POTD 1842

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Consider complex polynomials $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ with zeroes $x_1, \ldots, x_n$, and $Q(x) = x^n + b_1x^{n-1} + \cdots + b_n$ with zeroes $x_1^2, \ldots, x_n^2$. Prove that if $a_1 + a_3 + a_5 + \ldots$ and $a_2 + a_4 + a_6 + \ldots$ are real numbers, then $b_1 + b_2 + \cdots + b_n$ is also real.

Solution

We write down $P(x)$ and $Q(x)$.

$$P(x) = \prod_{r=1}^{n}(x - x_r) \qquad Q(x) = \prod_{r=1}^{n}(x - x_r^2)$$

Now if we consider $Q(x^2)$ we can use difference of 2 squares to relate $Q(x)$ and $P(x)$ namely,

$$Q(x^2) = \prod_{r=1}^{n}(x^2 - x_r^2) = \prod_{r=1}^{n}(x - x_r)(x + x_r) = P(x)P(-x)$$

Now we can deduce a relationship between the sequence $(b_i)_{i=1}^n$ and $(a_i)_{i=1}^n$. Precisely we find, using $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ and $Q(x) = x^n + b_1x^{n-1} + \cdots + b_n$

$$Q(1) - 1 = \sum_{r=1}^{n} b_r = \left[1 + \sum_{r=1}^{n} a_r\right]\left[(-1)^n + \sum_{r=1}^{n}(-1)^{n-r}a_r\right] - 1$$

That is,

$$\sum_{r=1}^{n} b_r = (-1)^n + \sum_{r=1}^{n}(-1)^{n-r}a_r + (-1)^n\sum_{r=1}^{n} a_r + \left[\sum_{r=1}^{n} a_r\right]\left[\sum_{j=1}^{n}(-1)^{n-j}a_j\right] - 1$$

We finish the problem by analysing the 2 cases:

1. When n is even,

$$\begin{aligned}\sum_{r=1}^{n} b_r &= \sum_{r=1}^{n}(-1)^r a_r + \sum_{r=1}^{n} a_r + \left[\sum_{r=1}^{n} a_r\right]\left[\sum_{j=1}^{n}(-1)^j a_j\right] \\ &= 2(a_2 + a_4 + \cdots + a_n) + (a_1 + a_2 + \cdots + a_n)(-a_1 + a_2 - \cdots + a_n)\end{aligned}$$

Now since the sum of the even and the sum of the odd indexed terms are both real, and the set of real numbers is closed under addition we conclude that $\sum_{r=1}^{n} b_r$ must be real.

2. When n is odd,

$$\begin{aligned}\sum_{r=1}^{n} b_r &= -2 - \sum_{r=1}^{n}(-1)^r a_r - \sum_{r=1}^{n} a_r - \left[\sum_{r=1}^{n} a_r\right]\left[\sum_{j=1}^{n}(-1)^j a_j\right] \\ &= -2 - 2(a_2 + a_4 + \cdots + a_{n-1}) - (a_1 + a_2 + \cdots + a_n)(-a_1 + a_2 - \cdots - a_n)\end{aligned}$$

which shows that $\sum_{r=1}^{n} b_r$ must be real, for the same reason.