Croatia IMO Team Selection Test 2009 — P1/4

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Find all real numbers x, y, z such that the following two equations are satisfied:

$$3\left(x^2+y^2+z^2\right)=1$$
$$x^2y^2+y^2z^2+z^2x^2=xyz(x+y+z)^3$$

Solution

Lemma: $xy+yz+zx=\frac{1}{2}(x+y+z)^2-\frac{1}{6}$.

Proof: Use the first equation to see that $(x^2+y^2+z^2)=(x+y+z)^2-2(xy+yz+zx)=\frac{1}{3}$ and so $xy+yz+zx=\frac{1}{2}(x+y+z)^2-\frac{1}{6}$ as desired.

Now using our lemma observe that,

$$\begin{aligned}(x+y+z)^3&=x^3+y^3+z^3+3(x+y+z)(xy+yz+zx)-3xyz\\&=x^3+y^3+z^3+3(x+y+z)\left(\frac{1}{2}(x+y+z)^2-\frac{1}{6}\right)-3xyz\\&=x^3+y^3+z^3+\frac{3}{2}(x+y+z)^3-\frac{1}{2}(x+y+z)-3xyz\\ \frac{1}{2}(x+y+z)^3&=\frac{1}{2}(x+y+z)+3xyz-(x^3+y^3+z^3)\end{aligned}$$

Now the second equation asserts that $\sum_{\text{sym}}x^2y^2=xyz((x+y+z)+6xyz-2(x^3+y^3+z^3))$ so

$$2xyz(x^3+y^3+z^3)+(x^2y^2+y^2z^2+z^2x^2)=6(xyz)^2+xyz(x+y+z)$$

If we have $xyz=0$ then the equation becomes $(x^2y^2+y^2z^2+z^2x^2)=0$ and so $xy=yz=zx=0$ and at least 2 of x, y, z must be 0. Suppose without loss of generality that $x=y=0$, then we just need $3z^2=1 \iff z=\pm\frac{1}{\sqrt{3}}$. So $(x,y,z)=\left(0,0,\pm\frac{1}{\sqrt{3}}\right)$ is a solution up to it's 6 permutations.

Now assuming that $x, y, z>0$ and $xyz>0$, we can say by AM-GM, $2xyz(x^3+y^3+z^3)\geq 6(xyz)^2$ and by Muirhead's inequality, $\sum_{\text{sym}}x^2y^2z^0=(x^2y^2+y^2z^2+z^2x^2)\geq xyz(x+y+z)=\sum_{\text{sym}}x^2yz$ since $(2,2,0)\succ(2,1,1)$. But that means we must have the equality case in both of these inequalities. AM-GM reaches equality exactly when $x=y=z$, so we must have $9x^2=1 \iff x=\pm\frac{1}{3}$ and $3x^4=27x^6 \iff 3x^4(9x^2-1)=3x^4(3x-1)(3x+1)=0 \iff x=\pm\frac{1}{3}$ or 0 so the only solutions here are $x=y=z=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. Now since we assumed that x, y, z are positive we know that in fact each one of x, y, z could be $\pm\frac{1}{3}$. But it is straightforward to check each case, we have up to permutations, $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ which we know works. $(\frac{1}{3},\frac{1}{3},-\frac{1}{3})$ but that makes the RHS of second equation <0 whilst the LHS is ≥ 0, so it's impossible. $(\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$ makes $x+y+z$ negative whilst $xyz>0$ so this is also impossible by the same reason as the last case. $(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$ which works since $3 \cdot \frac{1}{81} = -\frac{1}{27} \cdot -1$