Putnam 2003 — A1

Jonathan Kasongo

July 4, 2025

Putnam 2003 — A1

Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k$$

with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

Solution

We say that we have represented n *nicely* if we have written $n = a_1 + a_2 + \cdots + a_k$, with $a_1, a_2, ..., a_k$ satisfying the conditions given in the problem statement. We claim that for a fixed positive integer n, there are n ways to represent n nicely. We can actually show a clever way to prove this fact inductively.

The base case $n = 1$ holds trivially. Now assume the claim holds for some $n = m$. There are m ways to represent m nicely. Now in each of these representations take a_1 and replace it with $a_1 + 1$. Every $(a_r)_{r=1}^{k}$ is still either a_1 or $a_1 + 1$ so this is a way to nicely represent $m + 1$. The only representation that cannot be generated like this is

$$m + 1 = \underbrace{1 + 1 + \cdots + 1}_{m+1 \text{ repit(itions)}}$$

because this is the only representation, where you cannot take a different nicely represented number and replace a_1 with $a_1 + 1$, since that would imply that $a_1 + 1 = 1 \iff a_1 = 0$ which is impossible.

So for the number $m + 1$ we have shown that there are $m + 1$ ways to represent it nicely, and now since we have the truth of $n = 1$ and the truth of the $n = m$ statement implies the truth of the $n = m + 1$ statement, the desired result is true for all positive integers n. ■