· Unitary operator with small perturbation:

$\hat{U} = \hat{I} + i\varepsilon\hat{F}$

$\hat{F} = \hat{x} \rightarrow$ translation, $\hat{H} \rightarrow$ time evolution, $\hat{L} \Rightarrow$ rotation

$\hat{U}_z(\delta\alpha) = I - \frac{i}{\hbar}\delta\alpha\,\hat{L}_z$

In general, $\hat{U}_n(\delta\alpha) = I - \frac{i}{\hbar}\delta\alpha\,\hat{n}\cdot\hat{L}$

Finite rotation: $\hat{U}_n(\alpha) = \exp\left(-\frac{i}{\hbar}\alpha\,\hat{n}\cdot\hat{L}\right)$

Isolated system: $[\hat{L}, \hat{H}] = 0$, then $\hat{L}$ is conserved

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· Hermitian Operators (equi. def)

i) $\langle \chi | (A\psi) \rangle = \langle (A\chi) | \psi \rangle$ iii) $A^\dagger = A$

ii) $\langle \psi | (A\psi) \rangle = \langle (A\psi) | \psi \rangle$

$\Rightarrow \langle \psi | A | \psi \rangle$ is real.

A has real eigenvalues.

· Adjoint (Hermitian conjugate):

$\langle \chi | A^\dagger | \psi \rangle = \langle (A\chi) | \psi \rangle = \langle \psi | A | \chi \rangle^*$

$\langle \phi | := \langle \chi | A^\dagger \Rightarrow |\phi\rangle = A|\chi\rangle$

· Unitary: $U^{-1} = U^\dagger$. $U := e^{iA} \rightsquigarrow U^\dagger = (e^{iA})^\dagger = e^{-iA}$ $(A^\dagger = A)$

· $f(z) := \sum c_i z^i \rightsquigarrow f(A) := \sum c_i A^i$

$[f(A)]^\dagger = \sum c_i^* (A^i)^\dagger = \sum c_i^* (A^\dagger)^i = f^*(A^\dagger)$

· Projection Operator:

Idempotent: $\Lambda^2 = \Lambda$.

If $\Lambda$ is Hermitian and idempotent, then it is a projection operator.

$\rightarrow$ Decomposition: $\psi = \Lambda\psi + (I - \Lambda)\psi$.

$\langle \Lambda\psi | (I - \Lambda)\psi \rangle = \langle \psi | \Lambda - \Lambda^2 | \psi \rangle = 0$

· $\psi = \sum c_n \psi_n \rightarrow c_m = \langle \psi_m | \psi \rangle$

· $\psi(\vec{r}, t) = \sum \left[ \int \psi_n^*(\vec{r}')\,\psi(\vec{r}', t)\,d\vec{r}' \right] \psi_n(\vec{r})$

$= \int \left[ \sum \psi_n^*(\vec{r}')\,\psi_n(\vec{r}) \right] \psi(\vec{r}', t)\,d\vec{r}'$

$\Rightarrow \sum_n \psi_n^*(\vec{r}')\,\psi_n(\vec{r}) = \delta(\vec{r} - \vec{r}')$

$\rightsquigarrow \sum_n \psi_n^*(\vec{r}_1', \ldots, \vec{r}_N')\,\psi_n(\vec{r}_1, \ldots, \vec{r}_N) = \delta(\vec{r}_1 - \vec{r}_1') \cdots \delta(\vec{r}_N - \vec{r}_N')$

$\langle \chi | \psi \rangle = \int \chi^*(\vec{r}, t)\,\psi(\vec{r}, t)\,d\vec{r}$

$= \iint \chi^*(\vec{r}, t)\,\delta(\vec{r} - \vec{r}')\,\psi(\vec{r}', t)\,d\vec{r}\,d\vec{r}'$

$= \sum_n \int \chi^*(\vec{r}, t)\,\psi_n(\vec{r})\,d\vec{r} \int \psi_n^*(\vec{r}')\,\psi(\vec{r}', t)\,d\vec{r}'$

$= \sum \langle \chi | \psi_n \rangle \langle \psi_n | \psi \rangle$

$\Rightarrow \sum |\psi_n\rangle\langle\psi_n| = I.$

- $\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_m\sum_n c_m^* c_n \langle\psi_m|A|\psi_n\rangle$

$= \sum_m\sum_n c_m^* c_n a_n \langle\psi_m|\psi_n\rangle = \sum_n |c_n|^2 a_n$

- $\psi = \sum c_n\psi_n + \int c(a)\psi_a da$

$\langle A\rangle = \langle\psi|A|\psi\rangle$

$= \cancel{\sum_m\sum_n c_m^* c_n \langle\psi_m|A|\psi_n\rangle + \sum_m \int da\, c_m^* c(a)\langle\psi_m|A|\psi_a\rangle}$

$= \sum |c_n|^2 a_n + \int |c(a)|^2 a\, da$

$\langle\psi_m|\psi_n\rangle = \delta_{mn}$, $\langle\psi_{a'}|\psi_a\rangle = \delta(a-a')$

$c_n = \langle\psi_n|\psi\rangle$, $c(a) = \langle\psi_a|\psi\rangle$

Closure relation for a one-particle system:

$\sum \psi_n^*(\vec r')\psi_n(\vec r) + \int \psi_a^*(\vec r')\psi_a(\vec r)\,da = \delta(\vec r - \vec r')$

- Observable $\hat A, \hat B, \hat C$ are compatible if they possess a common set of eigenfunctions.
  - $AB\psi_n = a_n b_n \psi_n = b_n a_n\psi_n = BA\psi_n$
    $\Rightarrow (AB-BA)\psi = \sum c_n(AB-BA)\psi_n = 0$
    $\Rightarrow [A,B]=0$.
  - If $[A,B]=0$: $A(B\psi_n) = BA\psi_n = a_n(B\psi_n)$
    - where $a_n$ non-degenerated: then $B\psi_n = b_n\psi_n$
    - where $a_n$ is degenerate
      $B\psi_{nr} = \sum_{s=1}^{\alpha} c_{rs}\psi_{ns}$
      $B\sum_{r=1}^{\alpha} d_r\psi_{nr} = \sum_{r=1}^{\alpha}\sum_{s=1}^{\alpha} d_r c_{rs}\psi_{ns} = b_n\sum_{r=1}^{\alpha} d_r\psi_{nr}$
      $\Rightarrow \sum_{r=1}^{\alpha} d_r c_{rs} = b_n d_s \;\forall s$

- $[A,B] = -[B,A]$

$[A,B+C] = [A,B]+[A,C]$

$[A,BC] = [A,B]C + B[A,C]$

$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

- $\Delta A := [\langle(A-\langle A\rangle)^2\rangle]^{\frac12} = [\langle A^2\rangle - \langle A\rangle^2]^{\frac12}$

$\Delta A\,\Delta B \ge \frac12 |\langle[A,B]\rangle|$ where A, B are observables.

- Prf: $\bar A := A - \langle A\rangle$, $\bar B := B - \langle B\rangle$ (Linear Hermitian)
  $\Rightarrow (\Delta A)^2 = \langle\bar A^2\rangle$, $(\Delta B)^2 = \langle\bar B^2\rangle$
  $[\bar A,\bar B] = [A-\langle A\rangle, B-\langle B\rangle] = [A,B]$
  $C := \bar A + i\lambda\bar B$, $\lambda\in\mathbb R$. $C^\dagger = \bar A - i\lambda\bar B$
  $\langle CC^\dagger\rangle = \langle\psi|CC^\dagger|\psi\rangle = \langle C^\dagger\psi|C^\dagger\psi\rangle \in \mathbb{R}\setminus\mathbb{R}_-$
  $f(\lambda) := \langle CC^\dagger\rangle$ $\quad\because \langle i[A,B]\rangle \in \mathbb R$
  $= \langle(\bar A + i\lambda\bar B)(\bar A - i\lambda\bar B)\rangle$
  $= \langle\bar A^2\rangle + \lambda^2\langle\bar B^2\rangle - i\lambda\langle[\bar A,\bar B]\rangle$
  $= (\Delta A)^2 + \lambda^2(\Delta B)^2 - i\lambda\langle[A,B]\rangle \ge 0$
  $f\left(\frac{i\langle[\hat A,\hat B]\rangle}{2(\Delta B)^2}\right) = f(\lambda)_{min} = (\Delta A)^2 + \frac14\frac{(\langle[A,B]\rangle)^2}{(\Delta B)^2} \ge 0$
  $\Rightarrow (\Delta A)^2(\Delta B)^2 \ge -\frac14(\langle[A,B]\rangle)^2$
  $[\hat x,\hat p_x] = i\hbar \Rightarrow \Delta x\,\Delta p_x \ge \frac{\hbar}{2}$
- The minimum uncertainty holds when
  $\lambda = i[\langle A,B\rangle]/[2(\Delta B)^2]$, $\langle C^\dagger\psi\rangle = 0$
  Consider $\hat x, \hat p_x$: $\lambda = -\hbar/[2(\Delta p_x)^2]$
  $C^\dagger = \hat x - \langle x\rangle + \frac{i\hbar}{2(\Delta p_x)^2}(\hat p_x - \langle p_x\rangle)$
  $C^\dagger\psi = 0 \Rightarrow \frac{2i(\Delta p_x)^2}{\hbar}(\hat x - \langle x\rangle)\psi(x) = (\hat p_x - \langle p_x\rangle)\psi(x)$
  $\Rightarrow \psi(x) = C\exp\left(\frac{i}{\hbar}\langle p_x\rangle x\right)\exp\left[-(\Delta p_x)^2(x-\langle x\rangle)^2/\hbar^2\right]$
  Gaussian Wave packet

· Unitary transformation

$$\begin{tikzcd} \psi \arrow[r, "A"] \arrow[d, "U"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "U"] \\ \psi' \arrow[r, "A'"] & X' \end{tikzcd} \qquad \begin{aligned} A'U &= UA \\ \Rightarrow A' &= UAU^\dagger \\ A &= U^\dagger A' U \end{aligned}$$

• $A$ is Hermitian $\Leftrightarrow$ $A'$ is Hermitian.

• Operator equations remain unchanged in form

$A = c_1 B + c_2 CD \quad (c_1, c_2 \in \mathbb{C})$

$\Rightarrow A' = c_1 B' + c_2 C'D'$

$[A, B] = C \Rightarrow [A', B'] = C$

• $(a_n, \psi_n) = \mathrm{eig}(A)$

$\Leftrightarrow (a_n, U\psi_n) = \mathrm{eig}(A')$

• $\langle X|A|\psi\rangle = \langle X'|A'|\psi'\rangle$

• Fourier transform $\psi(x,t) \leftrightarrow \phi(p_x,t)$ is unitary:

$\phi(p_x,t) = U\psi(x,t)$

$= (2\pi\hbar)^{-\frac{1}{2}} \int e^{-ip_x x/\hbar} \psi(x,t)\, dx$

$\psi(x,t) = U^{-1}\phi(p_x,t)$

$= (2\pi\hbar)^{-\frac{1}{2}} \int e^{ip_x x/\hbar} \phi(p_x,t)\, dp_x$

· Infinitesimal unitary transformation

$U = I + i\varepsilon F, \ \varepsilon \in \mathbb{R}$, ~~F is Hermitian~~

$I = U^\dagger U = (I - i\varepsilon F^\dagger)(I + i\varepsilon F)$

$= I - i\varepsilon F^\dagger + i\varepsilon F + \cancel{O(\varepsilon^2)}$

$\Rightarrow F^\dagger = F \Rightarrow F$ is Hermitian.

$F$ is called the generator of the inf. unit. $U$

$\psi' = U\psi = \psi + i\varepsilon F\psi = \psi + \delta\psi$ where $\delta\psi \equiv i\varepsilon F\psi$

$\therefore A' = A + \delta A = UAU^\dagger$

$\Rightarrow A + \delta A = A + i\varepsilon[F, A] + \cancel{O(\varepsilon^2)}$

$\Rightarrow \delta A = i\varepsilon[F, A]$ to first order in $\varepsilon$.

· Matrix representation

$|\psi\rangle = \sum_n |\psi_n\rangle c_n, \ |X\rangle = \sum |\psi_m\rangle d_m, \ A_{mn} = \langle\psi_m|\hat{A}|\psi_n\rangle$

$X = A\psi \Rightarrow d_m = \langle\psi_m|X\rangle = \langle\psi_m|A|\psi\rangle = \sum_n A_{mn} c_n$

$\Rightarrow \vec{d} = A\vec{c}$

$\langle X|\psi\rangle = \langle X|\left[\sum_n |\psi_n\rangle\langle\psi_n|\right]|\psi\rangle$

$= \sum_n (\langle\psi_n|X\rangle)^* \langle\psi_n|\psi\rangle = \sum d_n^* c_n = \vec{d}^\dagger \cdot \vec{c}$

· If the complete set of orthonormal eigenfunctions $\{\psi_n\}$ of an observable $A$ is used as a basis, then $A$ is represented by a diagonal matrix.

· LHO: $\hat{H} = \hat{p}_x^2/2m + \frac{1}{2} m\omega^2 \hat{x}^2$

$\hat{a}_\pm := \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\, x \mp \frac{ip_x}{\sqrt{m\hbar\omega}}\right]$

$\Rightarrow [a_-, a_+] = 1$

$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-)$

$= \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2}) = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$

$= \hbar\omega(\hat{N} + \frac{1}{2})$

· $L^2$ and $L_z$ form a complete set for the specification of angular momentum states.

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \rightsquigarrow \begin{cases} L_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \\ L_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \\ L_z = -i\hbar\frac{\partial}{\partial\varphi} \end{cases}$$

$0 \le r \le \infty, \ 0 \le \theta \le \pi, \ 0 \le \varphi \le 2\pi$

$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$