

PROBABILISTIC ROBOTICS: ROBOT MOTION

Pierre-Paul TACHER

1

Evolution of the state will be approximated to the second and first order, for x and \dot{x} respectively:

$$\begin{aligned} x_{t+1} &= x_t + \Delta t \times \dot{x}_t + \frac{1}{2}(\Delta t)^2 \times \hat{\ddot{x}}_t + o(\Delta t^2) \\ \Rightarrow x_{t+1} &\approx x_t + \Delta t \times \dot{x}_t + \frac{1}{2}(\Delta t)^2 \times \hat{\ddot{x}}_t \end{aligned}$$

$$\begin{aligned} \dot{x}_{t+1} &= \dot{x}_t + \Delta t \times \hat{\ddot{x}}_t + o(\Delta t) \\ \Rightarrow \dot{x}_{t+1} &\approx \dot{x}_t + \Delta t \times \hat{\ddot{x}}_t \end{aligned}$$

where $\hat{\ddot{x}}_t$ is the actual acceleration. If we note \ddot{x}_t the commanded acceleration,

$$\hat{\ddot{x}}_t = \ddot{x}_t + \epsilon_t$$

where ϵ_t is gaussian $\epsilon_t \hookrightarrow \mathcal{N}(0, \sigma^2)$ We can modelize evolution of the state with matrix equality:

$$\begin{aligned} \begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix}}_G \hat{\ddot{x}}_t \\ \begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \ddot{x}_t + \underbrace{\begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix}}_{\epsilon'_t} \epsilon_t \end{aligned}$$

We have

$$G \times G^T = \begin{bmatrix} \frac{1}{4}(\Delta t)^4 & \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 & (\Delta t)^2 \end{bmatrix}$$

As a linear function of gaussian ϵ_t , the random variable ϵ'_t is known to be multivariate Gaussian $\mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\sigma^2 G G^T}_R\right)$. The conditional law of random variable $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ given $x_t, \dot{x}_t, \ddot{x}_t$ is then known to

be also Gaussian $\mathcal{N}\left(\underbrace{A \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \ddot{x}_t}_\mu, R\right)$. The corresponding density is

$$p(x_{t+1}, \dot{x}_{t+1} \mid x_t, \dot{x}_t, \ddot{x}_t) = \frac{1}{2\pi\sqrt{|R|}} \times e^{-\frac{1}{2} \left(\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} - \mu \right)^T R^{-1} \left(\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} - \mu \right)}$$

1

The fact that the covariance matrix R is not diagonal shows that x_{t+1} and \dot{x}_{t+1} are correlated. The coefficient of correlation is

$$\rho = \frac{\frac{1}{2}(\Delta t)^3}{\sqrt{\frac{1}{4}(\Delta t)^4 \times \Delta t^2}}$$

$$\rho = 1$$

showing position and speed are fully correlated.

2

Conditioned on $x_t, \dot{x}_t, \ddot{x}_t$, the r.v. $\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix}$ is gaussian; we can then show (cf. A.1) that the r.v. \dot{x}_{t+1} conditioned on $x_{t+1}, x_t, \dot{x}_t, \ddot{x}_t$ is also gaussian $\dot{x}_{t+1} \hookrightarrow \mathcal{N}(\nu, \Lambda)$, where

$$\begin{aligned} \nu &= E(\dot{x}_{t+1}) + \frac{\frac{1}{2}(\Delta t)^3}{\frac{1}{4}(\Delta t)^4} (x_{t+1} - E(x_{t+1})) \\ &= E(\dot{x}_{t+1}) + \frac{2}{\Delta t} (x_{t+1} - E(x_{t+1})) \\ \Lambda &= (\Delta t)^2 - \frac{(\frac{1}{2}(\Delta t)^3)^2}{\frac{1}{4}(\Delta t)^4} \\ \Lambda &= 0 \end{aligned}$$

showing again the speed is deterministic function of the position.

3

We have the model

$$\begin{bmatrix} x_{n+1} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}(\Delta t)^2 \\ \Delta t \end{bmatrix} \epsilon_t}_{\epsilon'_t}$$

with $\epsilon'_t \hookrightarrow \mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\sigma^2 G G^T}_R)$. If we assume the random variables $\begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix}$ and ϵ'_n to be independant and

$\begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} \hookrightarrow \mathcal{N}(\mu_n, \Sigma_n)$, then $\begin{bmatrix} x_{n+1} \\ \dot{x}_{n+1} \end{bmatrix}$ is Gaussian $\mathcal{N}(\underbrace{A\mu_n}_{\mu_{n+1}}, \underbrace{A\Sigma_n A^T + R}_{\Sigma_{n+1}})$. We will study the sequence of covariance matrices defined by the recurrence relation

$$\forall n \in \mathbb{N}, \quad \Sigma_{n+1} = A\Sigma_n A^T + R$$

It is convenient to write it in vectorized form

$$\begin{aligned} \begin{bmatrix} \sigma_{X,n+1}^2 \\ \sigma_{XY,n+1} \\ \sigma_{XY,n+1} \\ \sigma_{Y,n+1}^2 \end{bmatrix} &= \begin{bmatrix} 1 & \Delta t & \Delta t & (\Delta t)^2 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{X,n}^2 \\ \sigma_{XY,n} \\ \sigma_{XY,n} \\ \sigma_{Y,n}^2 \end{bmatrix} + \sigma^2 \begin{bmatrix} \frac{1}{4}(\Delta t)^4 \\ \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 \\ (\Delta t)^2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}A & a_{12}A \\ a_{21}A & a_{22}A \end{bmatrix} \begin{bmatrix} \sigma_{X,n}^2 \\ \sigma_{XY,n} \\ \sigma_{XY,n} \\ \sigma_{Y,n}^2 \end{bmatrix} + \sigma^2 \begin{bmatrix} \frac{1}{4}(\Delta t)^4 \\ \frac{1}{2}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^3 \\ (\Delta t)^2 \end{bmatrix} \\ \Leftrightarrow \text{vec}(\Sigma_{n+1}) &= A \otimes A \times \text{vec}(\Sigma_n) + \text{vec}(R) \end{aligned}$$

where \otimes is the Kronecker product. A little observation shows that

$$\forall n \in \mathbb{N}, \quad (A \otimes A)^n = \begin{bmatrix} 1 & n\Delta t & n\Delta t & n^2(\Delta t)^2 \\ 0 & 1 & 0 & n\Delta t \\ 0 & 0 & 1 & n\Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\forall n \in \mathbb{N}, \quad \text{vec}(\Sigma_n) = (A \otimes A)^n \text{vec}(\Sigma_0) + \left(\sum_{i=0}^{n-1} (A \otimes A)^i \right) \text{vec}(R)$$

Then, using the formulas of a sum of integers and sum of squared integers,

$$\sum_{i=0}^{n-1} (A \otimes A)^i = \begin{bmatrix} n & \frac{n(n-1)}{2} \Delta t & \frac{n(n-1)}{2} \Delta t & \frac{n(n-1)(2n-1)}{6} (\Delta t)^2 \\ 0 & n & 0 & \frac{n(n-1)}{2} \Delta t \\ 0 & 0 & n & \frac{n(n-1)}{2} \Delta t \\ 0 & 0 & 0 & n \end{bmatrix}$$

from which we can build a closed formula for Σ_n . What is of interest for us here is the asymptotic behavior,

$$\begin{aligned} \sigma_{X,n}^2 &\underset{n \rightarrow \infty}{\sim} \frac{n(n-1)(2n-1)}{6} (\Delta t)^2 \times (\Delta t)^2 \sigma^2 \\ &\underset{n \rightarrow \infty}{\sim} \frac{n^3}{3} (\Delta t)^4 \sigma^2 \\ \sigma_{XY,n} &\underset{n \rightarrow \infty}{\sim} \frac{n(n-1)}{2} \Delta t \times (\Delta t)^2 \sigma^2 \\ &\underset{n \rightarrow \infty}{\sim} \frac{n^2}{2} (\Delta t)^3 \sigma^2 \\ \sigma_{Y,n}^2 &\underset{n \rightarrow \infty}{\sim} n (\Delta t)^2 \sigma^2 \end{aligned}$$

leading to

$$\begin{aligned} \rho_{XY,n} &\underset{n \rightarrow \infty}{\sim} \frac{\frac{n^2}{2} (\Delta t)^3 \sigma^2}{\sqrt{\frac{n^3}{3} (\Delta t)^4 \sigma^2 \times n (\Delta t)^2 \sigma^2}} \\ &= \frac{\sqrt{3}}{2} < 1 \end{aligned}$$

Asymptotically position and speed do not get fully correlated.

4

The conventions used for this exercise and the following 2 are represented in figure 1. Let's study the movement of the center of front wheel B in frame $\mathcal{R} = (O, \vec{i}, \vec{j})$. We let $\mathcal{R}' = (A, \vec{i}', \vec{j}')$ the frame with origin the center of back wheel, translating with regards to \mathcal{R} . We can write

$$\begin{aligned} \vec{v}_{B/\mathcal{R}} &= \vec{v}_{B/\mathcal{R}'} + \vec{v}_{\mathcal{R}'/\mathcal{R}} \\ &= \vec{v}_{B/\mathcal{R}'} + \vec{v}_{A/\mathcal{R}} \end{aligned} \quad \textcircled{1}$$

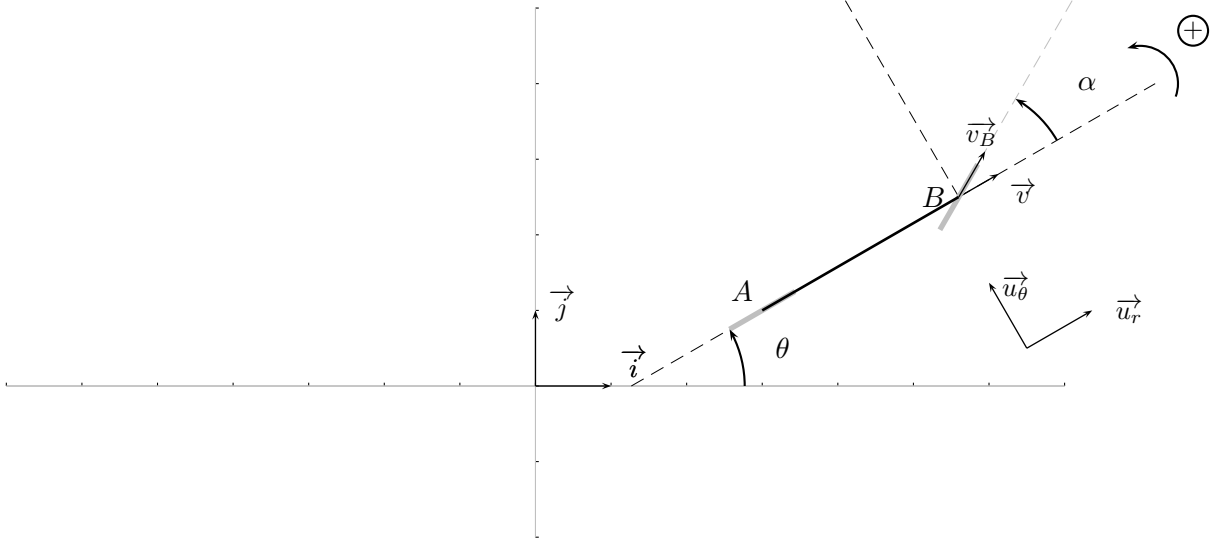


FIGURE 1. Ideal bicycle model

In frame \mathcal{R}' , the length AB being constant, B is in rotation

$$\begin{aligned} \forall t \geq 0, \quad & \left\| \overrightarrow{AB}(t) \right\|^2 = l \\ \Rightarrow \forall t \geq 0, \quad & \frac{d\overrightarrow{AB}}{dt}(t) \cdot \overrightarrow{AB}(t) = 0 \\ \Leftrightarrow \forall t \geq 0, \quad & \overrightarrow{v_{B/\mathcal{R}'}}(t) \cdot \overrightarrow{AB}(t) = 0 \end{aligned}$$

so

$$\overrightarrow{v_{B/\mathcal{R}'}} = l\dot{\theta}\vec{u}_{\theta}$$

A is restrained to move in the direction of the bicycle axis:

$$\overrightarrow{v_{A/\mathcal{R}}} = vu_r$$

$$\textcircled{1} \Rightarrow \overrightarrow{v_{B/\mathcal{R}}} = l\dot{\theta}\vec{u}_{\theta} + vu_r$$

B is restrained to move in the direction of front wheel, so if we project the previous relation we get

$$\begin{cases} v_B \sin \alpha = l\dot{\theta} \\ v_B \cos \alpha = v \end{cases} \Rightarrow v \tan \alpha = l\dot{\theta} \quad \textcircled{2}$$

Projecting now \vec{v}_B on (\vec{i}, \vec{j}) , we obtain

$$\begin{aligned} & \begin{cases} v_B \cos(\alpha + \theta) = \dot{x} \\ v_B \sin(\alpha + \theta) = \dot{y} \end{cases} \\ & \Leftrightarrow \begin{cases} \frac{v}{\cos \alpha} \cos(\alpha + \theta) = \dot{x} \\ \frac{v}{\cos \alpha} \sin(\alpha + \theta) = \dot{y} \end{cases} \\ & \Leftrightarrow \begin{cases} v(\cos \theta - \tan \alpha \sin \theta) = \dot{x} & \textcircled{3} \\ v(\tan \alpha \cos \theta + \sin \theta) = \dot{y} & \textcircled{4} \end{cases} \end{aligned}$$

We discretize time in ②,③ and ④ to obtain approximate motion equations:

$$\begin{cases} x_{t+1} = x_t + \hat{v}(\cos \theta_t - \tan \hat{\alpha} \sin \theta_t) \Delta t \\ y_{t+1} = y_t + \hat{v}(\tan \hat{\alpha} \cos \theta_t + \sin \theta_t) \Delta t \\ \theta_{t+1} = \theta_t + \frac{\hat{v}}{l} \tan \hat{\alpha} \times \Delta t \end{cases}$$

with

$$\begin{aligned} \hat{\alpha} &= \alpha + \epsilon_{\sigma_\alpha^2} \\ \hat{v} &= v + \epsilon_{\sigma_v^2} \end{aligned}$$

where $\epsilon_{\sigma^2} \hookrightarrow \mathcal{N}(0, \sigma^2)$.

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cf. file `bicyclespl.m` for the matlab code. In figure 2, I plotted one step of a simulation starting from state $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \pi/4 \end{bmatrix}$.

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This exercise consists in inverting formulas ②,③ and ④ for α , v and θ_{t+1} .

$$\begin{aligned} & \Leftrightarrow \begin{cases} v(\cos \theta - \tan \alpha \sin \theta) = \dot{x} \\ v(\tan \alpha \cos \theta + \sin \theta) = \dot{y} \end{cases} \\ & \Leftrightarrow \begin{cases} \cos \theta - \tan \alpha \sin \theta = \frac{\dot{x}}{v} \\ \tan \alpha \cos \theta + \sin \theta = \frac{\dot{y}}{v} \end{cases} \\ & \Leftrightarrow \begin{cases} \dot{y}(\cos \theta - \tan \alpha \sin \theta) = \dot{y} \frac{\dot{x}}{v} \\ \dot{x}(\tan \alpha \cos \theta + \sin \theta) = \dot{x} \frac{\dot{y}}{v} \end{cases} \\ & \Rightarrow \tan \alpha (\dot{x} \cos \theta + \dot{y} \sin \theta) = \dot{y} \cos \theta - \dot{x} \sin \theta \\ & \Leftrightarrow \tan \alpha = \frac{\dot{y} \cos \theta - \dot{x} \sin \theta}{\dot{x} \cos \theta + \dot{y} \sin \theta} \end{aligned}$$

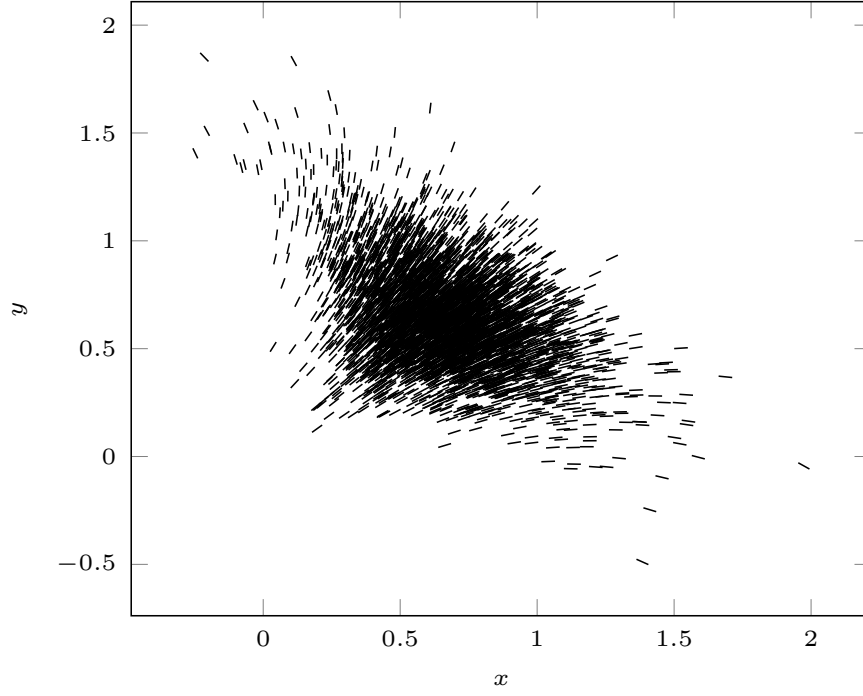


FIGURE 2. 5000 samples from motion model, $\alpha = 0$, $v = 90 \times 10^{-2}$, $\sigma_\alpha^2 = 15^\circ^2$, $\sigma_v^2 = 0.04v^2$.

In order to compute the most likely value of steering angle α , it suffices to insert in previous relation $\dot{x} = \frac{x_{t+1}-x_t}{\Delta t}$, $\dot{y} = \frac{y_{t+1}-y_t}{\Delta t}$ and $\theta = \theta_t$. From there we can compute the speed

$$\begin{aligned} \frac{v^2}{\cos^2 \alpha} &= \dot{x}^2 + \dot{y}^2 \\ \Leftrightarrow v^2 &= \cos^2 \alpha (\dot{x}^2 + \dot{y}^2) \end{aligned}$$

and finally the most likely final orientation will be given by

$$\theta_{t+1} = \theta_t + \frac{v}{l} \tan \alpha \times \Delta t$$

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The model is represented in figure 3. The forward and sideward speeds are designed by v_1 and v_2 respectively.

$$\begin{aligned} \dot{x} &= v_1 \cos \theta - v_2 \sin \theta \\ \dot{y} &= v_1 \sin \theta + v_2 \cos \theta \\ \dot{\theta} &= \omega \end{aligned}$$

We discretize to get

$$\begin{aligned} x_{t+1} &= x_t + (v_1 \cos \theta_t - v_2 \sin \theta_t) \Delta t \\ y_{t+1} &= y_t + (v_1 \sin \theta_t + v_2 \cos \theta_t) \Delta t \\ \theta_{t+1} &= \theta_t + \omega \Delta t \end{aligned}$$

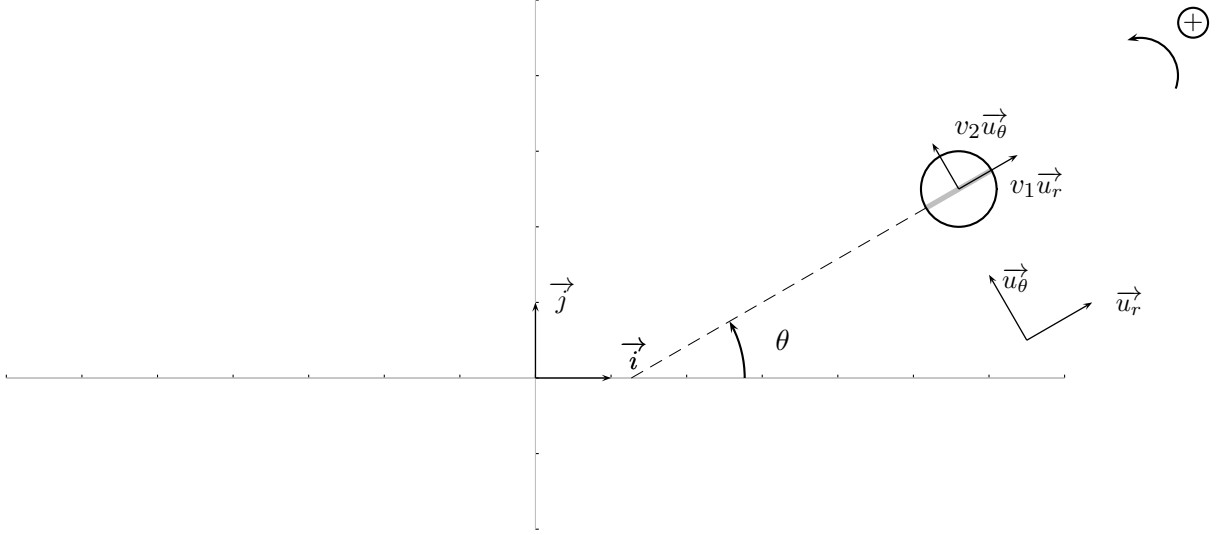


FIGURE 3. holonomic robot model

To compute the transition probability density, one needs to invert the previous system for the controls v_1 , v_2 and ω ; for that it suffices to notice that:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

so that

$$\begin{aligned} v_1 &= ((x_{t+1} - x_t) \cos \theta_t + (y_{t+1} - y_t) \sin \theta_t) \frac{1}{\Delta t} \\ v_2 &= (-(x_{t+1} - x_t) \sin \theta_t + (y_{t+1} - y_t) \cos \theta_t) \frac{1}{\Delta t} \\ \omega &= \frac{\theta_{t+1} - \theta_t}{\Delta t} \end{aligned}$$

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$$\text{Var}(\epsilon_{b^2}) = E[X^2] - (E[X])^2$$

$$\begin{aligned} \int_{\Omega} X^2 dP &= \int_{-\sqrt{6b}}^{\sqrt{6b}} x^2 \left(\frac{1}{\sqrt{6b}} - \frac{|x|}{6b^2} \right) dx \\ &= 2 \int_0^{\sqrt{6b}} x^2 \left(\frac{1}{\sqrt{6b}} - \frac{|x|}{6b^2} \right) dx \\ &= 2 \left(\frac{1}{\sqrt{6b}} \int_0^{\sqrt{6b}} x^2 dx - \frac{1}{6b^2} \int_0^{\sqrt{6b}} x^3 dx \right) \\ &= 2 \left(\frac{1}{\sqrt{6b}} \left[\frac{x^3}{3} \right]_0^{\sqrt{6b}} - \frac{1}{6b^2} \left[\frac{x^4}{4} \right]_0^{\sqrt{6b}} \right) \\ &= 2 \left(\frac{1}{\sqrt{6b}} \frac{6\sqrt{6b}^3}{3} - \frac{1}{6b^2} \frac{36b^4}{4} \right) \\ &= b^2 \end{aligned}$$

so,

$$\text{Var}(\epsilon_{b^2}) = b^2$$

Let now X and Y 2 independant uniform r.v. $\hookrightarrow \mathcal{U}([-b, b])$, and let $Z = \frac{\sqrt{6}}{2}[X + Y]$. We have

$$\text{Var}(Z) = \frac{3}{2}[\text{Var}(X) + \text{Var}(Y)]$$

and

$$\begin{aligned} \text{Var}(X) = \text{Var}(Y) &= E[X^2] - (E[X])^2 \\ &= E[X^2] \\ &= \int_{-b}^b x^2 \frac{1}{2b} dx \\ &= 2 \int_0^b x^2 \frac{1}{2b} dx \\ &= 2 \frac{1}{2b} \left[\frac{x^3}{3} \right]_0^b \\ &= \frac{1}{b} \frac{b^3}{3} \\ &= \frac{b^2}{3} \end{aligned}$$

so that

$$\text{Var}(Z) = b^2$$

Appendix A

A.1.

lemma. Let $Z = (X, Y)$ be a gaussian r.v. which takes its values in $\mathbb{R}^n \times \mathbb{R}^p$, $Z \hookrightarrow \mathcal{N}(\mu, \Lambda)$. We note

$$\Lambda = \begin{bmatrix} \Lambda_X & \Lambda_{XY}^T \\ \Lambda_{XY} & \Lambda_Y \end{bmatrix}$$

Then, the law of r.v. Y conditionned on $(X = x)$ is gaussian $\hookrightarrow \mathcal{N}(m, \Sigma)$, where

$$\begin{aligned} m &= EY + \Lambda_{XY} \Lambda_X^{-1} (x - EX) \\ \Sigma &= \Lambda_Y - \Lambda_{XY} \Lambda_X^{-1} \Lambda_{XY}^T \end{aligned}$$

Proof. We already know that X is gaussian, as linear transformation (namely projection) of gaussian Z . Let $Y' = Y - CX$. The r.v. (X, Y') is also gaussian and we can verify that

$$\Lambda_{XY'} = \Lambda_{XY} - CX$$

We can choose $C = \Lambda_{XY} \Lambda_X^{-1}$ so that X and Y' are decorrelated, but this is the same as independance since (X, Y') is gaussian. The law of $Y = Y' + CX$ conditionned on $(X = x)$ is then simply the law of $Y' + Cx$. This shows it is gaussian and

$$\begin{aligned} m &= EY' + Cx \\ &= (EY - \Lambda_{XY} \Lambda_X^{-1} EX) + \Lambda_{XY} \Lambda_X^{-1} x \\ &= EY + \Lambda_{XY} \Lambda_X^{-1} (x - EX) \\ \Sigma &= \Lambda_Y - C \Lambda_X C^T \\ &= \Lambda_Y - \Lambda_{XY} \Lambda_X^{-1} \Lambda_X \Lambda_X^{-1} \Lambda_{XY}^T \\ &= \Lambda_Y - \Lambda_{XY} \Lambda_X^{-1} \Lambda_{XY}^T \end{aligned}$$

where we have used the fact that Λ_X is symetric and Y' and CX are independant. □