Solving Problems from 101 Problems in Algebra — T Andreescu & Z Feng

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July 29, 2025

Recently I have been wanting to improve my algebra so I have been working through *101 Problems in Algebra* by *T Andreescu & Z Feng*.

1 Introductory Problems

Problem 1

Let a, b and c be real and positive parameters. Solve the equation

$$\sqrt{a+bx}+\sqrt{b+cx}+\sqrt{c+ax}=\sqrt{b-ax}+\sqrt{c-bx}+\sqrt{a-cx}$$

Solution 1

If x is a solution then that equation should hold for all triplets of parameters $a, b, c \in \mathbb{R}^+$. Consider the triplet $(a, b, c) = (1, 1, 1)$, we must have $3\sqrt{1+x} = 3\sqrt{1-x} \iff 1+x = 1-x \iff x = 0$. So $x = 0$ is the only candidate for a solution. Now we can easily plug in $x = 0$ into the above equation to see that it indeed works for all a, b, c. Hence $x = 0$ is the only such solution. ■

Remark: This one took me far longer than it should have!

Problem 2

Find the general term of the sequence defined by $x_0 = 3, x_1 = 4$ and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all $n \in \mathbb{N}$.

Solution 2

After a little exploration it is not hard to see that each term x_r seems to simply be $r + 3$. This guess is easily proved since $x_{n-1}^2 - nx_n = (n+2)^2 - n(n+3) = (n^2+4n+4) - (n^2+3n) = n+4 = x_{n+1}$. ■

Problem 3

Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that

(i) $-1 \le x_i \le 2$, for $i = 1, 2, \ldots, n$;

(ii) $x_1 + x_2 + \cdots + x_n = 19$;

(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.

Determine the minimum and maximum possible values of

$$x_1^3 + x_2^3 + \cdots + x_n^3.$$

Solution 3

Let a, b, c, d be the number of (-1)'s, 0's, 1's and 2's in the sequence, respectively. We have $-a + c + 2d = 19$ and $a + c + 4d = 99$ by condition (i) and (ii). We are interested in $x_1^3 + x_2^3 + \cdots + x_n^3 = -a + c + 8d = -a + (19 + a - 2d) + 8d = 19 + 6d$, using $c = 19 + a - 2d$. But since $2c + 6d = 118 \iff c = 59 - 3d$, from adding the first 2 equations and $c \geq 0 \iff 3d \leq 59$ but that means that $0 \leq d \leq 19$. So the maximum value is $19 + 6(19) = 133$ and the minimum is $19 + 6(0) = 19$. ■

Remark: I had to use a hint for this one.

Problem 4

The function f, defined by

$$f(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are non-zero real numbers, has the properties

$$f(19) = 19, \quad f(97) = 97, \quad f(f(x)) = x$$

for all x, except $-\frac{d}{c}$.

Find the range of f.

Solution 4

Clearly for all $x \neq -d/c$, x is in the image of f due to the third property. Using the third property we know that $x = \frac{af(x)+b}{cf(x)+d}$ we can re-arrange to find that $f(x) = \frac{b-dx}{cx-a} = \frac{ax+b}{cx+d}$ so there must exist $\lambda \in \mathbb{R}_{\neq 0}$ such that $cx - a = \lambda(cx + d)$ and $b - dx = \lambda(ax + b)$. From the second equation $b = b\lambda$ so $\lambda = 1$ since $b \neq 0$ therefore $a = -d$. Now using those 2 fixed points we know that

$$19a + b = 19^2c - 19a$$
$$97a + b = 97^2c - 97a$$

We can subtract these 2 equations to get rid of b and re-arrange to get $a = 58c$. Now $-d/c = 58c/c = 58$. So since 58 isn't in the domain of f and f is an involution, 58 isn't in the image of f either. So the range is $\mathbb{R} \setminus \{58\}$. ■

Remark: I had to use a hint for this one.

Problem 5

Prove that

$$\frac{(a-b)^2}{8a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{(a-b)^2}{8b}$$

for all $a \geq b > 0$.

Solution 5

The inequality is the same as

$$\frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{a}-\sqrt{b})^2}{8a} \leq \frac{(\sqrt{a}-\sqrt{b})^2}{2} \leq \frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{a}-\sqrt{b})^2}{8b}$$

That is

$$\frac{(\sqrt{a}+\sqrt{b})^2}{8a} \leq \frac{1}{2} \leq \frac{(\sqrt{a}+\sqrt{b})^2}{8b}$$
$$b(\sqrt{a}+\sqrt{b})^2 \leq 4ab \leq a(\sqrt{a}+\sqrt{b})^2$$

but since $b(\sqrt{a}+\sqrt{b})^2 \leq b(2\sqrt{a})^2 = 4ab$ and $4ab = a(2\sqrt{b})^2 \leq a(\sqrt{a}+\sqrt{b})^2$, the problem is finished. ■

Remark: I had to use a hint for this one.

Problem 6

Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say a and b, and then write the numbers $a + \frac{b}{2}$ and $b - \frac{a}{2}$ instead.
Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

Solution 6

Suppose we apply the operation on a and b. The sum of their squares is $a^2 + b^2$. Now after you apply the operation, the sum of squares becomes $(a^2 + ab + \frac{b^2}{4}) + (b^2 - ab + \frac{a^2}{4}) = \frac{5}{4}(a^2 + b^2)$. So the sum of squares all the numbers on the board will strictly increase after applying each operation. This means that after some number of operations we can never coincide with the original set because the sum of squares of the numbers on the board, before and after the operations, will be different. ■

Remark: I first came across this technique by watching Timothy Gowers solve invariance problems on YouTube!

Problem 7

The polynomial

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$

may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and a_is are constants. Find a_2.

Solution 7

The polynomial in terms of y is

$$1-(y-1)+(y-1)^2-(y-1)^3+\cdots+(y-1)^{16}-(y-1)^{17}$$

and we want the coefficient of y^2.

We can use binomial expansion to find that the coefficient is $a_2=\sum_{r=2}^{17}(-1)^r\binom{r}{2}(-1)^{r-2}=\sum_{r=2}^{17}\binom{r}{2}$. But by the Hockey stick identity this is just $\binom{18}{3}=816$. ■

Problem 8

Let a, b and c be distinct non-zero real numbers such that

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}$$

Prove that $|abc|=1$.

Solution 8

The equation is symmetric, so notice that over all cyclic rotations (a, b, c) we have $a-b=\frac{1}{c}-\frac{1}{b}=\frac{b-c}{bc}$ and we can say that $bc(a-b)=b-c$ holds over the cyclic rotations.

Now observe the following

$$\begin{aligned} b-c &= bc(a-b) \\ &= bc\left[ab(c-a)\right] \\ &= bc\left[ab(ac\left[b-c\right])\right] \end{aligned}$$

Now we can cancel $b-c\neq 0$ since we are told a, b, c are distinct giving $1=(abc)^2$ and so $|abc|=1$ follows as desired. ■

Problem 9

Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases}-1 & \text{if } x<-1 \\ 3x+2 & \text{if } -1\leq x\leq 0 \\ -2x+2 & \text{if } x>0.\end{cases}$$

Solution 9

It is natural to guess that the left hand side can be written of the form $a|x+1|+b|x|+cx+d$. Now if you plug in the results we have and solve the resulting simultaenous equation you end up with $\frac{3}{2}|x+1|-\frac{5}{2}|x|-x+\frac{1}{2}$. ■

Problem 10

Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

Solution 10

Let $a = 3^x$, $b = 2^x$. The equation becomes

$$\begin{aligned}\frac{b^3 + a^3}{ab^2 + a^2 b} &= \frac{7}{6}\\ 6(a^3 + b^3) &= 7ab(a + b)\\ 6(a^2 - ab + b^2) &= 7ab\\ 6a^2 - 13ab + 6b^2 &= 0\end{aligned}$$

We can use the quadratic formula to deduce that $a = \frac{3}{2}b \iff \frac{a}{b} = \left(\frac{3}{2}\right)^x = \frac{3}{2}$ and so $x = 1$ is the only solution. ∎

Problem 11

Find the least positive integer m such that

$$\binom{2n}{n}^{1/n} < m$$

holds for all positive integers n.

Solution 11

We claim that the answer is $m = 3$. Firstly to show that $m = 3$ works we have to show that

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n} < 3^n = (1 + 2)^n = \sum_{r=0}^{n} \binom{n}{r} 2^r$$

with the first equality being due to the Vandermonde Identity, and then since $\binom{n}{r} < \sum_{r=0}^{n} \binom{n}{r} = 2^r$ the claim is true.

Now we just have to show that $m = 2$ fails. Observe that $\binom{2}{1} = 2 < 2$ is a contradiction so indeed $m = 3$ is the smallest such m. ∎

Problem 12

Let a, b, c, d and e be positive integers such that

$$a + b + c + d + e = abcde$$

Find the maximum possible value of $\max\{a, b, c, d, e\}$

Solution 12

Suppose without loss of generality that $a \geq b \geq c \geq d \geq e$. We can see that

$$a = \frac{b+c+d+e}{bcde-1}$$

So for some small $\epsilon > 0$,

$$\begin{aligned} a + \epsilon &\geq \frac{b+c+d+e}{bcde} \\ &= \frac{1}{cde} + \frac{1}{bde} + \frac{1}{bce} + \frac{1}{bcd} \end{aligned}$$

So we want b, c, d, e to be small since we cannot have them all equal to 1 the next best permutation is $(2, 1, 1, 1)$ which achieves the maximum value of $a = 5$. ■

Problem 13

Evaluate

$$\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} + \cdots + \frac{2001}{1999!+2000!+2001!}$$

Solution 13

It is easier to solve the more general problem. Define

$$\begin{aligned} S_n &= \sum_{r=1}^{n} \frac{r+2}{r!+(r+1)!+(r+2)!} \\ &= \sum_{r=1}^{n} \frac{r+2}{r!\,[1+(r+1)+(r+1)(r+2)]} \\ &= \sum_{r=1}^{n} \frac{r+2}{r!(r+2)^2} \\ &= \sum_{r=1}^{n} \frac{1}{r!(r+2)} \end{aligned}$$

Now when you analyse the cases with small n, a sensible guess is that the required form is

$$\frac{\frac{1}{2}(n+2)!-1}{(n+2)!} = \frac{1}{2} - \frac{1}{(n+2)!}$$

We prove our guess with induction. The base case is true since $S_1 = \frac{1}{3} = \frac{1}{2} - \frac{1}{6}$. Now assume that the result is true for some $n = k$. Indeed

$$\begin{aligned}
S_{k+1} &= S_k + \frac{1}{(k+1)!(k+3)} \\
&= \frac{1}{2} - \frac{1}{(k+2)!} + \frac{1}{(k+1)!(k+3)} \\
&= \frac{1}{2} - \frac{1}{(k+1)!}\left[\frac{1}{k+2} - \frac{1}{k+3}\right] \\
&= \frac{1}{2} - \frac{1}{(k+1)!}\left[\frac{1}{(k+2)(k+3)}\right] \\
&= \frac{1}{2} - \frac{1}{(k+3)!}
\end{aligned}$$

So the result is true by induction. So the required answer is $S_{1999} = \frac{1}{2} - \frac{1}{2001!}$. ■