Recursive filter

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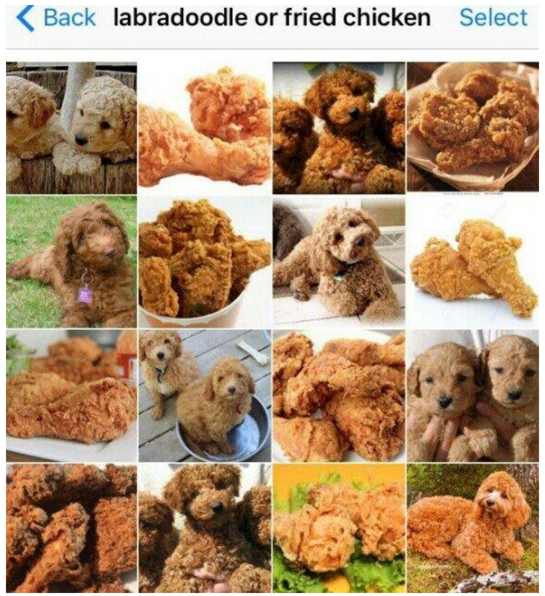
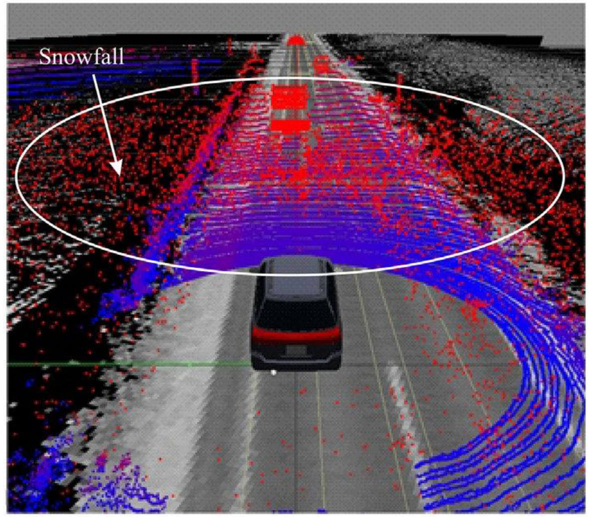
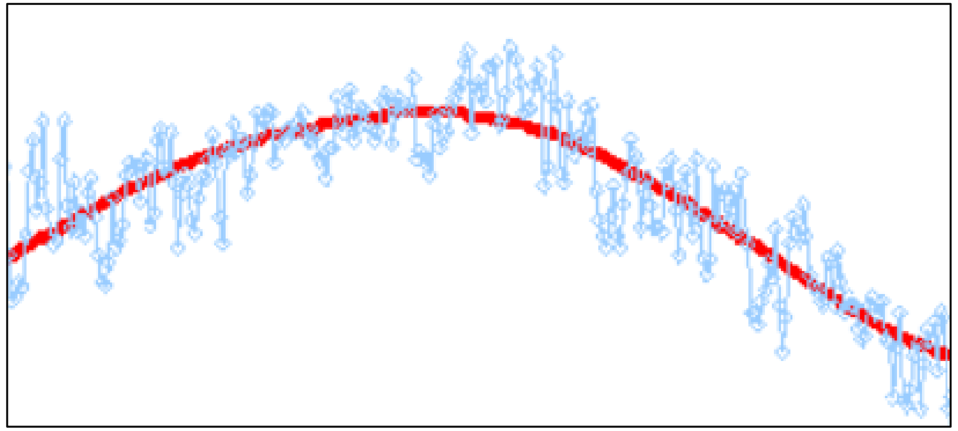
Contents

Introduction

- 믿을 수 있는 센서?



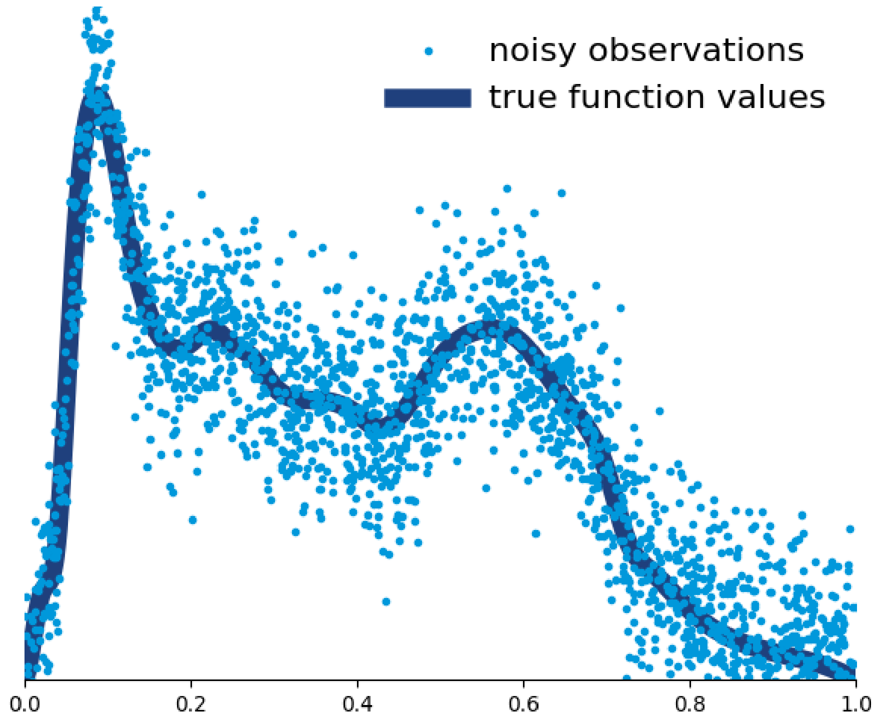




https://www.youtube.com/watch?v=g10fiQUhzN4

Introduction

- Recursive filter
  - Use previous results to estimate true value



Average filter

- 평균 필터
  - 가장 간단한 Recursive filter

$$\hat{x}_k = \frac{x_1 + x_2 + \cdots + x_k}{k}$$

$$\hat{x}_{k-1} = \frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1}$$

$$\hat{x}_k = \frac{(\hat{x}_{k-1} \cdot (k-1) + x_k)}{k}$$

$$\boldsymbol{\hat{x}_k = \frac{k-1}{k}\hat{x}_{k-1} + \frac{1}{k}x_k}$$

Moving average filter

- 이동 평균 필터
  - Main idea : 시스템의 동적인 변화를 반영하기 위해 최근 n 개의 데이터만 평균에 사용

$$\hat{x}_k = \frac{x_{k-(n-1)} + x_{k-(n-2)} + \cdots + x_k}{n}$$

$$\hat{x}_{k-1} = \frac{x_{k-n} + x_{k-(n-1)} + \cdots + x_{k-1}}{n}$$

$$\Rightarrow$$

$$\hat{x}_k - \hat{x}_{k-1} = \frac{x_k - x_{k-n}}{n}$$

$$\boldsymbol{\hat{x}_k = \hat{x}_{k-1} + \frac{1}{n}(x_k - x_{k-n})}$$

Low pass filter

- 저주파 통과 필터
  - Main idea : 최신 데이터를 오래된 데이터보다 더 높은 비중으로 사용(시상수 alpha의 의미)

$$\boldsymbol{\hat{x}_k = \alpha\hat{x}_{k-1} + (1-\alpha)x_k} \qquad (0 < \alpha < 1)$$

$$\hat{x}_{k-1} = \alpha\hat{x}_{k-2} + (1-\alpha)x_{k-1}$$

$$\vdots$$

$$\hat{x}_1 = \alpha\hat{x}_0 + (1-\alpha)x_1$$

$$\hat{x}_0 = x_0$$

$$\hat{x}_k = \alpha^2\hat{x}_{k-2} + \alpha(1-\alpha)x_{k-1} + (1-\alpha)x_k$$

$$\hat{x}_k = \alpha^3\hat{x}_{k-3} + \alpha^2(1-\alpha)x_{k-2} + \alpha(1-\alpha)x_{k-1} + (1-\alpha)x_k$$

$$\vdots$$

$$\hat{x}_k = \alpha^k x_0 + \alpha^{k-1}(1-\alpha)x_1 + \cdots + \alpha(1-\alpha)x_{k-1} + (1-\alpha)x_k$$

Kalman filter

- Basic Kalman filter
  - Main idea : 시스템 모델을 통해 예측 값을 구하고, 측정된 값을 통해 최적의 추정값을 결정
  - State space model of the system

$$\dot{x} = Ax + Bu \qquad\qquad \boldsymbol{x_k = Ax_{k-1} + Bu_k}$$
$$y = Cx \qquad or \qquad \boldsymbol{y_k = Cx_k}$$

  - Real world(Gaussian error)

$$x_k = Ax_{k-1} + Bu_k + w_k$$
$$y_k = Cx_k + v_k$$

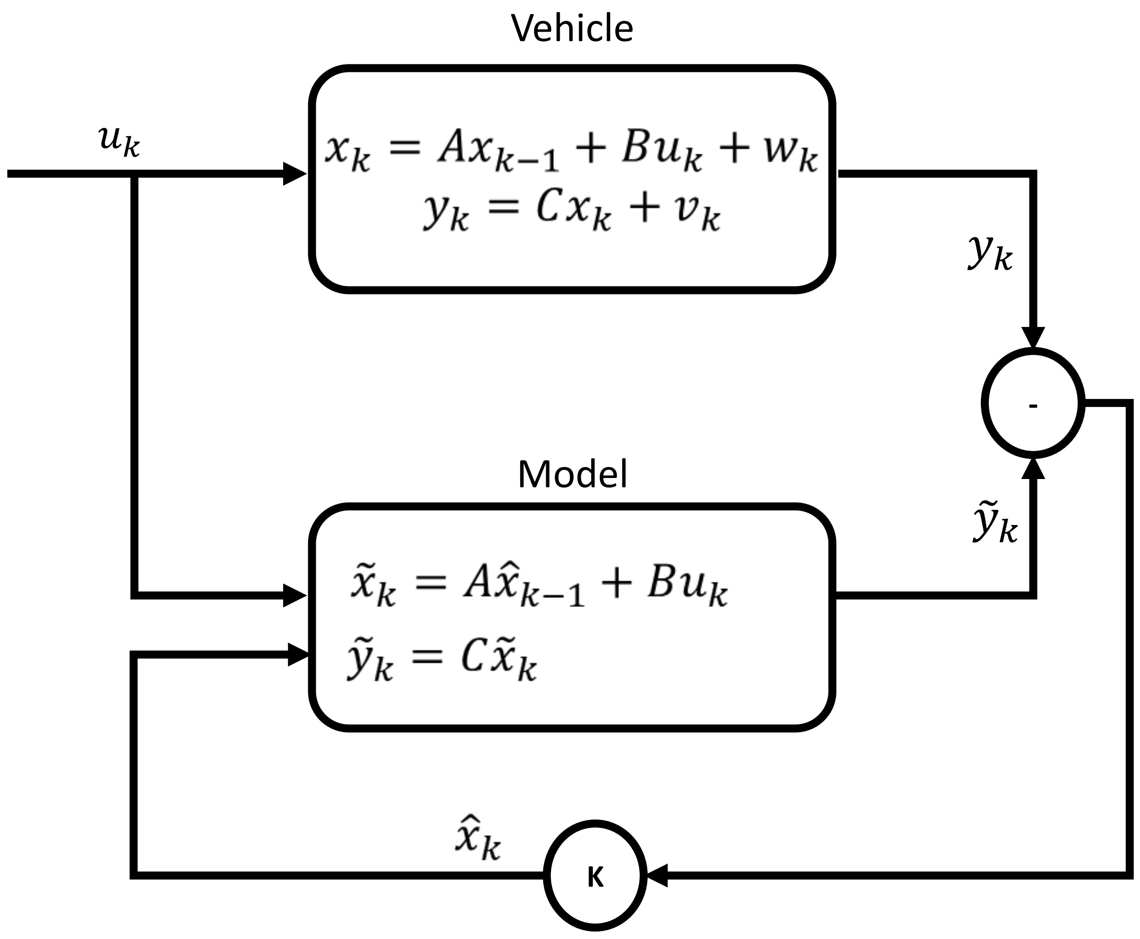
  - Optimal estimated x

$$\tilde{x}_k = A\hat{x}_{k-1} + Bu_k$$
$$\tilde{y}_k = C\tilde{x}_k$$

Prediction

$$\hat{x}_k = \tilde{x}_k + K_k(y_k - \tilde{y}_k)$$

Correction

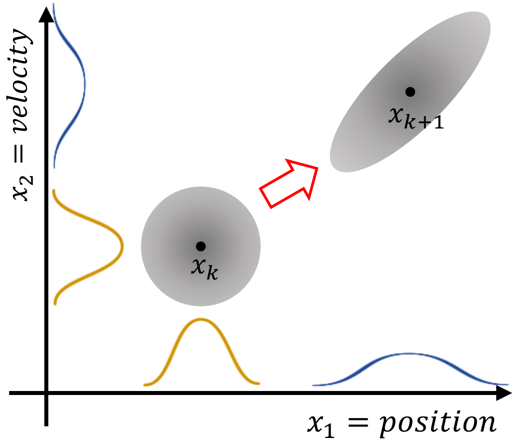


Kalman filter

- Basic Kalman filter
  - Assume well-estimated x_k

$$\hat{x}_k = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \qquad x_k = \begin{pmatrix} n(x_1, \sigma_1) \\ n(x_2, \sigma_2) \\ n(x_3, \sigma_3) \\ \vdots \end{pmatrix} \qquad e_k = x_k - \hat{x}_k = \begin{pmatrix} n(0, \sigma_1) \\ n(0, \sigma_2) \\ n(0, \sigma_3) \\ \vdots \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{pmatrix}$$

  - Covariance of e_k



e_1, e_2, ⋯ 들의 상관 관계 : Covariance(Correlation) of e_k

$$P_k = E\begin{pmatrix} e_1e_1 & e_1e_2 & \cdots & e_1e_m \\ e_2e_1 & e_2e_2 & \cdots & e_2e_m \\ \vdots & \vdots & \ddots & \vdots \\ e_me_1 & e_me_2 & \cdots & e_me_m \end{pmatrix} = E(e_k {e_k}^T)$$

Kalman filter

- Basic Kalman filter
  - The goal of Kalman filter : Optimize(minimize) the error!

$$\underset{K_k}{Minimize}\left(E(\|e_k\|^2)\right)$$

$$E(\|e_k\|^2) = E({e_k}^T e_k) = E\left(tr(e_k {e_k}^T)\right) = tr\left(E(e_k {e_k}^T)\right) = tr(P_k)$$

$$\frac{\partial}{\partial K_k}\left(E(\|e_k\|^2)\right) = \frac{\partial}{\partial K_k}\left(tr(P_k)\right) = 0$$

P_{k-1} 을 알고 있으므로, P_k를 P_{k-1}로 표현하면 끝!(점화식 형태)

Kalman filter

- Basic Kalman filter
  - Prediction step

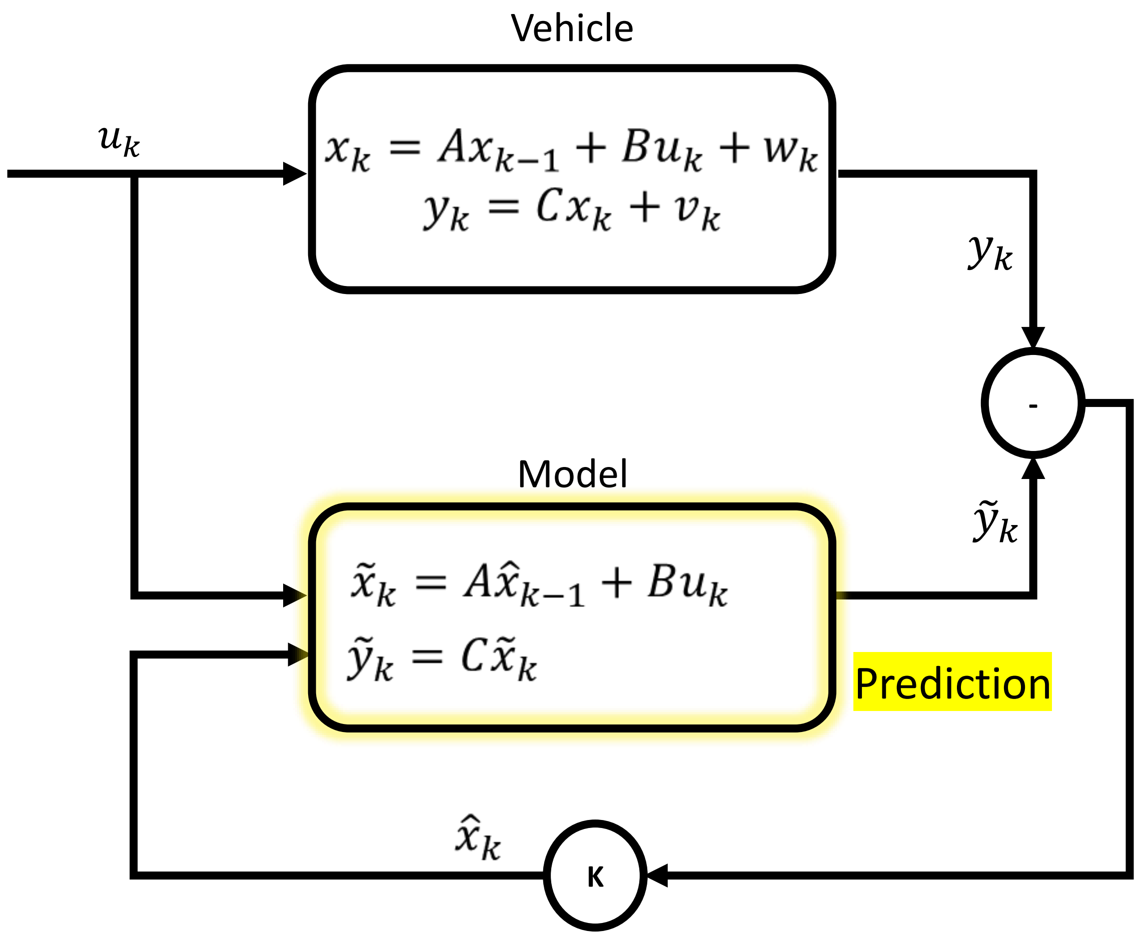
$$\tilde{x}_k = A\hat{x}_{k-1} + Bu_k$$
$$x_k = Ax_k + Bu_k + w_k$$

$$\Rightarrow \quad \tilde{e}_k = x_k - \tilde{x}_k = A(x_{k-1} - \hat{x}_{k-1}) + w_k$$

$$\tilde{P}_k = E\left(\tilde{e}_k {\tilde{e}_k}^T\right) = E\left(\left(A(x_{k-1} - \hat{x}_{k-1}) + w_k\right)\left((x_{k-1} - \hat{x}_{k-1})^T A^T + {w_k}^T\right)\right)$$

$$= E(A(x_{k-1} - \hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1})^T A^T)$$
$$\cancel{+E(A(x_{k-1} - \hat{x}_{k-1})w_k{}^T)}$$
$$\cancel{+E(w_k(x_{k-1} - \hat{x}_{k-1})^T A^T)}$$
$$\underbrace{+E(w_k w_k{}^T)}_{= \boldsymbol{Q_k}}$$

$$\tilde{P}_k = AP_{k-1}A^T + Q_k$$

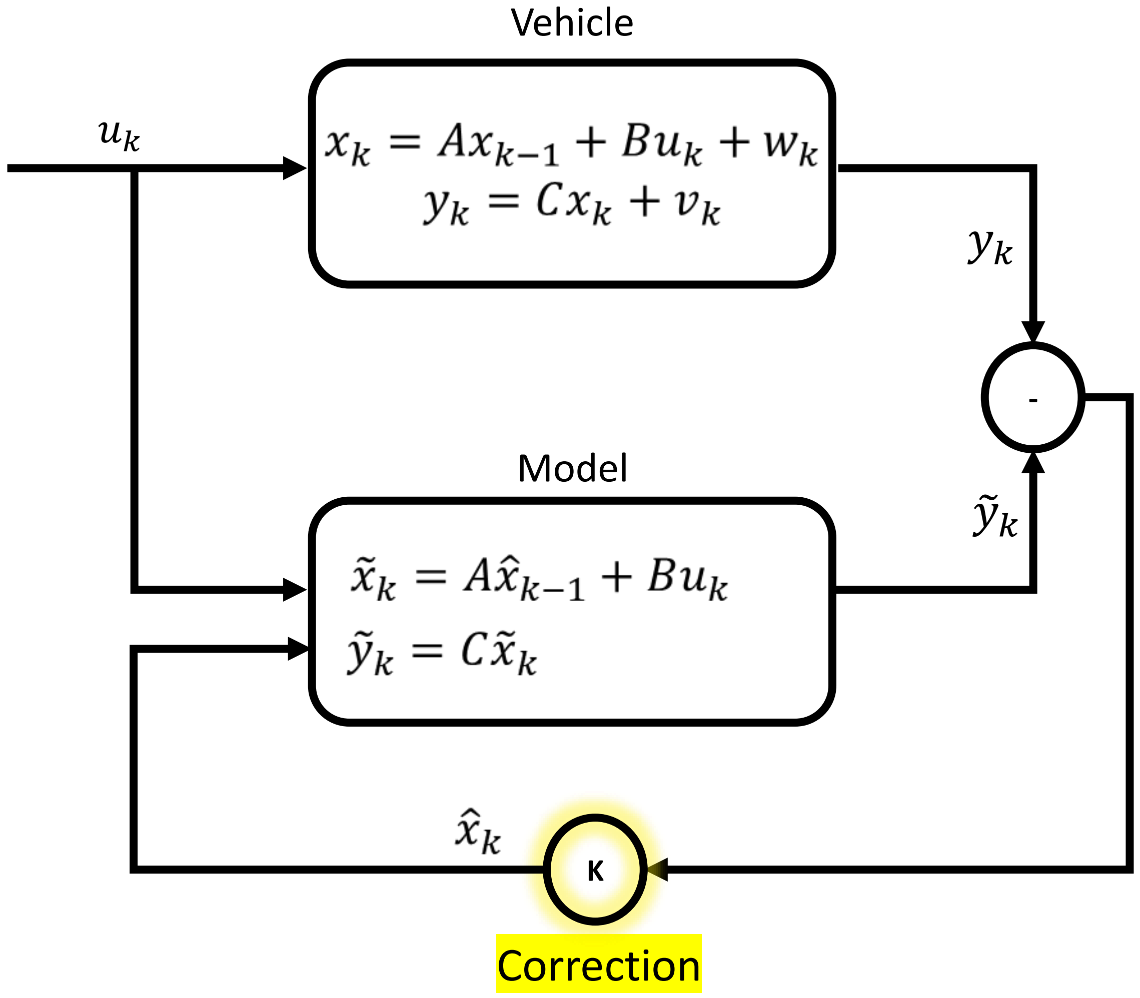


Kalman filter

- Basic Kalman filter
  - Correction step

$$\hat{x}_k = \tilde{x}_k + K_k(y_k - \tilde{y}_k)$$
$$\tilde{y}_k = C\hat{x}_k$$
$$y_k = Cx_k + v_k$$

$$\Rightarrow \quad x_k - \hat{x}_k = x_k - \tilde{x}_k - K_k(Cx_k + v_k - C\tilde{x}_k)$$
$$= (I - K_kC)(x_k - \tilde{x}_k) - K_kv_k$$



$$P_k = E(e_k{e_k}^T) = E\big((x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\big)$$
$$= E\Big(\big((I - K_kC)(x_k - \tilde{x}_k) - K_kv_k\big)\big((x_k - \tilde{x}_k)^T(I - K_kC)^T - (K_kv_k)^T\big)\Big)$$
$$= E\big((I - K_kC)(x_k - \tilde{x}_k)(x_k - \tilde{x}_k)^T(I - K_kC)^T\big)$$
$$+E\big((I - K_kC)\cancel{(x_k - \tilde{x}_k)}(-K_kV_k)^T\big)$$
$$+E\big((-K_kV_k)\cancel{(x_k - \tilde{x}_k)}^T(I - K_kC)^T\big)$$
$$+K_k\underbrace{E(v_k{v_k}^T)}_{= R_k}{K_k}^T$$

$$P_k = (I - K_kC)\tilde{P}_k(I - K_kC)^T + K_kR_k{K_k}^T$$

Kalman filter

- Basic Kalman filter
  - Kalman gain calculation(Optimization)

$$\frac{\partial}{\partial K_k}\left(E(\|e_k\|^2)\right) = \frac{\partial}{\partial K_k}\left(tr(P_k)\right) = 0$$

$$P_k = (I - K_k C)\tilde{P}_k (I - K_k C)^T + K_k R_k K_k{}^T$$

$$\Rightarrow \quad \frac{\partial}{\partial K_k}\left(tr\left((I - K_k C)\tilde{P}_k (I - K_k C)^T + K_k R_k K_k{}^T\right)\right) = 0$$

$$K_k = \tilde{P}_k C^T inv\left(C\tilde{P}_k C^T + R_k\right) \quad or \quad \frac{\tilde{P}_k C^T}{C\tilde{P}_k C^T + R_k}$$

$$P_k = (I - K_k C)\tilde{P}_k$$

HL Klemove

Kalman filter

- Basic Kalman filter
  - Flow & meaning

① Prediction

$$\tilde{x}_k = A\hat{x}_{k-1} + Bu_k$$

$$\tilde{y}_k = C\tilde{x}_k$$

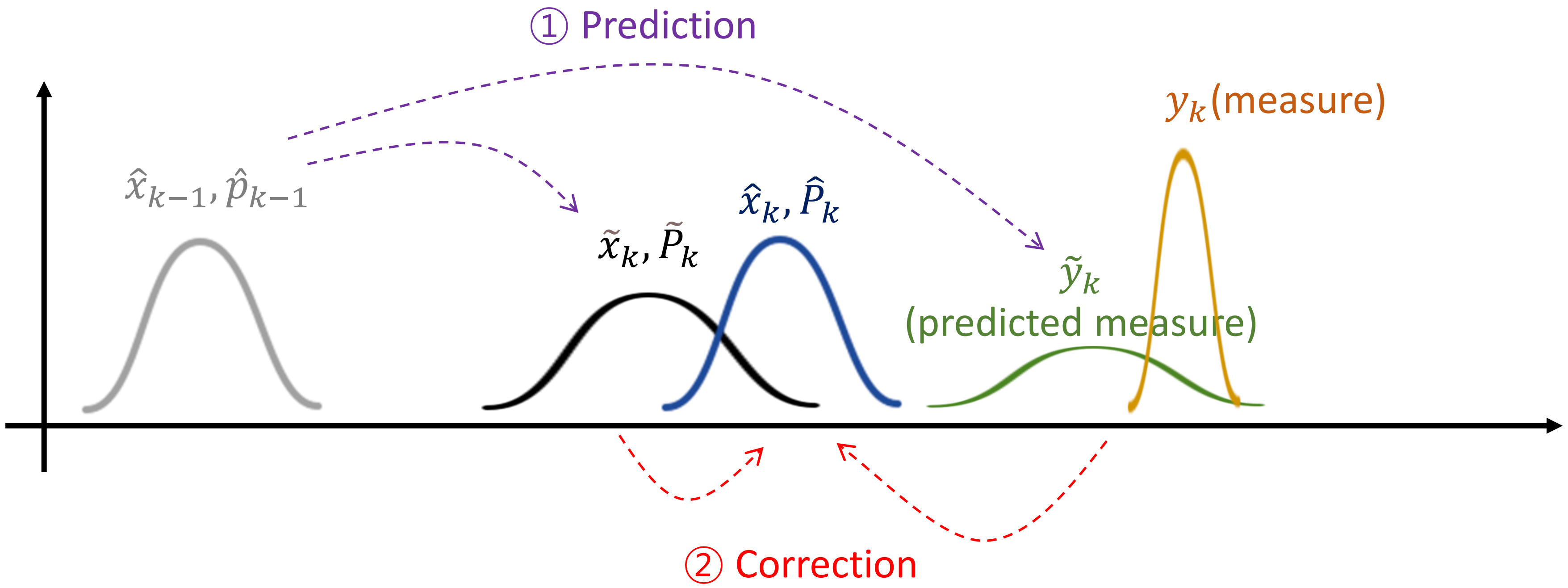
$$\tilde{P}_k = AP_{k-1}A^T + \underline{Q_k}$$

② Correction

$$\hat{x}_k = \tilde{x}_k + \underline{K_k}(y_k - \tilde{y}_k)$$

$$K_k = \frac{\tilde{P}_k C^T}{C\tilde{P}_k C^T + \underline{R_k}}$$

$$P_k = (I - K_k C)\tilde{P}_k$$



K_k(Kalman gain) : 예측된 값에 측정값과의 차이를 얼마나 반영할 것인지에 대한 값

Q_k(Model noise) : Kalman gain 을 크게 하는 요소(Tuning parameter)

R_k(Sensor noise) : Kalman gain 을 작게 하는 요소 (Tuning parameter)

HL Klemove

Thank You