Baltic Way 2020 — P4

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Find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + x + y) = f(x + y) + yf(y)$.

Solution

Let's call the equation in the problem statement $(*)$. Clearly $f(x) = 0$ is a solution, we prove that it is the only one. We first prove the following claim,

Claim: *If f isn't injective then $f(x) = 0$ is the only solution.*

Proof: Assume that f is not an injection. Then there must exist $a \neq b$ such that $f(a) = f(b)$, for some real a, b. Using $(*)$ and the property $f(a) = f(b)$, we must have $f(f(a)+a+b) = f(a+b)+bf(b) = f(b+a)+af(a) = f(f(b)+b+a)$. But that means that we must have $af(a) = bf(b) \iff f(a)[a-b] = 0$. So either $a = b$ (which is impossible since we assumed $a \neq b$) or $f(a) = 0$. Now put $x = a$ into $(*)$, $f(f(a) + a + y) = f(a + y) + yf(y)$. But since $f(a) = 0$ we have to have $yf(y) = 0$ for all real y, that is $f(y) = 0$ for all non-zero y. We now prove that $f(0) = 0$ by contradiction. Suppose that $f(0) = c \neq 0$. Then plugging in $x = y = 0$ into the equation in the problem statement gives $f(f(0)) = f(0) = c$, that is $f(c) = c \neq 0$, but we already know that only non-zero value in the image of $f(x)$ occurs at $x = 0$, so $c = 0$ a contradiction. So we have shown that $f(x) = 0$ for all real x under the assumption that f isn't injective. This proves the lemma.

Plug in $y = 0$ into $(*)$ to find that $f(f(x) + x) = f(x)$. Now if f were injective then we get that $f(f(x) + x) = f(x) \iff f(x) + x = x \iff f(x) = 0$. This finishes the proof that $f(x) = 0$ for all real x. ■