British Math Olympiad Round 1 2008 — P5/6

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August 6, 2025

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Determine the sequences $a_0, a_1, a_2, \ldots$ which satisfy all of the following conditions:

- $a_{n+1} = 2a_n^2 - 1$ for every integer $n \geq 0$,
- a_0 is a rational number and
- $a_i = a_j$ for some i, j with $i \neq j$.

Solution

Notice that if $a_0 > 1$ then the sequence is strictly increasing because $a_{n+1} = 2a_n^2 - 1 > a_n \iff (2a_n + 1)(a_n - 1) > 0 \iff a_n > 1$ or $a_n < -\frac{1}{2}$. Since a_0 is rational, and the set of rational numbers is closed under multiplication and subtraction, we conclude that a_n is always rational. Write $a_n = \frac{p}{q}$ for integers $|q| \geq |p|$ and $\gcd(p, q) = 1$ and $q \neq 0$. If $|q| = |p|$ then $a_n = \pm 1$ and it can easily be checked that the only 2 sequences that work are $-1, 1, 1, 1, \ldots$ and $1, 1, 1, \ldots$. Now if $|q| > |p|$ then notice that $a_{n+1} = \frac{2p^2 - q^2}{q^2}$. Because p^2 and q^2 are coprime the fraction can only simplify if q^2 has a factor of 2 and $a_{n+1} = \frac{p^2 - q^2/2}{q^2/2}$. Now observe that if $|q| > 2$ then the denominators of each of the terms are strictly increasing since $q^2 > \frac{q^2}{2} > |q| \iff q^2 - 2|q| = |q|(|q| - 2) > 0$ which is clearly true when $|q| > 2$. But if the denominators are strictly increasing then there can never be 2 different numbers that are the same in the sequence.

Thus $|q| \leq 2$. That means that a_0 is one of $0, \pm\frac{1}{2}$. Now one can easily check each case and find that the only sequences here are $0, -1, 1, 1, 1, \ldots$ and $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$ and $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$. $\blacksquare$

Remark: I got this idea from the field of combinatorics actually, once I was confident that $a_0 = 0, \pm 1, \pm\frac{1}{2}$ and tested other values of a_0 I noticed that the denominator increased very quickly and that this could probably be used as a monovariant. I spent around 3 hours thinking about this problem.