The Compactness Theorem via Ultraproducts

Zehan Xie

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Abstract

This report is going to give a proof of compactness theorem for the first-order logic via ultraproducts. We firstly introduce filters and ultrafilters; secondly prove an existence theorem and the ultrafilter lemma; thirdly state and prove Łoś's theorem, and finally prove the compactness theorem.

Contents

1 Introduction

Notations: To avoid confusion, I will use $\subseteq$ means subset, and $\subset$ means proper subset, and A^c means the complement for a set. Also, $0 \notin \mathbb{N}$.

Definition 1.1 (Satisfiability)**.** Let Σ be a set of $\mathcal{L}$-sentences.

- A structure $\mathcal{M}$ is a **model** of Σ, written as $\mathcal{M} \models \Sigma$, iff $\mathcal{M} \models \sigma$ for every $\sigma \in \Sigma$.
- And Σ is **satisfiable** iff it has a model.

Theorem 1.2 (Compactness theorem)**.** *Let Σ be a set of $\mathcal{L}$-sentences. Every finite subset of Σ is satisfiable, if and only if Σ is satisfiable. Equivalently, first-order logic is compact.* *[3]*

We will prove Theorem 1.2 by ultraproducts. The key tools are the existence theorem, the ultrafilter lemma, and the Łoś's theorem (the fundamental theorem of ultraproducts).

We are going to use I be an non-empty set (we use this notation is for the future notation of **index set** for the ultraproduct) for the base of filters.

Definition 1.3 (Filter)**.** A **filter** F on I is a non-empty collection $F \subseteq \mathcal{P}(I)$ satisfying:

(F1) $\emptyset \notin F, I \in F$.

(F2) If $A, B \in F$, then $A \cap B \in F$ (closed under intersection).

(F3) If $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$ (upward closure).

This definition can be found in Thomas Jech's Set Theory [8].

Intuitively, constructing a filter F will give us a collection of large subsets of I that is closed under taking supersets and intersections.

Example 1.4. The smallest filter is just $\{I\}$.

Example 1.5. Let I be any set, and let $i \in I$ be fixed. The collection

$$F_i = \{A \subseteq I \mid i \in A\}$$

is a filter on I.

Definition 1.6 (Maximal Filter)**.** A filter F on I is a **maximal** filter iff there is no proper filter F' such that $F \subset F'$.

Definition 1.7 (Ultrafilter)**.** A filter U on I is called an **ultrafilter** iff for every set $A \subseteq I$, we have either $A \in U$ or $A^c \in U$.

Theorem 1.8. *Every maximal filter is an ultrafilter.*

Proof. ($\Rightarrow$) Let F be a maximal filter, and assume it is not ultra. Then there exists an $A \subseteq I$, none of A and A^c is inside F. But this is absurd because it against the definition of maximal.

($\Leftarrow$) Let F be an ultrafiler. Assume it is not maximal, then there exists filter G such that $F \subset G$. Pick $A \in G \backslash F$, then $A \notin F$, since F is ultra, we must have $A^c \in F \subset G$. Thus by intersection closure, $A \cap A^c \in G$, but this is absurd. $\square$

This proof is by myself.

2 Existence Theorem and Ultrafilter Lemma

2.1 Existence Theorem of Filter

Definition 2.1 (Finite Intersection Property)**.** Let I be an non-empty set and let $\mathcal{A} \subseteq \mathcal{P}(I)$ be a family of subsets of I with the **finite intersection property**, that is, for every finite subset $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ we have $A_1 \cap \cdots \cap A_n \neq \emptyset$. If this holds, then we say $\mathcal{A}$ has the finite intersection property.

Theorem 2.2 (Filter existence theorem)**.** *Given $F \in \mathcal{P}(I)$, there exists a filter $\mathcal{F}$ containing F if and only if F has the finite intersection property.*

Proof. ($\Rightarrow$) This is obvious because intersection closure and $\emptyset$ is not inside a filter.

($\Leftarrow$) Suppose F has the finite intersection property. Set $F_0 = F$, and for $n \in \mathbb{N}$ define:

$$F_{2n+1} = \left\{ \bigcap_{k<m} A_k \;\middle|\; m \in \mathbb{N},\ A_k \in F_{2n} \text{ for all } k < m \right\},$$

$$F_{2n+2} = \{ B \subseteq X \mid \exists A \in F_{2n+1}\ (A \subseteq B) \}.$$

Finally, define

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} F_n.$$

□

It is a filter, because every F_i has the finite intersection property, and thus $\emptyset \notin \mathcal{F}$. This constructive proof is by OperatorP [2].

2.2 Ultrafilter Lemma

Theorem 2.3 (Ultrafilter lemma)**.** *Let F be a filter on I. Then there exists an ultrafilter U on I such that $F \subseteq U$.*

Proof. Consider the collection $\mathcal{P}$ the set of all proper filters on I that contain F, with order $\subseteq$. Notice $P_{\subseteq}$ is a partial ordered set. For each chain, by the upward closure property, the union of filters is still a filter, hence the upper bound exists. By Zorn's lemma, P has an maximal element, which is also the ultrafilter we want. □

The existence theorem and ultrafilter can be found in Banana Space [2].

3 Ultraproducts

We now define the important object: Ultraproduct.

3.1 Direct products and the ultraproduct equivalence relation

Let $\{\mathcal{M}_i \mid i \in I\}$ be a family of $\mathcal{L}$-structures.

Definition 3.1 (Direct product)**.** The **direct product** $\prod_{i\in I} \mathcal{M}_i$ is:

$$\prod_{i\in I} \mathcal{M}_i = \{s \mid s(i) \in \mathcal{M}_i \ \forall i \in I\},$$

To form an **ultraproduct** we then quotient the direct product by an equivalence relation given by an ultrafilter.

Definition 3.2 (Ultraproduct equivalence relation)**.** Let U be an ultrafilter on I and let $s, t \in \prod_{i\in I} M_i$. Define

$$s \sim_U t \leftrightarrow \{i \in I \mid s(i) = t(i)\} \in U.$$

We write $[s]$ for the equivalence class of s under $\sim_U$.

It is an equivalence relation, I will shortly show this: $s \sim_U s$ is because $I \in U$; symmetry is trivial; For transitivity: $a \sim_U b \sim_U c \implies a \sim_U$ is because $\{i \in I \mid a(i) = b(i)\} \cap \{i \in I \mid b(i) = c(i)\} \subseteq \{i \in I \mid a(i) = c(i)\} \in U$ by upward closure.

3.2 Definition of the ultraproduct

Definition 3.3 (Ultraproduct)**.** Let $\{\mathcal{M}_i \mid i \in I\}$ be a family of $\mathcal{L}$-structures, and let U be an ultrafilter on I. The **ultraproduct** of the $\mathcal{M}_i$ with respect to U is the

$\mathcal{L}$-structure

$$\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \sim_U$$

defined as follows:

- The domain of $\mathcal{M}$ is the direct product quotients the equivalence relation

$$\mathcal{M} = \Big(\prod_{i \in I} M_i \Big) / \sim_U \ .$$

- For n-ary function symbol f, define

$$\mathcal{M} \models f([x_1], ..., [x_n]) = [y] \leftrightarrow \{ i \mid \mathcal{M}_i \models f(x_1(i), ..., x_n(i)) = y_i \} \in \mathcal{U}.$$

- For n-ary relation symbol P, define

$$\mathcal{M} \models P\big([s_1], \ldots, [s_n]\big) \leftrightarrow \{ i \in I : \mathcal{M}_i \models P\big(s_1(i), \ldots, s_n(i)\big) \} \in U.$$

- For the constant symbol c, define

$$c^{\mathcal{M}} = [s_c], \quad \text{where } s_c(i) = c^{\mathcal{M}_i}.$$

One may verify that the interpretation is well-defined. The definition can be found in Zolin's Youtube video series: Advanced Course in the Classical Logic, episode 8, chapter 4 [1].

By chatting with my friends about the definition of ultraproduct [4], I had the intuition of how the ultrafilter works here. Also, they provided measure theory views of an ultraproduct to me, but unfortunately I could not fully understand.

4 Łoś's Theorem

The ultraproduct is a known good tool for many things, more applications can be found in [5, 7]! Now we shall introduce the Łoś Theorem.

Theorem 4.1 (Łoś's theorem)**.** *Let* $\{\mathcal{M}_i \mid i \in I\}$ *be a family of* $\mathcal{L}$*-structures and* U *be an ultrafilter on the set* I*. Let*

$$\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \sim_U$$

be the ultraproduct. For every $\mathcal{L}$*-formula* $\varphi(x_1, \ldots, x_n)$ *and elements* $[s_1], ..., [s_n] \in \mathcal{M}$*, we have*

$$\mathcal{M} \models \varphi([s_1], \ldots, [s_n]) \quad \leftrightarrow \quad \big\{ i \in I : \mathcal{M}_i \models \varphi\big(s_1(i), \ldots, s_n(i)\big) \big\} \in U.$$

Proof. The proof is by induction on the construction of φ.

Atomic formulas: For atomic formulas like $P(t_1, \ldots, t_m)$, where t_j are terms, the theorem holds directly from the definition of the ultraproduct.

Boolean operators: Assume the statement holds for formulas φ and ψ.

- For $\neg\varphi$, we have

  $$\mathcal{M} \models \neg\varphi([s_1], \ldots, [s_n]) \ \leftrightarrow \ \mathcal{M} \not\models \varphi([s_1], \ldots, [s_n]).$$

  By the induction hypothesis this just means

  $$\{i \mid \mathcal{M}_i \models \varphi(s_1(i), \ldots, s_n(i))\} \notin U,$$

  Notice that U is an ultrafilter, if a set is not in U, then the complement of it must in U

$$\{i : \mathcal{M}_i \models \neg\varphi(s_1(i), \ldots, s_n(i))\} \in U.$$

- For $\varphi \wedge \psi$, we have

$$\mathcal{M} \models \varphi\wedge\psi([s_1], \ldots, [s_n]) \;\leftrightarrow\; \mathcal{M} \models \varphi([s_1], \ldots, [s_n]) \text{ and } \mathcal{M} \models \psi([s_1], \ldots, [s_n]).$$

  By the induction hypothesis, this is equivalent to

$$A \in U \text{ and } B \in U,$$

  where for convenience I denote

$$A = \{i \mid \mathcal{M}_i \models \varphi(s_1(i), \ldots, s_n(i))\}, \quad B = \{i \mid \mathcal{M}_i \models \psi(s_1(i), \ldots, s_n(i))\}.$$

  Since U is a filter, by the intersection closure property $A \cap B \in U$, we have

$$\{i : \mathcal{M}_i \models \varphi \wedge \psi(s_1(i), \ldots, s_n(i))\} \in U.$$

  The cases of $\varphi \vee \psi$ and $\varphi \to \psi$ can be written as $\neg(\neg\varphi \wedge \neg\psi)$ and $\neg\varphi \vee \psi$ separately. So we automatically have Łoś's Theorem holds for them.

Existential quantifier: Let $\varphi(x, \vec{y})$ be a formula, we denote tuples or vectors $\vec{y} = (y_1, \ldots, y_n)$ and $\vec{s} = (s_1, \ldots, s_n)$.

($\Rightarrow$) Suppose $\mathcal{M} \models \exists x\, \varphi(x, [\vec{s}])$. Then we must have some $[t] \in M$ such that $\mathcal{M} \models \varphi([t], [\vec{s}])$. By the induction hypothesis, we have

$$S = \{i \in I : \mathcal{M}_i \models \varphi(t(i), \vec{s}(i))\} \in U.$$

then notice we can see there is one, so it exists

$$\{i : \mathcal{M}_i \models \exists x\, \varphi(x, \vec{s}(i))\} \supseteq S \in U,$$

by upward closure.

($\Leftarrow$) This is similar. Say

$$A = \{i \in I : \mathcal{M}_i \models \exists x\, \varphi(x, \vec{s}(i))\} \in U.$$

For each $i \in A$ we choose some $a_i \in M_i$ which has $\mathcal{M}_i \models \varphi(a_i, \vec{s}(i))$(by the way, for $i \notin A$, we can just choose an arbitrary $a_i \in M_i$, it does not matter). Let $t \in \prod_{i \in I} M_i$ be the sequence $t(i) = a_i$, and consider $[t] \in \mathcal{M}$, by the induction hypothesis applied to φ, we have

$$\{i : \mathcal{M}_i \models \varphi(t(i), \vec{s}(i))\} \supseteq A \in U,$$

so $\mathcal{M} \models \varphi([t], [\vec{s}])$, and we can see there is one, so there exists one x such that $\mathcal{M} \models \exists x\, \varphi(x, [\vec{s}])$.

The universal quantifier can be written as $\forall x\, \psi \equiv \neg\exists x\, \neg\psi$. This finishes the induction and the proof of Łoś's theorem. $\square$

The proof is mainly from Zolin's YouTube video [1] but the notations were different. You can also find this theorem in Banana Space by the user "OperatorP", although they said it was boring, so he left the proof to the readers [2].

5 Proof of the Compactness Theorem

Now we can use Łoś's theorem to prove the compactness of first-order logic.

Theorem 5.1 (Compactness). *Let Σ be a set of $\mathcal{L}$-sentences. Every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable iff Σ is satisfiable.*

Proof. ($\Leftarrow$) This is trivial.

($\Rightarrow$)Let I be the set of all finite subsets of Σ,

$$I = \{\, i \subseteq \Sigma : i \text{ is finite}\,\}.$$

Then for each $i \in I$ choose a $\mathcal{M}_i$ such that $\mathcal{M}_i \models i$.

Then for each sentence $\sigma \in \Sigma$ we can define

$$X_\sigma = \{\, i \in I : \mathcal{M}_i \models \sigma \,\} \subseteq I.$$

The collection $\{X_\sigma \mid \sigma \in \Sigma\}$ has the finite intersection property: let $\sigma_1, \ldots, \sigma_n \in \Sigma$ be arbitrary, and call

$$i_0 = \{\sigma_1, \ldots, \sigma_n\} \in I.$$

One may verify that $i_0 \in X_{\sigma_1} \cap ... \cap X_{\sigma_n}$.

By the Existence Theorem, there exists an filter F such that $\{X_\sigma : \sigma \in \Sigma\} \subseteq F$. By the Ultrafilter Lemma, there exists an ultrafiilter U such that $F \subseteq U$.

Now we can construct the ultraproduct

$$\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \sim_U .$$

Then by Łoś's theorem (Theorem 4.1), for each $\sigma \in \Sigma$ we have

$$\mathcal{M} \models \sigma \leftrightarrow \{\, i \in I : \mathcal{M}_i \models \sigma \,\} \in U.$$

Observe that the set on the right side is just X_σ, and $X_\sigma \in U$ by construction.

Hence $\mathcal{M} \models \sigma$ for every $\sigma \in \Sigma$, so $\mathcal{M}$ is a model of Σ. The proof is done. $\square$

References

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(He can play carillon! ← a very cool instrument :)

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