British Math Olympiad 2007 Round 1 — P3/6

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August 4, 2025

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The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

Solution

There are no conditions on where to place the numbers 7,8,9 so we can ignore them for now and just place them in the remaining 3 places, once we have placed the numbers 1 to 6. There are $\binom{9}{6}$ ways to select 6 spaces in our nine-digit number. The number 6 must go before the number 5 so that we can make the digits 1 to 6 be out of their natural order. Thus we can place that number 6 in any of the first $k \leq 5$ spaces such that this subsequence of 6 spaces looks like

$$1, 2, ..., k - 1, 6, k, ..., 5$$

clearly satisfying both of the given condiions. There are thus $5 \times \binom{9}{6}$ ways to arrange those 6 numbers and finally we can place the numbers 7,8,9 in the remaining 3 spaces in 3! ways. So the final answer is $5 \times \binom{9}{6} \times 3! = 2520$. ∎