Canadian Math Olymipad 2015 — P1/5

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Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$.

$$(n-1)^2 < f(n)f(f(n)) < n^2 + n$$

Solution

With $(n-1)^2 < f(n)f(f(n)) < n(n+1)$, we must have

$$\max\{f(n), f(f(n))\} > n - 1$$

This is because if we didn't have this condition then, we would have $f(n), f(f(n)) \leq n - 1$ and $f(n)f(f(n)) \leq (n-1)^2$ which gives a contradiction. However that means that,

$$\min\{f(n), f(f(n))\} < n + 1$$

because we have already established that one of those factors is at least n and we must have their product be $< n(n+1)$.

We can re-write our condition as

$$(n-1)^2 < \min\{f(n), f(f(n))\} \cdot \max\{f(n), f(f(n))\} < n(n+1)$$

and we can deduce both

$$\frac{(n-1)^2}{\max\{f(n), f(f(n))\}} < \frac{(n-1)^2}{n-1} = n - 1 < \min\{f(n), f(f(n))\} < n + 1$$

and (using $\min\{f(n), f(f(n))\} \leq n$)

$$n - 1 < \max\{f(n), f(f(n))\} < \frac{n(n+1)}{\min\{f(n), f(f(n))\}} \leq \frac{n(n+1)}{n} = n + 1$$

But this forces both $f(n), f(f(n)) = n$. So the only function that works is $f(n) = n$, for $n \in \mathbb{N}$ ■