Lecture 02
Bits, Bytes and Integers – part 2

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Slides adapted from Randy Bryant and Dave O'Hallaron: Introduction to Computer Systems, CMU

Two's Complement Arithmetic

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

- **4-bit Examples:**

HW	TC
0100	
+0011	
=	

HW	TC
1100	
+0011	
=	

HW	TC
0100	
+1101	
=	

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of } x \\ + \textit{ bit representation of } {-x} \\ \hline 0 \end{array} \quad \text{(ignoring the carry-out bit)}$$

  - What are the 8-bit negative encodings for the following?

$$\begin{array}{r} 00000001 \\ +\ ???????? \\ \hline 00000000 \end{array} \qquad \begin{array}{r} 00000010 \\ +\ ???????? \\ \hline 00000000 \end{array} \qquad \begin{array}{r} 11000011 \\ +\ ???????? \\ \hline 00000000 \end{array}$$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of } \; x \\ + \; \textit{bit representation of} -x \\ \hline 0 \end{array}$$ (ignoring the carry-out bit)

  - What are the 8-bit negative encodings for the following?

$$\begin{array}{r} 00000001 \\ + \; 11111111 \\ \hline 100000000 \end{array} \qquad \begin{array}{r} 00000010 \\ + \; 11111110 \\ \hline 100000000 \end{array} \qquad \begin{array}{r} 11000011 \\ + \; 00111101 \\ \hline 100000000 \end{array}$$

These are the bitwise complement plus 1!

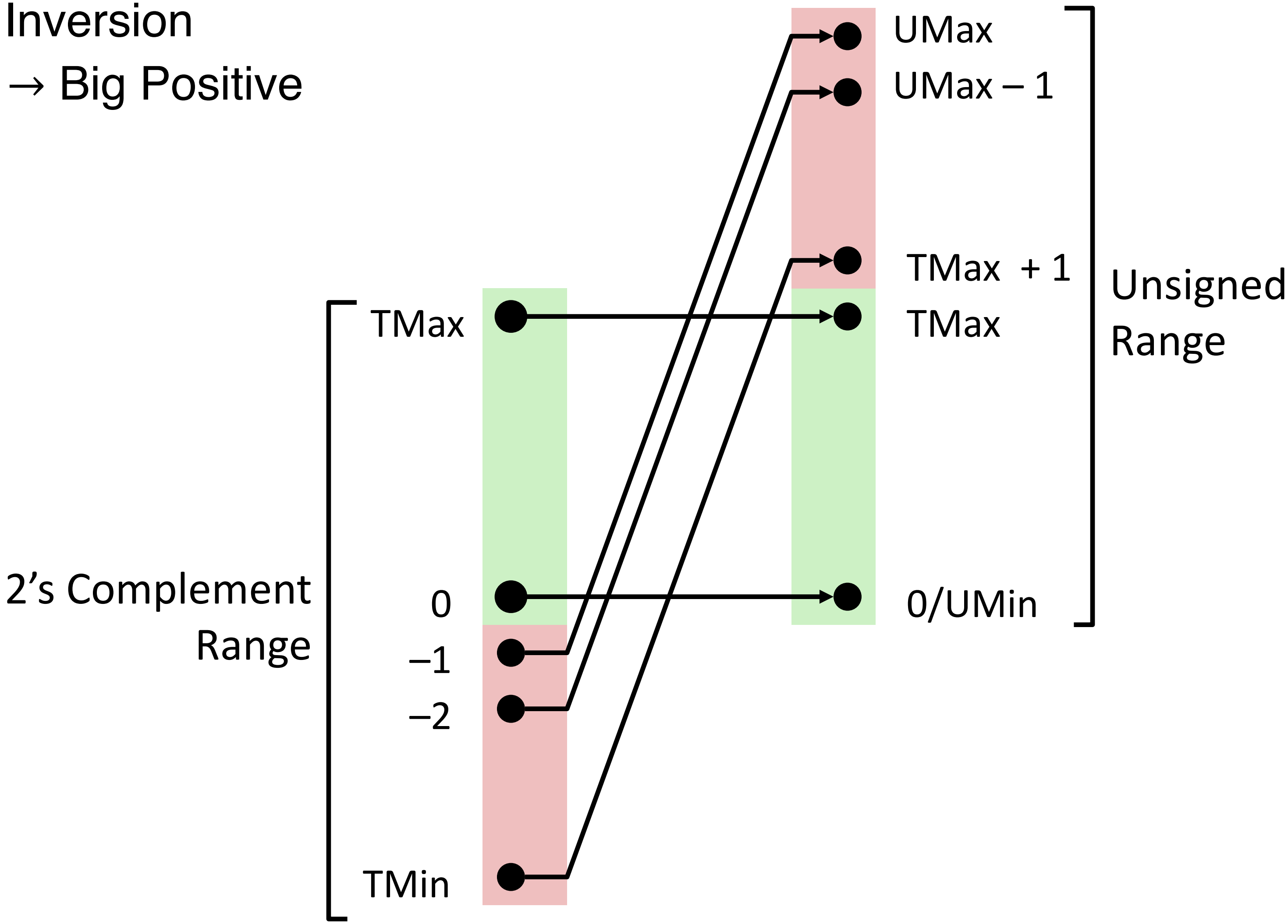
`-x == ~x + 1`

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Sign extension, overflow
- Shifting and arithmetic operations

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0` = 0
  - UMax = `0b11…1` = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0` = -2^{w-1}
  - TMax = `0b01…1` = $2^{w-1} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 64$

	Decimal	Hex
UMax	18,446,744,073,709,551,615	FF FF FF FF FF FF FF FF
TMax	9,223,372,036,854,775,807	7F FF FF FF FF FF FF FF
TMin	-9,223,372,036,854,775,808	80 00 00 00 00 00 00 00
-1	-1	FF FF FF FF FF FF FF FF
0	0	00 00 00 00 00 00 00 00

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;
  - *Implicit* casting can occur during assignments or function calls
    - tx = ux;
    - uy = ty;

Casting Surprises!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to unsigned**
  - Including comparison operators $<$, $>$, $==$, $<=$, $>=$

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Sign extension, overflow**
- Shifting and arithmetic operations

Sign Extension

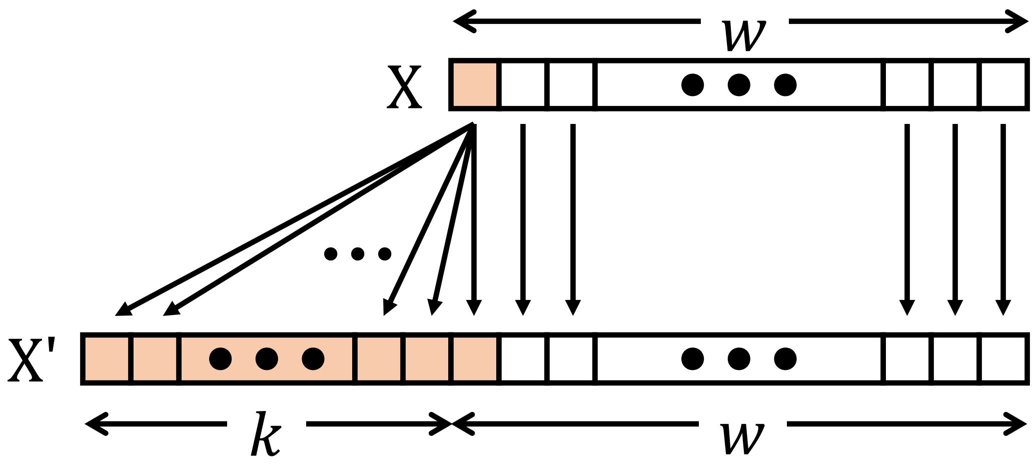
- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* `char` → `short` → `int` → `long`
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?

4-bit:	0010	=	+2
8-bit:	00000010	=	+2

  - Negative Case?

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X' *with the same value*
- **Rule: Add k copies of sign bit**
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \ldots, x_1, x_0}_{\text{original } \mathrm{X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

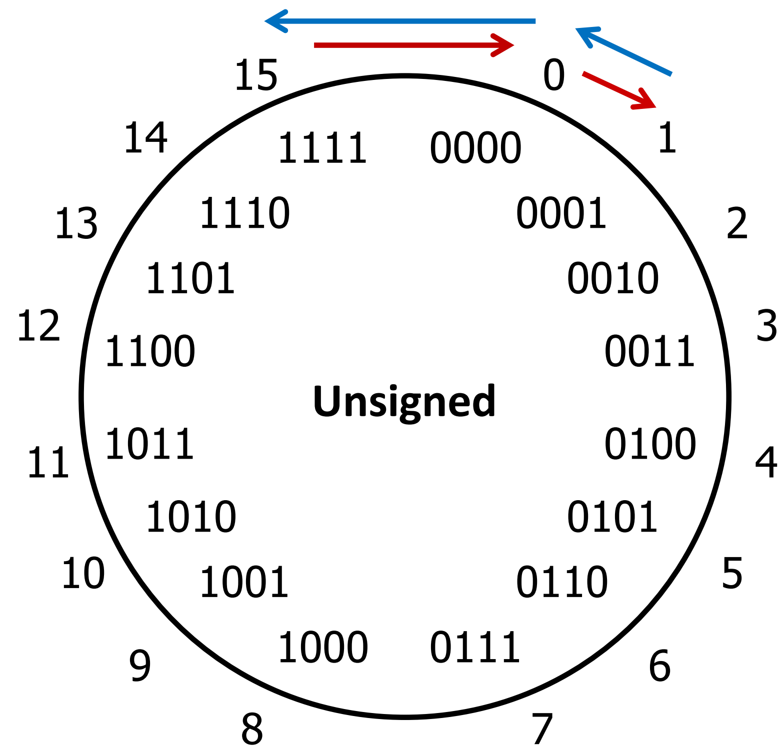
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

$$\begin{array}{r} 15 \\ +\ \ 2 \\ \hline \cancel{17} \\ 1 \end{array} \qquad \begin{array}{r} 1111 \\ +\ \ 0010 \\ \hline \cancel{1}0001 \end{array}$$

- **Subtraction:** borrow ($+2^N$)

$$\begin{array}{r} 1 \\ -\ \ 2 \\ \hline \cancel{-1} \\ 15 \end{array} \qquad \begin{array}{r} 10001 \\ -\ \ 0010 \\ \hline 1111 \end{array}$$



$\pm 2^N$ because of modular arithmetic

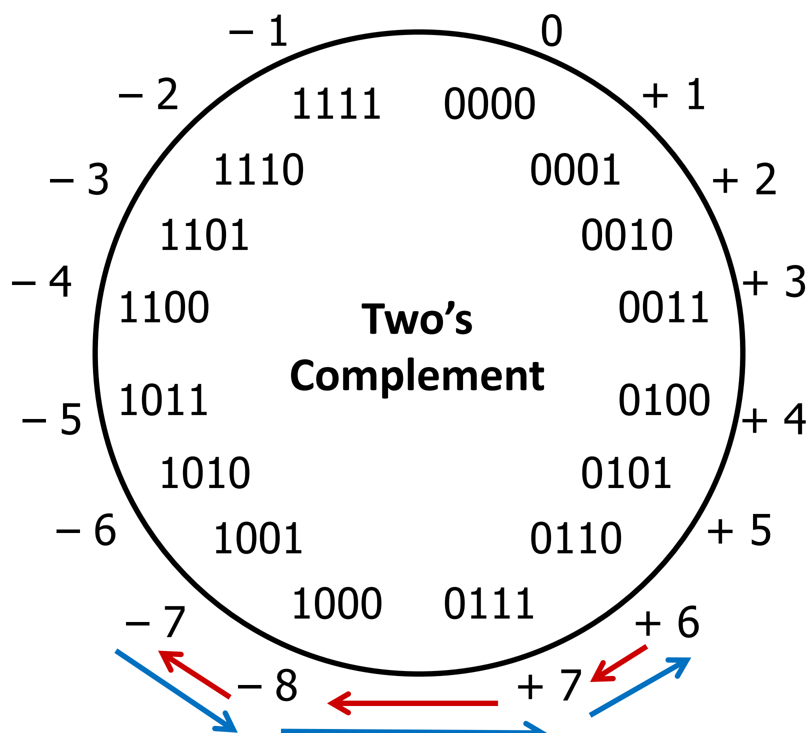
Overflow: Two’s Complement

- **Addition:** (+) + (+) = (–) result?

$$\begin{array}{r} 6 \\ +\ \ 3 \\ \hline \cancel{9} \\ {\color{red}-7} \end{array} \qquad \begin{array}{r} 0110 \\ +\ \ 0011 \\ \hline 1001 \end{array}$$

- **Subtraction:** (–) + (–) = (+)?

$$\begin{array}{r} -7 \\ -\ \ 3 \\ \hline \cancel{-10} \\ {\color{blue}6} \end{array} \qquad \begin{array}{r} 1001 \\ -\ \ 0011 \\ \hline 0110 \end{array}$$



For signed: overflow if operands have same sign and result’s sign is different

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Sign extension, overflow
- **Shifting and arithmetic operations**

Shift Operations

- Throw away (drop) extra bits that “fall off” the end
- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Fill with 0’s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Logical shift (for unsigned values)
    - Fill with 0’s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left (maintains sign of `x`)

Shift Operations

- Left shift (`x<<n`)
  - Fill with 0's on right
- Right shift (`x>>n`)
  - Logical shift (for unsigned values)
    - Fill with 0's on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

- Notes:
  - Shifts by `n<0` or `n≥w` (`w` is bit width of `x`) are *undefined*
  - **In C:** behavior of `>>` is determined by compiler
    - In gcc / C lang, depends on data type of `x` (signed/unsigned)
  - **In Java:** logical shift is `>>>` and arithmetic shift is `>>`

Shifting Arithmetic?

- What are the following computing?
  - `x>>n`
    - `0b 0100 >> 1 = 0b 0010`
    - `0b 0100 >> 2 = 0b 0001`
    - Divide by 2^n
  - `x<<n`
    - `0b 0001 << 1 = 0b 0010`
    - `0b 0001 << 2 = 0b 0100`
    - Multiply by 2^n

Shifting is faster than general multiply and divide operations!

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: `x*2`^{n}?

			Signed	Unsigned
`x = 25;`	`00011001`	=	25	25
`L1=x<<2;`	`0000110010000`	=	100	100
`L2=x<<3;`	`00011001000`	=	-56	200
`L3=x<<4;`	`000110010000`	=	-112	144

signed overflow

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: $x/2^n$?

```
xu = 240u;  11110000           = 240

R1u=xu>>3;  00011110000        =  30

R2u=xu>>5;  0000011110000      =   7
```

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator `>>` does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: `x/2^n`?

```
xs = -16;     11110000          = -16
R1s=xs>>3; 11111110000      =  -2
R2s=xs>>5; 1111111110000  =  -1
```

rounding (down)

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)

- We can only represent so many numbers in w bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding

- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking