Australian Math Olympiad 2024 — P3/8

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Let $a_1, a_2, \ldots, a_n$ be positive real numbers, where $n \geq 2$. For each permutation $(b_1, b_2, \ldots, b_n)$ of $(a_1, a_2, \ldots, a_n)$, define its *score* to be

$$\frac{b_1^2}{b_2} + \frac{b_2^2}{b_3} + \cdots + \frac{b_{n-1}^2}{b_n}.$$

Show that there exist two permutations of $(a_1, a_2, \ldots, a_n)$ whose scores differ by at least $3|a_1 - a_n|$.

Solution

We claim that it is possible to say WLOG $a_1 \leq a_2 \leq \cdots \leq a_n$, this is because any permutation $(b_r)_{r=1}^n$ can still be obtained and $a_n - a_1$ will be the largest difference of any two terms in $(a_r)_{r=1}^n$ so if we show that the score is $\geq 3(a_n - a_1)$ we also solve the original problem.

Claim: For positive real numbers $x \geq y$ we have $\frac{x^2}{y} - \frac{y^2}{x} \geq 3(x - y)$.

Proof: We need

$$\frac{x^3 - y^3}{xy} = \frac{(x-y)(x^2 + xy + y^2)}{xy} \geq 3(x - y)$$

so it suffices to show that $x^2 + y^2 \geq 2xy$ which is true by AM-GM. □

Consider,

$$\begin{aligned}
&\left(\frac{a_n^2}{a_{n-1}} + \frac{a_{n-1}^2}{a_{n-2}} + \cdots + \frac{a_2^2}{a_1}\right) - \left(\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_{n-1}^2}{a_n}\right) \\
&= \left(\frac{a_n^2}{a_{n-1}} - \frac{a_{n-1}^2}{a_n}\right) + \left(\frac{a_{n-1}^2}{a_{n-2}} - \frac{a_{n-2}^2}{a_{n-1}}\right) + \cdots + \left(\frac{a_2^2}{a_1} - \frac{a_1^2}{a_2}\right) \\
&\geq 3\left[(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1)\right] \\
&= 3(a_n - a_1)
\end{aligned}$$

as desired. ■

Remark: The main challenge here is to guess the permutation that works, and once you've seen it the problem becomes routine.