Canadian Junior Mathematical Olympiad 2023 — P1/5

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Let a and b be non-negative integers. Consider a sequence $s_1, s_2, s_3, \ldots$ such that $s_1 = a$, $s_2 = b$ and $s_{i+1} = |s_i - s_{i-1}|$ for $i \geq 2$. Prove that there is some i for which $s_i = 0$.

Solution

Suppose, for the sake of contradiction, that $s_i \neq 0$. Notice that $\max(s_i, s_{i-1}) = \min(s_i, s_{i-1}) + s_{i+1}$ and since none of the terms are ever 0, we have $s_{i+1} < \min(s_i, s_{i-1})$. That means that the sequence $(s_r)_{r=1}$ contains a decreasing non-contigous subsequence $s_{\sigma(1)} > s_{\sigma(2)} > \ldots$. This means there must eventually exist some i such that $s_{\sigma(i)} = 0$. ■