A PROOFS

Proposition A.1 (Proposition 3.3 restated). *Let $X_1, \ldots, X_n$ denote some keys over (R, Σ). Then $\{X_1, \ldots, X_n\}$ satisfies the composite independence property if and only if all of the following hold:*

(1) For all $i = 1, \ldots, n$, X_i is a minimal key for (R, Σ).
(2) If X denotes some minimal key for (R, Σ), then there is some $j \in \{1, \ldots, n\}$ such that $X = X_j$.
(3) (R, Σ) is in Boyce-Codd Normal Form. □

Proof. We show first that the three conditions are sufficient for the composite independence property for $\{X_1, \ldots, X_n\}$ to hold.

Let r denote a relation over R that satisfies Σ, and for all $i = 1, \ldots, n$, let $\mu_i \in dom(X_i)$ such that $\mu_i \notin r(X_i)$. Let $\nu \in dom(R - X_1 \cdots X_n)$. We claim that $r' = r \cup \{t'\}$ satisfies Σ for $t' = \mu_1 \cdots \mu_n \nu$. For consider $X \to A \in \Sigma^+$ such that $A \notin X$ and assume that r' violates $X \to A$. In particular, $t(X) = t'(X)$ for some $t \in r$. The BCNF condition (3) implies that $X \to R \in \Sigma^+$. Hence, X is a key for (R, Σ). If X is even a minimal key, then by (1) and (2) there is some $j \in \{1, \ldots, n\}$ such that $X = X_j$. If X is not a minimal key, then X is a superset of some minimal key, and by (1) and (2), there must be some $j \in \{1, \ldots, n\}$ such that $X_j \subseteq X$. Consequently, $\mu_j = t'(X_j) = t(X_j) \in r(X_j)$, which is a contradiction. That means our assumption that r' violates $X \to A$ must have been wrong, which means that r' satisfies Σ. This proves the composite independence property for $\{X_1, \ldots, X_n\}$.

We show now that the composite independence property for $\{X_1, \ldots, X_n\}$ is sufficient for (1), (2), and (3) to hold.

Firstly, since the composite independence property for $\{X_1, \ldots, X_n\}$ implies the weak independence property for $\{X_1, \ldots, X_n\}$, Proposition 2.1 implies (1).

We are now going to show that (2) holds as well. Assume there is some other minimal key X, different from every $X_1, \ldots, X_n$. For $i = 1, \ldots, n$ we then know that $X - X_i \neq \emptyset$ and $X_i - X \neq \emptyset$ holds. Let $r := \{t\}$ with any tuple t over R. It follows that r satisfies Σ. We now define a tuple t' over R such that for all $i = 1, \ldots, n$, $t'(X_i - X) \neq t(X_i - X)$ and $t'(X - X_i) = t(X - X_i)$, and $t'(R - X_1 \cdots X_n) = t(R - X_1 \cdots X_n)$. In particular, for all $i = 1, \ldots, n$ we have $t'(X_i) \notin r(X_i)$. Due to the composite independence property for $\{X_1, \ldots, X_n\}$ it follows that $r' := r \cup \{t'\}$ satisfies Σ. However, it follows that $t'(X) = t(X)$, which means that r does not satisfy $X \to R \in \Sigma^+$, a contradiction to the assumption that X is another minimal key. Consequently, (2) must hold.

It remains to show (3). For consider $X \to A \in \Sigma^+$ such that $A \notin X$. We distinguish between two cases.

Case 1: There is some $j \in \{1, \ldots, n\}$ such that $X_j \subseteq X$. Since X_j is a key for (R, Σ) we have $X \to R \in \Sigma^+$.

Case 2: For all $i = 1, \ldots, n$ there is some $A_i \in X_i - X$. We will show that this case cannot occur. Indeed, let $r := \{t\}$ be some relation over R. Then r satisfies Σ. Define a tuple t' over R such that, for all $i = 1, \ldots, n$, $t'(X \cap X_i) := t(X \cap X_i)$, $t'(X - X_i) := t(X - X_i)$, for all $B \in X_i - X$, $t'(B) \neq t(B)$, and for all $B \in R - X_1 \cdots X_n$, $t'(B) \neq t(B)$. It follows for all $i = 1, \ldots, n$, $t'(X_i) \notin r(X_i)$. By the independence property for $(X_1, \ldots, X_n)$ we conclude that $r' := r \cup \{t'\}$ satisfies Σ. However, by construction of t', we have $t'(X) = t(X)$ and, since $A \notin X$, $t'(A) \neq t(A)$, meaning that r' does not satisfy $X \to A$. This is a contradiction, and Case 2 cannot occur. □

Theorem A.2 (Theorem 4.3 restated). *Let Σ denote a set of FDs over relation schema R. For every positive integer n the following holds: (R, Σ) is in Composite Object Normal Form of order n if and only if (R, Σ) is in Boyce-Codd Normal Form and there are exactly n minimal keys.*

Proof. *(If).* Let $X \in LHS$. Since (R, Σ) is in BCNF, X must satisfy the uniqueness property. Since there are exactly n minimal keys, and X is a minimal key, X is part of the n minimal keys $X_1, \ldots, X_n$. Due to Proposition 3.3, $\{X_1, \ldots, X_n\}$ satisfies the independence property and is, therefore, a composite object of order n. Consequently, (R, Σ) is in Composite Object Normal Form of order n.

(Only if). Let $X \to A \in \Sigma^+_{\mathfrak{A}}$ such that $A \notin X$. Since (R, Σ) is in composite object normal form of order n, it follows that there is some minimal key X_i for (R, Σ) such that $X_i \subseteq X$ such that X_i participates in a composite object of order n, say $\{X_1, \ldots, X_n\}$. Proposition 3.3 then yields the assertion that (R, Σ) is in Boyce-Codd Normal Form and $X_1, \ldots, X_n$ form the minimal keys. □