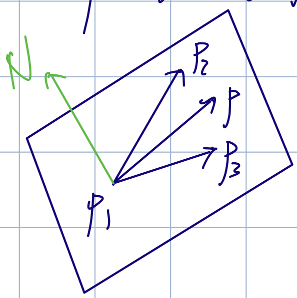
Lecture 3 plane

3 points to form a plane

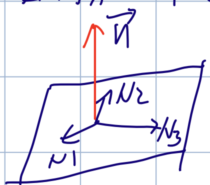


(1) $\det(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P}) = 0$ $\quad \det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(2) $N = \overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}$ 解 $\overrightarrow{P_1P} \cdot \vec{N} = 0$

$Ax + By + Cz = D$ (A, B, C) 即为 normal vector $\vec{N}$

若 $Ax = b$ 中 $\det(A) = 0$，则 $\vec{N_1}, \vec{N_2}, \vec{N_3}$ are coplanar



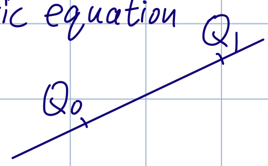
找出一个特解 x_p

$x = x_p + c\vec{n}$

Lecture 5 Equation of line

1. intersection of 2 planes
2. trajectory of a moving point parametric equation

ex. line through $Q_0 = (-1,2,2)$ $Q_1 = (1,3,-1)$



$l: \overrightarrow{Q_0 Q(t)} = t\,\overrightarrow{Q_0 Q_1} = t\langle 2,1,-3\rangle$

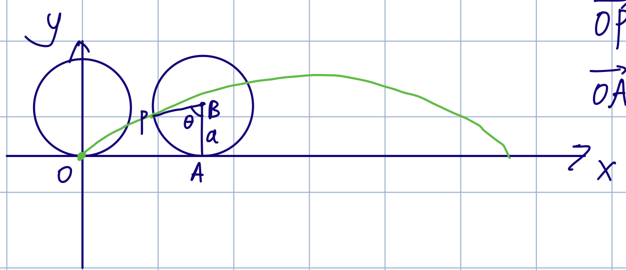
$$\begin{cases} x(t) = -1+2t \\ y(t) = 2+t \\ z(t) = 2-3t \end{cases} \qquad Q(t) = Q_0 + t\,\overrightarrow{Q_0 Q_1}$$

Application:

Intersection with plane? ex. $x+2y+4z=7$ line through $(-1,2,2),(1,3,-1)$

the plane divide the space into 2 part: $x+2y+4z>7$ and $x+2y+4z<7$ opposite side

$x(t)+2y(t)+4z(t) = -8t+11 = 7$ $t=\frac{1}{2}$ intersection point: $(0,\frac{5}{2},\frac{1}{2})$



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$

$\overrightarrow{OA} = \langle a\theta, 0\rangle$ $\overrightarrow{AB} = \langle 0, a\rangle$ $\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle$

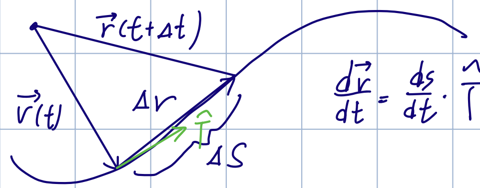
$\overrightarrow{OP} = \langle \underbrace{a\theta - a\sin\theta}_{=x(\theta)}, \underbrace{a - a\cos\theta}_{=y(\theta)}\rangle$

velocity ex $\vec{r}(t) = \langle t-\sin t, 1-\cos t\rangle$

$\vec{v} = d\vec{r}/dt = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle = \langle 1-\cos t, \sin t\rangle$ speed $= |\vec{v}|$

unit tangent vector $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt} = |\vec{v}|$, $\therefore \frac{d\vec{r}}{ds} = \hat{T}$



$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\cdot\hat{T}$

Velocity has $\begin{cases} \text{direction: tangent to traj}, \hat{T} \\ \text{length: speed}, \frac{ds}{dt} \end{cases}$

Lecture 6 Partial derivative

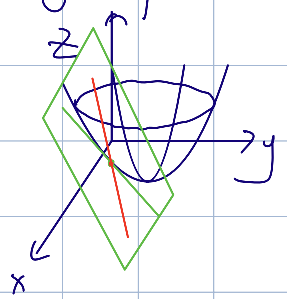
$f = f(x,y)$ $\quad \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$

$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x)}{\Delta x} = f_x \quad \frac{\partial f}{\partial y} = \lim_{\Delta y\to 0}\frac{f(x,y+\Delta y) - f(x,y)}{\Delta y} = f_y$

Approximation:

If we change $x\to x+\Delta x$, $y\to y+\Delta y$, $z = f(x,y)$ then $\Delta z \approx f_x\Delta x + f_y \Delta y$

Tangent plane



$L_1 = \begin{cases} z = z_0 + a(x-x_0) \\ y = y_0 \end{cases} \quad \frac{\partial f}{\partial x}(x_0,y_0) = a$

$L_2 = \begin{cases} z = z_0 + b(y-y_0) \\ x = x_0 \end{cases} \quad \frac{\partial f}{\partial y}(x_0,y_0) = b$

plane: $z = z_0 + a(x-x_0) + b(y-y_0)$

Application. Optimization problems

At a local min or max: $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$ tangent plane is horizental

Def: (x_0,y_0) is a critical point if $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$

ex: $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

$\begin{cases} f_x = 2x - 2y + 2 = 0 \\ f_y = -2x + 6y - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 0 \end{cases}$ critical point $(x,y) = (-1,0)$

Lecture 8 second derivative

critical point $\begin{cases} \text{local max} \\ \text{local min} \\ \text{saddle} \end{cases}$ global max/min: either critical point or boundary

ex. $w = ax^2 + bxy + cy^2$

$w = \frac{1}{4a}\left[4a^2\left(x + \frac{b}{2a}y\right)^2 + (4ac - b^2)y^2\right]$

3 cases: (1) $4ac - b^2 < 0 \Rightarrow$ saddle point

(2) $4ac - b^2 = 0$ $\quad w = x^2$

(3) $4ac - b^2 > 0$ $\quad$ if $a > 0$, minimum; if $a < 0$, maximum

In general: look at second derivative

$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$

second derivative test

At a critical point (x_0, y_0) at f

let $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$ $\quad C = f_{yy}(x_0, y_0)$

If $AC - B^2 > 0$ and $A > 0$, $f(x_0, y_0)$ is a local minimum

If $AC - B^2 > 0$ and $A < 0$, local maximum

If $AC - B^2 < 0$ saddle

If $AC - B^2 = 0$ can't conclude

proof: quadraic approximation

$\Delta f = \overset{=0}{f_x} \cdot (x - x_0) + \overset{=0}{f_y}(y - y_0) + \frac{1}{2} f_{xx}(x - x_0)^2 + f_{xy}(x - x_0)(y - y_0) + \frac{1}{2} f_{yy}(y - y_0)^2$

$= \frac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2$

In degenerate case, depends on higher derivative

Lecture 9 Least Square

$y = \hat{a}x + \hat{b}$

deviation for each point: $y_i - (ax_i + b)$

Minimise $D(a,b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2 = \sum_{i=1}^{n} [y_i^2 + (ax_i+b)^2 - 2y_i(ax_i+b)]$

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} [2(ax_i+b)\cdot x_i - 2x_i y_i] = 0 \Leftrightarrow a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} [2(ax_i+b) - 2y_i] = 0 \Leftrightarrow a\sum x_i + nb = \sum y_i$$

solve for (a,b) $\begin{cases} \hat{a} = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} \\ \hat{b} = \dfrac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2} = \dfrac{\sum y_i - a\sum x_i}{n} \end{cases}$

Lecture 11 Differential

Total differential

$f(x,y,z)$ $df = f_x dx + f_y dy + f_z dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

Important: df is NOT Δf

can do ① encode how change on x, y, z affect f

② placeholder for small variation $\Delta x, \Delta y, \Delta z$

to get approximation formula $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$

③ divide by something like dt to get a rate of change when $x(t), y(t), z(t)$

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

ex. $w = x^2 y + z$, $x = t$, $y = e^t$, $z = \sin t$

$$\frac{dw}{dt} = 2xy \cdot \frac{dx}{dt} + x^2 \cdot \frac{dy}{dt} + 1 \cdot \frac{dz}{dt}$$

$$= 2te^t + t^2 e^t + \cos t = (t^2 + 2t)e^t + \cos t$$

Prove chain rule with more variables

$w = f(x,y)$ $x = x(u,v)$ $y = y(u,v)$

$w = f(x(u,v), y(u,v))$, what about $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

$$dw = f_x dx + f_y dy = f_x (x_u du + x_v dv) + f_y (y_u du + y_v \cdot dv)$$

$$= \underbrace{(f_x x_u + f_y y_u)}_{\text{has to be } \frac{\partial w}{\partial u}} du + \underbrace{(f_x x_v + f_y y_v)}_{\text{has to be } \frac{\partial w}{\partial v}} dv$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

Lecture 12 Gradient

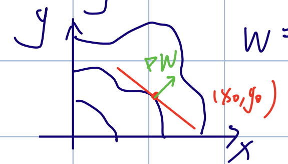
$w = f(x,y,z)$

$\nabla w = \langle w_x, w_y, w_z \rangle$ gradient of w at some point (x,y,z)

$\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}$

Theory: $\nabla w \perp$ level surface $\{w = \text{constant}\}$



$w = f(x,y)$

沿 ∇w 方向函数值变化最快

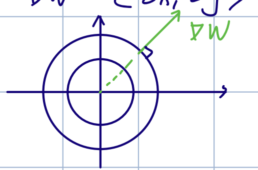
ex1. $w = a_1 x + a_2 y + a_3 z$

$\nabla w = \langle a_1, a_2, a_3 \rangle$

level surface: $a_1 x + a_2 y + a_3 z = C$ plane with normal vector ∇w

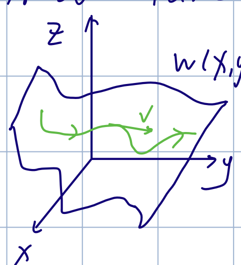
ex2. $w = x^2 + y^2$ level curve $w = c$ is a circle $x^2 + y^2 = C$

$\nabla w = \langle 2x, 2y \rangle$



∇w always points at higher value of a function

Proof: Take a curve $\vec{r} = \vec{r}(t)$ that stays on the level $w = C$



$w(x,y,z) = C$ Take a motion on the surface $w(x,y,z) = C$

$v = \frac{d\vec{r}}{dt}$ is tangent to the plane $w(x,y,z) = C$

By chain rule: $\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt} = \nabla w \cdot \vec{v} = 0$ because $w(t) = C =$ constant

so $\nabla w \perp \vec{v}$, This is true for any motion on $w = C$

$\vec{v}$ can be any vector tangent to the surface

so ∇w is prependicular to the tangent plance to the level

Application: find the tangent plane to surface $x^2+y^2-z^2=4$ at $(2,1,1)$?

Level set $w=4$ when $w=x^2+y^2-z^2$

gradient $\nabla w = \langle 2x, 2y, -2z\rangle = \langle 4,2,-2\rangle$, which is the normal vector to the plane $4(x-2)+2(y-1)-2(z-1)=0 \Leftrightarrow 4x+2y-2z=8$

Another way: $dw = 2x\,dx + 2y\,dy - 2z\,dz = 4\,dx + 2\,dy - 2\,dz$ at $(2,1,1)$

$\Delta w \approx 4\Delta x + 2\Delta y - 2\Delta z$

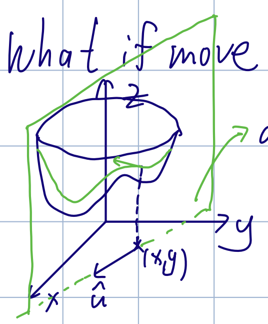
when $\Delta w = 0$ its tangent plane: $4\Delta x + 2\Delta y - 2\Delta z = 0$

$4(x-2)+2(y-1)-2(z-1)=0$

Directional derivatives

$w = w(x,y) \rightsquigarrow$ know $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

what if move in direction of $\hat{u}$ = unit vector



a straight line trajectory $\vec{r}(s)$, $\frac{d\vec{r}}{ds} = \hat{u}$

$\frac{dw}{ds} = \nabla w \cdot \frac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$

Def: If $\hat{u} = \langle a, b\rangle$ (unit vector)

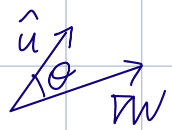
$$\begin{cases} x(s) = x_0 + as \\ y(s) = y_0 + bs \end{cases}$$

$\boxed{\frac{dw}{ds}\Big|_{\hat{u}}} = \nabla w \cdot \hat{u}$ components of ∇w in direction of $\hat{u}$

Directional derivative

Geometric view: slice the graph by the plane of the unit vector, slope $= \frac{dw}{ds}\Big|_{\hat{u}}$

$$\frac{dw}{ds}\Big|_{\hat{u}} = \nabla w \cdot \hat{u} = |\nabla w| \cdot \underbrace{|\hat{u}|}_{=1} \cdot \cos\theta$$



$$= |\nabla w| \cdot \cos\theta$$

$\Rightarrow$ be maximal when $\cos\theta = 1$ $\Rightarrow$ $\hat{u} = \operatorname{dir}(\nabla w)$

Conclusion: ∇w = dir of fatest increase of w

$|\nabla w| = \frac{dw}{ds}\Big|_{\hat{u} = \operatorname{dir}(\nabla w)}$

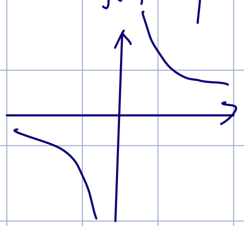
minimal when $\cos\theta = -1$ $\quad \hat{u} = \operatorname{dir}(\nabla w)$

when $\frac{dw}{ds}\Big|_{\hat{u}} = 0$ $\quad \cos\theta = 0$ $\quad \theta = 90°$ $\quad \hat{u} \perp \nabla w \Leftrightarrow \hat{u}$ tangent to level

Lecture 13 Lagrange multipliers

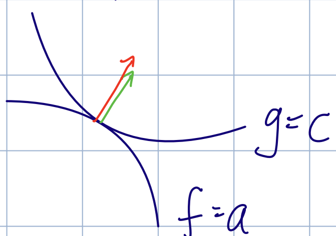
min/max a function $f(x,y,z)$ where x,y,z are not independent, $g(x,y,z)=c$

ex: find point cloest to origin on $xy=3$



Minimize $f(x,y)=x^2+y^2$ subject to constraint $\overset{g(x,y)}{xy}=3$

Observe: at minimum, level curve of f is tangent to $g=3$



When $f(x,y)$'s level curve is tangent to $g(x,y)$'s level curve, normal vectors parallel

$\nabla f \parallel \nabla g$ so $\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ xy = 3 \end{cases} \quad \begin{matrix} \text{from } \nabla f = \lambda \nabla g \\ \\ \text{from the constraint} \end{matrix}$$

Why is this method valid?

At constraint min/max, in any direction along $g=c$, the rate of change of f must be 0

For any $\hat{u}$ tangent to $g=c$, we must have $\frac{df}{ds}\big|_{\hat{u}}=0$

$\nabla f \cdot \hat{u} = 0$, so any such $\hat{u}$ is $\perp \nabla f$, so $\nabla f \perp$ level set of g

Know $\nabla g \perp$ level set of g, so $\nabla f \parallel \nabla g$

WARNING: method don't tell whether it's maximum or minimum (or a saddle)

CAN'T use second derivative test

Lecture 14 Non-independent variables

$f(x,y,z)$ where $g(x,y,z) = C$

then $z = z(x,y)$ $\quad \frac{\partial z}{\partial x}$? $\frac{\partial z}{\partial y}$?

ex $x^2+yz+z^3=8$ at $(2,3,1)$

Take differential: $2x\,dx + z\,dy + (y+3z^2)\,dz = 0$

at $(2,3,1)$ $\quad 4dx + dy + 6dz = 0$

If we view $z = z(x,y)$ $\quad dz = -\frac{1}{6}(4dx + dy)$

$\frac{\partial z}{\partial x} = -\frac{2}{3}/1 = -\frac{2}{3}$ $\qquad \frac{\partial z}{\partial y} = -\frac{1}{6}/1 = -\frac{1}{6}$

(y constant $dy=0$) $\qquad$ (x constant $dx=0$)

In general:

$g(x,y,z) = C$ then $dg = g_x\,dx + g_y\,dy + g_z\,dz = 0$

Solve for dz: $dz = -\frac{g_x}{g_z}dx - \frac{g_y}{g_z}dy$

so. for $\frac{\partial z}{\partial x}$: set $dy = 0$ $\quad dz = -\frac{g_x}{g_z}dx$ $\quad \frac{\partial z}{\partial x} = \frac{-g_x}{g_z}$

ex. $f(x,y) = x+y$ $\quad \frac{\partial f}{\partial x} = 1$

$x = u$ $\;y = u+v$, then $f = 2u+v$ $\quad \frac{\partial f}{\partial u} = 2$

$x = u$, but $\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial u}$, because $\frac{\partial f}{\partial x}$ means keeping y constant, $\frac{\partial f}{\partial u}$ means keeping v constant

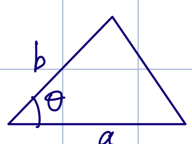
Need clearer Notations

$\left(\frac{\partial f}{\partial x}\right)_y$ = keep y constant

$\left(\frac{\partial f}{\partial u}\right)_v$ = keep v constant

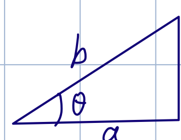
so $\underbrace{\left(\frac{\partial f}{\partial x}\right)_y}_{1} \neq \underbrace{\left(\frac{\partial f}{\partial x}\right)_v}_{2} = \underbrace{\left(\frac{\partial f}{\partial u}\right)_v}_{2}$

ex. area of a triangle



$A = \frac{1}{2}ab\sin\theta$

Assume it's a right triangle



$a = b\cos\theta$ constraint

Rate of A respect to θ

(1) treat a,b,θ as independent $\frac{\partial A}{\partial \theta} = \left(\frac{\partial A}{\partial \theta}\right)_{a,b} = \frac{1}{2}ab\cos\theta$ loss the constraint

(2) keep a constant $b = b(a,\theta)$ $\left(= \frac{a}{\cos\theta}\right)$ to keep right angle $\left(\frac{\partial A}{\partial \theta}\right)_a$

(3) keep b constant, $a = a(b,\theta)$ $\quad \left(\frac{\partial A}{\partial \theta}\right)_b$

Compute $\left(\frac{\partial A}{\partial \theta}\right)_a$

Method 0: solve for b then substitute

$$= \frac{\partial}{\partial \theta}\left(\frac{1}{2} a \cdot \frac{a}{\cos\theta} \cdot \sin\theta\right) = \frac{1}{2} a^2 \sec^2\theta$$

2 systematic methods

Method 1: differentials

keep a fixed: $da = 0$

constraint $a = b\cos\theta$ $\quad da = \cos\theta\, db - b\sin\theta\, d\theta = 0$

$db = b\tan\theta\, d\theta$ (To find when θ change, how b change)

function $A = \frac{1}{2} ab\sin\theta$

$$dA = \frac{1}{2} b\sin\theta \underbrace{da}_{=0} + \frac{1}{2} a\sin\theta \underbrace{db}_{\text{substitute}} + \frac{1}{2} ab\cos\theta\, d\theta$$

$\sin\theta\tan\theta + \cos\theta = \sec\theta$

$$= \frac{1}{2} a\sin\theta \cdot b\tan\theta\, d\theta + \frac{1}{2} ab\cos\theta\, d\theta = \left(\frac{1}{2} ab\sin\theta\tan\theta + \frac{1}{2} ab\cos\theta\right) d\theta = \frac{1}{2} ab\sec\theta\, d\theta$$

so $\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2} ab\sec\theta$

Summary: (1) write dA in terms of $da, db, d\theta$

(2) set $da = 0$

☆ (3) differentiate constraint $\Rightarrow$ solve for db in terms of $d\theta$

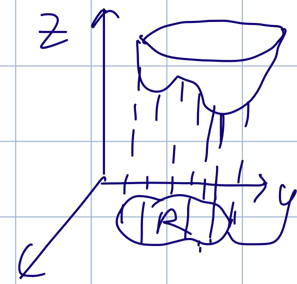
plug in

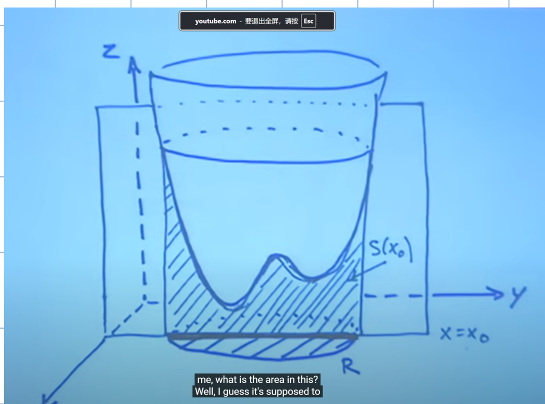
Method 2: Chain Rule $\left(\frac{\partial}{\partial \theta}\right)_a$ in formula for A

$$\left(\frac{\partial A}{\partial \theta}\right)_a = A_\theta \underbrace{\left(\frac{\partial \theta}{\partial \theta}\right)_a}_{1} + A_a \underbrace{\left(\frac{\partial a}{\partial \theta}\right)_a}_{0} + A_b \underbrace{\left(\frac{\partial b}{\partial \theta}\right)_a}_{\text{use constraint}}$$

Lecture 15 Double integral

one-variable Integral: area of graph
Double integral: volumn below the graph
over a region R is xy plane $\iint_R f(x,y)\,dA$

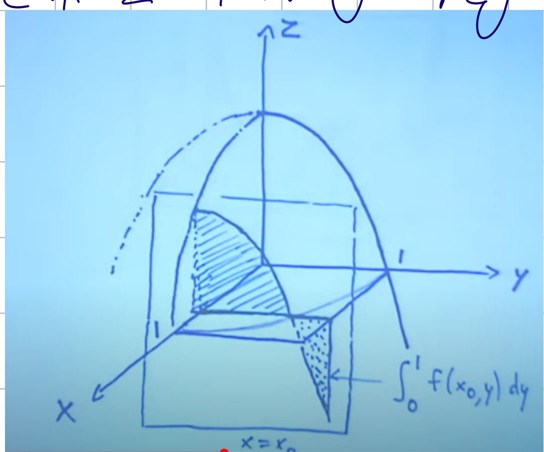




move a plane from one axis, take slices
add all the $A \cdot dx$ $\quad V = \int_{x_{min}}^{x_{max}} S(x)\,dx$
For given x, $S(x) = \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy$

$$\iint_R f(x,y)\,dA = \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy \right] dx$$

ex1. $z = 1 - x^2 - y^2$ region $0 \le x \le 1$ $\quad 0 \le y \le 1$



$\int_0^1 \int_0^1 (1 - x^2 - y^2)\,dy\,dx$
$= \int_0^1 \left[(1-x^2)y - \frac{1}{3}y^3 \right]_0^1 dx$
$= \int_0^1 \left(\frac{2}{3} - x^2 \right) dx$
$= \left(\frac{2}{3}x - \frac{1}{3}x^3 \right) \Big|_0^1$
$= \frac{1}{3}$
$dA = dy\,dx$

ex2: Integral in such R

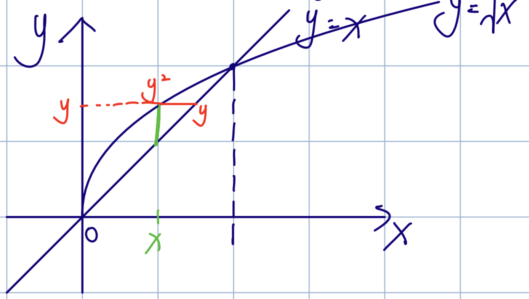
$\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx = \int_0^1 \left[(1-x^2)\sqrt{1-x^2} - \frac{1}{3}(1-x^2)\sqrt{1-x^2} \right] dx$
$= \int_0^1 \frac{2}{3}(1-x^2)^{\frac{3}{2}}\,dx = \frac{2}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta$
$= \frac{2}{3} \cdot \frac{3}{16}\pi = \frac{1}{8}\pi$

Easier in Polar coordinates

Exchanging order

ex1. $\int_0^1\int_0^2 dx\,dy = \int_0^2\int_0^1 dy\,dx$

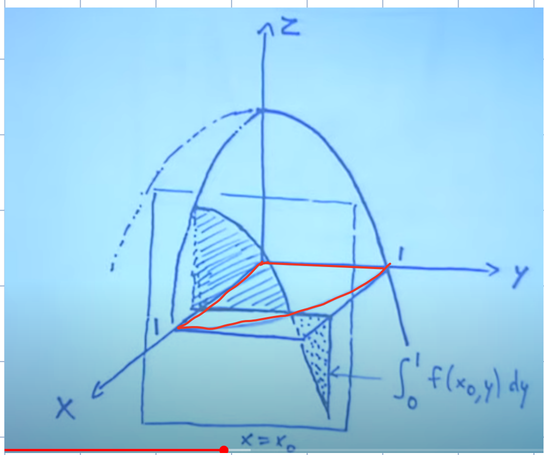
ex2. $\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$



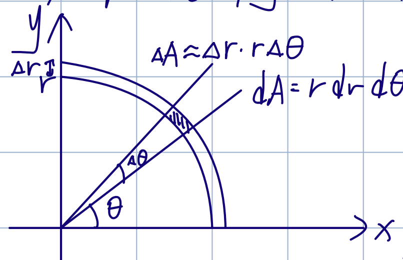
$$\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$$
$$= \int_0^1\int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy = \int_0^1 (1-y)e^y\,dy$$
$$= \left[(2-y)e^y\right]_0^1$$
$$= e-2$$

$$\bar{x} = \frac{1}{Mass}\iint x f(\theta, r)\,dy\,dx$$
$$\bar{y} = \frac{1}{Mass}\iint y f(\theta, r)\,dy\,dx$$

Lecture 17 Polar Coordinates



$x = r\cos\theta,\ y = r\sin\theta$

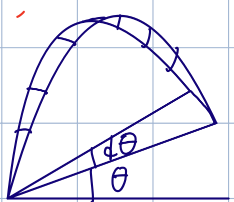


$\frac{1}{2}\Delta\theta\left((r+\Delta r)^2 - r^2\right)$
$= \Delta\theta\, r\, dr$

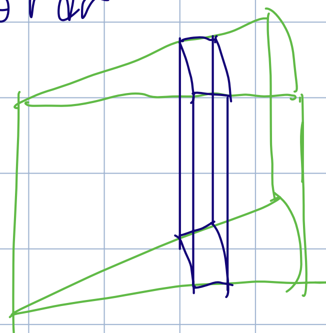
$$\int_0^{\frac{\pi}{2}}\int_0^1 f(\theta,r)\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left(\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right)\Big|_0^1 d\theta$$
$$= \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{\pi}{8}$$

???

?



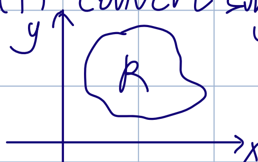
$$\underbrace{\underbrace{\iint f(\theta,r)\,dr}_{\text{角积}A}\cdot r\,d\theta\cdot\frac{1}{2}}_{\text{体积}dV}$$



$$\iint \underbrace{\underbrace{f(\theta,r)\,dr}_{S}\ \frac{1}{2}r\,d\theta}_{dV}$$

Application

(1) convert single integral to double integral

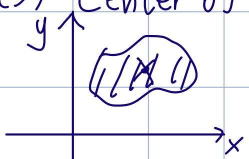


$\text{Area} = \iint 1\cdot dA$

(2) Average value of $f = \bar{f} = \frac{1}{\text{Area}(R)}\iint_R f\,dA$

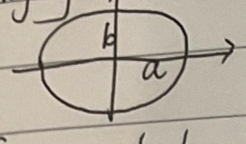
weighted average of f with density δ: $\bar{f} = \frac{1}{\text{Mass}(R)}\iint_R f\delta\,dA$

(3) Center of mass of a (planar) object (with density δ)



$(\bar{x},\bar{y})$ $\quad \bar{x} = \frac{1}{\text{Mass}}\iint x\delta\,dA,\ \bar{y} = \frac{1}{\text{Mass}}\iint y\delta\,dA$

changing variables in $\iint$

ex1. 

$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$

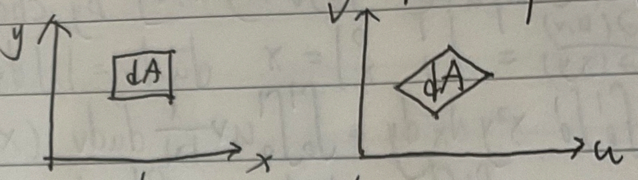
$\iint_{(\frac{x}{a})^2+(\frac{y}{b})^2\le 1} dx\,dy = ab\iint_{u^2+v^2\le 1} du\,dv = \pi ab$

set $\frac{x}{a}=u$ $\frac{y}{b}=v$

In general: finding scaling factor between $dx\,dy \leftrightarrow du\,dv$

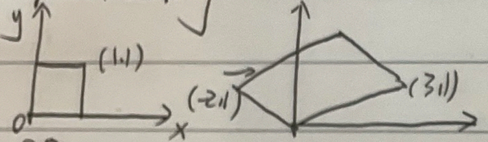
become a parallelogram

$\begin{cases} u=3x-2y \\ v=x+y \end{cases}$



$dA = dx\,dy$ and $dA' = dv\,du$

The scaling factor here doesn't depend ~~cho~~ on coice of rectangle



$A' = \left|\begin{matrix}3 & 1\\ -2 & 1\end{matrix}\right| = 5$

so $dA' = 5dA$ $du\,dv = 5dx\,dy$

$\iint f(x,y)\,dx\,dy = \iint g(u,v)\frac{1}{5}\,du\,dv$, bound also needs change

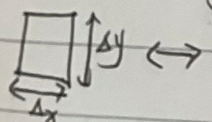
↑ Only for linear transformation

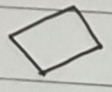
General case: $u=u(x,y)$ $v=v(x,y)$

$\Delta u \approx \cancel{du} \; u_x\cdot\Delta x + u_y\cdot\Delta y$

$\Delta v \approx v_x\cdot\Delta x + v_y\cdot\Delta y$

Small rectangle in x-y coordinates



$\leftrightarrow$ small uv-coordinate parallelogram 

$A' = \left|\begin{matrix}u_x & v_x\\ u_y & v_y\end{matrix}\right|$

can be seen as linear transformation, which changing scaler depends on u_x, u_y, v_x, v_y (x_0, y_0)

$dA' = \left|\begin{matrix}u_x & v_x\\ u_y & v_y\end{matrix}\right| dA$ or $dA' = \left|\begin{matrix}u_x & u_y\\ v_x & v_y\end{matrix}\right| dA$

Jacobian: $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$du\,dv = |J|\,dx\,dy = \left\| \begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix} \right\| dx\,dy$

ex1. Polar coordinate

$x = r\cos\theta \quad y = r\sin\theta$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$dx\,dy = |r|\,dr\,d\theta = r\,dr\,d\theta$

ex2. Compute $\int_0^1\int_0^1 x^2 y\,dx\,dy$ by changing to $u=x$ $v=xy$

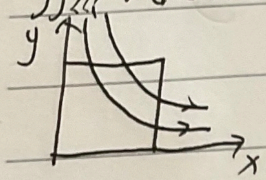
$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x \quad du\,dv = |x|\,dx\,dy = x\,dx\,dy$

$\int_0^1\int_0^1 x^2 y\,dx\,dy = \int_0^1\int_0^1 uv\frac{1}{|x|}\,du\,dv$ (×)

$= \int_0^1\int_0^1 v\,du\,dv = \int_0^1 v\,dv = \frac{1}{2}$ (×)

For bounds

$\iint_{???} v\,du\,dv$ or $dv\,du$?



$\underbrace{\iint v\,du}_{u \text{ change}, V \text{ constant}}\,dv$

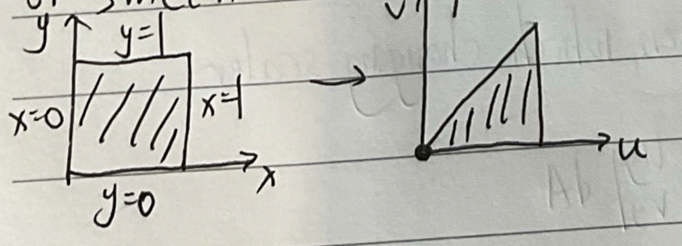
$xy = C$

when V is constant, find scope of u

$y=1$ $V=x$, $u=V$ $\int_V^1 v\,du$

$= \int_0^1\int_V^1 v\,du\,dv$

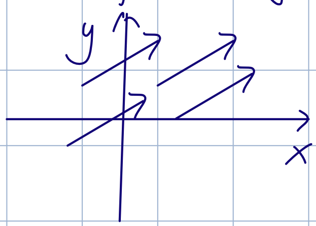
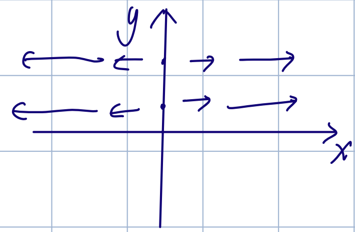
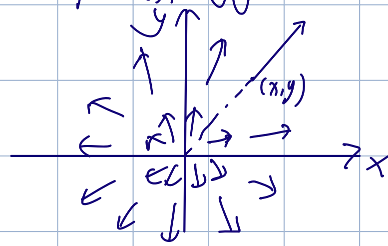
or switch to u-v picture



Lecture 19 Vector fields

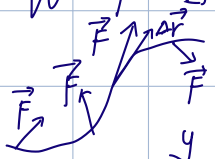
$\vec{F} = M\vec{i} + N\vec{j}$, M, N are functions of x, y

ex: $\vec{F} = 2\vec{i} + \vec{j}$ $\qquad$ $\vec{F} = x\vec{i}$ $\qquad$ $\vec{F} = x\vec{i} + y\vec{j}$

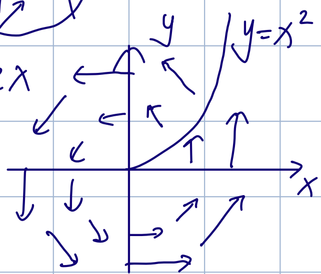
line integra

$W = \vec{F} \cdot \Delta\vec{r}$ $\quad$ Along a trajectory C $\quad$ $\vec{F} = M\hat{i} + N\hat{j}$



$$W = \int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} \cdot dt = \int_C M\,dx + N\,dy$$

ex



$\vec{F} = -y\vec{i} + x\hat{j}$

$C: x = t$

$\quad y = t^2 \quad (0 \le t \le 1)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (-y, x) \cdot (1, 2t)\,dt$$
$$= \int_0^1 (-t^2, t) \cdot (1, 2t)\,dt$$
$$= \int_0^1 t^2\,dt = \frac{1}{3}$$

Special case, when $\vec{F}$ is gradient of a function $f(x,y)$

Fundamental theorem of Calculus for line integral

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$

proof: $\int_C \nabla f \cdot d\vec{r} = \int_C f_x dx + f_y dy = \int_C df = f(P_1) - f(P_0)$

proof: $\int_C \nabla f \cdot d\vec{r} = \int_C f_x dx + f_y dy \qquad C: x = x(t) \quad y = y(t)$

$$= \int_C \left(f_x \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}\right) dt = \int_{t_0}^{t_1} \frac{df}{dt}\,dt = f(x(t), y(t)) \Big|_{t_0}^{t_1} = f(P_1) - f(P_0)$$

Only works if $\vec{F}$ is a gradient field

(1) Path-independence $\quad \int_{C_1} \nabla f\, d\vec{r} = \int_{C_2} \nabla f\, d\vec{r}$ if C_1 and C_2 have same start/end points

$\Leftrightarrow$ (2) $\vec{F}$ is a gradient field

$\Leftrightarrow$ (3) $M\,dx + N\,dy$ is an exact differential (can be expressed in df)

Lecture 21 Exact differential

if $\vec{F} = \nabla f$, $M = f_x$ $N = f_y$, then $f_{xy} = f_{yx} \Rightarrow M_y = N_x$

Then: if $\vec{F} = \langle M, N \rangle$ defined, differentiable everywhere and $M_y = N_x$, then $\vec{F}$ is a gradient field

ex. $\vec{F} = -y\hat{i} + x\hat{j}$

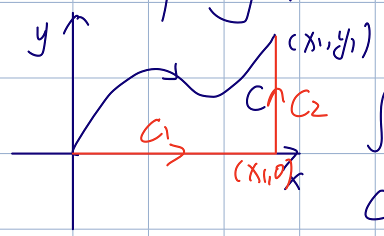
$\frac{\partial M}{\partial y} = -1$ $\frac{\partial N}{\partial x} = 1$ are not the same, not a gradient field

ex. $\vec{F} = (4x^2 + axy)\hat{i} + (3y^2 + 4x^2)\hat{j}$

$\frac{\partial M}{\partial y} = ax = \frac{\partial N}{\partial x} = 8x$ $a = 8$

Then, how do we find the potential (only if $M_y = N_x$)

(1) Computing line integral



$\int_C \vec{F} \, d\vec{r} = f(x_1, y_1) - \underset{\sim\sim\sim}{f(0,0)}$ choose a constant

$\int_C \vec{F} \, d\vec{r} = (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$

C_1: $y = 0$ $dy = 0$ $\int_{C_1} \vec{F} \, d\vec{r} = \int_0^{x_1} 4x^2 \, dx = \frac{4}{3} x_1^3$

C_2: $y: 0 \to y_1$ $x = x_1$ $dx = 0$ $\int_{C_2} \vec{F} \, d\vec{r} = \int_0^{y_1} (3y^2 + 4x_1^2)\,dy = y_1^3 + 4x_1^2 y$

$f(x_1, y_1) = \frac{4}{3} x_1^3 + y_1^3 + 4x_1^2 y_1 + C$ $f(x, y) = \frac{4}{3}x^3 + y^3 + 4x^2 y + C$

(2) Antiderivatives

Want to solve $\begin{cases} f_x = 4x^2 + 8xy & ① \\ f_y = 3y^2 + 4x^2 & ② \end{cases}$

①: $f = \frac{4}{3}x^3 + 4x^2 y + g(y)$

$\Rightarrow f_y = \quad 4x^2 + g'(y) \xrightarrow{\text{match with ②}} g'(y) = 3y^2$ $g(y) = y^3 + C$

$f = \frac{4}{3}x^3 + 4x^2 y + y^3 + C$

$\vec{F}$ is a consecutive field, the $\oint_C \vec{F} \, d\vec{r} = 0$

Definition $\operatorname{curl}(\vec{F}) = N_x - M_y$ and $N_x = M_y \Leftrightarrow \operatorname{curl}(\vec{F}) = 0$

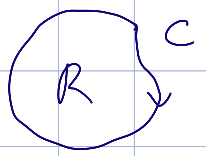
For a velocity field, curl measures rotation component of motion

ex. $\vec{F} = \langle -y, x \rangle$ $\operatorname{curl}(\vec{F}) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2$

curl measures $(2\times)$ angular velocity of rotation component of velocity field

in a force field, curl measure torque $\frac{\text{torque}}{\text{moment of inertia}} = \frac{d}{dt}(\text{angular velocity})$

Lecture 22 Green Theorem

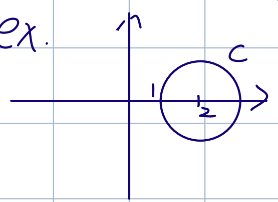


If C is a closed curve enclosing a region R, counterclockwise
$\vec{F}$ vector field defined and differentiable in R

Then $\oint_C \vec{F}\, d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA$

$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$

ex.



$\oint_C y e^{-x} dx + (\frac{1}{3}x^2 - e^{-x}) dy$

$= \iint_R (x + e^{-x} - e^{-x})\, dA$

$= 2\pi$

Proof: Special case: $\operatorname{curl}(\vec{F}) = 0$

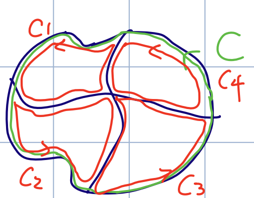
$\oint_C \vec{F}\, d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA = 0$ if $\operatorname{curl}(\vec{F}) = 0$

Proof: $\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$

Observe: first improve $\oint_C M\,dx = \iint -M_y\, dA$ (special care when $N = 0$)

Similarly: $\oint_C N\,dy = \iint N_x\, dA$ ($M = 0$)

Observe 2: decompose R into simple regions



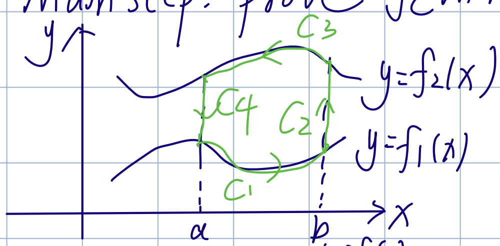
$\oint_C M\,dx + N\,dy = \sum_{i=1}^{n} \oint_{C_i} M\,dx + N\,dy$

$\iint_R (N_x - M_y)\, dA = \sum_{i=1}^{n} \iint_{r_i} (N_x - M_y)\, dy$

(go twice along boundary, cancelled out)

Cut R into pieces that for $a \le x \le b$, $f_1(x) \le y \le f_2(x)$

Main step: prove $\oint_C M\,dx = \iint_R -M_y\, dx$ if R vertically simple



$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\, dx$ $\quad \int_{C_3} M\,dx = \int_b^a M(x, f_2(x))\, dx$

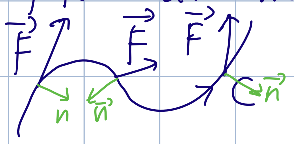
$\int_{C_2} M\,dx = \int_{C_4} M\,dx = 0$

$\int_C M\,dx = \int_a^b M(x, f_1(x))\, dx - \int_a^b M(x, f_2(x))\, dx$

$\iint -M_y\, dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx = -\int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\, dx = \int_a^b [M(x, f_1(x)) - M(x_1, f_2(x))]\, dx$

Lecture 23 Flux

Flux: another line integral : $\int_C \vec{F}\cdot\hat{n}\cdot ds$



$\vec{n}$ lies right to the curve

Flux: $\int_C \vec{F}\,\vec{n}\,ds$ measure normal component of $\vec{F}$

Interpretation: For $\vec{F}$ a velocity field, Flux measure how much fluid passes throught C

Calculation: $d\vec{r} = \hat{T}\,ds = \langle dx, dy\rangle$

$\hat{n}$ is $\hat{T}$ rotate 90° clockwise: $\hat{n}\,ds = \langle dy, -dx\rangle$

if $\vec{F} = \langle P, Q\rangle$ $\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \langle P,Q\rangle\cdot\langle dy, -dx\rangle = \int_C -Q\,dx + P\,dy$

Green Theorem for flux

If C encloses a region R counterclockwise and $\vec{F} = \langle P, Q\rangle$ defined in R

then $\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$ (divergence of $\vec{F}$)

$\operatorname{div}\langle P, Q\rangle = P_x + Q_y$

$\oint_C \underbrace{-Q}_{M}\,dx + \underbrace{P}_{N}\,dy = \iint_R (\underbrace{P_x}_{N_x} + \underbrace{Q_y}_{-M_y})\,dA$ the same as Green Theorem

ex. $\vec{F} = x\hat{i} + y\hat{j}$ C = circle of $r = a$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial x}(x) = 2$

$\oint_C \vec{F}\,\hat{n}\,ds = \iint_R 2\,dA = 2\pi a^2$

Interpretation

(1) how much the flow is "expanding"

(2) the source rate = amount of fluid added to the system per unit time and area

line integral: $\int_C \vec{F}\,d\vec{r} = \int_C M\,dx + N\,dy$ — $\langle dx, dy\rangle$ $\hat{T}\cdot ds$

line integral Flux: $\int_C \vec{F}\cdot\hat{n}\cdot ds = \int_C -Q\,dx + P\,dy$ — $\langle dy, -dx\rangle$ $\hat{n}\cdot ds$

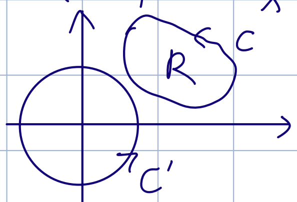
curl F = $N_x - M_y$ $\oint_C \vec{F}\,d\vec{r} = \iint_R (N_x - M_y)\,dA$

div F = $P_x + Q_y$ $\oint_C \vec{F}\cdot\hat{n}\cdot ds = \iint_R (P_x + Q_y)\,dA$

More validity of Green's Theorem

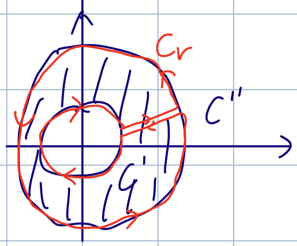
Only work if $\vec{F}$ (and derivative) defined everywhere

ex. $\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2+y^2}$ For not $x=y=0$, curl $\vec{F} = 0$



$\oint_C \vec{F}\,d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA = 0$

$\oint_{C'} \vec{F}\,d\vec{r} = ?$ can't use Green theorem directly

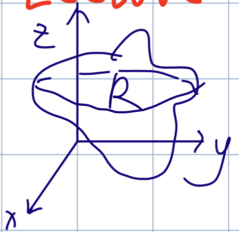


$\oint_{C''} \vec{F}\,d\vec{r} - \oint_{C'} \vec{F}\,d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$

take line integral for Cr to prove

If curl $\vec{F} = 0$ and domain where $\vec{F}$ defined is simply connected, $\vec{F}$ is conservative and is a gradient

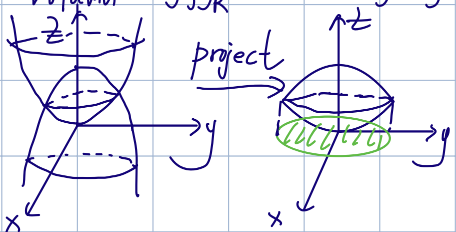
Lecture 25 Triple Integrals



$$\iiint_R f(x,y,z)\,dz\,dy\,dx$$

ex. region between $z=x^2+y^2$, $z=4-x^2-y^2$

volumn $=\iiint_R 1\,dV = \int\int\int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$ find wherever $z_{bottom} < z_{top}$ $x^2+y^2 \le 4-x^2-y^2$; $x^2+y^2 \le 2$



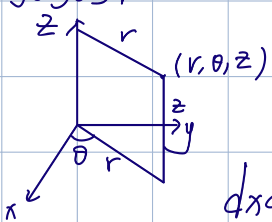
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (4-2x^2-2y^2)\,dy\,dx$$

switch to polar coordinate

Better: Polar coordinates

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} dz \cdot r\,dr\,d\theta$$ cylindrical coordinate



$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy\,dz = dV = r\,dr\,d\theta\,dz$

Average value of $f(x,y,z)$ in x,y,z

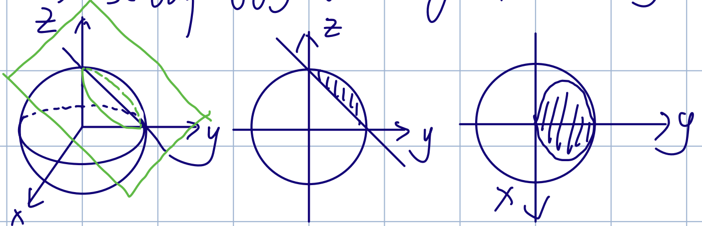
$\bar{f} = \frac{1}{Vol(R)}\iiint_R f\,dV$ or with weighted $\bar{f} = \frac{1}{Mass(R)}\iiint_R f\delta\,dV$

Center of Mass $(\bar{x},\bar{y},\bar{z})$, $\bar{x} = \frac{1}{Mass}\iiint x\delta\,dV$ $\bar{y} = \frac{1}{Mass}\iiint_R y\delta\,dV$ $\bar{z} = \frac{1}{Mass}\iiint_R z\delta\,dz$

Moments of inertia

$I = \iiint_R (\text{distance to the axis})^2\delta\,dV$ ex. $I_z = \iiint_R r^2\delta\,dV$

ex3. setup $\iiint$ for region $z > 1-y$ inside unit ball

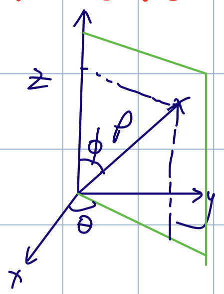


$$= \int_0^1 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz\,dx\,dy$$

$(1-y) < \sqrt{1-x^2-y^2}$

$(1-y)^2 < 1-x^2-y^2$

Lecture 26 Spherical coordinate



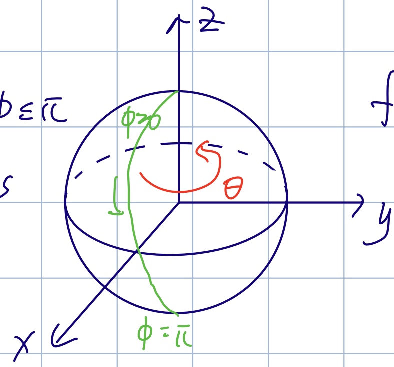
ϕ: go down from z-axis $0 \le \phi \le \pi$

θ: counterclockwise from x-axis

$z = \rho\cos\phi$ $\quad \rho = \sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$

$r = \rho\sin\phi$; $x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

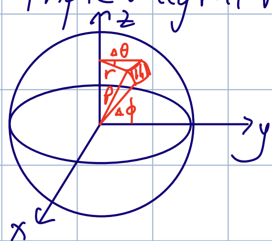


fix $\rho = a$

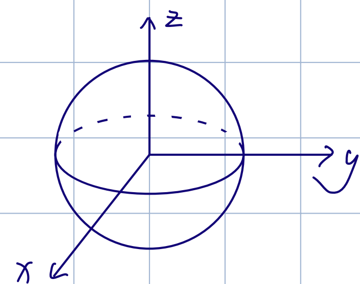
$\phi \leftrightarrow$ latitude

θ: longitude

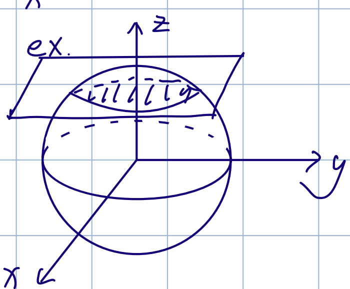
Triple integral in spherical coordinate



$$dV = \rho\, d\phi\, r\, d\theta\, d\rho$$
$$= \rho\,\rho\sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



ex.

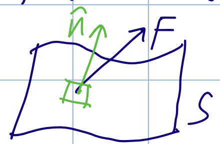


$$= \int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{\sqrt{2}}{2}\sec\phi}^{1} \left(1-\frac{\sqrt{2}}{2}\sec(\theta)\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}$$

Lecture 27 Flux in space

flux through a surface; surface integral

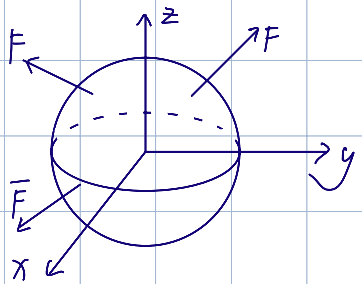
$\vec{F}$ vector field, S a surface



2 choices for $\hat{n}$ (orientation)

$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\,dS$ Notation $d\vec{S} = \hat{n}\,dS$

ex1. Flux of $\vec{F} = \langle x, y, z\rangle$ through sphere of a centered at O



$\hat{n} = \frac{1}{a}\langle x, y, z\rangle$

$\vec{F}\cdot\hat{n} = |\vec{F}| = a$

$\iint_S \vec{F}\hat{n}\,dS = \iint_S a\,dS = 4\pi a^3$

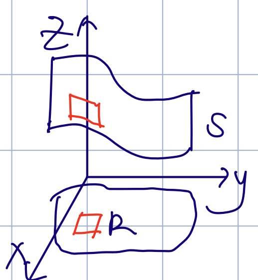
ex2. Same sphere $\vec{H} = z\hat{k}$

$\vec{H}\cdot\vec{n} = \langle 0, 0, z\rangle\cdot\langle x, y, z\rangle\cdot\frac{1}{a} = \frac{z^2}{a}$

$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \frac{z^2}{a}\,dS$

$dS = \rho^2\sin\phi\,d\phi\,d\theta = a^2\sin\phi\,d\phi\,d\theta \quad z = a\cos\phi$

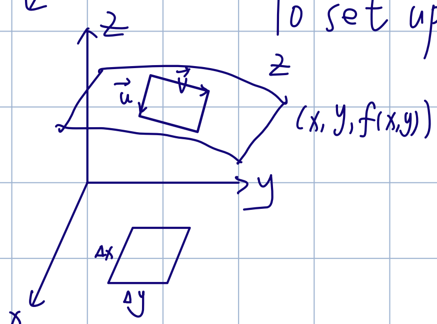
$\iint_S \frac{z^2}{a}\,dS = \int_0^{2\pi}\int_0^{\pi}\frac{a^2\cos^2\phi}{a}\cdot a^2\sin\phi\,d\phi\,d\theta = \frac{4}{3}\pi a^3$



graph $z = f(x,y)$

$\hat{n}\,dS = \pm\underbrace{\langle -f_x, -f_y, 1\rangle}_{\text{Not } \hat{n}}\underbrace{dx\,dy}_{\text{Not } dS}$ simplifies magically!

To set up bounds on $\iint \cdots dx\,dy$, look at shadow on x-y plane



$\pm\vec{u}\times\vec{v} = \Delta S\cdot\hat{n}$

$\vec{u}$: from $(x, y, f(x,y))$ to $(x+\Delta x, y, f(x+\Delta x, y))$ $\quad f(x+\Delta x, y) \approx f(x,y) + \Delta x\cdot f_x$

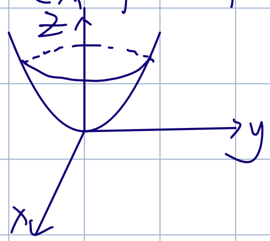
$\vec{u} = \langle \Delta x, 0, f_x\Delta x\rangle \quad \vec{v} = \langle 0, \Delta y, f_y\Delta y\rangle$

$\quad = \langle 1, 0, f_x\rangle\Delta x \quad\quad = \langle 0, 1, f_y\rangle\Delta y$

$\hat{n}\Delta S = \vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}\Delta x\Delta y = \langle -f_x, -f_y, 1\rangle\Delta x\Delta y$

$\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$

ex. $\vec{F} = z\hat{k}$, $z = x^2 + y^2$ above the unit circle



$= \iint \langle 0, 0, z \rangle \cdot \langle -2x, -2y, 1 \rangle \, dy\, dx$ choose the unit vector upward

$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2 \, dy\, dx$ get rid of z

$= \int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}$