STEP 3 2014 — P2

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(i) Show, by means of the substitution $u = \cosh x$, that

$$\int \frac{\sinh x}{\cosh 2x}\,\mathrm{d}x = \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1}\right| + C.$$

(ii) Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x}\,\mathrm{d}x.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1+u^4}\,\mathrm{d}u = \frac{\pi + 2\ln\left(\sqrt{2}+1\right)}{4\sqrt{2}}.$$

Solution

Parts (i) and (ii) will be omitted here since these problems are just straightforward u-subs. The real challenge here is part (iii), here was my solution. We will call the left hand side I.

After a lot of attempts I couldn't see a clear substitution or algebraic method that yielded the answer. But after studying the right hand side of that equality, I guessed that the integral is probably just some linear combination of the 2 previous integrals. This thought motivated the following claim.

Claim: For some choice of real p, q and $k \neq 0$,

$$I = \int_p^q \frac{a\sinh x + b\cosh x}{\cosh 2x}\mathrm{d}x$$

Proof: Observe that right hand side becomes

$$\begin{aligned} I &= \int_p^q \frac{a\sinh x + b\cosh x}{\cosh 2x}\mathrm{d}x \\ &= \int_p^q \frac{a(e^x - e^{-x}) + b(e^x + e^{-x})}{e^{2x} + e^{-2x}}\mathrm{d}x \\ &= \int_p^q \frac{a(e^{3x} - e^x) + b(e^{3x} + e^x)}{e^{4x} + 1}\mathrm{d}x \end{aligned}$$

Put $u = e^x$, $\mathrm{d}u = e^x\mathrm{d}x$.

$$I = \int_{e^p}^{e^q} \frac{(au^3 - au) + (bu^3 + bu)}{u^4 + 1} \frac{\mathrm{d}u}{u}$$

Now if we pick $a = -b$ we are left with

$$I = 2b \int_{e^p}^{e^q} \frac{1}{u^4 + 1} \mathrm{d}u$$

Now clearly if we use $b = \frac{1}{2}, a = \frac{-1}{2}, q = 0, p \to -\infty$ the right hand side becomes I.

Now the rest of the problem is just computation, it is left as an excercise to the reader (I'm too lazy to do it.)