A Solution to MA312

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Problem statement: Determine all pairs of integers (x, y) that satisfy $1 + x + x^2 + x^3 + x^4 = y^4$

Claim: The only solutions are $(0, -1), (0, 1), (-1, 1)$ and $(-1, -1)$.

Notice that if y satisfies the equation then so does $-y$ since $y^4 = (-y)^4$. Reduce modulo 2 to show that

$$4x + 1 \equiv 1 \equiv y \pmod{2}$$

So y must be odd. Put $y = 2m + 1$ for some integer m. Then,

$$1 + x + x^2 + x^3 + x^4 = 16m^4 + 32m^3 + 24m^2 + 8m + 1$$

After some routine algebra we get,

$$x(1 + x + x^2 + x^3) = m(16m^3 + 32m^2 + 24m + 8)$$

Now since $m \mid x$ and $x \mid m$, we can conclude that $x = \pm m$.

If $x = m$, then substituting and rearranging gives us,

$$x(15x^3 + 31x^2 + 23x + 7) = 0$$

Clearly $x = 0$ is a solution with $y = \pm 1$. Then we want to know if that cubic has anymore integer solutions. It is a consequence of Vieta's formulae, that if a polynomial has integer roots they must divide the constant term. After checking we find that $x = -1$ with $y = \pm 1$ is the only other solution here.

Now if $x = -m$, we end up with,

$$x(15x^3 - 33x^2 + 23x - 9) = 0$$

We now use the same strategy to find that the only integer root here is $x = 0$.

So finally we conclude that the only solutions are indeed $(x, y) = (0, \pm 1), (-1, \pm 1)$.