VE401, Probabilistic Methods in Eng.

Recitation Class - Week 3

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   - Connection of Distributions
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Pascal Distribution

Definition. A random variable (X, f_X) has the ***Pascal distribution*** with parameters $p, 0 < p < 1$ and $r \in \mathbb{N} \setminus \{0\}$ if the probability density function is given by

$$f_X : \{r, r+1, \ldots\} \to \mathbb{R}, \qquad f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}.$$

Interpretation. $f_X(x)$ is the probability of obtaining the r-th success in the x-th Bernoulli trail, given the probability of success for each trail is p.

Pascal Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{r}{p}.$$

- Variance. Let $q = 1 - p$,

$$\mathrm{Var}[X] = \frac{rq}{p^2}.$$

- M.G.F.

$$m_X : (-\infty, -\ln q) \to \mathbb{R}, \qquad m_X(t) = \frac{(pe^t)^r}{(1 - qe^t)^r}.$$

Negative Binomial Distribution

Definition. A random variable (X, f_X) has the ***negative binomial distribution*** with parameters r and p if the probability density function is given by

$$f_X : \mathbb{N} \to \mathbb{R}, \qquad f_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x.$$

Interpretation. $f_X(x)$ is the probability of x failures before first obtaining r successes in Bernoulli trials, given the probability for each success is p.

Negative Binomial Distribution

Mean, variance and M.G.F.

- Mean. Let $q = 1 - p$,

$$\mathrm{E}[X] = \frac{rp}{q}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{rp}{q^2}.$$

- M.G.F.

$$m_X : (-\infty, -\ln q) \to \mathbb{R}, \qquad m_X(t) = \frac{p^r}{(1 - qe^t)^r}.$$

1. Common Distributions of Discrete Random Variables (Continue)
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Poisson Distribution

Definition. A random variable X has the ***Poisson distribution*** with parameter $k > 0$ if probability density function is given by

$$f_X : \mathbb{N} \to \mathbb{R}, \qquad f_X(x) = \frac{k^x e^{-k}}{x!}$$

Interpretation. $f_X(x)$ is the probability of x arrivals in the time interval $[0, t]$ with arrival rate $\lambda > 0$, and $k = \lambda t$.

"...which describes the occurrence of events that occur at a <u>constant rate</u> and <u>continuous environment</u>."

Poisson Distribution

Interpretation. *Constant rate* and *continuous environment*?

- Continuous environment. Not limited to time intervals, but also subregions of two- or three-dimensional regions or sublengths of a linear distance, and any regions that can be divided into arbitrarily small pieces.
- Constant rate. The probability of an occurrence during each very short interval (region) must be approximately proportional to the length (area, volume) of that interval (region).

Poisson Distribution

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Examples. Poisson process can be used to model

a) the number of particles that strike a certain target at a constant rate in a particular period;

b) the number of oocysts that occur in a water supply system given constant rate of occurrence per liter;

and many more.

Poisson Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = k.$$

- Variance.

$$\mathrm{Var}[X] = k.$$

- M.G.F.

$$m_X : \mathbb{R} \to \mathbb{R}, \qquad m_X(t) = e^{k(e^t - 1)}.$$

1. Common Distributions of Discrete Random Variables (Continue)
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Distributions Based on Bernoulli Trials

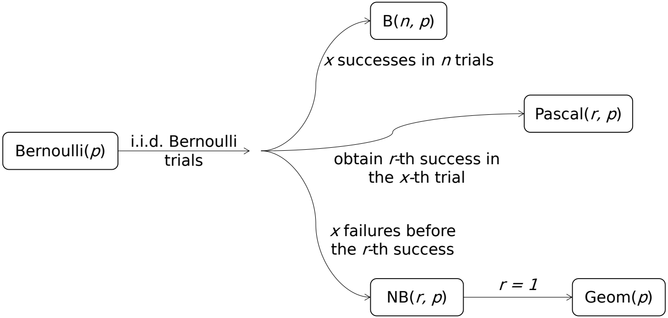

Figure: Connections of distributions based on Bernoulli trials.

Connections of Distributions

- Bernoulli $\to$ Binomial. $X_1, \ldots, X_n$ are independent random variables,

$$X_i \sim \text{Bernoulli}(p) \quad \Rightarrow \quad X = X_1 + \cdots + X_n \sim \text{B}(n, p).$$

- Binomial $\to$ Binomial. $X_1, \ldots, X_k$ are independent random variables,

$$X_i \sim \text{B}(n_i, p) \quad \Rightarrow \quad X = X_1 + \cdots + X_k \sim \text{B}(n, p),$$

  where $n = n_1 + \cdots + n_k$.

- Geometric $\to$ Negative binomial. $X_1, \ldots, X_r$ are independent random variables,

$$X_i \sim \text{Geom}(p) \quad \Rightarrow \quad X = X_1 + \cdots + X_r \sim \text{NB}(r, p).$$

Connections of Distributions

- Negative binomial → Negative binomial. $X_1, \ldots, X_n$ are independent random variables,

$$X_i \sim \mathrm{NB}(r_i, p) \quad \Rightarrow \quad X = X_1 + \cdots + X_n \sim \mathrm{NB}(r, p),$$

  where $r = r_1 + \cdots + r_n$.

- Poisson → Poisson. $X_1, \ldots, X_n$ are independent random variables,

$$X_i \sim \mathrm{Poisson}(k_i) \quad \Rightarrow \quad X = X_1 + \cdots + X_n \sim \mathrm{Poisson}(k),$$

  where $k = k_1 + \cdots + k_n$.

Closeness of Binomial Distribution and Poisson Distribution

Theorem. For $n \in \mathbb{N} \setminus \{0\}, 0 < p < 1$, suppose $f(x; n, p)$ denotes the probability density function of binomial distribution with parameters n and p, while $f(x; k)$ denotes the probability density function of Poisson distribution with parameter k. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers between 0 and 1 such that

$$\lim_{n \to \infty} np_n = k,$$

then

$$\lim_{n \to \infty} f(x; n, p_n) = f(x; k), \qquad \text{for all } x = 0, 1, \ldots$$

This means we can approximate the binomial distribution with Poisson distribution when n is large.

Closeness of Binomial Distribution and Poisson Distribution I

Proof. For $n \in \mathbb{N} \setminus \{0\}, 0 < p < 1$, suppose $f(x; n, p)$ denotes the probability density function of binomial distribution with parameters n and p, while $f(x; k)$ denotes the probability density function of Poisson distribution with parameter k. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers between 0 and 1 such that

$$\lim_{n\to\infty} np_n = k,$$

then

$$\lim_{n\to\infty} f(x; n, p_n) = f(x; k), \qquad \text{for all } x = 0, 1, \ldots$$

Closeness of Binomial Distribution and Poisson Distribution II

Proof. The probability density function for binomial distribution is given by

$$f(x;n,p_n) = \frac{n(n-1)\cdots(n-x+1)}{x!} p_n^x (1-p_n)^{n-x}.$$

Suppose $k_n = np_n$ so that $\lim_{n\to\infty} k_n = k$. Then the expression above can be rewritten as

$$f(x;n,p_n) = \frac{k_n^x}{x!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \left(1 - \frac{k_n}{n}\right)^n \left(1 - \frac{k_n}{n}\right)^{-x}.$$

Furthermore, for each $x \geq 0$,

$$\lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \left(1 - \frac{k_n}{n}\right)^{-x} = 1,$$

Closeness of Binomial Distribution and Poisson Distribution III

and

$$\lim_{n\to\infty}\left(1-\frac{k_n}{n}\right)^n = e^{-k}.$$

Therefore,

$$\lim_{n\to\infty} f(x;n,p_n) = \frac{k^x e^{-k}}{x!} = f(x;k).$$

1 Common Distributions of Discrete Random Variables (Continue)

2 Continuous Random Variables
- Features of Continuous Random Variables
- Transformation of Random Variables

3 Common Distributions of Continuous Random Variable

4 Supplementary Discussions

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Continuous Random Variables

Definition. Let S be a sample space. A ***continuous random variable*** is a map $X : S \to \mathbb{R}$ together with a function $f_X : \mathbb{R} \to \mathbb{R}$ with the properties that

(i) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ and

(ii) $\displaystyle\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1.$

The integral of f_X is interpreted as the probability that X assumes values x in a given range, i.e.,

$$P[a \leq X \leq b] = \int_a^b f_X(x)\mathrm{d}x.$$

The function f_X is called the ***probability density function*** of random variable X.

Cumulative Distribution

Definition. Let (X, f_X) be a continuous random variable. The cumulative distribution function for X is defined by $F_X : \mathbb{R} \to \mathbb{R}$,

$$F_X(x) := P[X \le x] = \int_{-\infty}^{x} f_X(y) \mathrm{d}y.$$

By the fundamental theorem of calculus, we can obtain the density function from F_X by

$$f_X(x) = F_X'(x).$$

Expectation, Variance, and M.G.F.

- Expectation.

$$\mathrm{E}[X] := \int_{\mathbb{R}} x \cdot f_X(x)\mathrm{d}x.$$

- Variance.

$$\mathrm{Var}[X] := \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - \mathrm{E}[X]^2.$$

- Moment-generating function.

$$m_X(t) = \mathrm{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\mathrm{d}x.$$

Note. All previous properties about expectation, variance and M.G.F. hold for continuous random variables.

Location of Continuous Distributions

Definitions.

- The ***median*** M_X is defined by $P[X \leq M_X] = 0.5$.
- The ***mean*** is given by $\mathrm{E}[X]$.
- The ***mode*** x_0, is the location of the maximum of f_X.

1 Common Distributions of Discrete Random Variables (Continue)

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Transformation of Random Variables

Theorem. Let X be a continuous random variable with density f_X. Let $Y = \varphi \circ X$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly monotonic and differentiable. The density for Y is then given by

$$f_Y(y) = f_X(\varphi^{-1}(y)) \cdot \left| \frac{\mathrm{d}\varphi^{-1}(y)}{\mathrm{d}y} \right|, \qquad \text{for } y \in \operatorname{ran} \varphi$$

and

$$f_Y(y) = 0, \qquad \text{for } y \notin \operatorname{ran} \varphi.$$

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Transformation of Random Variables

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$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Solution.

1. Identify and calculate φ^{-1} and $\dfrac{\mathrm{d}\varphi^{-1}(y)}{\mathrm{d}y}$.
2. Substitute x with $\varphi^{-1}(y)$ in the density function of X.

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Solution. We have $\varphi(x) = ve^{xt}$, and thus

$$\varphi^{-1}(y) = \frac{1}{t}\log\left(\frac{y}{v}\right), \qquad \frac{\mathrm{d}\varphi^{-1}(y)}{\mathrm{d}y} = \frac{1}{ty}.$$

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Solution (continued). Therefore,

$$f_Y(y) = \begin{cases} 3\left(1 - \dfrac{1}{t}\log\left(\dfrac{y}{v}\right)\right)^2 \cdot \dfrac{1}{ty}, & v < y < ve^t, \\ 0 & \text{otherwise.} \end{cases}$$

1 Common Distributions of Discrete Random Variables (Continue)

2 Continuous Random Variables

3 Common Distributions of Continuous Random Variable
- Exponential Distribution
- Gamma Distribution
- Normal Distribution
- Connections of Distributions

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Exponential Distribution

Definition. A continuous random variable (X, f_β) a ***exponential distribution*** with parameter β if the probability density function is defined by

$$f_\beta(X) = \begin{cases} \beta e^{-\beta x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Interpretation. The time between successive arrivals of a Poisson process with rate λ follows exponential distribution with parameter $\beta = \lambda$. (Recall $P[T > t] = e^{-\beta t}$.)

Note. Memoryless property:

$$P[X > x + s | X > x] = P[X > s].$$

Exponential Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{1}{\beta}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{1}{\beta^2}.$$

- M.G.F.

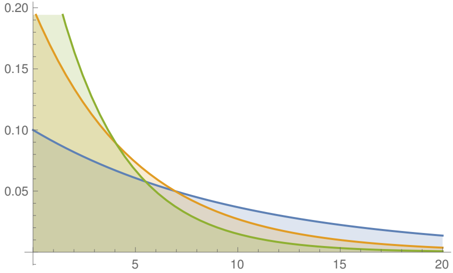
$$m_X : (-\infty, \beta) \to \mathbb{R}, \qquad m_X(t) = \frac{1}{1 - t/\beta}.$$

Exponential Distribution

Plots.

In[ ]:=
```
Plot[Table[PDF[ExponentialDistribution[β], x], {β, {0.1, 0.2, 0.3}}] // Evaluate, {x, 0, 20},
 Filling → Axis]
```

Out[ ]=



Exponential Distribution

Example 2. n light bulbs are burning simultaneously and independently, and the lifetime for each bulb follows the $\text{Exp}(\beta)$.

(i) What is the distribution of the length of time Y_1 until the first failure in one of the n bulbs?

(ii) What is the distribution of the length of time Y_2 after the first failure until a second bulb fails?

Exponential Distribution

Example 2. n light bulbs are burning simultaneously and independently, and the lifetime for each bulb follows the $\text{Exp}(\beta)$.

1. (i) What is the distribution of the length of time Y_1 until the first failure in one of the n bulbs?
2. (ii) What is the distribution of the length of time Y_2 after the first failure until a second bulb fails?

Solution (i). Suppose random variables $X_1, \ldots, X_n$ satisfies that $X_i \sim \text{Exp}(\beta)$, and $Y_1 = \min\{X_1, \ldots, X_n\}$. Then for any $t > 0$,

$$\begin{aligned} P[Y_1 > t] &= P[X_1 > t, \ldots, X_n > t] \\ &= P[X_1 > t] \times \cdots \times P[X_n > t] \\ &= e^{-n\beta t}, \end{aligned}$$

indicating an exponential distribution with parameter $n\beta$. What about (ii)?

1. Common Distributions of Discrete Random Variables (Continue)

2. Continuous Random Variables

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Gamma Distribution

Definition. Let $\alpha, \beta \in \mathbb{R}, \alpha, \beta > 0$. A continuous random variable $(X, f_{\alpha,\beta})$ follows a ***gamma distribution*** with parameters α and β if the probability density function is given by

$$f_{\alpha,\beta}(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} \mathrm{d}z, \alpha > 0$ is the Euler gamma function.

Interpretation. The time needed for the next r arrivals in a Poisson process with rate λ follows a Gamma distribution with parameters $\alpha = r, \beta = \lambda$.

Gamma Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{\alpha}{\beta}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{\alpha}{\beta^2}.$$

- M.G.F.

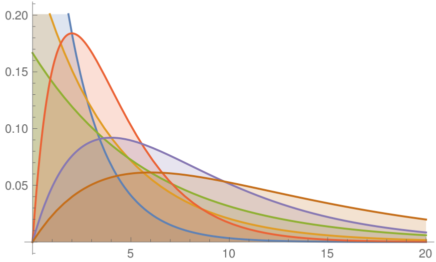
$$m_X : (-\infty, \beta) \to \mathbb{R}, \qquad m_X(t) = \frac{1}{(1 - t/\beta)^{\alpha}}.$$

Gamma Distribution

Plots.

In[∘]:= `Plot[Table[PDF[GammaDistribution[α, β], x], {α, {1, 2}}, {β, {2, 4, 6}}] // Evaluate, {x, 0, 20}, Filling → Axis]`

Out[∘]=



Chi-Squared Distribution

Definition. Let $\gamma \in \mathbb{N}$. A continuous random variable (X^2_γ, f_X) follows a ***chi-squared distribution*** with γ degrees of freedom if the probability density function is given by

$$f_\gamma(x) = \begin{cases} \dfrac{1}{2^{\gamma/2}\Gamma(\gamma/2)} x^{\gamma/2-1} e^{-x/2}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

which is a gamma distribution with $\alpha = \gamma/2, \beta = 1/2$. Therefore,

$$\mathrm{E}[X^2_\gamma] = \gamma, \qquad \mathrm{Var}[X^2_\gamma] = 2\gamma.$$

Chi-Squared Distribution

Plots.

```
In[•]:= Plot[Table[PDF[ChiSquareDistribution[γ], x], {γ, {3, 5, 7}}] // Evaluate, {x, 0, 20},
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```

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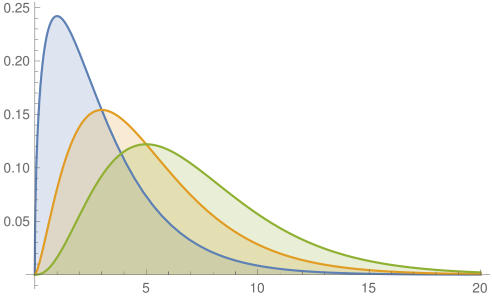

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- Exponential Distribution
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Normal Distribution

Definition. A continuous random variable (X, f_{μ,σ^2}) has the ***normal distribution*** with mean $\mu \in \mathbb{R}$ and variance $\sigma^2, \sigma > 0$ if the probability density function is given by

$$f_{\mu,\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \qquad x \in \mathbb{R}.$$

Normal Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \mu.$$

- Variance.

$$\mathrm{Var}[X] = \sigma^2.$$

- M.G.F.

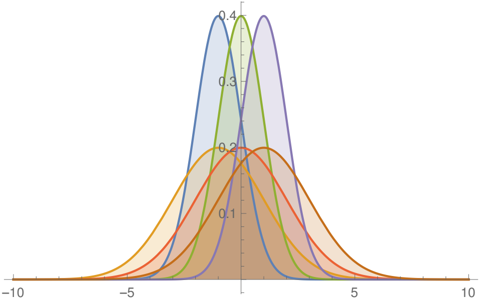
$$m_X : \mathbb{R} \to \mathbb{R}, \qquad m_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Normal Distribution

Plots.

In[∘]:=
```
Plot[Table[PDF[NormalDistribution[μ, σ], x], {μ, {-1, 0, 1}}, {σ, {1, 2}}] // Evaluate,
  {x, -10, 10}, Filling → Axis]
```

Out[∘]=



Standardizing Normal Distribution

Suppose $X \sim \text{Normal}(\mu, \sigma^2)$. Then

$$Z = \frac{X - \mu}{\sigma} \sim \text{Normal}(0, 1),$$

where the normal distribution with mean μ and variance σ^2 is the ***standard normal distribution***. Furthermore, the cumulative distribution function of X is given by

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right), \quad F^{-1}(p) = \mu + \sigma\Phi^{-1}(p),$$

where Φ is the cumulative distribution function for the standard normal distribution function.

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Distributions based on of Poisson Process

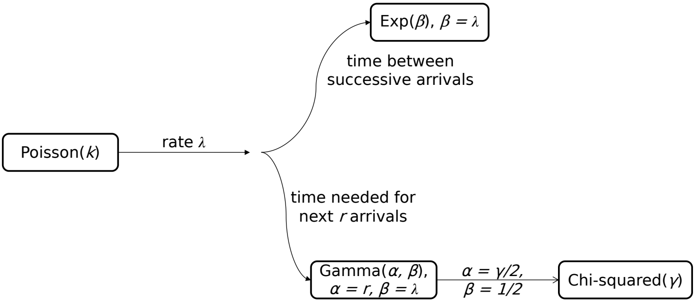

Figure: Connections of distributions based on Poisson process.

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   - Aircraft Maintenance Problem (1960s)
5. Exercises

Aircraft Maintenance Problem (1960s)

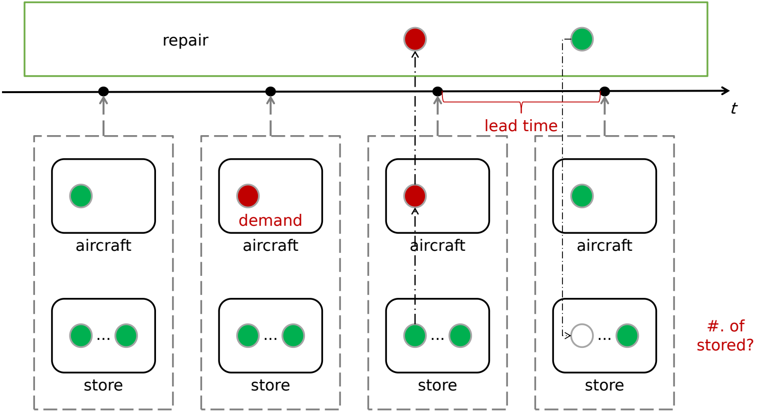
[1]In order to ensure a high standard of serviceability without incurring unnecessary aircraft delay, an airline adopts the following replacement policy for its repairable aircraft components:

1. An unserviceable component is removed from an aircraft and sent for inspection to repair.
2. An immediate demand for a serviceable replacement is made to the stores. If available, the replacement is at once fitted to the aircraft.
3. When the previously unserviceable component has been repaired to be serviceable, it is placed in the stores.

[1]The Application of the Negative Binomial Distribution to Stock Control Problems by C.J. Taylor: https://www.jstor.org/stable/3007410

Aircraft Maintenance Problem (1960s)

Sample



Qustion: What is the probability p_{rk} of exactly r demands during the lead time when the parameter value is k ?

Aircraft Maintenance Problem (1960s)

Assumptions We have the following models.

- The demand is a random variable R that follows a Poisson distribution

$$f_R(r|t) = \frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)},$$

where λ is the average number of demands per unit time, and t denotes the lead time.

- The lead time t follows a gamma distribution

$$f_T(t) = \frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)}, \qquad k > 0,$$

which can be seen from $\alpha = k, \beta = \mu$.

Aircraft Maintenance Problem (1960s)

Solution Then the probability of r demands during the lead time with parameter k is

$$
\begin{aligned}
p_{rk} &= \int_0^\infty f_R(r|t) f_T(t) \mathrm{d}t \qquad \text{(recall } P[A] = \sum P[A|B]P[B]\text{)} \\
&= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)} \times \frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)} \mathrm{d}t \\
&= \frac{\lambda^t \mu^k}{\Gamma(r+1)\Gamma(k)} \int_0^\infty t^{r+k-1} e^{-(\lambda+\mu)t} \mathrm{d}t \qquad \text{(let } z = (\lambda+\mu)t\text{)} \\
&= \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}\Gamma(r+1)\Gamma(k)} \int_0^\infty z^{r+k-1} e^{-z} \mathrm{d}z \\
&= \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}} \times \frac{\Gamma(r+k)}{\Gamma(r+1)\Gamma(k)} = \binom{r+k-1}{k-1} \frac{(\lambda/\mu)^r}{(1+\lambda/\mu)^{r+k}},
\end{aligned}
$$

implying r follows a negative binomial distribution with mean $\lambda k/\mu$ and variance $\lambda k/\mu(1+\lambda/\mu)$.

1 Common Distributions of Discrete Random Variables (Continue)

2 Continuous Random Variables

3 Common Distributions of Continuous Random Variable

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- Exercise 1.
- Exercise 2.
- Exercise 3.
- Exercise 4.
- Exercise 5.

Exercise 1.

Exercise 1. Suppose Keven plays a game where he has probability p to win in each play. When he wins, his fortune is doubled, and when he loses, his fortune is cut in half. If he begins playing with a given fortune $c > 0$, what is the expected value of his fortune after n independent plays?

Exercise 1. Sol

Solution. Define a random variable $X_i, i = 1, \ldots, n$, where $X_i = 2$ if Keven's fortune is doubled on the i-th play and $X_i = 1/2$ if his fortune is cut in half on the i-th play. Then

$$\mathrm{E}[X_i] = 2p + \frac{1}{2}(1 - p) = \frac{1}{2}(1 + 3p).$$

Because the plays are independent from each other, the expected final fortune is

$$c\mathrm{E}[X_1 \cdots X_n] = c \times \mathrm{E}[X_1] \cdots \mathrm{E}[X_n] = \frac{c}{2^n}(1 + 3p)^n.$$

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Exercise 2.

Exercise 2. For $0 < p < 1$ and $n = 2, 3, \ldots$, determine the value of

$$\sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}.$$

Exercise 2. Sol I

Solution. Changing the lower limit of the summation from $x = 2$ to $x = 0$, we obtain

$$\sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} - \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}.$$

If X has the binomial distribution with parameters n and p, then the two terms in the summation are $\mathrm{E}[X^2]$ and $\mathrm{E}[X]$, respectively. Therefore,

$$\begin{aligned} \mathrm{E}[X^2] - \mathrm{E}[X] &= \mathrm{Var}[X] + \mathrm{E}[X]^2 - \mathrm{E}[X] \\ &= np(1-p) + (np)^2 - np \\ &= n(n-1)p^2. \end{aligned}$$

<u>**Note**</u>. Not limited to this example, it is useful in general to calculate some integrals (especially in statistics) if we know the probability density functions, expectations and variances of some common distributions.

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Exercise 3.

Exercise 3. Suppose that a book with n pages contains on the average λ misprints per page. What is the probability that there will be at least m pages which contain more than k misprints?

Exercise 3. Sol I

Solution. Let Y denote the number of misprints on a given page. Then the probability p that a given page will contain more than k misprints is

$$p = P[Y > k] = \sum_{i=k+1}^{\infty} f(i;\lambda) = \sum_{i=k+1}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!},$$

and

$$1 - p = \sum_{i=0}^{k} f(i;\lambda) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}.$$

Let X denote the number of pages among n pages in the book, where there are more than k misprints. Then for $x = 0, 1, \ldots, n$,

$$P[X = x] = \binom{n}{x} p^x (1-p)^{n-x},$$

Exercise 3. Sol II

which gives

$$P[X \geq m] = \sum_{x=m}^{n} \binom{n}{x} p^x (1-p)^{n-x},$$

where p is given by the formula above.

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Exercise 4.

Exercise 4. Suppose that a certain system contains three components C_1, C_2, C_3 that function independently of each other and are connected as series, so that the system fails as soon as one of the components fails. Suppose that the length of life of the C_1, C_2, C_3 has the exponential distribution with parameters

$$\beta_1 = 0.001, \qquad \beta_2 = 0.003, \qquad \beta_3 = 0.006,$$

respectively, all measured in hours. Determine the probability that the system will not fail before 100 hours.

Exercise 4. Sol

Solution. Let Y denote the length of life of the system, and X_1, X_2, X_3 are the lifetime of each component. Then for any $y > 0$,

$$\begin{aligned} P[Y > y] &= P[X_1 > y, X_2 > y, X_3 > y] \\ &= P[X_1 > y]P[X_2 > y]P[X_3 > y] \\ &= e^{-(\beta_1+\beta_2+\beta_3)y}. \end{aligned}$$

Therefore,

$$P[Y > 100] = e^{-100\times 0.01} = \frac{1}{e}.$$

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Exercise 5.

Exercise 5. Let X_1, X_2, X_3 be independent lifetimes of memory chips. Suppose each X_i follows the normal distribution with mean 300 hours and standard deviation 10 hours. Compute the probability that at least one of the three chips lasts at least 290 hours.

Exercise 5. Sol

Solution. Let A_i be the event that chip i lasts at most 290 hours. Then the probability of interest is given by

$$\begin{aligned} P &= \bigcup_{i=1}^{3} A_i^c \\ &= 1 - P\left[\bigcap_{i=1}^{3} A_i\right] \\ &= 1 - \prod_{i=1}^{3} P[A_i]. \end{aligned}$$

Since the lifetime of each chip has the normal distribution with mean 300 and standard deviation 10, each A_i has probability

$$F\left(\frac{290-300}{10}\right) = F(-1) = 1 - 0.8413 = 0.1587.$$

Therefore,

$$P = 1 - 0.1587^3 = 0.996.$$