Canadian Math Olympiad 2013 — P1/5

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July 13, 2025

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Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

Solution

The answer is $P(x)=c\in\mathbb{R}$. One can easily check that this works. Assume for the sake of contradiction that $n:=\deg P>0$, then one can write $P(x)=ax^n+bx^{n-1}+O(x^{n-2})$ for some real $a\neq 0, b$. Substituting this defintion into the given equation yields

$$\begin{aligned}
&= (x+1)\left[a(x-1)^n+b(x-1)^{n-1}+O(x^{n-2})\right]-(x-1)\left[ax^n+bx^{n-1}+O(x^{n-2})\right]\\
&= \left[ax^{n+1}+bx^n+(b-a)x^{n-1}+O(x^{n-2})\right]-\left[ax^{n+1}+(b-a)x^n-bx^{n-1}+O(x^{n-2})\right]\\
&= ax^n+(2b-a)x^{n-1}+O(x^{n-2})
\end{aligned}$$

But that is a polynomial of degree n so we have to have $n=0$, contradiction. ■