Discussion 6

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1 Bijection (again)

- f is one-to-one or injective if $x \neq y \Rightarrow f(x) \neq f(y)$. When we try to prove f is injective, it may be easier to prove the contraposition: $f(x) = f(y) \Rightarrow x = y$.
- f is onto or surjective if $(\forall y, \exists x)(f(x) = y)$.

2 Cardinality

2.1 Writing Proofs

- To prove set A and B are of the same "size" or cardinality, you may try:
  1. Give a bijective function from A to B.
  2. First show $|A| \leq |B|$, by giving an injective function f from A to B, or by arguing A is a subset of B; then show $|B| \leq |A|$, by giving an injective function g from B to A.
- In particular, to show a set S is countable:
  1. Explicitly enumerate the elements (e.g. show $\{0,1\}^*$ is countable).
  2. Find an injective function from S to some countable set T (e.g. $\{0,1,2\}^*$), or $\mathbb{N}$, or some subset of $\mathbb{N}$:
- To show a set S is uncountable:
  1. Cantor's Diagonalization (e.g. show $\mathbb{R}[0,1]$ and $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$ is uncountable).

2.2 Some facts worth knowing

- $\mathbb{Q}$ is countable; $\mathbb{R}$ is uncountable.
- Functions from $\mathcal{N}$ to $\mathcal{N}$ (proved by Cantor's diagonal trick).

- Computer programs are countable. This is because computer programs are just strings of finite length, though the length could be different. If $\mathcal{A} = \{0, 1, \ldots, 256\}$ is the list of characters, then we can enumerate any program by listing $\mathcal{A}, \mathcal{A} \times \mathcal{A}, \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \ldots$ which correspond to programs of length $1, 2, 3, \ldots$. More concisely, $\mathcal{A}^n$ repesents all possible programs of length n, and computer programs are countable because $\cup_{n=1}^{\infty} \mathcal{A}^n$ is countable (see homework6).