British Math Olympiad Round 2 2025 — P1/4

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Prove that if n is a positive integer, then $\frac{1}{n}$ can be written as a finite sum of reciprocals of different triangular numbers.

Solution

Let $t_n = \frac{n(n+1)}{2}$ denote the n^{th} triangular number.

Claim: For any positive integer n, $\frac{1}{t_{2n}} + \frac{1}{t_{2n+1}} = \frac{1}{2}\frac{1}{t_n}$.

Proof: The left hand side reads

$$\frac{2}{2n(2n+1)} + \frac{2}{(2n+1)(2n+2)} = \frac{2}{2n+1}\left(\frac{1}{2n} + \frac{1}{2n+2}\right) = \frac{1}{2n+1}\left(\frac{2n+1}{n(n+1)}\right) = \frac{1}{n(n+1)}$$

as desired.

Now using our claim we have

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n+1} + \left(\frac{1}{t_{2n}} + \frac{1}{t_{2n+1}}\right)$$

By repeatedly substituting say k times, we can show that

$$\frac{1}{n} = \frac{1}{n+k} + \sum_{r=1}^{k}\left(\frac{1}{t_{2r}} + \frac{1}{t_{2r+1}}\right)$$

Now if we pick $k = t_n - n = \frac{n(n-1)}{2}$ we have completed the proof for all $n > 1$, since all those fractions are pairwise distinct since $t_n < t_{n+1}$. But obviously $n = 1$ can be written as a finite sum of reciprocals of different triangular numbers, since 1 is a triangular number. ■