British Math Olympiad Round 1 2003 — P3/5

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Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.
Prove that Alice can always obtain a score at least as great as Barbara's.

Solution

Label the numbers on the cards as $a_1, a_2, ..., a_{2n}$ respectively.
Define $P_n := a_1 + a_3 + \cdots + a_{2n-1}$, $Q_n := a_2 + a_4 + \cdots + a_{2n}$.

Notice that Alice always has a strategy to make her score P_n or Q_n. She should start by removing the number a_1 or a_{2n} respectively, then Barbara will be forced to remove a number a_k where the index k is of the opposite parity to the index of the number Alice chose, since if Alice chose a_1 then Barbara must choose one of a_2, a_{2n} and if Alice chose a_{2n} then Barbara must choose one of a_1, a_{2n-1}. Now once Barbara has made her choice a_k, Alice can remove the number on the same side that Barbara made her choice. This number $a_{k\pm1}$ will have an index of the opposite parity to the index of the number Barbara just chose. Now if you just keep repeating this strategy then you can clearly remove all card numbers that have the same index parity, whilst Barbara will remove all card numbers that have an opposite index parity to the cards Alice chose, proving our claim.

Now since Alice can simply use one of the strategies above to get a score of $\max(P_n, Q_n) \geq \min(P_n, Q_n)$, Alice can always obtain a score at least as great as Barbara's. ■