The Compactness Theorem

A Proof via Ultraproducts

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Notation: $\subset$ means "subset", $\subsetneq$ means "proper subset".

Outline

Compactness

From MATH 320 (or other analysis course), we have the following definition for the compactness:

Definition (Compactness from MATH 320)

A space X is called compact iff every open cover of it has a finite sub-cover.

Example

The interval $[0, 1]$ is compact.

Theorem (Compactness)

A set of first order sentences Σ has a model iff every finite subset $\Sigma_0 \subseteq \Sigma$ has a model.

What we want to prove: **FOL is compact**.

Filter

Definition (Filter)

A filter F on the set X is a subset of $\mathcal{P}(X)$, has the following properties:

- Non-trivial: $\emptyset \notin F$, $\emptyset \neq F$.
- Upward closure: $A \in F$, $A \subset B \implies B \in F$.
- Intersection closure: $A \in F$, $B \in F \implies A \cap B \in F$.

Note: some books define that $X \in F$, but I think it is redundant since we can immediately prove it for every non-empty filter from the second property. (maybe im wrong)
Note': intuitively thinking, a filter gives us some large subsets.

Ultrafilter

Definition (Ultrafilter/Maximal Filter)

On X,
A filter F is called maximal, if there is no filter F' such that $F \subsetneq F'$.
A filter F is called an ultrafilter if for all $A \subset X$, we have $A \in F$ or $X \backslash A \in F$.

Theorem

They are the same.

Proof.

We know $\emptyset \notin F$. Notice $A \cap (X \backslash A) = \emptyset$, so at most one of them is in F. $\square$

Warm-up: Filters on a Finite Set

Let $I = \{1, 2, 3\}$. Which of the followings are **filters** on I?
(You can raise hands / vote: A, B, C, D.)

Options

(A) $\mathcal{F}_A = \{\{1, 2, 3\}\}$
(B) $\mathcal{F}_B = \{X \subseteq I : 1 \in X\} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$
(C) $\mathcal{F}_C = \{X \subseteq I : |X| \geq 2\}$
(D) $\mathcal{F}_D = \{\emptyset, \{1, 2, 3\}\}$

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Answer: (A) and (B).

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Answer: (A) and (B).

- (A) is the trivial filter $\{\{1, 2, 3\}\}$.
- (B) is closed under intersection and supersets, very good.
- (C): $\{1, 2\}, \{2, 3\} \in \mathcal{F}_C$ but their intersection $\{2\}$ has size 1.
- (D) contains $\emptyset$, so it is against the definition.

Which Filters Are Ultrafilters?

We have two filters on $I = \{1, 2, 3\}$:

(A) $\mathcal{F}_A = \{\{1, 2, 3\}\}$

(B) $\mathcal{F}_B = \{X \subseteq I : 1 \in X\}$

Recall: $\mathcal{U}$ is an ultrafilter on I if for every $X \subseteq I$, exactly one of X and $I \setminus X$ is in $\mathcal{U}$.

Which one(s) are(is) ultrafilter(s)?

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Answer: Only (B) is an ultrafilter.
Reason:

- For $\mathcal{F}_B$: for any $X \subseteq I$, either $1 \in X$ (so $X \in \mathcal{F}_B$) or $1 \notin X$ (so $I \setminus X \in \mathcal{F}_B$).
- For $\mathcal{F}_A$: neither $\{1\}$ nor $\{2, 3\}$ is in $\mathcal{F}_A$. It's bad.

Existence Theorem of Filter

Theorem (Existence)

Let $G \subseteq \mathcal{P}(X)$ be a collection of subsets.
There exists a filter F on X such that $G \subset F$ if and only if G has the finite intersection property(FIP), that is, for all $X_1, X_2, ..., X_n \in G$, $\bigcap_{i=1}^{n} X_i \neq \emptyset$.

Proof.

The part of $(\Rightarrow)$ is obvious (closed under intersection, so it's still inside G, and $\emptyset \notin G$); for the part of $(\Leftarrow)$, trust me (we can directly construct a filter F, by extending G manually, then do the union). $\square$

Lemma

Lemma (Ultrafilter Lemma)

Every filter is contained by an ultrafilter. (In **ZFC***)*

Proof.

Consider the set of all filters containing a given filter F, this forms a partial order set with $\subset$; any chain in this partial ordered set has a union which is still a filter containing F (e.g. $A \subset B$, $A \cup B = B$ still filter), hence an upper bound. By Zorn there is a maximal filter, and a maximal filter is an ultrafilter extending F. □

Ultraproduct

Definition (Ultraproduct)

Let $\mathcal{M}_i$ be a family of $\mathcal{L}$-structures $\{M_i \mid i \in I\}$ and $\mathcal{U}$ be an ultrafilter on I. Let $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \sim_{\mathcal{U}}$, where

$$s \sim_{\mathcal{U}} t \leftrightarrow \{i \in I \mid s(i) = t(i)\} \in \mathcal{U}.$$

Predicate: $\mathcal{M} \models P(\bar{t}_1, ..., \bar{t}_n) \leftrightarrow \{i \mid \mathcal{M}_i \models P(\bar{t}_1(i), ..., \bar{t}_n(i))\} \in \mathcal{U}$.
Function: $\mathcal{M} \models f(\bar{x}_1, ..., \bar{x}_n) = \bar{y} \leftrightarrow \{i \mid \mathcal{M}_i \models f(\bar{x}_1(i), ..., \bar{x}_n(i)) = y_i\} \in \mathcal{U}$.
Constant: $\mathcal{M} \models c = \bar{b} \leftrightarrow \{i \mid \mathcal{M}_i \models c = b(i)\} \in \mathcal{U}$.
Note: bar means equivalent class in $\sim_{\mathcal{U}}$.
Note': Intuitively, filters give us BIG subsets. Elements in the ultrafilter $\mathcal{U}$ contain almost all elemnts in I. This $\sim_{\mathcal{U}}$ means: suppose we have s and t two arrays (from ΠM_i), then we ask $\mathcal{U}$, are they the same in "almost all" i indices? If this set of i is in $\mathcal{U}$ then Yes; otherwise No.

Theorem (Exercise)

The things above are well-defiend; hence $\mathcal{M}$ is a structure.

Łoś's Theorem

A.K.A. the fundamental theorem of ultraproducts. ("Ł" sounds like "w" not "l")

Theorem (Łoś)

For any formula $\varphi(x_1, \ldots, x_n)$ *and elements* $s_1, \ldots, s_n \in \prod_{i \in I} \mathcal{M}_i$ *we have*

$$\mathcal{M} \models \varphi(\bar{s_1}, \ldots, \bar{s_n}) \leftrightarrow \{i \in I : \mathcal{M}_i \models \varphi(s_1(i), \ldots, s_n(i))\} \in \mathcal{U}.$$

Proof part 1.

We manually do induction on φ.
If φ is atomic, it is trivial (by definition of ultraproduct). □

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Proof part 2.

If $\varphi = \neg\psi$, then

$$\begin{aligned} \mathcal{M} \models \neg\psi &\iff \mathcal{M} \not\models \psi \\ &\iff \{i : \mathcal{M}_i \models \psi\} \notin \mathcal{U} \quad \text{(by I.H.)} \\ &\iff \{i : \mathcal{M}_i \not\models \psi\} \in \mathcal{U} \quad \text{(property of ultrafilter)} \\ &\iff \{i : \mathcal{M}_i \models \neg\psi\} \in \mathcal{U} \end{aligned}$$

□

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Proof part 3.

If $\varphi = \psi \wedge \theta$, then

$$\begin{aligned} \mathcal{M} \models \psi \wedge \theta &\iff \mathcal{M} \models \psi \text{ and } \mathcal{M} \models \theta \\ &\iff \{i : \mathcal{M}_i \models \psi\} \in \mathcal{U} \text{ and } \{i : \mathcal{M}_i \models \theta\} \in \mathcal{U} \quad \text{(I.H.)} \\ &\iff \{i : \mathcal{M}_i \models \psi \wedge \theta\} \in \mathcal{U} \quad \text{(intersection closure)} \end{aligned}$$

□

Proof part 4.

If $\varphi(x_1, \ldots, x_n) = \exists z\, \psi(z, x_1, \ldots, x_n)$ and denote:

$$A := \{i \in I : \mathcal{M}_i \models \exists z\, \psi(z, s_1(i), \ldots, s_n(i))\}.$$

We want $\mathcal{M} \models \exists z \psi(z, \bar{s}_1, ..., \bar{s}_n) \leftrightarrow A \in \mathcal{U}$.

($\Rightarrow$): If $\mathcal{M} \models \exists z\, \psi(z, \bar{s}_1, \ldots, \bar{s}_n)$, choose $\bar{t} \in \mathcal{M}$ with $\mathcal{M} \models \psi(\bar{t}, \bar{s}_1, \ldots, \bar{s}_n)$. Pick a representative $t \in \prod_{i \in I} \mathcal{M}_i$ of $\bar{t}$. By I.H., set $B := \{i : \mathcal{M}_i \models \psi(t(i), s_1(i), \ldots, s_n(i))\} \in \mathcal{U}$. Notice $B \subset A$, so $A \in \mathcal{U}$ (by upward closure).

($\Leftarrow$): say $A \in \mathcal{U}$. For each $i \in A$, choose $a_i \in \mathcal{M}_i$ with $\mathcal{M}_i \models \psi(a_i, s_1(i), \ldots, s_n(i))$, and define $t(i) = a_i$ for $i \in A$ manually. Let $\bar{t}$ be the class of t. Then set $C := \{i : \mathcal{M}_i \models \psi(t(i), s_1(i), \ldots, s_n(i))\} \supset A \in \mathcal{U}$, so $C \in \mathcal{U}$, and by I.H. $\mathcal{M} \models \psi(\bar{t}, \bar{s}_1, \ldots, \bar{s}_n)$, hence $\mathcal{M} \models \exists z\, \psi(z, \bar{s}_1, \ldots, \bar{s}_n)$. $\square$

Proof

Theorem

A set of first-order sentences Σ has a model if and only if every finite subset $\Sigma_0 \subset \Sigma$ has a model.

We gonna do from right to left but all steps are iff so we are fine:
Assume every finite subset of Σ has a model. We construct a model for Σ.
WLOG let's say Σ is a theory.

1. Let I be the set of all finite subsets of Σ, call each finite subset $i = \Sigma_i$.

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4. By Existence theorem, there is a filter F such that $\{S_\sigma\} \subset F$.
5. By Ultrafilter Lemma, there is an ultrafilter $\mathcal{U}$ on I such that $\{S_\sigma\} \subset F \subset \mathcal{U}$..

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6. Define $\mathcal{M}^* = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$.

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5. By Ultrafilter Lemma, there is an ultrafilter $\mathcal{U}$ on I such that $\{S_\sigma\} \subset F \subset \mathcal{U}$..
6. Define $\mathcal{M}^* = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$.
7. For every $\sigma \in \Sigma$, because $\{i : \mathcal{M}_i \models \sigma\} \supset S_\sigma \in \mathcal{U}$, by Łoś, we have $\mathcal{M}^* \models \sigma$.

□

1. Propositional logic is compact.
2. Second order logic is NOT compact.
3. ...

Evgeny Zolin
Advanced Course in Classical Logic
YouTube.
[https://www.youtube.com/watch?v=COmEj6hCO08t=1187s]

OperatorP.
Lecture Notes: Set Theory Foundations.
Banana Space.
[https://www.bananaspace.org/]

Todd Trimble.
Compactness theorem.
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[https://ncatlab.org/nlab/history/compactness+theorem]

Minghui OuYang, Longke Tang
Chatting with my friends.

Thank You

Good luck with your finals! :3