STEP 2 2017/4 — A lesson in motivation

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The Schwarz inequality is

$$\left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x\right)^2 \leqslant \left(\int_a^b (\mathrm{f}(x))^2\ \mathrm{d}x\right)\left(\int_a^b (\mathrm{g}(x))^2\ \mathrm{d}x\right)$$

(i) By setting $\mathrm{f}(x) = 1$ in the inequality above, and choosing $\mathrm{g}(x), a$ and b suitably, show that for $t > 0$,

$$\frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \leqslant \frac{t}{2}$$

Comments:

This is a nicely crafted question on the integral form of the *Cauchy-Schwarz inequality*. If you look at the solution right away you may think that the problem isn't too challenging, but behind the short statement lies an unforgiving trap.

My first instinct was to notice the similarity of the left hand side to the formula for $\tanh(x)$, and perhaps use this or some variation of the hyperbolic tan as my $\mathrm{g}(x)$.

However if you try to work out the details of this approach yourself, you too will see that the hyperbolic tangent leads to nothing but a dead end. Perhaps the inequality was written that way on purpose.

After falling for the $\tanh(x)$ trap and wasting around an hour and a half I found the elegant intended (probably) solution.

Solution:

It suffices to show that,

$$\begin{aligned} 2(e^t - 1) &\leq t(e^t + 1) \\ e^t - 1 &\leq \frac{t}{2}(e^t + 1) \\ (e^t - 1)^2 &\leq \frac{t}{2}(e^{2t} - 1) \end{aligned}$$

Now pick $a = 0$, $b = t/2$ and $\mathrm{g}(x) = e^{2x}$. The Cauchy-Schwarz inequality then tells us that

$$\left(\int_0^{t/2} e^{2x}\ \mathrm{d}x\right)^2 \leq \frac{t}{2}\int_0^{t/2} e^{4x}\ \mathrm{d}x$$

giving $(e^t - 1)^2 \leq \frac{t}{2}(e^{2t} - 1)$ as desired. ■