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- $Ax \le b \in \mathbb{R}^m$ defines a region which is the intersection of m half-space.
  It is a polyhedron.
  If bounded, it is a polytope.
  - Probability simplex:
    $$P = \{x \in \mathbb{R}^n : x \ge 0, \textstyle\sum x_i = 1\} \quad \text{(polytope)}$$
    $$= \text{co}\{e^{(1)} \ldots e^{(n)}\} \quad n \text{ vertices}$$
  - l_1-norm ball
    $$D = \{x \in \mathbb{R}^n : \|x\|_1 \le 1\}$$
    $$= \{x \in \mathbb{R}^n : \textstyle\sum |x_i| \le 1\}$$
    $$= \{x \in \mathbb{R}^n : \max_{s \in \{-1,1\}^n} s^T x \le 1\} \quad \text{(polytope)}$$

- LP Model:
  $$p^* = \min \; c^T x + \cancel{d}$$
  $$\text{s.t. } A_{eq} x = b_{eq}$$
  $$Ax \le b$$

- $f : \mathbb{R}^n \to \mathbb{R}$ is polyhedral if its epigraph is polyhedron
  i.e. $\text{epi} f = \{(x,t) \in \mathbb{R}^{n+1} : f(x) \le t\}$
  $$= \{(x,t) \in \mathbb{R}^{n+1} : C\begin{bmatrix} x \\ t \end{bmatrix} \le d\}$$
  for some $C \in \mathbb{R}^{m, n+1}$, $d \in \mathbb{R}^m$

  - $f(x) = \max_{i \in [m]} a_i^T x + b_i$ is polyhedral
    $$\text{epi} f = \{(x,t) \in \mathbb{R}^{n+1} : a_i^T x + b_i \le t, \forall i \in [n]\}$$
  - $f(x) = \|x\|_\infty = \max_{i \in [n]} \max(x_i, -x_i)$
    maximum of $2n$ affine functions
  - $f(x) = \sum_{j=1}^{q} \max_{i \in [m]} a_{ij}^T x + b_{ij}$ projection of a polyhedron $\to$ polyhedron.
    $$\text{epi} f = \{(x,t) \in \mathbb{R}^{n+1} : \exists u \in \mathbb{R}^q, \mathbf{1}^T u \le t, a_{ij}^T x + b_{ij} \le u_j\}$$

- If f is polyhedral, then
  $\min_x f(x)$ s.t.: $Ax \le b$
  is cast as LP:
  $\min_{x,t} t$ s.t.: $Ax \le b$, $(x,t) \in \text{epi} f$

- $\min_x \|Ax-b\|_\infty$

  $\Leftrightarrow \min_{x,t} t$ s.t.: $\|Ax-b\|_\infty \le t$

  $\Leftrightarrow \min_{x,t} t$
  s.t.: $a_i^T x - b_i \le t$,
  $a_i^T x - b_i \ge -t$

- $\min_x \|Ax-b\|_1$

  $\Leftrightarrow \min_{x,u} \mathbf{1}^T u$ s.t.: $a_i^T x - b_i \le u_i$
  $a_i^T x - b_i \ge -u_i$

- Quadratic function:
  $f(x) = \frac{1}{2} x^T H x + c^T x + d$, H symmetric
  ($x^T A x = x^T \frac{A+A^T}{2} x$, $\frac{A+A^T}{2}$ symmetric)
  If $H \succeq 0$, then $f(x)$ is convex.

- QP model:
  $\min \frac{1}{2} x^T H x + c^T x + d$
  s.t.: $Ax \le b$
  where $H = H^T \succeq 0$
  - $H \succ 0$: $x^* = -H^{-1} c$, $p^* = d + (x^*)^T H x^*$
  - $H \succeq 0$; $c \in R(H)$: $x^* = -H^{\dagger} c + \xi$, $\xi \in N(H)$, $p^* = d - (x^*)^T H x^*$
  - $H \succeq 0$; $c \notin R(H)$: unbounded

- $\|x\|_1 = n_z(x)^T |x| \le \|n_z(x)\|_2 \|x\|_2 = \|x\|_2 \sqrt{\text{card}(x)}$
  $\le n_z(x)^T \mathbf{1} \|x\|_\infty = \|x\|_\infty \text{card}(x)$
  $\therefore \text{card}(x) \le k \Rightarrow \|x\|_1^2 \le k\|x\|_2^2$; $\|x\|_1 \le k\|x\|_\infty$

- Second-order cone (SOC):
  $K_n = \{(x,t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R} : \|x\|_2 \le t\}$
  Rotated second-order cone:
  $K_n^r = \{(x,y,z),\ x \in \mathbb{R}^n,\ y,z \in \mathbb{R} : x^T x \le 2yz;\ y \ge 0,\ z \ge 0\}$

  $$w := \begin{bmatrix} x \\ \frac{1}{\sqrt{2}}(y-z) \end{bmatrix} \quad t := (y+z)/\sqrt{2}$$

  $$(x,y,z) \in K_n^r \Leftrightarrow (w,t) \in K_{n+1}$$

- $x^T Q x + c^T x \le t$

  $$\Leftrightarrow \left\| \begin{bmatrix} \sqrt{2} Q^{\frac{1}{2}} \\ -c^T \end{bmatrix} x + \begin{bmatrix} 0 \\ t - \frac{1}{2} \end{bmatrix} \right\|_2 \le -c^T x + \left(t + \frac{1}{2}\right)$$

  $\rightsquigarrow (y,t) \in K_m$, y, t are affine fn of x

- SOCP: $\min c^T x$
  s.t.: $\|A_i x + b_i\|_2 \le c_i^T x + d_i$

- QP $\to$ SOCP:

  $\min_x x^T Q x + c^T x$ s.t.: $a_i^T x \le b_i$, where $Q = Q^T \succeq 0$

  $\Leftrightarrow \min_{x,y} c^T x + y$
  s.t.: $\left\| \begin{bmatrix} 2Q^{\frac{1}{2}} x \\ y-1 \end{bmatrix} \right\|_2 \le y+1$
  $a_i^T x \le b_i$

- Quadratic-constrained QP $\to$ SOCP:

  $\min_x x^T Q_0 x + a_0^T x$
  s.t.: $x^T Q_i x + a_i^T x \le b_i$
  where $Q_i = Q_i^T \succeq 0$

  $\Leftrightarrow \min_{x,t} a_0^T x + t$
  s.t.: $\left\| \begin{bmatrix} 2Q_0^{\frac{1}{2}} x \\ t-1 \end{bmatrix} \right\|_2 \le t+1$

  $\left\| \begin{bmatrix} 2Q_i^{\frac{1}{2}} x \\ b_i - a_i^T x - 1 \end{bmatrix} \right\|_2 \le b_i - a_i^T x + 1$

- Box uncertainty:

  $\mathcal{U} = \{a : \|a - \hat{a}\|_\infty \le \rho\}$

  $= \{\hat{a} + \rho u : \|u\|_\infty \le 1\}$

  $\max_{a\in\mathcal{U}} a^T x = \hat{a}^T x + \rho \cdot \left(\max_{\|u\|_\infty\le 1} u^T x\right) = \hat{a}^T x + \rho\|x\|_1$

  polyhedron with 2^n vertices

- Sphere uncertainty:

  $\mathcal{U} = \{a : \|a - \hat{a}\|_2 \le \rho\}$

  $= \{\hat{a} + \rho u : \|u\|_2 \le 1\}$

  $\max_{a\in\mathcal{U}} a^T x = \hat{a}^T x + \rho \left(\max_{\|u\|_2\le 1} u^T x\right) = \hat{a}^T x + \rho\|x\|_2$

  SOCP.

- Ellipsoidal uncertainty:

  $\mathcal{U} = \{a : (a-\hat{a})^T P^{-1} (a-\hat{a}) \le \rho\}$

  where $P \succ 0$ $\therefore \exists R$ s.t. $P = R^T R$.

  $\mathcal{U} = \{a = \hat{a} + Ru : \|u\|_2 \le 1\}$

  $\max_{a\in\mathcal{U}} a^T x = \hat{a}^T x + \max_{\|u\|_2\le 1} (Ru)^T x = \hat{a}^T x + \|R^T x\|_2$

- Chance-constrained LP:

  $\min_x c^T x$

  s.t. $a_i^T x \le b_i \rightarrow$ ~~where~~ $a_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\bar{a}_i, \overset{\succ 0}{\Sigma_i})$

  $\therefore E(a_i^T x) = \bar{a}_i^T x, \quad Var(a_i^T x) = x^T \Sigma_i x$

  $\rightsquigarrow \min c^T x$

  s.t. $P[a_i^T x \le b_i] \ge p_i, \quad p_i > 0.5$

  $\Leftrightarrow \min_x c^T x$

  s.t. $\bar{a}_i^T x \le b_i - \Phi^{-1}(p_i) \|\Sigma_i^{\frac{1}{2}} x\|_2$ (SOCP)

  Prf: $z_i(x) := (a_i^T x - \bar{a}_i^T x) / \sigma_i(x) \sim \mathcal{N}(0,1)$

  $\sigma_i(x) := \|\Sigma_i^{\frac{1}{2}} x\|_2$

  $\tau_i(x) := (b_i - \bar{a}_i^T x) / \sigma_i(x)$

  $P[a_i^T x \le b_i] = P[z_i(x) \le \tau_i(x)] \ge p_i$

  $\Leftrightarrow \cancel{p_i \ge} \tau_i(x) \ge \Phi^{-1}(p_i)$

- Robust Least Squares

  $\mathcal{U} = \{A : \|A - \hat{A}\| \le \rho\}$

  $\min_x \max_{A\in\mathcal{U}} \|Ax - y\|_2$

  $\Leftrightarrow \min_x \max_{\|\Delta\|\le\rho} \|(\hat{A}+\Delta)x - y\|_2$

  - $\|(\hat{A}+\Delta)x - y\|_2 \le \|\hat{A}x - y\|_2 + \|\Delta x\|_2 \quad \Delta x = \alpha(\hat{A}x - y), \alpha > 0$

    $\|\Delta x\|_2 \le \|\Delta\| \cdot \|\Delta x\|_2 \le \rho\|x\|_2 \quad \Delta = v x^T, \|\Delta\| = \rho$

    $\therefore \max_{\|\Delta\|\le\rho} \|(\hat{A}+\Delta)x - y\|_2 \le \|\hat{A}x - y\|_2 + \rho\|x\|_2$

    the upper bound is obtained when

    $\Delta = \frac{\rho}{\|\hat{A}x - y\|_2 \cdot \|x\|_2} (\hat{A}x - y) x^T$

  $\Leftrightarrow \min_x \|\hat{A}x - y\|_2 + \rho\|x\|_2$

  $\Leftrightarrow \min_{x,u,v} u + \rho v$

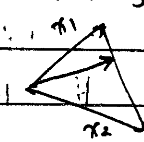
  s.t. $u \ge \|\hat{A}x - y\|_2, \quad v \ge \|x\|_2$ (SOCP)

- Consider $P = \{x^{(1)}, \dots, x^{(m)}\}$
  - linear hull $\{x : x = \sum \lambda_i x^{(i)}, \forall \lambda_i \in \mathbb{R}\}$
  - affine hull $\{x : x = \sum \lambda_i x^{(i)}, \sum \lambda_i = 1, \forall \lambda_i \in \mathbb{R}\}$

    aff P is the smallest affine set containing P.
  - convex combination: $x = \sum \lambda_i x^{(i)}, \sum \lambda_i = 1, \lambda_i \ge 0 \ \forall i$
  - convex hull co $P = \{x : x = \sum \lambda_i x^{(i)}, \sum \lambda_i = 1, \lambda_i \ge 0\}$
  - conic hull conic $P = \{x : x = \sum \lambda_i x^{(i)}, \lambda_i \ge 0\}$

- Set $C \subseteq \mathbb{R}^n$ is convex if

  $x_1, x_2 \in C, \lambda \in [0,1] \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in C$



- Set $C \subseteq \mathbb{R}^n$ is a cone if

  $x \in C, \alpha \ge 0 \Rightarrow \alpha x \in C$

- A conic hull of a set is a convex cone

- C_i is convex, $\forall i \in I \Rightarrow C = \bigcap_{i \in I} C_i$ is convex
  - S_+^n : symmetric, PSD, $n \times n$. S^n: symmetric. convex
    $S_+^n = \bigcap_{u \in \mathbb{R}^n} \{ X \in S^n : u^T X u \geq 0 \}$ is also convex.
    It is a convex cone.
- If $f: \mathbb{R}^n \to \mathbb{R}^m$ is affine, $C \subset \mathbb{R}^n$ is convex,
  then $f(C) = \{ f(x) : x \in C \}$ is convex.
  - The projection of a convex set C onto a subspace is represented by a linear map.
    Thus the projected set is convex.
- $f: \mathbb{R}^n \to \mathbb{R}$ is convex if
  $f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$,
  $\forall \lambda \in [0,1]$, $x, y \in \mathrm{dom} f = \{ x \in \mathbb{R}^n : -\infty < f(x) < \infty \}$
  - f is concave if $-f$ is convex
  - Convex functions must be $+\infty$ outside domains
- $\mathrm{epi} f := \{ (x,t), x \in \mathrm{dom} f, t \in \mathbb{R} : f(x) \leq t \}$
  f is a convex function if and only if epi f is a convex set
- α-sublevel set of f: $S_\alpha := \{ x \in \mathbb{R}^n : f(x) \leq \alpha \}$
  - f is convex $\Rightarrow$ S_α is convex set
- If f_i are convex, then $f(x) = \sum \alpha_i f_i(x)$, $\alpha_i \geq 0$ is convex
  - log-sum-exp function: $f(x) = \log(\sum e^{x_i})$
    $\mathrm{epi} f = \{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : \sum e^{x_i - t} \leq 1 \}$
    $g_i(x) := e^{x_i - t}$ is convex
    $\therefore \mathrm{epi} f = S_1 = \{ x \in \mathbb{R}^n : \sum g_i(x) \leq 1 \}$ is a convex set
  - $f(x) = \sum x_i \log x_i$ is convex
- If $f: \mathbb{R}^n \to \mathbb{R}$ is convex $\Rightarrow g(x) := f(Ax+b)$ is convex.
  (affine var. transf. preserves convexity)

- If f is differentiable, — describes tangent hyperplane
  f is convex $\Leftrightarrow$ $f(y) \geq f(x) + \nabla f(x)^T (y-x)$, $\forall x, y \in \mathrm{dom} f$
- If f is twice differentiable,
  f is convex $\Leftrightarrow$ $\nabla^2 f \succeq 0$ $\forall x \in \mathrm{dom} f$
- f is convex $\Leftrightarrow$ $g(t) := f(x_0 + tv)$ is convex, $\forall x_0, v$
- $(f_\alpha)_{\alpha \in A}$ is a family of convex functions,
  then $f(x) = \max f_\alpha(x)$ is convex
  over $\{ \cap_{\alpha \in A} \mathrm{dom} f_\alpha \} \cap \{ x : f(x) < \infty \}$
  - $x \mapsto \|Ax+b\|_2 - (c^T x + d)$ is convex
  - $x \mapsto \sum_{i=1}^{k} x_{[i]} = \max_u u^T x : u \in \{0,1\}^n$, $\mathbf{1}^T u = k$
    is convex
  - $X \mapsto \lambda_{max}(X) = \max_u u^T X u : \|u\|_2 = 1$
  - $X \mapsto \|AX+B\| + tr(C^T X)$
    $\|M\| := \max_{\|u\|_2 = 1} \|Mu\|_2$
    $f(X) = \max_{\|u\|_2 \leq 1} \|(AX+B)u\|_2 + tr(C^T X)$
    $\forall u$, $f(\cdot, u)$ is convex.

Convex problem:
  $p^* = \min f_0(x)$ s.t. $f_i(x) \leq 0$, $Ax = b$
  where f_0, f_i convex
  - feasible set $\mathcal{X}$ is convex
  - local optimal $\Rightarrow$ global optimal
  - $\mathcal{X}_{opt}$ is convex

· Primal Problem:

$p^* = \min f_0(x)$

s.t.: $f_i(x) \le 0$

$h_i(x) = 0$

$\mathcal{L}(x,\lambda,\nu) := f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x)$

$\therefore \max_{\lambda \ge 0, \nu} \mathcal{L}(x,\lambda,\nu) = \begin{cases} f_0(x) & \text{if } x \text{ is feasible} \\ +\infty & \text{otherwise} \end{cases}$

· Minimax Inequality:

$F: X \times Y \to \mathbb{R}$

$\underbrace{\min_{x\in X} \max_{y \in Y} F(x,y)}_{g(x)} \ge \max_{y \in Y} \underbrace{\min_{x \in X} F(x,y)}_{h(y)}$

$g(x)$: x plays first

$h(y)$: y plays first

Prf: $\forall (x_0, y_0) \in X \times Y$

$h(y_0) \le F(x_0, y_0) \le g(x_0)$

$\Rightarrow \max_y h(y) \le \min_x g(x)$

· $p^* = \min_x \max_{\lambda \ge 0, \nu} \mathcal{L}(x,\lambda,\nu)$

$d^* := \max_{\lambda \ge 0, \nu} \underbrace{\min_x \mathcal{L}(x,\lambda,\nu)}_{g(\lambda,\nu)}$

convex set $\leftarrow \mathbb{R}^m_+ \times \mathbb{R}$

· $g(\lambda,\nu)$ is a pointwise minimum of affine functions $\to g(\lambda,\nu)$ is concave

$\max_x g(\lambda,\nu)$ is convex

· $S_k(w) := \sum_{i=1}^k w_{[i]}$, $w_{[i]}$ is the i-th largest element in w.

· $S_k(w) = \max_{u \in \{0,1\}^n} u^T w : \mathbb{1}^T u = k$

pointwise maximum $\therefore S_k(w)$ is convex.

· Claim: $S_k(w) \le \theta(\mathbb{1}^T w)$, $w \ge 0$

$\Leftrightarrow \exists \nu$ s.t. $k\nu + \sum_{i=1}^n \max(0, w_i - \nu) \le \theta(\mathbb{1}^T w)$

Prf: Note that $k\nu + \sum \max(0, w_i - \nu)$

$= \underbrace{\max(\nu, w_{[1]}) + \cdots + \max(\nu, w_{[k]})}_{A(w)} + \underbrace{\max(0, w_{[k+1]} - \nu) + \max(0, w_{[n]} - \nu)}_{B(w)}$

$A(w) \ge S_k(w)$, $B(w) \ge 0$.

($\Rightarrow$) Let $\nu = w_{[k+1]}$. Then $A(w) = S_k(w)$, $B(w) = 0$.

$\therefore S_k(w) = A(w) + B(w) \le \theta(\mathbb{1}^T w)$.

($\Leftarrow$) $S_k(w) \le A(w) + B(w) \le \theta(\mathbb{1}^T w)$

· Slaters condition for $\boxed{\text{convex}}$ programs:

If $p^* = \min_x f_0(x)$ s.t.: $f_i(x) \le 0$ ($f_0(\cdot), f_i(\cdot)$ convex)

$f_0: D \to \mathbb{R}$, $Ax = b$.

If the problem is strictly feasible i.e. $\exists x \in \text{relint} D$ s.t. $Ax_0 = b$, $f_i(x_0) < 0$, $\forall i$ (including implicit ineq. related to the problem's domain),

then $p^* = d^*$

· LASSO: $p^* = \min_x \|Ax - b\|_2^2 + \lambda \|x\|_1$

To make use of duality, transformed into

$p^* = \min_{x,z} \frac{1}{2} \|z\|_2^2 + \lambda \|x\|_1$ s.t.: $z = Ax - b$

$d^* = \max g(\nu)$. $g(\nu) = \frac{1}{2}\|z\|_2^2 + \lambda \|x\|_1 + \nu^T(z - Ax + b)$

$= \nu^T b + \min_x \lambda \|x\|_1 - x^T(A^T\nu) \rightsquigarrow \alpha(x)$

$+ \min_z \frac{1}{2}\|z\|_2^2 + \nu^T z \rightsquigarrow \beta(z)$

$\alpha(x) = \sum \lambda|x_i| - x_i(A^T\nu)_i \ge \sum [\lambda - |(A^T\nu)_i|] |x_i|$

i, $\|A^T\nu\|_\infty \le \lambda$: RHS ≥ 0, $\alpha(0) = 0 \Rightarrow \alpha^*(x) = 0$, $x^* = 0$

ii, $\|A^T\nu\|_\infty > \lambda$: $\alpha^*(x) = +\infty$.

$\beta^*(z) = -\frac{1}{2}\|\nu\|_2^2$

$\therefore$ Dual: $d^* = \max_\nu \nu^T b - \frac{1}{2}\|\nu\|_2^2$ s.t.: $\|A^T\nu\|_\infty \le \lambda$

- Logistic Regression

$p^* \triangleq \max_{\omega, b} \log L(\omega, b)$

$= \sum_{i=1}^{m} \log(1 + \exp(-y_i(\omega^T x_i + b)))$ set to 0

introduce slack var. to make duality meaningful

$p^* = \max_{\omega, \nu} -\sum_{i=1}^{m} f(\nu_i)$ s.t. $\nu = A^T \omega$

$A = [y_1 x_1 \cdots y_m x_m]$

$f(\xi) := \log(1 + e^{-\xi})$

The constraint is linear equality, feasible

$\therefore$ By Slater's thm

$p^* = d^* = \min_{\nu} g(\nu)$

$g(\nu) := \max_{\omega} -\sum f(\nu_i) + \nu^T(A^T\omega - \nu)$

$= \max_{\omega} -\sum_i (f(\nu_i) + \nu_i \nu_i) + \nu^T A^T \omega$

i) $A\nu \neq 0$, $g(\nu) = +\infty$

ii) $A\nu = 0$: with $0 \log 0 = 0$:

$\min_{\xi} f(\xi) + \alpha\xi = \begin{cases} \alpha \log \alpha + (1-\alpha)\log(1-\alpha), & 0 \le \alpha \le 1 \\ +\infty, & \text{otherwise} \end{cases}$

$\therefore p^* = d^* = \min_{\nu} -H(\nu)$, s.t. $\nu \in [0,1]^m$, $A\nu = 0$

$H(\nu) := -\sum_i (\nu_i \log \nu_i + (1-\nu_i)\log(1-\nu_i))$

concave.

- $\boxed{\text{Sion's minimax thm}}$

Let $X \subseteq \mathbb{R}^n$ be concave, $Y \subseteq \mathbb{R}^m$ compact set.

$F: X \times Y \to \mathbb{R}$ s.t. $\forall y \in Y$, $F(\cdot, y)$ is convex, cont. over X; $\forall x \in X$, $F(x, \cdot)$ is concave, cont. over Y.

Then $\max_{y \in Y} \min_{x \in X} F(x,y) = \min_{x \in X} \max_{y \in Y} F(x,y)$.

- $d^* = \max_{\lambda \geq 0} g(\lambda)$

$g(\lambda) := \min_{x} L(x, \lambda, \nu)$

$L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$

- KKT conditions for (x, λ) pair:

i) Primal feasibility

$x \in D$, $f_i(x) \le 0$

ii) Dual feasibility

$\lambda \geq 0$

iii) Complementary slackness

$\lambda_i f_i(x) = 0$

iv) Lagrangian stationarity

$x \in \arg\min L(\cdot, \lambda)$

if $f_i(\cdot)$ differentiable, then

$\nabla_x f_0(x) + \sum \lambda_i \nabla_x f_i(x) = 0$.

- Assume that the primal problem is attained i.e. $\exists x^* \in D$ s.t. $p^* = f_0(x^*)$; the dual is attained; the strong duality holds.

Then a primal pair (x, λ) is optimal

$\Leftrightarrow$ (x, λ) satisfies KKT conditions.

- Primal Problem:

$p^* = \min_{x} f_0(x)$ s.t.: $f_i(x) \le 0$, $i \in [m]$.

$Ax = b$

where $f_0, f_1 \cdots f_m$ are convex differentiable

$D := \cap_{i=0}^{m} \operatorname{dom} f_i$

Assume i) $D = \mathbb{R}^n$ ii) Strictly feasible (so Slater's holds)

iii) is attained i.e. $\exists x^* \in D$ s.t. $p^* = f_0(x^*)$.

$\Leftrightarrow p^* = \min_{x \in X} f_0(x)$. $X \subseteq D$ denotes the feasible set.

~~can show that optimal x^*: $Ax^* = b$~~, $\exists \nu$ s.t. $\nabla f_0(x)^T + A^T \nu = 0$

$\Rightarrow (x, \lambda)$ is optimal iff KKT holds.

· Recover primal solution from dual:

~~If $L(x, \lambda^*, \nu^*)$ has a unique minimizer (λ^*, ν^*):~~

If $L(\cdot, \lambda^*, \nu^*)$ has a unique minimizer x^*:

if x^* is feasible, then it is the primal-optimal solution.

if not, then no primal-optimal solution exists.

· SAFE (Safe feature eliminations):

primal: $p^* = \min_x \|Ax - b\|_2^2 + \mu\|x\|_1$, col. of A

dual: $d^* = \max_u b^T u : \|u\|_2 \le 1, |a_i^T u| \le \mu, i \in [n]$.

If $\|a_i\|_2 < \mu$, ~~then~~ $|a_i^T u| \le \mu$, $\forall \|u\|_2 \le 1$

∴ ith constraint is not active in the dual, could be removed.

It amounts to solve primal with ith feature removed. i.e. safe to set $x_i = 0$.

137A Final

· Unitary: $U^{-1} = U^+$

can be expressed as $U = e^{iA}$, A is Hermitian.

· Idempotent: $\Lambda^2 = \Lambda$.

· If Λ is also Hermitian, then it is a projection operator.

Let ψ be any ~~operator~~ function: $\psi = \Lambda\psi + (I-\Lambda)\psi$

then $\langle \Lambda\psi | (I-\Lambda)\psi \rangle$

$= \langle \psi | \Lambda^+ | I-\Lambda | \psi \rangle = \langle \psi | 0 | \psi \rangle = 0$

Note that $I-\Lambda$ is also a projection operator.

· $\psi = \sum c_n \psi_n$, $c_m = \langle \psi_m | \psi \rangle$

$\Rightarrow$ ~~$\psi(r, t) =$~~ $\sum_n \psi_n^*(r') \psi_n(r) = \delta(r - r')$ (closure) → Dirac delta

⤳ expresses completeness of $\{\psi_n\}$

$\Rightarrow \langle \chi | \psi \rangle = \sum_n \langle \chi | \psi_n \rangle \langle \psi_n | \psi \rangle$

∴ Closure relation can be written as $\sum_n |\psi_n\rangle\langle\psi_n| = I$.

· Including continuous eigenvalues:

$\psi = \sum_n c_n \psi_n + \int c(a) \psi_a da$

$c_n = \langle \psi_n | \psi \rangle$, $c(a) = \langle \psi_a | \psi \rangle$

Closure relation:

$\sum \psi_n^*(r') \psi_n(r) + \int \psi_a^*(r') \psi_a(r)\, da = \delta(r - r')$

· Fundamental commutation relation:

$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$

· ~~$(\Delta A)^2 (\Delta B)^2 \ge -\frac{1}{4}(\langle [A,B] \rangle)^2$~~

· $\Delta A \Delta B \ge \frac{1}{2} |\langle [A,B] \rangle|$