T2

数学建模三次研究作业

T1

(1) 共 1:0　2:0　2:1 胜

设 A　B　C 分别表 先罚胜，后罚胜，持平（两局判）

π为现规　π'为新规，

$P_\pi(A) = 2p(1-p)(1-q)^2 + p^2(1-q)^2 + 2p^2q(1-q)$

$P_\pi(B) = 2q(1-q)(1-p)^2 + q^2(1-p)^2 + 2q^2p(1-p)$

$P_\pi(C) = (1-p)^2(1-q)^2 + 4pq(1-p)(1-q) + (pq)^2$

代入 $p=\frac{3}{4}, q=\frac{2}{3}$

$P_\pi(A) = \frac{17}{48} \longrightarrow \frac{51}{144}$

$P_\pi(B) = \frac{2}{9} \longrightarrow \frac{32}{144}$

$P_\pi(C) = \frac{61}{144}$

新规中：

~~1:0　2:1~~

$P_{\pi'}(A) = -3p^2q^2 + 3p^2q + 3pq^2 - p^2 - q^2 - 3pq + p + q$

$P_{\pi'}(B) = -3p^2q^2 + 3q^2p + 3qp^2 - p^2 - q^2 - 3pq + p + q$

$P_{\pi'}(C) = 6p^2q^2 - 6pq^2 - 6p^2q + 2p^2 + 2q^2 + 6pq - 2p - 2q + 1$

代入 $p=\frac{3}{4}, q=\frac{2}{3}$

$P_{\pi'}(A) = \frac{41}{144}$

$P_{\pi'}(B) = \frac{41}{144}$

$P_{\pi'}(C) = \frac{31}{72}$

(2) 设先罚赢凡 A'，后罚赢凡 B'（不限两局）

设 $\mu = (1-p)(1-q) + pq = 1 - p - q + 2pq$ $\longrightarrow \mu^2 = 1 + (p+q)^2 + 4p^2q^2 - 4pq(p+q) + 4pq - 2(p+q)$

旧规： $P_\pi(A'|C) = [1 + \mu + \mu^2 + \cdots]\, p(1-q)$

$= \frac{p(1-q)}{1-\mu} = \boxed{\frac{p - pq}{p + q - 2pq}}$

$P_\pi(B'|C) = \frac{q(1-p)}{1-\mu} = \boxed{\frac{q - pq}{p + q - 2pq}}$

~~$((1-p-q+2pq)(q-pq)$~~

~~$q - pq - pq + p^2q - q^2 + pq^2 + pq^2 - 2p^2q^2$~~

~~$= q - pq$~~

新规：

$P_{\pi'}(A'|C)$

$= p(1-q) + q(1-p)\mu + \cdots$

$= p(1-q)[1 + \mu^2 + \mu^4 + \cdots] + q(1-p)[\mu + \mu^3 + \cdots]$

$= \frac{1}{1-\mu^2}[p(1-q) + \mu q(1-p)]$

~~$P_{\pi'}(A'|C) = p - pq + \ldots$~~

~~$= -4p^2q^2 + 4p^2q + 4q^2p - p^2 - q^2 - 6pq + 2p + 2q$~~

$= \frac{-2p^2q^2 + 3pq^2 + p^2q - 3pq - q^2 + p + q}{-4p^2q^2 + 4p^2q + 4q^2p - p^2 - q^2 - 6pq + 2p + 2q}$

$P_{\pi'}(B'|C)$

$= \frac{1}{1-\mu^2}[q(1-p) + \mu p(1-q)]$

$= \frac{-2p^2q^2 + 3qp^2 + pq^2 - 3pq - p^2 + p + q}{-4p^2q^2 + 4p^2q + 4q^2p - p^2 - q^2 - 6pq + 2p + 2q}$

本题这几乎同 T_1 T_2

(1) $P_{\pi}(A\ win) = \dfrac{\alpha+\beta}{1-\gamma^2}$

(2) ① $a = \gamma + \beta\dfrac{\alpha+\beta}{1-\gamma^2} = \boxed{\gamma + \dfrac{\alpha\beta+\beta^2}{1-\gamma^2}}$

$b = \boxed{\dfrac{\gamma(\alpha+\beta)}{1-\gamma^2}}$

③ $b = P(A\text{后胜} \mid \text{一回无分}) = \dfrac{P(A\text{后胜} \cap \text{一回无分})}{P(\text{一回无分})} = \dfrac{\frac{\gamma^2(\alpha+\beta)}{1-\gamma^2}}{\gamma}$

(3) A胜利概率：（π为旧策，π'为新策）

$P_{\pi'} = \alpha + \alpha\beta$ $\quad P_{\pi'}(A\ win) = \alpha + a\beta + b\gamma$

$$= \frac{1-\beta+\beta\gamma^2+\beta^2}{1+\gamma}$$

$$P_{\pi}(A\ win) = \frac{\alpha+\beta}{1-\gamma^2} = \frac{1}{1+\gamma} > \frac{1}{2}$$

$$P_{\pi}(A\ win) - P_{\pi'}(A\ win) = \frac{\alpha\beta}{1+\gamma} > 0.$$

解 $P_{\pi'}(A\ win) = \frac{1}{2}$

$\Longleftrightarrow 2\beta(\beta+\gamma^2-1) = \gamma-1$ 无解 $(\beta, \gamma \in (0,1))$

$\Delta = \cancel{1 - 8\beta(2\beta^2-2\beta+1) > 0}$

故 $P_{\pi'}(A\ win) > \frac{1}{2}$

有 $|P' - \frac{1}{2}| = P' - \frac{1}{2} = (P - \frac{1}{2}) - (P - P') < |P - \frac{1}{2}|$

故新赛制更公平