New Zealand Math Olympiad Round 2 2023 — P2/5

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Let a, b, c be positive reals such that $a + b + c = abc$. Prove that at least one of a, b, c is greater than $\frac{17}{10}$.

Solution

Without loss of generality assume that $0 < c \le b \le a$. Now for the sake of contradiction assume that $a \le \frac{17}{10}$. By AM-GM we have $\frac{1}{3}(a + b + c) \ge (abc)^{1/3} \iff \frac{1}{27}(abc)^3 \ge abc$. But since we know that $abc \ge c^3$ we can deduce that $\frac{1}{27}(abc)^3 \ge abc \ge c^3$ so $\frac{1}{27}(ab)^3 \ge 1$ and so $(ab)^3 \ge 27 \iff ab \ge 3$. However since we assumed that $a \le \frac{17}{10}$, we have $\frac{17^2}{10^2} \ge ab \ge 3$, that means that $\frac{289}{100} \ge 3$, but since $\frac{289}{100} < 3$ we have a contradiction. This completes the proof. ■

Remark: Actually the tightest bound we can get is that one of a, b, c *must be greater than* $\sqrt{3} \approx 1.73205\ldots$ *which is where the number* 1.7 *comes from.*