Singapore Junior Math Olympiad 2025 — P3/5

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Jack and Jill play the following game: Jack throws 3 dice and Jill can select some of them, possibly none, and turn each of them to the opposite side. Jill wins if the sum of the values on the dice is a multiple of 4. Can Jill always win? (Note the game is played with standard dice where the sum of the numbers on opposite sides is 7.)

Solution

We call Jill turning a die to the opposite side a *flip*. Suppose Jack rolls the numbers a, b and c. Jill wins if, after a series of flips, $a + b + c \equiv 0 \pmod 4$. After one flip a number x is sent to $7 - x \equiv 3 - x$ working modulo 4. Let Σ denote $a + b + c$. Let $f(x)$ denote the operation of flipping the number x. Observe the following table (from here on working modulo 4 is assumed).

x	$f(x)$	$f(x) - x$
0	-1	-1
1	2	1
2	1	-1
3	0	1

So if we flip an even number we can decrease Σ by 1, and if we flip an odd number we can increase Σ by 1.

If we initially have $\Sigma \equiv 0 \pmod 4$ then Jill wins without having to do anything.

If we initially have $\Sigma \equiv 1 \pmod 4$ there is either 2 even numbers and 1 odd number, or there is 3 odd numbers, due to parity issues. Then if there is any even number among a, b, c Jill can flip it to decrease Σ by 1 and Jill will win. Otherwise there must be 3 odd numbers and we can flip all 3 of them to increase Σ by 3 and Jill wins.

If we initially have $\Sigma \equiv 2 \pmod 4$ then we either have 1 even number, or 2 even numbers, due to parity issues. If there is 1 even number we can flip the other 2 odd numbers to increase Σ by 2 and Jill wins. If there are 2 even numbers then we can flip those 2 even numbers to decrease Σ by 2 and Jill will win.

If we initially have $\Sigma \equiv 3 \pmod 4$ then there is at least one odd number among a, b, c and we can flip it to increase Σ by 1 and Jill will win.

So Jill always has a strategy to win. ■