Math 104 Final

- ~~Acc~~ Def: Let $A \subseteq \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is an accumulation point of A if $\forall \delta > 0$, $\exists x \in A$ s.t. $0 < |x - x_0| < \delta$.
  Note that it does not require $x \in A$.
  $x_0 \in A$ (not just in $\mathbb{R}$) is a isolated point of A if it is not an accumulation point of A.
  i.e. $\exists \delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \cap A = \{x_0\}$
- Def: Let $f: A (\subseteq \mathbb{R}) \to \mathbb{R}$, x_0 an accumulation point of A. $\lim_{x \to x_0} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in A$ s.t. $0 < |x - x_0| < \delta$, $|f(x) - \varepsilon| < l$.
- Characterization of accumulation point:
  Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$. $x_0 \in \mathbb{R} \cup \{+\infty, +\infty\}$.
  The following are equivalent:
  i) x_0 is an accumulation point of A.
  ii) Every (punctuated) neighborhood of x_0 contains infinitely many elements of A.
  iii) $\exists (x_n)_{n \in \mathbb{N}}$ in A ($x_{n_1} \neq x_{n_2}$, $\forall n_1 \neq n_2$, i.e. pairwise distinct) s.t. $x_n \to x_0$.
- Characterization of limits via limit of sequences:
  Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R} \cup \{+\infty, -\infty\}$
  x_0 is an accumulation point of A.
  The following are equivalent:
  i) $\lim_{x \to x_0} f(x) = l$
  ii) For any sequence (x_n) in $\mathbb{R}$ with $\boxed{x_n \neq x_0}$, $\forall n$, and $x_n \to x_0$, we have $f(x_n) \to l$.
- Continuity: Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, $x_0 \in A$. (x_0 doesn't have to be an accumulation point of A).
  f is continuous at x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ ($\delta := \delta(\varepsilon, x_0)$) s.t. $\forall x \in A$ s.t. $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$.

- Let $f: A \to \mathbb{R}$, $x_0 \in A$.
  - i) If x_0 is an isolated point of A, then f is continuous at x_0.
  - ii) If x_0 is an accumulation point of A, then f is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.
    i.e. for any sequence (x_n) in A, $x_n \neq x_0$, $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.
- Characterization of continuity via limits of sequences:
  Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in A$. Then
  f continuous at x_0 $\iff$ For every sequence (x_n) in A, with $x_n \to x_0$ (not requiring $x_n \neq x_0$), we have $f(x_n) \to f(x_0)$.
- Local properties of continuous functions:
  - I. Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $\tilde{f}: A \cap (x_0 - \theta, x_0 + \theta) \to \mathbb{R}$, $\tilde{f}: x \mapsto f(x)$ for some $\theta > 0$. If $\tilde{f}$ is continuous at x_0, then f is continuous at x_0.
  - II. Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $x_0 \in A$. Then $\exists \delta > 0, M > 0$ s.t. $|f(x)| \leq M$, $\forall x \in A \cap (x_0 - \delta, x_0 + \delta)$
  - III. Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $x_0 \in A$, $f(x_0) \neq 0$. $\exists \delta > 0$ s.t. $\forall x \in A \cap (x_0 - \delta, x_0 + \delta)$, $\operatorname{sgn}(f(x)) = \operatorname{sgn}(f(x_0))$
- Every continuous function $f: [a,b] \to \mathbb{R}$ is bounded.
- Let $f: [a,b] \to \mathbb{R}$ be continuous. Then f has minimum and maximum value. i.e. $\exists y, y' \in [a,b]$ s.t. $f(y) \leq$ ~~min f(x)~~ $f(x) \leq f(y')$, $\forall x \in [a,b]$
- Intermediate value theorem (Bolzano):
  Let $f: [a,b] \to \mathbb{R}$ continuous, $f(a) < 0 < f(b)$. Then $\exists \xi \in (a,b)$ s.t. $f(\xi) = 0$

- Def: $I \subseteq \mathbb{R}$, $I \neq \emptyset$ is an interval if $\forall x, y \in I$ with $x < y$, we have $[x,y] \subseteq I$.
- Let $f: I \to \mathbb{R}$ be continuous, $I \subseteq \mathbb{R}$ is an interval. Then $f(I)$ is an interval.
- Let $f: I \to \mathbb{R}$ be continuous, $I \subseteq \mathbb{R}$ is an interval. Then f is one to one $\Rightarrow$ f is strictly monotonic.
  ~~Indeed~~ $f^{-1}: f(I) \to I$ is continuous and one to one, and has the same monocity as f.
- Def: Let (X, d_X), (Y, d_Y) be metric space. $f: X \to Y$. f is continuous at $x_0 \in X$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f(B_{d_X}(x_0, \delta)) \subseteq B_{d_Y}(f(x_0), \varepsilon)$
- Let (X, d_X), (Y, d_Y) be metric space. $f: X \to Y$. Then f is continuous $\iff$ f inverts open sets to open sets i.e. $f^{-1}(U)$ is open in (X, d_X), $\forall U \subseteq Y$ (open in (Y, d_Y)).
- Def: Let (X,d) be a metric space ~~and $K \subseteq X$~~. (X,d) is compact if every open cover of X (in (X,d)) has a finite subcover.
  $K \subseteq X$, K is a compact subcover of X if every open cover of K in (X,d) has a finite subcover.
- $K \subseteq X$, (X,d), K is compact $\Rightarrow$ K is closed and bounded
- Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $f: X \to Y$ continuous w.r.t. $\mathcal{T}_X$, $\mathcal{T}_Y$. Then $f(K)$ is compact in $(Y, \mathcal{T}_Y)$, $\forall K$ compact in $(X, \mathcal{T}_X)$.
- Def: Let (X,d) be a metric space. (X,d) is sequentially compact if every sequence in X has a converging subsequence.
- Compact $\iff$ sequentially compact.

$(a,b) \to \mathbb{R}$ $\quad x_0 \in (a,b)$

- Def. f is differentiable at x_0 if $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists in $\mathbb{R}$. $f'(x_0) := \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.
- If f is differentiable at x_0, then f is continuous at x_0.
- Rolle's theorem:
  Let $f: [a,b] \to \mathbb{R}$, with:
  - f continuous on $[a,b]$
  - f differentiable on (a,b)
  - $f(a) = f(b)$

  Then $\exists\, \xi \in (a,b)$ s.t. $f'(\xi) = 0$
- Mean-value theorem
  $f: [a,b] \to \mathbb{R}$. conti on $[a,b]$, diff on (a,b).
  Then $\exists\, \xi \in (a,b)$ s.t. $f'(\xi) = \frac{f(b)-f(a)}{b-a}$

  Generalized:
  $\exists\, \xi \in (a,b)$ s.t. $(f(b)-f(a))\, g'(\xi) = (g(b)-g(a))\, f'(\xi)$
- Let $f: (X, d_X) \to (Y, d_Y)$ be a continuous function and suppose that (X, d_X) is compact. Then f is uniformly continuous.
- $f: \mathbb{R} \to \mathbb{R}$ is a step function w.r.t. $\{x_0, \dots, x_n\}$
  $\iff \exists\, c_1, \dots, c_n$ s.t. $\phi(x) = \sum_{i=1}^{n} c_i \chi_{(x_{i-1}, x_i)}(x)$, $\forall x \neq x_0, \dots, x_n$

  Define $\int \phi = \sum_{i=1}^{n} c_i \int \chi_{(x_{i-1}, x_i)} = \sum c_i (x_i - x_{i-1})$
- Def: Let $f: \mathbb{R} \to \mathbb{R}$. f is Riemann integrable if $\forall \varepsilon > 0$, there exist step functions ϕ and ψ s.t. $\phi \le f \le \psi$ $\Rightarrow \int\phi \le \int\psi$.
  and $\int\psi - \int\phi < \varepsilon$.
- If f is Riemann integrable, then f is bounded and has bounded support

bdd & bdd support $f: \mathbb{R} \to \mathbb{R}$ is Riemann integrable
$\iff \sup\{\int\phi : \phi \text{ is a step function s.t. } \phi \le f\}$ $\leftarrow \int f$
$= \inf\{\int\psi : \psi \text{ is a step function s.t. } \psi \ge f\}$

- $f: \mathbb{R} \to \mathbb{R}$ is Riemann integrable
  $\iff$ there exists sequences $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ of step functions, with $\phi_n \le f \le \psi_n$, $\forall n \in \mathbb{N}$.
  s.t. $\int\psi_n - \int\phi_n \to 0$.
  (It follows that $\int\psi_n \to \int f$, $\int\phi_n \to \int f$)
- Def: $f: \mathbb{R} \to \mathbb{R}$ is bounded, with bounded support in $[a,b]$. For any partition $P = \{x_0, \dots, x_n\}$ where $a = x_0 < x_1 < \dots < x_n = b$, define
  $f_*(P) := \sum_{i=1}^{n} m_i \chi_{(x_{i-1}, x_i)}(x) + \sum_{i=0}^{n} f(x_i) \chi_{\{x_i\}}(x)$
  $f^*(P) := \sum M_i$ — — — — —
  where $m_i := \inf\{f(x) : x \in (x_{i-1}, x_i)\}$
  $M_i := \sup\{f(x) : x \in (x_{i-1}, x_i)\}$

  f is Riemann integrable $\Rightarrow \exists P$ s.t. $\int f^*(P) - \int f_*(P) < \varepsilon$
  $\iff \exists$ sequence (P_n) s.t. $\int f^*(P_n) - \int f_*(P_n) \to 0$
  it follows that $\int f = \lim_{n\to+\infty} f_*(P_n) = \lim_{n\to+\infty} f^*(P_n)$
  $\iff \sup\{\int f_*(P)\} = \inf\{\int f^*(P)\} = \int f$
- f, g is Riemann integrable
  $\Rightarrow |\int f| \le \int |f|$, $f \cdot g$, $\lambda f + \mu g$ also R-integrable.
- Every continuous / monotone function $f: [a,b] \to \mathbb{R}$ is R-integrable
- f R-integrable $\Rightarrow$ $F(x) = \int_a^x f(t)\,dt$ ~~R-integrable~~.
  $\Rightarrow F$ is Lipschitz continuous. i.e. $\exists M > 0$ s.t.
  $\forall x, y \in [a,b]$: $|F(x) - F(y)| < M|x-y|$