3D ROTATIONS

Here we discuss, mostly through derivations, 4 different representations of rotations in 3D. We begin with matrix representation, which is the most common representation in computer graphics, as they compose well and GPUs are optimized for matrix computation. We will discuss other representations as well, and derive matrix representations from them.

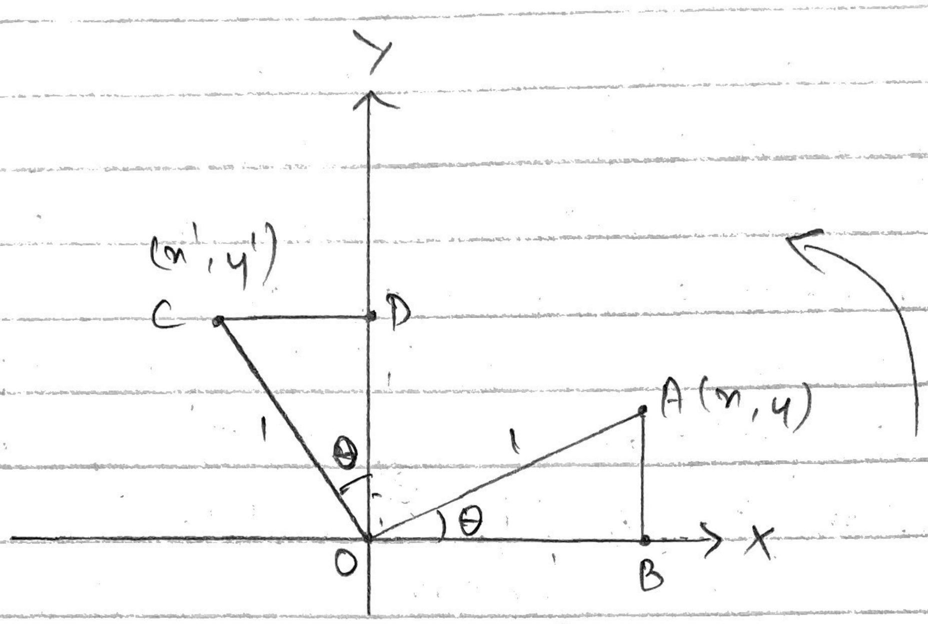
3D rotation representation: Matrix

Here we take the approach that transforming basis vectors and writing the transformation as column vectors of matrix, we arrive at transformation matrix.

For 3D rotation, we assume right hand coordinate system, where direction of rotation about axis is given by aligning our thumb to the axis and curling our fingers. The curl of the fingers gives the direction of rotation.

We warm up with 2D rotations:

~~2D roto~~



Here, (1, 0) is transformed to $A(x, y)$ and (0, 1) to $C(x', y')$.

In Δ OAB,

$$x = \cos\theta$$

$$y = \sin\theta$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

In Δ OCD,

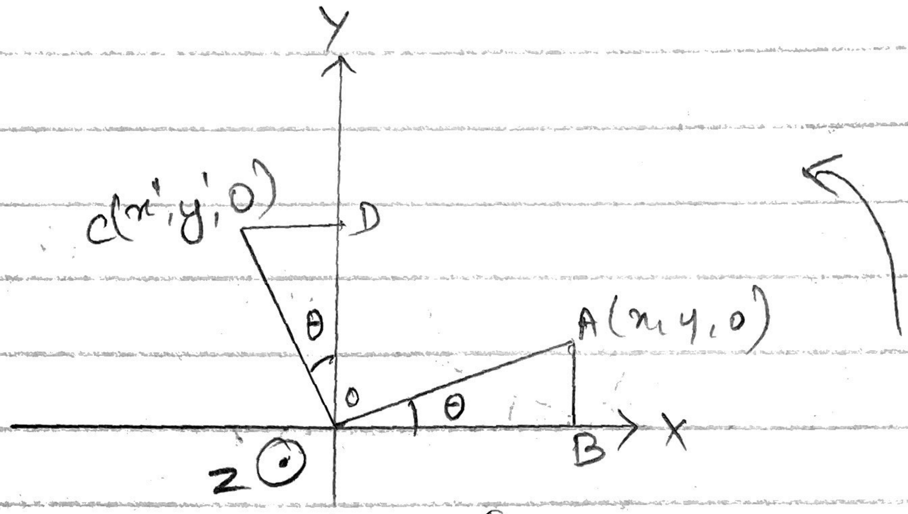
$$x' = -\sin\theta$$

$$y' = \cos\theta$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Thus, 2D rotation about origin, $T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

3D rotation about Z axis



Here, (1, 0, 0) is transformed to A(x, y, 0) and (0, 1, 0) to C(x', y', 0)

In ΔOAB,

$$x = \cos\theta \qquad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$$
$$y = \sin\theta ,$$
$$z = 0$$

In ΔOCD,

$$x' = -\sin\theta , \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$$
$$y' = \cos\theta$$
$$z = 0$$

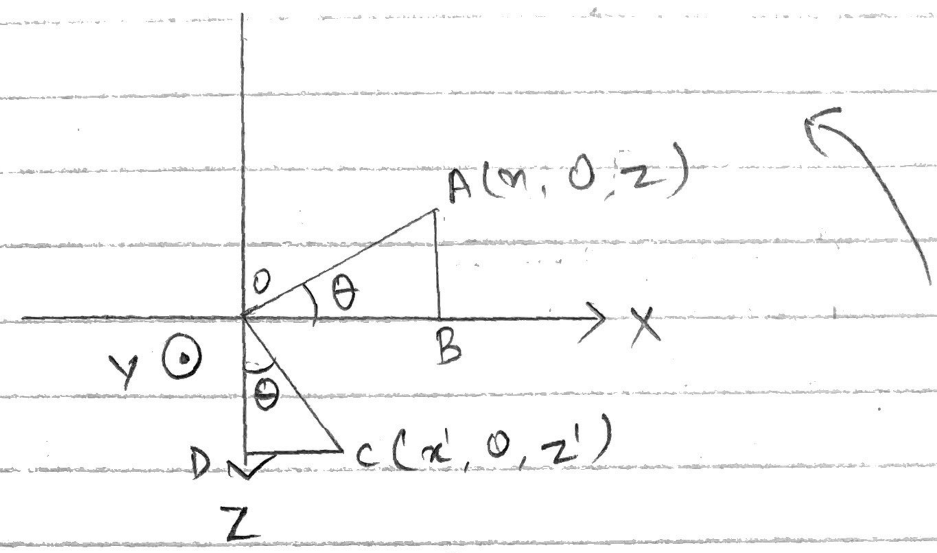
And since Z doesn't change with rotation, we have

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Thus, 3D rotation around Z axis is given as,

$$T = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

3D rotation about Y axis,



Here, (1, 0, 0) is transformed to A(x, 0, z)
and (0, 0, 1) is transformed to C(x', 0, z')

In Δ OAB,

$x = \cos\theta$

$y = 0$

$z = -\sin\theta$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\0\\-\sin\theta\end{pmatrix}$$

In Δ OCD,

$x' = \sin\theta$

$y' = 0$

$z' = \cos\theta$

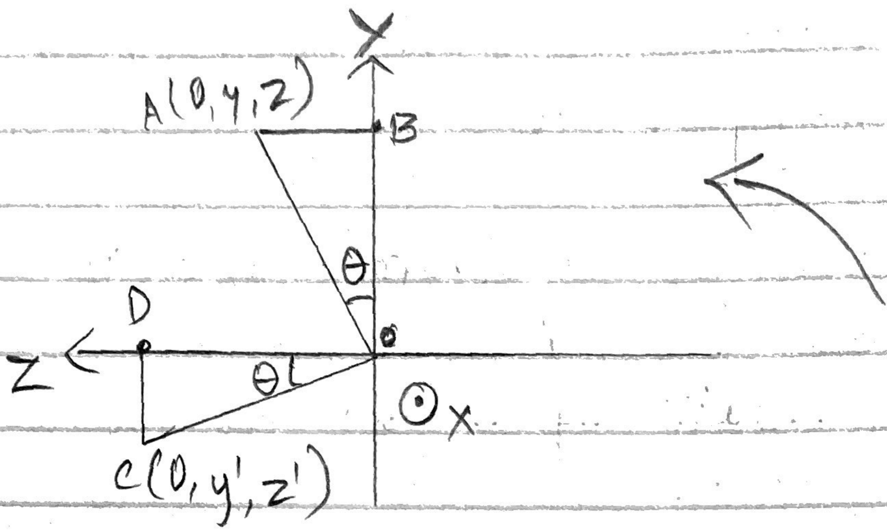
$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix}$$

And since basis vector (0, 1, 0) doesn't change, we have: $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

Thus 3D rotation around Y axis is given as:

$$T = \begin{pmatrix}\cos\theta & 0 & \sin\theta\\0 & 1 & 0\\-\sin\theta & 0 & \cos\theta\end{pmatrix}$$

3D rotation about X axis

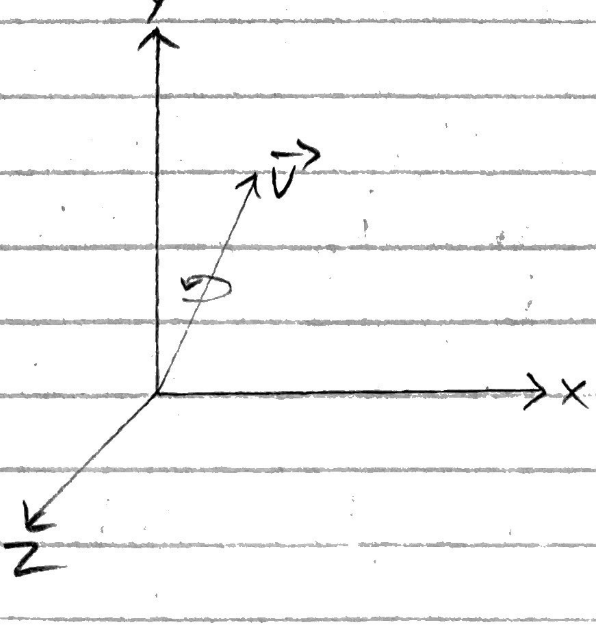


This would be derived same as we derived for Z axis, The final transformation, T =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

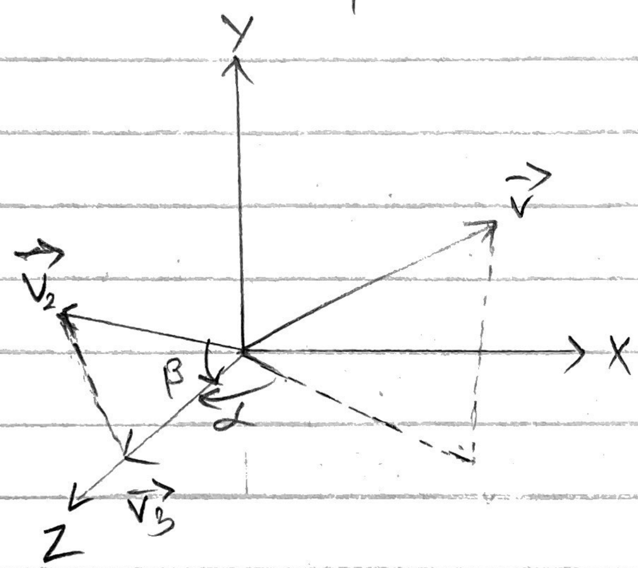
3D rotation about arbitrary axis

Here we assume rotation axis to be a vector $\vec{v}$, at origin, aligned arbitrarily. We are supposed to rotate by θ around this axis



As before, the direction of rotation is given by aligning thumb of right hand along axis and curling fingers.

The approach here is to rotate axis to align with one of the basis vectors, in this case Z, and then rotate by θ and reverse earlier rotations.



So, first we rotate vector $\vec{v}$ by α to $\vec{v_2}$ in YZ plane, then we rotate vector $\vec{v_2}$ by β onto Z axis. Note that we rotate instead of projecting since rotation is an inversible transformation, while projection at times isn't.

Notice that α rotation is more formally, rotation about Y axis by $-\alpha$, since its against mapping direction.
i.e. α Rotation $= R_{X,-\alpha}$

Similarly, β Rotation is rotation about X axis by +ve β. i.e. β Rotation $= R_{X,\beta}$

Then, the rotation around Z axis by θ follows : $R_{Z,\theta}$

Finally we reverse earlier 2 rotations. So, final transformation is given as :

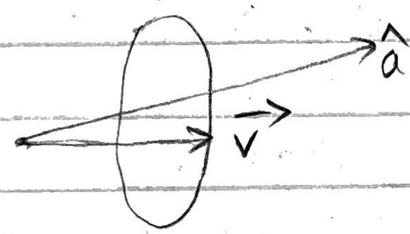
$$T = R_{Y,\alpha} * R_{X,-\beta} * R_{Z,\theta} * R_{X,\beta} * R_{Y,-\alpha}$$

Note that vectors are multiplied to the right.

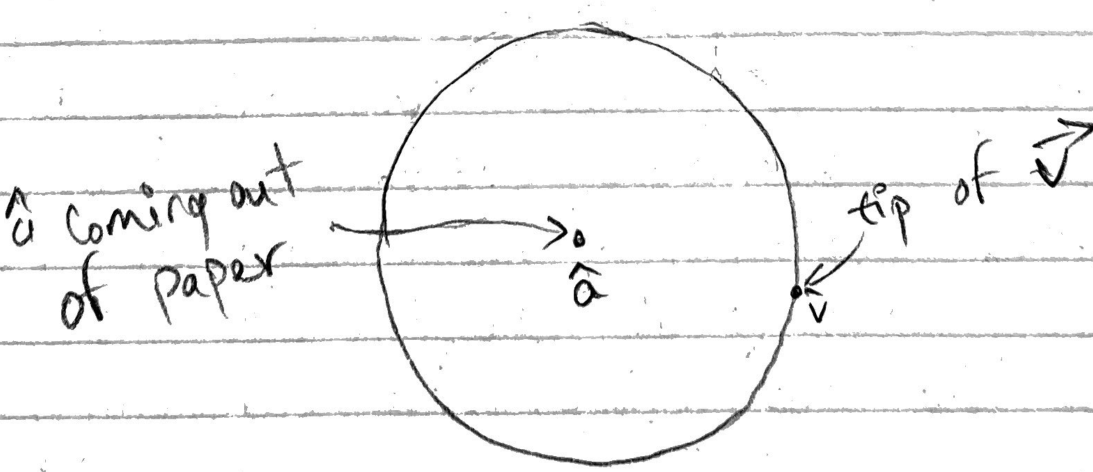
3D rotation representation : Axis Angle.

Axis Angle representation uses a pair $(\hat{a}, \theta)$ where $\hat{a}$ is a unit 3D vector and θ is the angle in range $[0, \pi]$, for a total of 4 parameters. This is much efficient when compared to matrice's 9 parameters. Also, this turns out to be a natural i.e. human friendly way of communicating orientation. How to neatly describe earth's rotation? Not Matrix! Rather, stick thumb of right hand parallel to axis of rotation and curl the finger. The curl gives the direction and thumb gives the axis of rotation. This representation is based on the same right hand rule. Also, the final transformation term above is rather expensive to compute. Instead we have another way of deriving rotation about arbitrary axis, and the final matrix that results is actually what is used in practise.

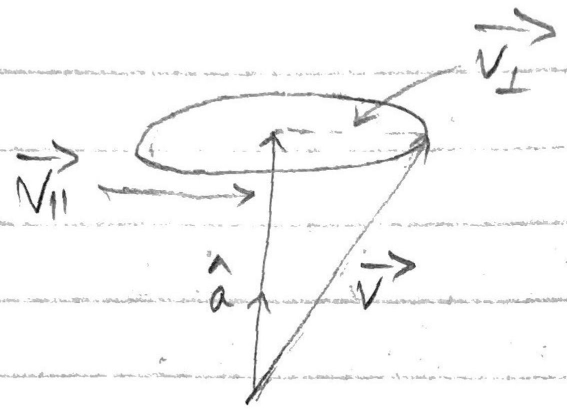
Here, let us consider a 3D unit vector $\hat{a}$. Let $\vec{v}$ be a vector we are rotating about $\hat{a}$. Rotation of $\vec{v}$ will form circular path, perpendicular to and centered about $\hat{a}$.



Seeing this from top down view, tip of $\vec{v}$ rotates around $\hat{a}$, as if 2D rotation around a point.



From yet another perspective, we can draw tip perpendicular to $\hat{a}$.



Notice that we have obtained decomposition of $\vec{v}$ into $\vec{V_{\parallel}}$ and $\vec{V_\perp}$ and $\vec{V} = \vec{V_{\parallel}} + \vec{V_\perp}$.

If R is the rotation transformation,

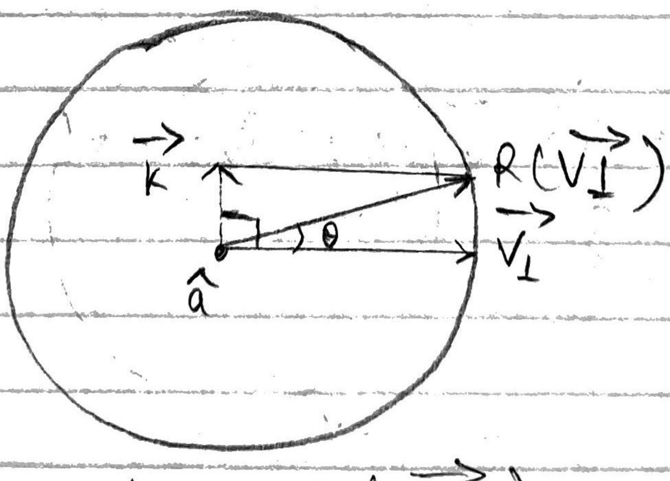
$$R(\vec{v}) = R(\vec{V_\perp}) + R(\vec{V_{\parallel}})$$

$\vec{V_{\parallel}}$ stays fixed during rotation. So,

$$R(\vec{v}) = R(\vec{V_\perp}) + \vec{V_{\parallel}} \qquad (1)$$

where $\vec{V_{\parallel}} = (\vec{v} \cdot \hat{a})\,\hat{a}$

$R(\vec{V_\perp})$ in the top down view looks as follows:



And notice that $R(\vec{V_\perp}) = \vec{V_\perp} + \vec{k}$.
But depending on θ i.e. the rotation $\vec{V_\perp}$ and $\vec{k}$ have varying effect on $R(\vec{V_\perp})$. for instance if θ were 0 then $R(\vec{V_\perp}) = \vec{V_\perp}$ and if θ were 90°, $R(\vec{V_\perp}) = \vec{k}$. This tells us that $R(\vec{V_\perp})$

is parameterized by θ as:

$$R(\vec{V_\perp}) = \vec{V_\perp} \cos\theta + \vec{k} \sin\theta .$$

So, we can substitute this into eqn (1)

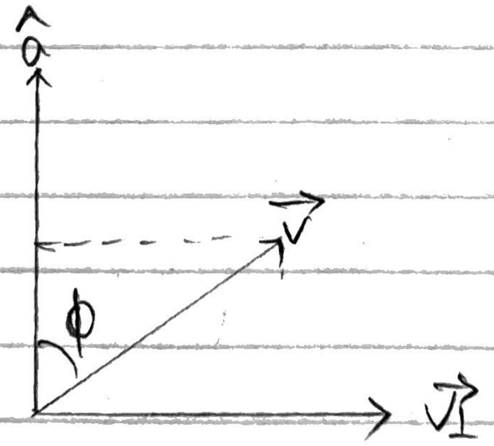
$$R(\vec{V}) = \vec{V_\perp} \cos\theta + \vec{k} \sin\theta + \vec{V_\parallel} \quad - (2)$$

$\vec{k}$ is still a mystery vector however. Notice from the figure however that since $\hat{a}$ is perpendicular to rotation, $\hat{a}$ is perpendicular to $\vec{k}$, which tells us that,

$$\vec{k} = \hat{a} \times \vec{V_\perp} \quad - (3)$$

Final step in this derivation is to convince you that $\hat{a} \times \vec{V_\perp}$ equals $\hat{a} \times \vec{v}$. First we'll prove direction of $\hat{a} \times \vec{v}$ equals $\hat{a} \times \vec{V_\perp}$.

Remember that $\hat{a}$, $\vec{v}$ and $\vec{V_\perp}$ all lie in same plane.



Thus, the direction of $a \times \vec{V_\perp}$ and $a \times \vec{v}$ are same.

Secondly for magnitude, $|\hat{a} \times \vec{v}| = |\hat{a}||\vec{v}| \sin\phi$
$= |\vec{v}| \sin\phi$
$= |\vec{v_\perp}|$

and,
$$|\hat{a} \times \vec{v_\perp}| = |\hat{a}| \, |\vec{v_\perp}| \sin 90^\circ$$
$$= |\vec{v_\perp}|$$

Thus, we have proven that, $\hat{a} \times \vec{v}$ equals $\hat{a} \times \vec{v_\perp}$ in both magnitude and direction. Thus we can write eqn 3 as:

$$\vec{k} = \hat{a} \times \vec{v}$$

Substituting this into eqn (2),

$$R(\vec{v}) = \vec{v_\perp} \cos\theta + (\hat{a} \times \vec{v}) \sin\theta + \vec{v_\parallel}$$

To express this as matrix, we transform basis vector, and write them as column vectors.

Transforming $\hat{x} = (1, 0, 0)$,
$$\vec{x_\parallel} = (\hat{x} \cdot \hat{a})\, \hat{a}$$
$$= a_x \, \hat{a}$$
$$= (a_x^2, \; a_x a_y, \; a_x a_z)$$
$$\vec{x_\perp} = \hat{x} - \vec{x_\parallel}$$
$$= (1, 0, 0) - (a_x^2, \; a_x a_y, \; a_x a_z)$$
$$= (1 - a_x^2, \; -a_x a_y, \; -a_x a_z)$$

$$\hat{a} \times \hat{x} = (a_x, a_y, a_z) \times (1, 0, 0)$$
$$= (0, \; a_z, \; -a_y)$$

So, the first column of the matrix,

$$R(\hat{x}) = \vec{x_{\perp}} \cos\theta + (\hat{a} \times \hat{x}) \sin\theta + \vec{x_{\parallel}}$$

$$= \begin{pmatrix} 1 - a_x^2 \\ -a_x a_y \\ -a_x a_z \end{pmatrix} \cos\theta + \begin{pmatrix} 0 \\ a_z \\ -a_y \end{pmatrix} \sin\theta + \begin{pmatrix} a_x^2 \\ a_x a_y \\ a_x a_z \end{pmatrix}$$

$$= \begin{pmatrix} a_x^2 (1 - \cos\theta) + \cos\theta \\ a_x a_y (1 - \cos\theta) + a_z \sin\theta \\ a_x a_z (1 - \cos\theta) + a_y \sin\theta \end{pmatrix}$$

Similarly we could derive 2nd and 3rd column by finding $R(\hat{y} = (0,1,0))$ and $R(\hat{z} = (0,0,1))$. Our final matrix R would look like:

$$R = \begin{bmatrix} a_x^2 (1-c) + c & a_x a_y (1-c) - a_z s & a_x a_z (1-c) + a_z s \\ a_x a_y (1-c) + a_z s & a_y^2 (1-c) + c & a_y a_z (1-c) - a_x s \\ a_x a_z (1-c) - a_y s & a_y a_z (1-c) + a_x s & a_z^2 (1-c) + s \end{bmatrix}$$

where, $c = \cos\theta$ and $s = \sin\theta$.

This gives us matrix representation from axis and angle of rotation.

Deriving axis angle representation from rotation matrix

We take a detour at begining to show that 3D rotation matrices are similar to rotation about standard Z. With this result we derive our angle of rotation. Deriving axis of rotation is easy with this basic fact that if v is axis of rotation, and R is rotation matrix, then $Rv = v$. But first, we derive change of basis matrix, and few other derivation that serve toward final proof that 3D rotation matrices are similar to rotation about standard z.

Change of Basis

If vector $v \in \mathbb{R}^n$, and basis of $\mathbb{R}^n$ is changed from B to B', then how are coordinate vectors $(v)_B$ and $(v)_{B'}$ related?

We will solve this for $\mathbb{R}^2$; solution for n dimensional spaces are similar.

Let $B = \{u_1, u_2\}$ and $B' = \{w_1, w_2\}$. Coordinate vectors for B' w.r.t B is, say:

$$(w_1)_B = \begin{pmatrix} a \\ b \end{pmatrix}, \quad (w_2)_B = \begin{pmatrix} c \\ d \end{pmatrix}.$$ That is,

$$w_1 = au_1 + bu_2 \quad \text{—— } (1)$$

$$w_2 = cu_1 + du_2 \quad \text{—— } (2)$$

Let v be any vector in R^2 and let

$$(v)_{B'} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$ so that,

$$v = k_1 w_1 + k_2 w_2 \quad — \quad (3)$$

from 1, 2 and 3:

$$v = k_1(au_1 + bu_2) + k_2(cu_1 + du_2)$$

or, $v = (k_1 a + k_2 c)u_1 + (k_1 b + k_2 d)u_2$

which is expressed in terms of basis B. So,

$$(v)_B = \begin{pmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{pmatrix}$$

$$(v)_B = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$(v)_B = \begin{pmatrix} a & c \\ b & d \end{pmatrix} (v)_{B'} \quad — \quad (4)$$

In 4, columns of matrix $P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ are coordinates of basis vectors of B' w.r.t. B. Thus we have following conclusion:

If we change basis of R^n from $B = \{u_1, u_2, ...\}$ to $B' = \{w_1, w_2, ...\}$ then for each $v \in R^n$, $(v)_B$ is related to $(v)_{B'}$ by, $(v)_B = P(v)_{B'}$

where columns of P are:

$$(w_1)_B, (w_2)_B \ldots (w_n)_B$$

In literature, B is also called old basis and B' is also called new basis and P is change of basis matrix. Columns of change-of-basis matrix from old to new basis are coordinate vectors of new basis relative to old basis.

Let $T: R^n \to R^n$ is a linear operator on R^n. Let "B" and "C" are 2 different sets of basis vectors. Let $(A)_B$ matrix define T for basis B and $(A)_C$ matrix define T for basis C. Give relation between $(A)_B$ and $(A)_C$.

If P is change of basis matrix from old basis B to new basis C and $x \in R^n$ then,

$$(x)_B = P(x)_C \quad \text{—— } ①$$

$$\text{or, } P^{-1}(x)_B = (x)_C \quad \text{—— } ②$$

Also, we have,

$$T(x)_B = (A)_B (x)_B \quad \text{—— } ③$$

$$T(x)_C = (A)_C (x)_C \quad \text{—— } ④$$

from 1:

$$(x)_B = P(x)_C$$

$$\text{or, } (A)_B (x)_B = (A)_B P(x)_C$$

$$\text{or, } T(x)_B = (A)_B P(x)_C \quad [\text{from } ③]$$

or, $P^{-1} T(x)_B = P^{-1} (A)_B \, P \, (x)_c$

or, $T(x)_c = (P^{-1} A_B P) (x)_c$ [from ②]

or, $(A)_c = P^{-1} A_B P$ [from ④]

This gives the relation between A_B and A_c. Note also that A_B is similar to A_c. This shows that if T is linear transformation from R^n to R^n, then different matrices that define 'T' in different basis, are all similar to one another.

<u>3D rotation matrices are similar to rotation about standard Z axis.</u>

Rotation about standard Z axis is well known:

$$R_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What about rotation about any unit vector $\hat{v}(a,b,c)$. By cross product and normalization, we can come up with 3 orthonormal vectors $\{\hat{t}, \hat{u}, \hat{v}\}$ and notice that this forms basis for R^3, say B.

Then rotating θ around this vector v in terms of basis B is, (Since $(v)_B = (0,0,1)$)

$$(R_{v,\theta})_B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Or, $(R_{v,\theta})_B = I^{-1} R_{z,\theta} I$.

Thus, we've shown that any 3D rotation matrices are similar to rotation about standard z axis.

Axis Angle Representation from rotation matrices

Let $R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be 3D rotation Matrix and v be axis of rotation,

We know that rotation matrix is an identity operation on axis of rotation i.e.

$Rv = v$

This is true for any real number k in

$$k \begin{pmatrix} h-f \\ c-g \\ d-b \end{pmatrix}.$$

Thus, we can choose k as 1 and normalize vector above to get axis of rotation as unit vector. However, if $\begin{pmatrix} h-f \\ c-g \\ d-b \end{pmatrix}$ is 0 vector then more work is needed to find axis of rotation.

Also we have established that 3D rotation are similar to rotation about standard Z. It follows that they have same trace i.e.

$$tr(R) = 1 + 2\cos\theta$$

$$\theta = \arccos\left(\frac{tr(R) - 1}{2}\right)$$

Notice that we end up with 2 values for θ. For e.g. $\arccos\left(\frac{1}{\sqrt{2}}\right)$ is either 45° or 315°.

To find out which θ to consider, plugin rotation axis and both θ's to earlier derived axis angle to rotation matrix function, which gives back 2 matrices, the θ that produced correct matrix, is the θ to consider.

Thus, we have derived axis angle representation from matrix

3D rotation representation: Euler Angles

Align thum, ring and middle finger of right hand perpendicular to one another. Now attach thumb to your chest, and stand upright. The thumb should be pointing to the chin middle finger to left chest and index finger away from the chest. We could similarly describe the orientation of a rigid body, via a frame of reference.

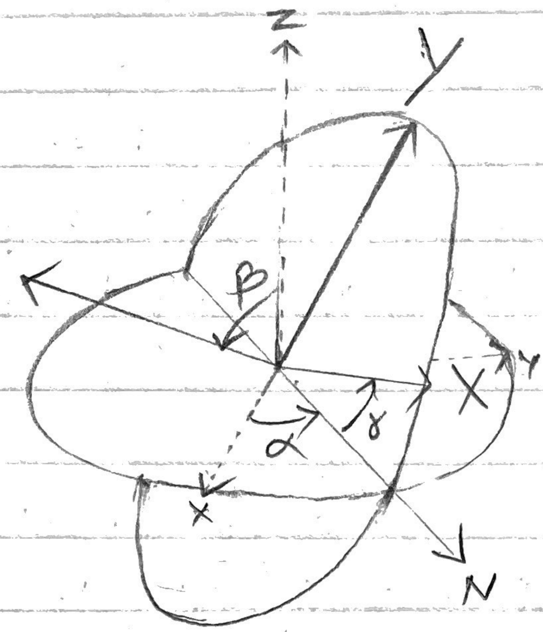
Any target orientation would be reached, starting from a known reference orientation, using specific sequence of rotations. Furthermore, 3 sequence of rotations about the axes of frame of reference are sufficient to reach any target orientation. Euler angles describe the magnitude of these rotations.

These rotations may be extrinsic or intrinsic. With extrinsic, we rotate about xyz of the original frame. With intrinsic, we rotate about XYZ frame, and this frame rotates per sequence of rotation. In either case, there are 6 different sequences of rotations:

X-Z-X; Y-Z-Y; Z-X-Z; Y-X-Y; Z-Y-Z; X-Y-X

Here we rotate about 2 distinct axes. There exists 6 other different sequences, that rotate all 3 axis, that we don't cover here!

Geometrically the original and target frame could be drawn as follows:



xyz (drawn as dotted) are axes of original frame and XYZ are axes of target frame. N axis, a.k.a line of Nodes, is the intersection of plane xy and XY. Thus, euler angles are the triplet of angles between 2 frame of references where:

α : Angle between x and N

β : Angle between z and Z

γ : Angle between N and X.

Intrinsic rotations

Here we rotate about the rotating frame in Z-X-Z sequence. Successive orientation may be denoted as follows:

$x-y-z$ (initial)
$x'-y'-z'$ (after first rotation)
$x''-y''-z''$ (after 2^{nd} rotation)
$X-Y-Z$ (final)

For above sequence of rotations, N is simply x', and since final rotation is rotation about z'', Z is z''. Thus, the euler angles for $z-x-z$ intrinsic rotations are:

α: Rotation about z
β: Rotation about x'
γ: Rotation about z''

Extrinsic Rotations

Here we rotate about the original frame in $z-x-z$ sequence. Euler angles for $z-x-z$ extrinsic rotations are:

γ: Rotation about z. X is now at angle γ with x.
β: Rotation about x. Z is now at angle β with z.
α: Rotation about z.

How this sequence of extrinsic rotation results in orientation as shown in figure is hard to visualize. Key is that after second rotation, N is collinear with x, and angle between X and x is γ. After 3^{rd} rotation, N rotates α from x.

Extrinsic euler angle rotation to rotation matrix

z-x-z extrinsic euler angles could be represented as matrix:

$R = R_{z,\alpha} * R_{x,\beta} * R_{z,\gamma}$

If $_{xyz}$ are basis (axes) for original frame then, $R_{x,\beta}$ is rotation about that x by β and $R_{z,\alpha}$ is rotation about that $_z$ by α.

If c is coordinate vector of $\mathbb{R}^3$ vector w.r.t this frame, then Rc results in rotated coordinate vector w.r.t same basis.

Relating Extrinsic with Intrinsic Rotations

To prove:

z-x-z extrinsic rotations with euler angles γ, β, α is same as z-x'-z'' intrinsic rotation with angles α, β, γ.

let, $E_1 \rightarrow R_{z,\gamma}$
$E_2 \rightarrow R_{x,\beta}$.
$E_3 \rightarrow R_{z,\alpha}$

$I_1 \rightarrow R_{z,\alpha}$
$I_2 \rightarrow R_{x',\beta}$
$I_3 \rightarrow R_{z'',\gamma}$

We ought to show $E_3 E_2 E_1 = I_3 I_2 I_1$, assuming coordinates are multiplied to the right

Here,

$$E_3 = I_1 \quad \text{(from definition)}$$

After I_1 say the orientation becomes $x'-y'-z'$. In this case I_2, rotation about x', would equal E_2, if $x'-y'-z'$ aligned with original $x-y-z$. So, below we rotate $x'-y'-z'$ back to $x-y-z$ and then apply E_2:

$$I_2 = I_1 E_2 I_1^{-1}$$

Similarly we have,

$$I_3 = (I_2 I_1) E_1 (I_2 I_1)^{-1}$$

So,

$$I_3 I_2 I_1 = (I_2 I_1) E_1 (I_2 I_1)^{-1} I_2 I_1$$

$$= (I_2 I_1) E_1$$

$$= (I_1 E_2 I_1^{-1} I_1) E_1$$

$$= I_1 E_2 E_1$$

or, $$I_3 I_2 I_1 = E_3 E_2 E_1$$

This equation relates extrinsic and intrinsic rotation

Gimbal Lock

Consider z-x-z extrinsic rotation, with 0 β, in that case there would be no rotation about x and that would align z and Z axis. The first and the final rotation by z, would therefore be redundant and ultimately this composition of rotations would only amount to rotation about z axis. The same happens with z-x'-z'' intrinsic rotation when β is 0.

Whenever rotations are composed, gimbal lock is always a possibility. Generally, for euler angle representation of 3D rotations, if the 1st and the 3rd axis are collinear then we have gimbal lock. To fix it, we have to further rotate the second axis, which breaks the collinearity between 1st and 3rd.

Matrix composition and euler angle representations suffer from gimbal lock, as they are composed. Other representations where we can describe orientation/rotation by single value, e.g. axis angle and quaternion don't suffer from this.

TODO:

Lerp and non constant angular velocity
Slerp
3D rotation representations: Quaternions.