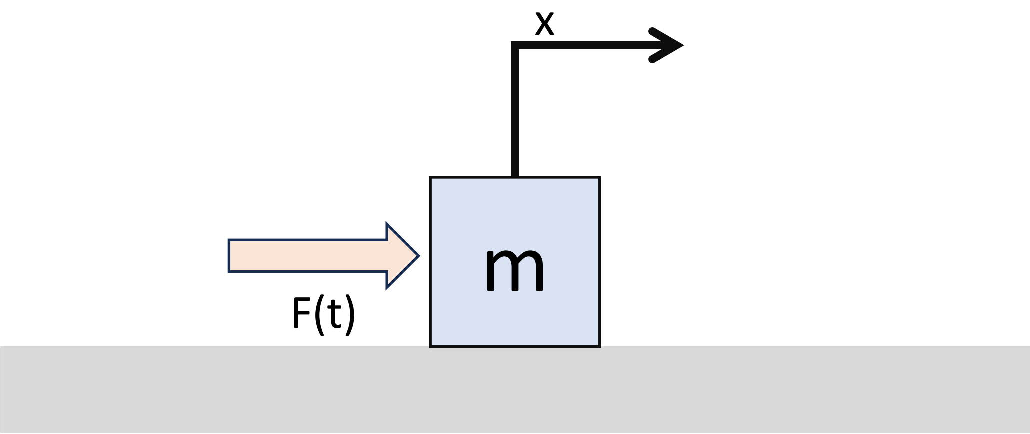
$$\dot{x} = \frac{x_k - x_{k-1}}{\Delta t} = Ax_{k-1} + Bu_k$$

$$x_k = (I + A\Delta t)x_{k-1} + B\Delta t u_k$$

State space equation

- 상태 공간 방정식
  - Example: Particle mass system

Measure(Sensor) : x

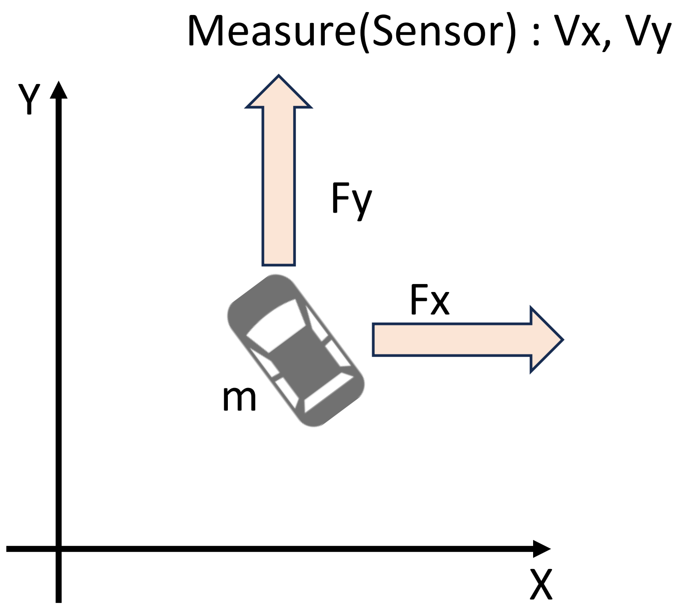


$$F = ma = m\ddot{x}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \quad \\ \quad \end{pmatrix} F$$

State space equation

- 상태 공간 방정식
  - Example: Vehicle (Global coordinate)



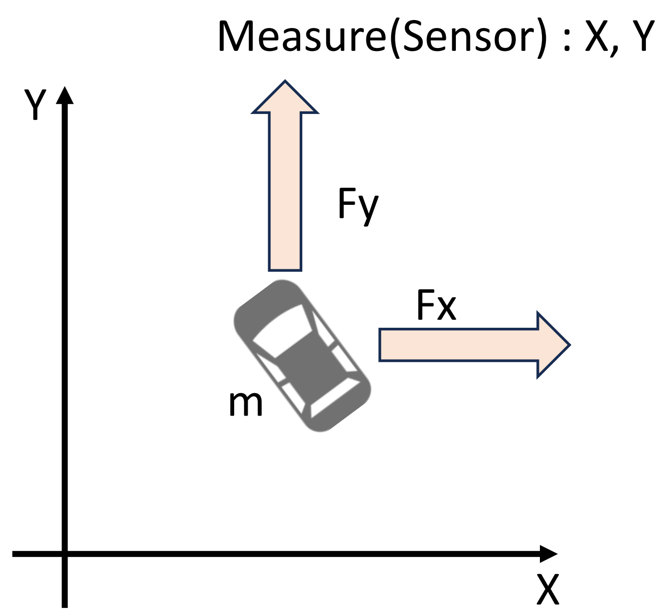
$$F = ma_x = m\ddot{x} \qquad F = ma_y = m\ddot{y}$$

$$\frac{d}{dt}\begin{pmatrix} Vx \\ Vy \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Vx \\ Vy \end{pmatrix} + \begin{pmatrix} \frac{Fx}{m} \\ \frac{Fy}{m} \end{pmatrix}$$

$$\begin{pmatrix} Vx \\ Vy \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Vx \\ Vy \end{pmatrix}$$

State space equation

- 상태 공간 방정식
  - Example: Vehicle (Global coordinate)



$$F = ma_x = m\ddot{x} \qquad F = ma_y = m\ddot{y}$$

$$\frac{d}{dt}\begin{pmatrix} Vx \\ Vy \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Vx \\ Vy \end{pmatrix} + \begin{pmatrix} \frac{Fx}{m} \\ \frac{Fy}{m} \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = ???$$

HL Klemove

Thank You