Australian Math Olympiad 2015 — P6/8

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Determine the number of distinct real solutions of the equation

$(x-1)(x-3)(x-5)\cdots(x-2015) = (x-2)(x-4)(x-6)\cdots(x-2014)$

Solution

Let $L(x)$ be the left hand side and $R(x)$ be the right hand side. Define $f(x) = L(x) - R(x)$ then we are interested in the number of distinct real roots of f. Since $\deg f = 1008$, $f(x)$ can't have more than 1008 distinct real roots. We will use the Intermediate Value Theorem to show that f has exactly 1008 distinct real solutions.

Lemma: For even integers $2 \le k \le 2014$, f has a root in the open interval $(k, k+1)$.

Proof: We split the proof into 2 cases, when $k \equiv 0 \pmod 4$ and when $k \equiv 2 \pmod 4$.

Case 1: $k \equiv 0 \pmod 4$. — Since k is even we know that $R(k) = 0$ when $2 \le k \le 2014$. Then since k is a multiple of 4, there are $k/2$ odd positive integers less than k, because the r^{th} positive odd integer less than k satisfies, $2r - 1 \le k \iff r \le (k+1)/2 \iff r \le \lfloor (k+1)/2 \rfloor = k/2$. Since $2 \mid k/2$, that means that an even number of brackets in the expression $(k-1)(k-3)(k-5)\cdots(k-2015)$ will be positive, and the remaining brackets will be negative. Since there are 1008 brackets, there will be an even number of brackets that are negative, meaning that $L(k)$ will be positive. So we have showed that $f(k) > 0$ is positive for $2 \le k \le 2014$. Now consider $f(k+1)$. Indeed $L(k+1) = 0$ since $k+1$ is an odd integer in the range $[1, 2015]$. Now since $k+1$ is one more than a multiple of 4, there are $k/2$ even positive integers less than $k+1$, because there are $k/2+1$ odd positive integers less than $k+1$ due to the reasoning we used last time, and therefore there must be $(k+1) - (k/2+1) = k/2$ even positive integers less than $k+1$. But since $2 \mid k/2$ that means that an even number of brackets in the expression $((k+1)-2)((k+1)-4)((k+1)-6)\cdots((k+1)-2014)$ will be positive. Now since there are 1007 brackets, we know that an odd number of brackets will be negative, therefore $R(k+1) < 0$ and $f(k+1) < 0$. Finally since f is continous and f exhibits a change in sign between inputs k and $k+1$ we can use the Intermediate Value Theorem to conclude that f has a root in the interval $(k, k+1)$.

Case 2: $k \equiv 2 \pmod 4$. — Indeed $R(k) = 0$ since k is an even positive integer ≤ 2014, and there are again $k/2$ odd positive integers less than k. But since k isn't a multiple of 4, $k/2$ will be odd. Hence an odd number of brackets in the expression $(k-1)(k-3)(k-5)\cdots(k-2015)$ will be positive, now since there are 1008 brackets an odd number of brackets will be negative and so $L(k) < 0$ and $f(k) < 0$. Now we know that there are $k/2+1$ odd integers that are less than $k+1$, but since $k/2+1$ is even, an even number of brackets in the expression $((k+1)-1)((k+1)-3)((k+1)-5)\cdots((k+1)-2015)$ will be positive and therefore an even number of brackets will be negative and $L(k+1) > 0$ so $f(k+1) > 0$. Finally since f is continous and f exhibits a change in sign between inputs k and $k+1$ we can use the Intermediate Value Theorem to conclude that f has a root in the interval $(k, k+1)$. This concludes the proof of the lemma.

Now since there are 1007 even integers in the range $[0, 2014]$, and $f(2015) = -(2013!!) < 0$, $f(2016) =$

$2015!! - 2014!! > 0$ implies that there is a root in the interval $(2015, 2016)$, so we have shown that f has exactly 1008 distinct roots. This finishes the problem. ■