POTD 1254

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If $a, b, c > 0$ and $a + b + c = 6$, show that

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq \frac{75}{4}.$$

Solution

In this solution any summation terms $\sum$ are assumed to be cyclic over a, b, c, that is for a function f, $\sum f(a, b, c) = f(a, b, c) + f(b, c, a) + f(c, a, b)$. The inequality is symmetric so we may assume, without loss of generality, that $a \geq b \geq c$. We start of by showing some useful inequalities. Firstly using QM-AM, we get that $\frac{1}{3}(a + b + c) = 2 \leq \sqrt{(a^2 + b^2 + c^2)/3}$ and so $12 \leq a^2 + b^2 + c^2$. Using the Re-arragement inequality on the sequences $a \geq b \geq c$ and $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$ we can show that $2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq 2\left(\frac{a}{a} + \frac{b}{b} + \frac{c}{c}\right) = 6$. We can now use these 2 results. Expand the LHS to get that $\sum a^2 + 2\sum \frac{a}{b} + \sum \frac{1}{a^2} \geq \frac{75}{4}$. By our 2 results we can see that it suffices to show that $\sum \frac{1}{a^2} \geq \frac{75}{4} - 12 - 6 = \frac{3}{4}$. By AM-GM we know that $a + b + c = 6 \geq 3(abc)^{1/3} \iff 2 \geq (abc)^{1/3} \iff 8 \geq abc \iff \frac{1}{8} \leq \frac{1}{abc}$. Now by the Re-arragement inequality on the sequence $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$ and itself coupled with the fact that $a + b + c = 6$ we know that $\sum \frac{1}{a^2} \geq \sum \frac{1}{ac} = \frac{1}{abc}(a + b + c) = \frac{6}{abc} \geq \frac{6}{8} = \frac{3}{4}$ as desired. This finishes the problem. ■