Australian Math Olympiad 2017 — P5/8

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July 6, 2025

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Determine the number of positive integers n less than 1000000 for which the sum

$$\frac{1}{2\times\lfloor\sqrt{1}\rfloor+1}+\frac{1}{2\times\lfloor\sqrt{2}\rfloor+1}+\cdots+\frac{1}{2\times\lfloor\sqrt{n}\rfloor+1}$$

is an integer.

Solution

We claim that the answer is 999. Denote that sum in the problem statement by S_n. We first prove the following lemma,

Lemma: *S_n is an integer if and only if n is 1 less than a square.*

Proof: The quantity $1/(2\lfloor\sqrt{k}\rfloor+1)$ is exactly $1/(2a+1)$ for when $a^2 \leq k \leq (a+1)^2-1$, for positive integers k, a, because once k becomes $\geq (a+1)^2$ the value of $\lfloor\sqrt{k}\rfloor \geq a+1$. In order to ensure that S_n is an integer we have to have $2a+1$ copies of the fraction $1/(2\lfloor\sqrt{k}\rfloor+1)$. There are $2a+1$ integers in the range $[a^2, (a+1)^2-1]$, so for S_n to be an integer all of the values of k in that range $(a^2 \leq k \leq (a+1)^2-1)$ have to occur in the summation. But clearly this can only happen if n itself is 1 less than a square, since that last fraction must complete the collection of the final $2a+1$ copies of the fraction $1/(2\lfloor\sqrt{k}\rfloor+1)$.

So the problem reduces to counting the number of positive integer n such that n is 1 less than a square. n must be one of the numbers $\{2^2-1, 3^2-1, 4^2-1, \ldots, (10^3)^2-1\}$. There are 999 elements in this set, this closes the problem. ■