Canadian Junior Math Olympiad 2025 — P1/5

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Suppose an infinite non-constant arithmetic progression of integers contains 1 in it. Prove that there are an infinite number of perfect cubes in this progression. (A *perfect cube* is an integer of the form k^3, where k is an integer. For example, -8, 0 and 1 are perfect cubes.)

Solution

Suppose the arithmetic progression has a fixed difference $d \in \mathbb{Z}_{\neq 0}$. Notice that if we can prove that any perfect cube $k^3 \in \mathbb{Z} \setminus \{-1, 0, 1\}$ exists in the sequence we automatically get that there are an infinite number of perfect cubes in the progression since there must exist some non-zero integer λ such that $1 + \lambda d = k^3$. Then this implies that $k^3 + (k^3\lambda)d = k^6$ is also part of the progression, but since $k^3 \neq k^{3n}$ for any integer $n \geq 2$ repeating this process will demonstrate that there are an infinite number of perfect cubes in the sequence.

Hence it suffices to show that there exists $\mu \in \mathbb{Z}_{\neq 0}$ and $m^3 \in \mathbb{Z} \setminus \{-1, 0, 1\}$ such that $1 + \mu d = m^3$ and so $\mu d = (m-1)(m^2 + m + 1)$. Now notice that we can let $m - 1 = d$ and $\mu = m^2 + m + 1$, since the discriminant of that quadratic $\Delta = 1 - 4 = -3 < 0$ shows us that $\mu \neq 0$. So this construction works for any d except $m = d + 1 \neq -1, 0, 1 \iff d \neq -2, -1, 0$. We can rule out the possibility of $d = 0$ since the progression is non-constant, $d = -1$ will make the progression simply run through all of the perfect cubes that are less than the initial value, and for $d = -2$ the number $(-3)^3 = -27 = 1 - 2(14)$ will occur in the sequence so it will contain an infinite number of perfect cubes by our arguement in the first paragraph. ■