# Australian Math Olympiad 2021 — P5/8

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Find the solutions to

$$a^{2n} + b^{2n} = 1 \quad \text{and} \quad a^{2n+1} + b^{2n+1} = 1$$

for positive integer $n$ and real numbers $a, b$.

**Solution**

Firstly we can assume without loss of generality that $a \geq b$, since the equations have symmetry in $a, b$ and any solution $(a, b),\ a > b$ also has $(b, a)$ as a solution. Indeed by re-arranging the second equation and multiplying the second equation by $a$ then re-arranging we derive that $a^{2n+1} = 1 - b^{2n+1}$ and also $a^{2n+1} = a - ab^{2n}$ so $a - 1 = ab^{2n} - b^{2n+1} \iff a = b^{2n}(a - b) + 1$ and so $a \geq 1 \implies a^{2n} \geq 1$. But since $a^{2n}, b^{2n}$ are both non-negative numbers and $a^{2n} + b^{2n} = 1$ and $a \geq 1$ we have to have $a = 1, b = 0$. Now clearly we can see that $(a, b, n) = (1, 0, n)$ or $(0, 1, n)$ work for any positive integer $n$. ■