Calculation of three modes interferometer with displacement operators

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In this manuscript I try to describe the whole physical system of three modes interferometer with displacement operators. I successfully finish the calculation that input state $|1,1,1\rangle$ passes the tritter(three ports beam-splitters). And I also try to apply displament operators to the following states. I didn't know how to slove it at first, but later on I found a solution to the problem and probably solved the problem correctly.

A. Analyzation of tritter with the input state $|1,1,1\rangle$

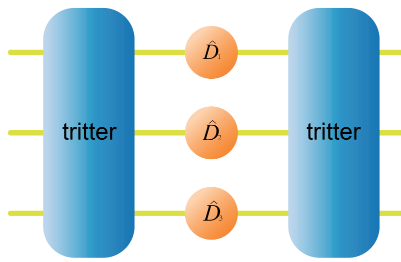

FIG. 1. Three modes interferometer with tritter (three ports beam splitters) and displacement transformation.

Like many articles do, the unitary transformation of the tritter (three ports beam-splitters) has the following form:

$$\hat{U} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \\ e^{i\frac{2\pi}{3}} & 1 & e^{i\frac{2\pi}{3}} \\ e^{i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} & 1 \end{pmatrix} . \tag{1}$$

This unitary operator and creation operator have the following connection:

$$\hat{b_i}^\dagger = \sum_{i,j} U_{ij} \hat{a_j}^\dagger , \tag{2}$$

where the $\hat{a}^\dagger$ and $\hat{b}^\dagger$ are creation operator of the input and output modes respectively.
Or we can rewrite it in matrix form:

$$\overrightarrow{\hat{b}^\dagger} = \hat{U}\overrightarrow{\hat{a}^\dagger},\ and\ \overrightarrow{\hat{a}^\dagger} = \hat{U}^{-1}\overrightarrow{\hat{b}^\dagger} = \hat{U}^\dagger\overrightarrow{\hat{b}^\dagger} . \tag{3}$$

For the input state $|1,1,1\rangle$, we have:

$$|1,1,1\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_3^\dagger |0,0,0\rangle . \tag{4}$$

Because tritter doesn't change the number of photon, so the state $|0,0,0\rangle$ has a special character:

$$|0,0,0\rangle_{in} \xrightarrow{Tritter} |0,0,0\rangle_{out} . \tag{5}$$

Therefore, for the input state $|1,1,1\rangle$, changing the creation operators, note that the conjugate of $\hat{U}$ should be taken here:

$$
\begin{aligned}
\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_3^\dagger\left|0,0,0\right\rangle_{in} \xrightarrow{Tritter} & \frac{1}{3\sqrt{3}}\left(\hat{b}_1^\dagger + e^{-i\frac{2\pi}{3}}\hat{b}_2^\dagger + e^{-i\frac{2\pi}{3}}\hat{b}_3^\dagger\right)\left(e^{-i\frac{2\pi}{3}}\hat{b}_1^\dagger + \hat{b}_2^\dagger + e^{-i\frac{2\pi}{3}}\hat{b}_3^\dagger\right)\left(e^{-i\frac{2\pi}{3}}\hat{b}_1^\dagger + e^{-i\frac{2\pi}{3}}\hat{b}_2^\dagger + \hat{b}_3^\dagger\right)\left|0,0,0\right\rangle_{out}. \\
=&\frac{1}{3\sqrt{3}}\Big[e^{i\frac{2\pi}{3}}\hat{b}_1^{\dagger 3} + e^{i\frac{2\pi}{3}}\hat{b}_2^{\dagger 3} + e^{i\frac{2\pi}{3}}\hat{b}_3^{\dagger 3} + \left(1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)\hat{b}_1^{\dagger 2}\hat{b}_2^\dagger + \left(1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)\hat{b}_1^{\dagger 2}\hat{b}_3^\dagger \\
&+\left(1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)\hat{b}_1^\dagger\hat{b}_2^{\dagger 2} + \left(1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)\hat{b}_1^\dagger\hat{b}_3^{\dagger 2} + \left(1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)\hat{b}_2^{\dagger 2}\hat{b}_3^\dagger \\
&+\left(1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)\hat{b}_2^\dagger\hat{b}_3^{\dagger 2}\Big]\left|0,0,0\right\rangle_{out} \\
=&\frac{1}{3\sqrt{3}}\left[e^{i\frac{2\pi}{3}}\hat{b}_1^{\dagger 3} + e^{i\frac{2\pi}{3}}\hat{b}_2^{\dagger 3} + e^{i\frac{2\pi}{3}}\hat{b}_3^{\dagger 3} + 3\left(1+e^{i\frac{2\pi}{3}}\right)\hat{b}_1^\dagger\hat{b}_2^\dagger\hat{b}_3^\dagger\right]\left|0,0,0\right\rangle_{out} \\
=&\left[\left(-\frac{1}{6\sqrt{3}}+\frac{i}{6}\right)\left(\hat{b}_1^{\dagger 3}+\hat{b}_2^{\dagger 3}+\hat{b}_3^{\dagger 3}\right) + \left(\frac{1}{2\sqrt{3}}+\frac{i}{2}\right)\hat{b}_1^\dagger\hat{b}_2^\dagger\hat{b}_3^\dagger\right]\left|0,0,0\right\rangle_{out} \\
=&\left(-\frac{\sqrt{2}}{6}+\frac{i}{\sqrt{6}}\right)\left(\left|3,0,0\right\rangle + \left|0,3,0\right\rangle + \left|0,0,3\right\rangle\right) + \left(\frac{1}{2\sqrt{3}}+\frac{i}{2}\right)\left|1,1,1\right\rangle.
\end{aligned} \tag{6}
$$

B. Displacement for the following states

After passing the tritter, photons are effected by the displacement operators in three modes, $\hat{D}_1, \hat{D}_2$and$\hat{D}_3$, respectively. With the identity(the disentangling theorem):

$$
e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]}. \tag{7}
$$

when $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\alpha^*\hat{a}$, we can the rewrite displacement operator:

$$
\hat{D}(\alpha) = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})} = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}. \tag{8}
$$

Because of the superposition principle, we can consider the state $|1,1,1\rangle$ and $|3,0,0\rangle\ldots$ separately.

1. $|n,0,0\rangle$ *states*

Firstly, we consider a more general case, the $\hat{D}_1(\alpha_1)$ operator act on $|n,0,0\rangle$:

$$
\begin{aligned}
\hat{D}_1(\alpha_1)\left|n,0,0\right\rangle =& e^{-\frac{1}{2}|\alpha_1|^2}e^{\alpha_1\hat{a}_1^\dagger}e^{-\alpha_1^*\hat{a}_1}\left|n,0,0\right\rangle \\
=& e^{-\frac{1}{2}|\alpha_1|^2}e^{\alpha_1\hat{a}_1^\dagger}\sum_{k=0}^{+\infty}\frac{\left(-\alpha_1^*\hat{a}_1\right)^k}{k!}\left|n,0,0\right\rangle \\
=& e^{-\frac{1}{2}|\alpha_1|^2}e^{\alpha_1\hat{a}_1^\dagger}\sum_{k=0}^{n}\frac{\left(-\alpha_1^*\right)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k,0,0\rangle \\
=& e^{-\frac{1}{2}|\alpha_1|^2}\sum_{l=0}^{+\infty}\frac{\left(\alpha_1\hat{a}_1^\dagger\right)^l}{l!}\sum_{k=0}^{n}\frac{\left(-\alpha_1^*\right)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k,0,0\rangle \\
=& e^{-\frac{1}{2}|\alpha_1|^2}\sum_{k=0}^{n}\sum_{l=0}^{+\infty}\alpha_1^l\left(-\alpha_1^*\right)^k\frac{\sqrt{n!(n-k+l)!}}{k!\ l!\ (n-k)!}|n-k+l,0,0\rangle
\end{aligned} \tag{9}
$$

So, for the $|3,0,0\rangle$ we have:

$$
\hat{D}_1(\alpha_1)|3,0,0\rangle = e^{-\frac{1}{2}|\alpha_1|^2}\sum_{k=0}^{3}\sum_{l=0}^{+\infty}\alpha_1^l\left(-\alpha_1^*\right)^k\frac{\sqrt{3!(3-k+l)!}}{k!\ l!\ (3-k)!}|3-k+l,0,0\rangle. \tag{10}
$$

other circumstances are similar to this.

2. $|1,1,1\rangle$ state

We want to calculate the $\hat{D}_1(\alpha_1)\hat{D}_2(\alpha_2)\hat{D}_3(\alpha_3)|1,1,1\rangle$. Because $\hat{D}_1$, $\hat{D}_2$ and $\hat{D}_3$ is commute, so which of them are calculated first doesn't matter. Here we start from $\hat{D}_1(\alpha_1)$:

$$\begin{aligned}\hat{D}_1(\alpha_1)|1,1,1\rangle =& e^{-\frac{1}{2}|\alpha_1|^2}e^{\alpha_1\hat{a}_1^\dagger}e^{-\alpha_1^*\hat{a}_1}|1,1,1\rangle \\ =& e^{-\frac{1}{2}|\alpha_1|^2}e^{\alpha_1\hat{a}_1^\dagger}\sum_{k=0}^{+\infty}\frac{\left(-\alpha_1^*\hat{a}_1\right)^k}{k!}|1,1,1\rangle \\ =& e^{-\frac{1}{2}|\alpha_1|^2}e^{\alpha_1\hat{a}_1^\dagger}\left(|1,1,1\rangle-\alpha_1^*|0,1,1\rangle\right) \\ =& e^{-\frac{1}{2}|\alpha_1|^2}\sum_{k=0}^{+\infty}\frac{\left(\alpha_1\hat{a}_1^\dagger\right)^k}{k!}\left(|1,1,1\rangle-\alpha_1^*|0,1,1\rangle\right) \\ =& e^{-\frac{1}{2}|\alpha_1|^2}\left(\sum_{k=0}^{+\infty}\frac{{\alpha_1}^k}{\sqrt{k!}}|k+1,1,1\rangle-\alpha_1^*\sum_{k=0}^{+\infty}\frac{{\alpha_1}^k}{\sqrt{k!}}|k,1,1\rangle\right) \qquad (11)\\ =& \sum_{k=0}^{+\infty}C_1(k)\left(|k+1,1,1\rangle-\alpha_1^*|k,1,1\rangle\right) \qquad (12)\end{aligned}$$

where the coefficient $C_1(k)$:

$$C_1(k)=e^{-\frac{1}{2}|\alpha_1|^2}\frac{\alpha_1^k}{\sqrt{k!}} \tag{13}$$

So we get:

$$\hat{D}_1(\alpha_1)|1,1,1\rangle=\sum_{k=0}^{+\infty}C_1(k)\left(|k+1,1,1\rangle-\alpha_1^*|k,1,1\rangle\right) \tag{14}$$

Then, considering the effect from $\hat{D}_2$ and $\hat{D}_3$:

$$\begin{aligned}&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\hat{D}_1(\alpha_1)|1,1,1\rangle \\ =&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\left[\sum_{k=0}^{+\infty}C_1(k)\left(|k+1,1,1\rangle-\alpha_1^*|k,1,1\rangle\right)\right] \\ =&\hat{D}_3(\alpha_3)\left[\sum_{k=0}^{+\infty}C_1(k)\left(\hat{D}_2(\alpha_2)|k+1,1,1\rangle-\alpha_1^*\hat{D}_2(\alpha_2)|k,1,1\rangle\right)\right] \\ =&\hat{D}_3(\alpha_3)\sum_{k=0}^{+\infty}C_1(k)\left[\sum_{l=0}^{\infty}C_2(l)\left(|k+1,l+1,1\rangle-\alpha_2^*|k+1,l,1\rangle\right)-\alpha_1^*\sum_{l=0}^{\infty}C_2(l)\left(|k,l+1,1\rangle-\alpha_2^*|k,l,1\rangle\right)\right] \\ =&\hat{D}_3(\alpha_3)\sum_{k=0}^{+\infty}\sum_{l=0}^{\infty}C_1(k)C_2(l)\Big[|k+1,l+1,1\rangle-\alpha_2^*|k+1,l,1\rangle-\alpha_1^*|k,l+1,1\rangle+\alpha_1^*\alpha_2^*|k,l,1\rangle\Big] \\ =&\sum_{k,l,m=0}^{+\infty}C_1(k)C_2(l)C_3(m)\Big[|k+1,l+1,m+1\rangle-\alpha_1^*|k,l+1,m+1\rangle-\alpha_2^*|k+1,l,m+1\rangle-\alpha_3^*|k+1,l+1,m\rangle \\ &+\alpha_1^*\alpha_2^*|k,l,m+1\rangle+\alpha_1^*\alpha_3^*|k,l+1,m\rangle+\alpha_2^*\alpha_3^*|k+1,l,m\rangle-\alpha_1^*\alpha_2^*\alpha_3^*|k,l,m\rangle\Big] \qquad (15)\end{aligned}$$

the coefficient $C_2(l)$ and $C_3(m)$:

$$C_2(l)=e^{-\frac{1}{2}|\alpha_2|^2}\frac{\alpha_2^l}{\sqrt{l!}},\ C_3(l)=e^{-\frac{1}{2}|\alpha_3|^2}\frac{\alpha_3^m}{\sqrt{m!}} \tag{16}$$

If we change the way of calculation from Eq. (11), reform the coefficients, we will get:

$$
\begin{aligned}
&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\hat{D}_1(\alpha_1)|1,1,1\rangle \\
=&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)e^{-\frac{1}{2}|\alpha_1|^2}\left(\sum_{k=0}^{+\infty}\frac{{\alpha_1}^k}{\sqrt{k!}}|k+1,1,1\rangle-\alpha_1^*\sum_{k=0}^{+\infty}\frac{{\alpha_1}^k}{\sqrt{k!}}|k,1,1\rangle\right) \\
=&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)e^{-\frac{1}{2}|\alpha_1|^2}\left[\sum_{k=1}^{+\infty}\left(\frac{{\alpha_1}^{k-1}}{\sqrt{(k-1)!}}-\frac{{\alpha_1}^k}{\sqrt{k!}}\right)|k,1,1\rangle-\alpha_1^*|0,1,1\rangle\right] \\
=&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\left(\sum_{k=1}^{+\infty}G_1(k)|k,1,1\rangle+\mu_1|0,1,1\rangle\right) \\
=&\hat{D}_3(\alpha_3)\left[\sum_{k=1}^{+\infty}G_1(k)\left(\sum_{l=1}^{+\infty}G_2(l)|k,l,1\rangle+\mu_2|k,0,1\rangle\right)+\mu_1\left(\sum_{l=1}^{+\infty}G_2(l)|0,l,1\rangle+\mu_2|0,0,1\rangle\right)\right] \\
=&\sum_{k,l,m=1}^{+\infty}G_1(k)G_2(l)G_3(m)|k,l,m\rangle+\sum_{k,l=1}^{+\infty}G_1(k)G_2(l)\mu_3|k,l,0\rangle+\sum_{k,m=1}^{+\infty}G_1(k)\mu_2G_3(m)|k,0,m\rangle+ \\
&\sum_{l,m=1}^{+\infty}\mu_1G_2(l)G_3(m)|0,l,m\rangle+\sum_{k=1}^{+\infty}G_1(k)\mu_2\mu_3|k,0,0\rangle+\sum_{l=1}^{+\infty}\mu_1G_2(l)\mu_3|0,l,0\rangle+\sum_{m=1}^{+\infty}\mu_1\mu_2G_3(m)|0,0,m\rangle+ \\
&\mu_1\mu_2\mu_3|0,0,0\rangle
\end{aligned}
\tag{17}
$$

where:

$$
\begin{aligned}
G_1(k)&=e^{-\frac{1}{2}|\alpha_1|^2}\left(\frac{{\alpha_1}^{k-1}}{\sqrt{(k-1)!}}-\frac{{\alpha_1}^k}{\sqrt{k!}}\right),\ \mu_1=-\alpha_1^*e^{-\frac{1}{2}|\alpha_1|^2} \\
G_2(l)&=e^{-\frac{1}{2}|\alpha_2|^2}\left(\frac{{\alpha_1}^{l-1}}{\sqrt{(l-1)!}}-\frac{{\alpha_2}^l}{\sqrt{l!}}\right),\ \mu_2=-\alpha_2^*e^{-\frac{1}{2}|\alpha_2|^2} \\
G_3(m)&=e^{-\frac{1}{2}|\alpha_3|^2}\left(\frac{{\alpha_3}^{m-1}}{\sqrt{(m-1)!}}-\frac{{\alpha_3}^m}{\sqrt{m!}}\right),\ \mu_3=-\alpha_3^*e^{-\frac{1}{2}|\alpha_3|^2}
\end{aligned}
\tag{18}
$$

C. Further Analyzation of the displament transformation in a delicate way

After finishing the previous calculation, although it should be correct anyway, I feel the results is so sophiscated that it's worthless for the following stage to continue the calculation or perceive the essence of the problem. So I refer to the method from Caves, and rewrite the results in Section B. When I first read the method, I felt that its method had little to do with my problem. But in fact it is critical to this problem. From the Eq. (42) to Eq. (44) in Caves, passing the displacement transformation, the amplitude $\langle m|\hat{D}|n\rangle$ is:

$$
\begin{aligned}
\langle m|\hat{D}|n\rangle=&\sqrt{\frac{n!}{m!}}e^{-\frac{1}{2}|\alpha|^2}\alpha^{m-n}\sum_{k=0}^{n}\frac{(n+m-n)!}{k!(n-k)!(m-n+k)!}(-|\alpha|^2)^k \\
=&\sqrt{\frac{n!}{m!}}e^{-\frac{1}{2}|\alpha|^2}\alpha^{m-n}L_n^{(m-n)}(|\alpha|^2)\quad(m\geqslant n)
\end{aligned}
\tag{19}
$$

The Associated Laguerre polynomials:

$$
L_n^k(x)=\frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{i!}\binom{k+n}{n-i}(-x)^i \tag{20}
$$

The complete formula is:

$$
\Lambda(m,n)=\langle m|\hat{D}|n\rangle=\begin{cases}\sqrt{\frac{n!}{m!}}e^{-\frac{1}{2}|\alpha|^2}\alpha^{m-n}L_n^{(m-n)}(|\alpha|^2), & m\geqslant n\\ \sqrt{\frac{m!}{n!}}e^{-\frac{1}{2}|\alpha|^2}(-\alpha^*)^{n-m}L_m^{(n-m)}(|\alpha|^2), & m<n\end{cases} \tag{21}
$$

Apply this results to our calculation:

$$\begin{aligned}\hat{D}_1|n,0,0\rangle =&\sum_{m=0}^{+\infty}|m,0,0\rangle\langle m,0,0|\hat{D}_1|n,0,0\rangle\\ =&\sum_{m=0}^{+\infty}\Lambda(m,n)|m,0,0\rangle\end{aligned} \tag{22}$$

Rewriting the Eq.(9), we can see that the two results are same:

$$\begin{aligned}\hat{D}_1(\alpha_1)\,|n,0,0\rangle =&e^{-\frac{1}{2}|\alpha_1|^2}\sum_{k=0}^{n}\sum_{l=0}^{+\infty}\alpha_1^l(-\alpha_1^*)^k\frac{\sqrt{n!(n-k+l)!}}{k!\;l!\;(n-k)!}|n-k+l,0,0\rangle\\ &(\text{ Set } m=n-k+l. \text{ Here we assume } m\geqslant n,\ m<n \text{ is same })\\ =&\sum_{m=0}^{+\infty}\sum_{k=0}^{n}e^{-\frac{1}{2}|\alpha_1|^2}\alpha_1^{k+m-n}(-\alpha_1^*)^k\frac{\sqrt{n!m!}}{k!\;(n-k)!\;(k+m-n)!}|m,0,0\rangle\\ =&\sum_{m=0}^{+\infty}\sqrt{\frac{n!}{m!}}e^{-\frac{1}{2}|\alpha_1|^2}\alpha_1^{m-n}\sum_{k=0}^{n}\frac{(n+m-n)!}{k!\;(n-k)!\;(k+m-n)!}\left(-|\alpha_1|^2\right)^k|m,0,0\rangle\\ =&\sum_{m=0}^{+\infty}\sqrt{\frac{n!}{m!}}e^{-\frac{1}{2}|\alpha_1|^2}\alpha_1^{m-n}L_n^{(m-n)}(|\alpha_1|^2)|m,0,0\rangle\\ =&\sum_{m=0}^{+\infty}|m,0,0\rangle\langle m,0,0|\hat{D}_1|n,0,0\rangle\\ =&\sum_{m=0}^{+\infty}\Lambda(m,n)|m,0,0\rangle\end{aligned} \tag{23}$$

And:

$$\hat{D}_1(\alpha_1)\,|3,0,0\rangle=\sum_{m=0}^{+\infty}\Lambda(m,3)|m,0,0\rangle \tag{24}$$

So the result is same as the Eq.(22). Now this preblem has a more satisfactory explanation. Similar to this, for the $|1,1,1\rangle$ state:

$$\begin{aligned}&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\hat{D}_1(\alpha_1)|1,1,1\rangle\\ =&\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\sum_{k=0}^{+\infty}\Lambda(k,1)|k,1,1\rangle\\ =&\hat{D}_3(\alpha_3)\sum_{k=0}^{+\infty}\Lambda(k,1)\sum_{l=0}^{+\infty}\Lambda(l,1)|k,l,1\rangle\\ =&\sum_{k,l,m=0}^{+\infty}\Lambda(k,1)\Lambda(l,1)\Lambda(m,1)|k,l,m\rangle\end{aligned} \tag{25}$$

This result can easily be generalized. For the general state $|K,L,M\rangle$:

$$\hat{D}_3(\alpha_3)\hat{D}_2(\alpha_2)\hat{D}_1(\alpha_1)|K,L,M\rangle=\sum_{k,l,m=0}^{+\infty}\Lambda(k,K)\Lambda(l,L)\Lambda(m,M)|k,l,m\rangle \tag{26}$$