Bangladesh Math Olympiad Higher Secondary 2024 — P6/10

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Find all polynomials $P(x)$ for which there exists a sequence $a_1, a_2, a_3, \ldots$ of real numbers such that

$$a_m + a_n = P(mn)$$

for any positive integers m, n.

Solution

Set $P(x) := \sum_{r=1}^{N} b_r x^r$ for some $b_r \in \mathbb{C}$. Put $m = n = 1$ to see that $P(1) = \sum_{r=1}^{N} b_r = 2a_1$.

Now put $n = 1$ to see that $a_m = P(m) - a_1 = \left(\sum_{r=1}^{N} b_r m^r\right) - a_1 = \sum_{r=1}^{N} b_r (m^r - 1/2)$.

So now we can re-write the equation given in the problem statement as

$$P(m) + P(n) - 2a_1 = P(mn)$$
$$\frac{1}{2}(P(m) + P(n) - P(mn)) = a_1$$

So now we can re-write a_m,

$$a_m = P(m) - \frac{1}{2}\left[P(mn) - P(m) - P(n)\right]$$
$$a_m = \frac{3}{2}P(m) + \frac{1}{2}P(n) - \frac{1}{2}P(mn)$$

Now substitute this new definition into the equation given in the problem statement,

$$P(mn) = \frac{3}{2}\left[P(m) + P(n)\right] + \frac{1}{2}\left[P(n) + P(m)\right] - P(mn)$$
$$2P(mn) = 2\left[P(m) + P(n)\right]$$

So the polynomial P obeys the equation $P(mn) = P(m) + P(n)$ for $m, n \in \mathbb{N}$. For $m = n$ that is $P\left(m^2\right) = 2P(m)$. But now recall that $\deg P(x) = N$ so we must have $2N = N$, but that forces $N = 0$. Hence $P(x) = c$ for some constant c. We note that $c \in \mathbb{R}$ because $P(mn) = c = a_m + a_n \in \mathbb{R}$.

Finally we conclude that any c works because there exists a sequence $\frac{1}{2}c = a_1 = a_2 = \cdots$ such that the desired condition is always fulfilled $\frac{1}{2}c + \frac{1}{2}c = P(mn) = c$ ■.

Well that was my solution, but I then came across this much nicer solution,

Alternative solution

Put $m = n$ to find that $2a_m = P\left(m^2\right)$. So now we can write

$$a_m + a_n = \frac{1}{2}\left(P(m^2) + P(n^2)\right) = P(mn)$$

Let $N := \deg P(x)$ and the equation we have above forces $2N = N$, so $N = 0$. You can then check that all constants $c \in \mathbb{R}$ work like in the other solution.