Linear Algebra HW1

#1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} .25 & .25 & .25 & .25 \\ .25 & .25 & -.25 & -.25 \\ .25 & -.25 & .25 & -.25 \\ .25 & -.25 & -.25 & .25 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \\ 1 \\ -1.5 \end{bmatrix}$$

Hence, $x = 0$, $y = 2.5$, $z = 1$, $w = -1.5$

#2

1. since $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 4 & 4 \\ -2 & -3-\lambda & -6 \\ 1 & 3 & 6-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -3-\lambda & -6 \\ 3 & 6-\lambda \end{vmatrix} - 4\begin{vmatrix} -2 & -6 \\ 1 & 6-\lambda \end{vmatrix} + 4\begin{vmatrix} -2 & -3-\lambda \\ 1 & 3 \end{vmatrix}$$

$$= (4-\lambda)\cdot\lambda\cdot(\lambda-3) - 8(\lambda-3) + 4(\lambda-3)$$

$$= (\lambda-3)(4\lambda-\lambda^2-4)$$

$$= -(\lambda-3)(\lambda-2)^2$$

$\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 2$

2. when $\lambda = 3$

$(A - \lambda I)\cdot v_1 = 0$

$$\begin{bmatrix} 1 & 4 & 4 \\ -2 & -8 & -6 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{cases} v_{12} + v_{13} = 0 \\ v_{11} = 0 \end{cases}$ Therefore, $v_1 = (0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^T$

similarly, when $\lambda = 2$, $v_2 = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$

3.

a) Known that $X = a \cdot V_1 + b V_2 + c V_3$

$$AX = A(a \cdot V_1 + b \cdot V_2 + c V_3)$$
$$= a\lambda_1 V_1 + b\lambda_2 V_2 + c\lambda_3 V_3$$

similarly. $A^2 X = a\lambda_1^2 V_1 + b\lambda_2^2 V_2 + c\lambda_3^2 V_3$

$$A^{35} X = a\lambda_1^{35} V_1 + b\lambda_2^{35} V_2 + c\lambda_3^{35} V_3$$

b) $\lim_{k\to\infty} A^k X = \lim_{k\to\infty} (a\lambda_1^k V_1 + b\lambda_2^k V_2 + c\lambda_3^k V_3)$

when $\lambda_1 = 1$. $|\lambda_2| < 1$, $|\lambda_3| < 1$,

$\lim_{k\to\infty} (a\lambda_1^k V_1 + b\lambda_2^k V_2 + c\lambda_3^k V_3)$

$$= a \cdot V_1 + 0 \cdot V_2 + 0 \cdot V_3 = a \cdot V_1$$

4.

a) Due to eigen value decomposition

$$A = Q \Lambda Q^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ -4 & -4 & 1 \end{bmatrix}$$

where $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $\Lambda = diag(-1, 0, 1)$

b) $A^{20} = Q\Lambda^{20} Q^{-1} = Q \cdot diag(1, 0, 1) \cdot Q^{-1}$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$