# New Zealand Math Camp Selection Test 2016 — P2/9

Jonathan Kasongo

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We consider $5 \times 5$ tables containing a real number in each of the 25 cells. The same number may occur in different cells, but no row or column contains five equal numbers. Such a table is *balanced* if the number in the middle cell of every row and column is the average of the numbers in that row or column. A cell is called *small* if the number in that cell is strictly smaller than the number in the cell in the very middle of the table. What is the least number of small cells that a balanced table can have?

**Solution**

We will show that the answer is 3. We re-interpret the problem as follows. Let the very center cell contain the number 0. Then have every other cell contains an arbitrary positive or negative number. If a cell contains a negative number then it is small, so we want to maximise the number of positive numbers that occur in the grid. We justify this re-interpretation now.

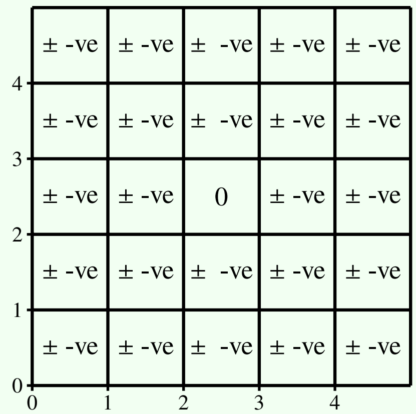
*Lemma: For any number in the middle cell of a row or column, we can construct a sequence of arbitrary positive or negative numbers to satisfy the 2 conditions.*

1. *The middle number must be the average of all the numbers in it's row or column.*
2. *We cannot have every number be the same in a row or column.*

*If the middle number is positive we show that we can have every cell in that row or column be positive. If the middle number is not positive we show that we must have at least one of the other cells in that row or column be negative, and that it is possible to have only one negative number in that row or column.*

*Proof:* We prove this lemma constructively. If the middle number is positive $x > 0$ then I can fill in the remaining 4 cells with the positive numbers $x, x, \alpha x, \beta x$ with $\alpha + \beta = 2, \alpha, \beta \in \mathbb{R}^+ \setminus \{1\}$in any order since $\frac{1}{5}(x + x + x + \alpha x + \beta x) = x$. We know that we don't have all of the numbers being equal since $\alpha, \beta \neq 1$. Now this sequence satisfies both criteria. Now we consider when the middle number is not positive $x \leq 0$. Suppose the other 4 cells contain the numbers $a_1, a_2, a_3, a_4$, then we must have their sum $S$ satisfying $\frac{1}{5}(S + x) = x \leq 0 \iff S = 4x \leq 0$. If $(a_r)_{r=1}^4 \geq 0$ then $S \geq 0$ and then we must have $a_1 = a_2 = a_3 = a_4 = 0$ but this is impossible since it violates condition 2. So we must have at least one of the numbers be negative, now we show that it is possible to have only one negative number. Fill in the remaining 4 cells in the necessary row or column with the numbers $c, c, c, 4x - 3c$ for some real $c > 0, c \neq x$, then indeed we have $\frac{1}{5}(x + c + c + c + (4x - 3c)) = x$ and $4x - 3c \neq c \iff x \neq c$ so we know that we definitely don't have all of the numbers in the row or column being equal. We can see that both condition 1 and 2 are satisfied. This completes the proof of the lemma.

By our construction we can see that the value of these arbitrary positive or negative numbers doesn't matter, since there are infinitely many selections of $\alpha, \beta, c$. Now this lemma allows us to re-interpret the problem: What is the minimum number of negative numbers in the grid such that the grid is balanced. The idea can be visualised using the figure below,




From our lemma we know that at least one of the cells in row 2 must be negative and at least one of the cells in column 2 must be negative. The 2 negative numbers can go in positions $(2, a)$ and $(b, 2)$, where $(i, j)$ denotes the cell in row $i$ and column $j$. We then must have a negative number in column $a$ and one in row $b$, since if we didn't then the grid wouldn't be balanced due to the lemma. By simply placing a negative number in cell $(b, a)$ we satisfy both those requirements in one cell. All the other cells can be positive by construction, so in total we have placed 3 negative numbers. All we need to do to finish the problem is show that there exists a construction with 2 negative numbers in a row/column and 3 positive numbers where the middle number is $x < 0$, since both row $b$ and column $a$ will have 2 negative numbers. For some let the other 4 cells be $7x, -x, -x, -x$ in some order, clearly all 5 numbers in the sequence are different since $x \neq 0$ and $\frac{1}{5}(x + 7x - x - x - x) = x$. This finishes the problem. ■