British Mathematical Olympiad Round 2 2022 — P2/4

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Find all functions f from the positive integers to the positive integers such that for all x, y we have:

$$2yf(f(x^2)+x) = f(x+1)f(2xy)$$

Solution

Let $P(x, y)$ denote the above assertion. $P(x, f(x+1))$ yields $2f(x+1)f(f(x^2)+x) = f(x+1)f(2xf(x+1))$ but since the image of f is the natural numbers we can cancel $f(x+1)$ to write $2f(f(x^2)+x) = f(2xf(x+1))$. Substitute this result into the given equation to find that $yf(2xf(x+1)) = f(x+1)f(2xy)$, call this result $(*)$. Plug $y = 1$ into $(*)$ to find that $f(2xf(x+1)) = f(x+1)f(2x)$ and so $f(x+1) = f(2xf(x+1))/f(2x)$. Now substitute this definition of $f(x+1)$ into $(*)$ to find that $y = f(2xy)/f(2x)$ and so $f(2xy) = yf(2x)$. Now if $x = 1$ then $f(2y) = yf(2)$. So we have found that f is a linear for even positive integer inputs. $P(2x, f(2x+1))$ gives $2f(f(4x^2)+2x) = f(4xf(2x+1))$. But since $f(2xy) = yf(2x)$ we can write $2f(f(4x^2)+2x) = 2f(2x+1)f(2x)$ that becomes $f(2xf(2x)+2x) = f(2x(f(2x)+1)) = f(2x+1)f(2x)$ and using $f(2y) = yf(2)$ we have $x(f(2x)+1)f(2) = f(2x+1)f(2x) = f(2x+1)xf(2)$. Finally cancel $xf(2)$ to find that $f(2x+1) = f(2x)+1 = xf(2)+1$. So we have shown that f is affine for all even positive integers, and all odd positive integers that are > 1. We can write $f(x) = ax+b$ where $x > 1$, then $f(2x) = 2ax+b = xf(2) = 2ax+2b$ and so $b = 0$. So $f(x) = ax$ for all positive integers $x > 1$. Plug this into the equation in the problem statement to get $2yf(ax^2+x) = (ax+a)2xya$ and so $a(ax^2+x) = (ax+a)xa \iff (ax+1) = ax+a \iff a = 1$ so we have found that $f(x) = x$ for $x > 1$. Finally $P(1, 1)$ gives $f(f(1)+1) = 2$, but since $f(1)+1 \geq 2 \iff f(1) \geq 1$ which is true since the image of f is the natural numbers, we can write $f(f(1)+1) = f(1)+1 = 2 \iff f(1) = 1$. So the final solution is $f(x) = x$ for all $x \in \mathbb{N}$.