

Higher-Order Functions

Class outline:

- Iteration example
- Designing functions
- Generalization
- Higher-order functions
- Lambda expressions
- Conditional expressions

Iteration example

Virahaṅka-Fibonacci numbers

Discovered by Virahanka in India, 600-800 AD, later re-discovered in Western mathematics and commonly known as Fibonacci numbers.

0 1 1 2 3 5 8 13 21 34 ...

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0 + 1 = 1 2 3 5 8 13 21 34 ...

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0 1 + 1 = 2 3 5 8 13 21 34 ...

Virahaṅka-Fibonacci numbers

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0 1 1 + 2 = 3 5 8 13 21 34 ...

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0 1 1 2 + 3 = 5 8 13 21 34 ...

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0 1 1 2 3 + 5 = 8 13 21 34 ...

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0 1 1 2 3 5 8 13 + 21 = 34 ...

Virahanka's question

How many poetic meters exist for a total duration?

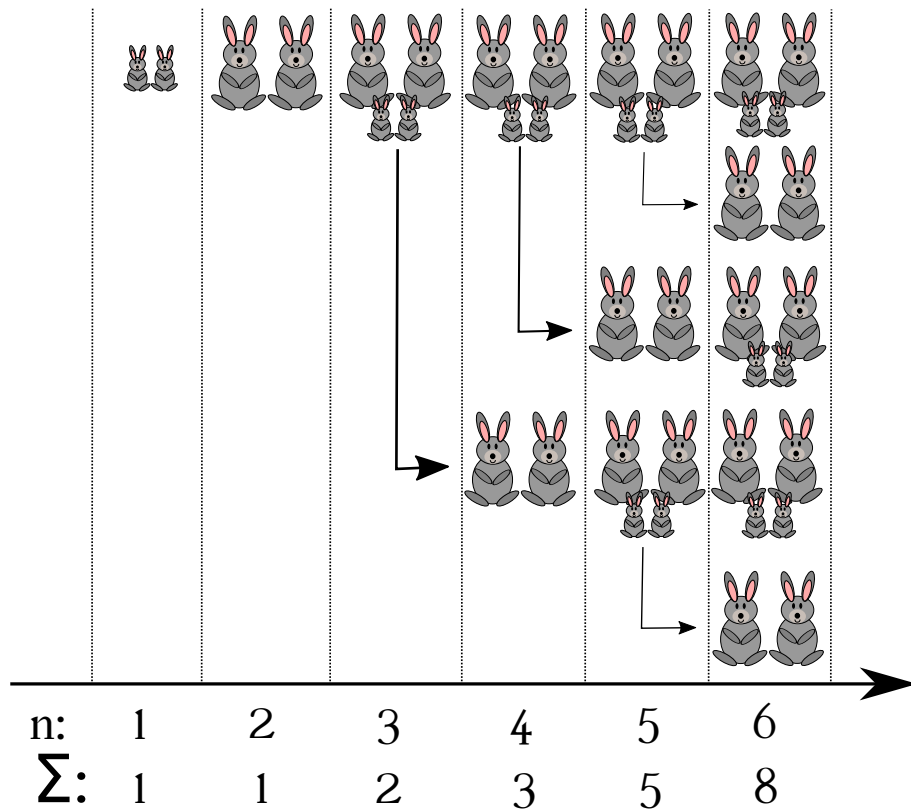
S = short syllable, L = long syllable

Duration	Meters	Total
1	S	1
2	SS, L	2
3	SSS, SL, LS	3
4	SSSS, SSL, SLS, LSS, LL	5
5	SSSSS, SSSL, SSLS, SLSS, SLL, LLS, LSL, LSSS	8

The So-called Fibonacci Numbers in Ancient and Medieval India

Fibonacci's question

How many pairs of rabbits can be bred after N months?



Attribution: [Fschwarzentruber, Wikipedia](#)

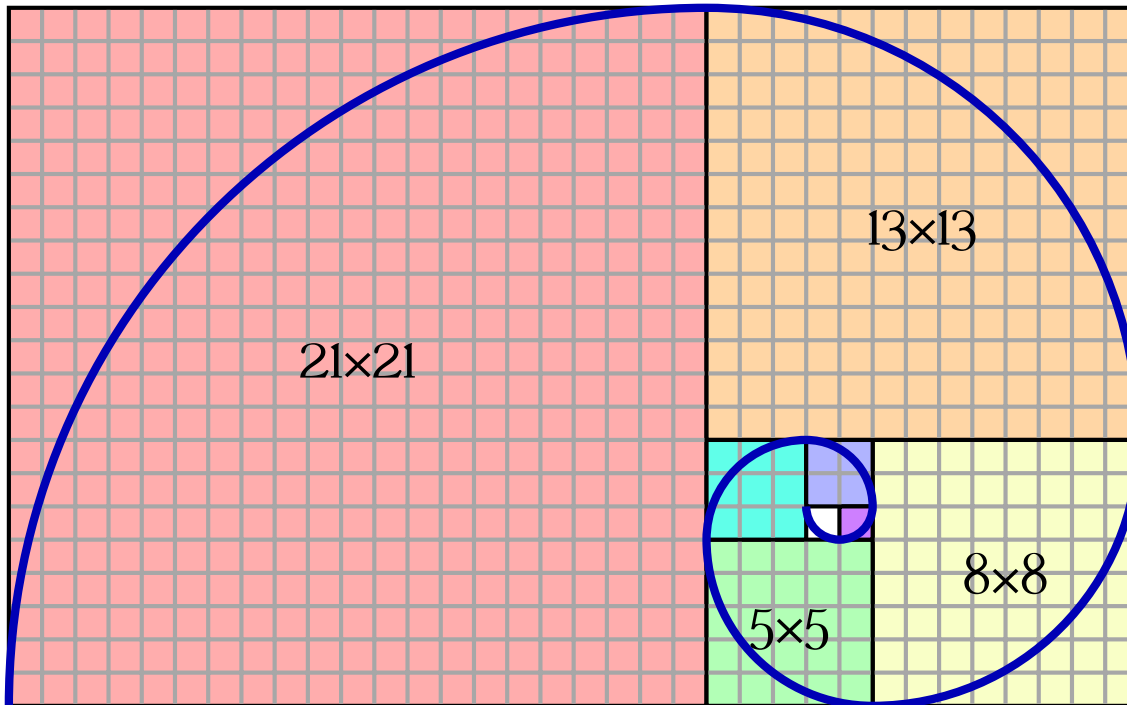
Virahanka-Fibonacci number generation

VF	0	1	1	2	3	5	8	13	21	34	55	...
N	0	1	2	3	4	5	6	7	8	9	10	...

```
def vf_number(n):  
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.  
    >>> vf_number(2)  
    1  
    >>> vf_number(6)  
    8  
    """  
    prev = 0 # First Fibonacci number  
    curr = 1 # Second Fibonacci number  
    k = 1  
    while k < n:  
        (prev, curr) = (curr, prev + curr)  
        k += 1  
    return curr
```


Golden spiral

The Golden spiral can be approximated by Virahanka-Fibonacci numbers.



Go bears!

The Golden spiral is found everywhere in nature...



Designing Functions

Describing Functions

```
def square(x):  
    """Returns the square of X."""  
    return x * x
```

Aspect	Example
A function's domain is the set of all inputs it might possibly take as arguments.	<code>x</code> is a number
A function's range is the set of output values it might possibly return.	<code>square</code> returns a non-negative real number
A pure function's behavior is the relationship it creates between input and output.	<code>square</code> returns the square of <code>x</code>

Designing a function

Give each function exactly one job, but make it apply to many related situations.

```
round(1.23)      # 1  
round(1.23, 0)   # 1  
round(1.23, 1)   # 1.2  
round(1.23, 5)   # 1.23
```

Don't Repeat Yourself (DRY): Implement a process just once, execute it many times.

Generalization

Generalizing patterns with arguments

Geometric shapes have similar area formulas.

Shape

Area

$$1 * r^2$$

$$\pi * r^2$$

$$\frac{3\sqrt{3}}{2} * r^2$$

A non-generalized approach

```
from math import pi, sqrt

def area_square(r):
    return r * r

def area_circle(r):
    return r * r * pi

def area_hexagon(r):
    return r * r * (3 * sqrt(3) / 2)
```

How can we generalize the common structure?

Generalized area function

```
from math import pi, sqrt

def area(r, shape_constant):
    """Return the area of a shape from length measurement R."""
    if r < 0:
        return 0
    return r * r * shape_constant

def area_square(r):
    return area(r, 1)

def area_circle(r):
    return area(r, pi)

def area_hexagon(r):
    return area(r, 3 * sqrt(3) / 2)
```

Higher-order functions

What are higher-order functions?

A function that either:

- Takes another function as an argument
- Returns a function as its result

All other functions are considered first-order functions.

Generalizing over computational processes

$$\sum_{k=1}^5 k = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{k=1}^5 k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$\sum_{k=1}^5 \frac{8}{(4k-3) \cdot (4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323}$$

The common structure among functions may be a computational process, not just a number.

Functions as arguments

```
def cube(k):  
    return k ** 3  
  
def summation(n, term):  
    """Sum the first N terms of a sequence.  
    >>> summation(5, cube)  
    225  
    """  
    total = 0  
    k = 1  
    while k <= n:  
        total = total + term(k)  
        k = k + 1  
    return total
```

Functions as return values

Locally defined functions

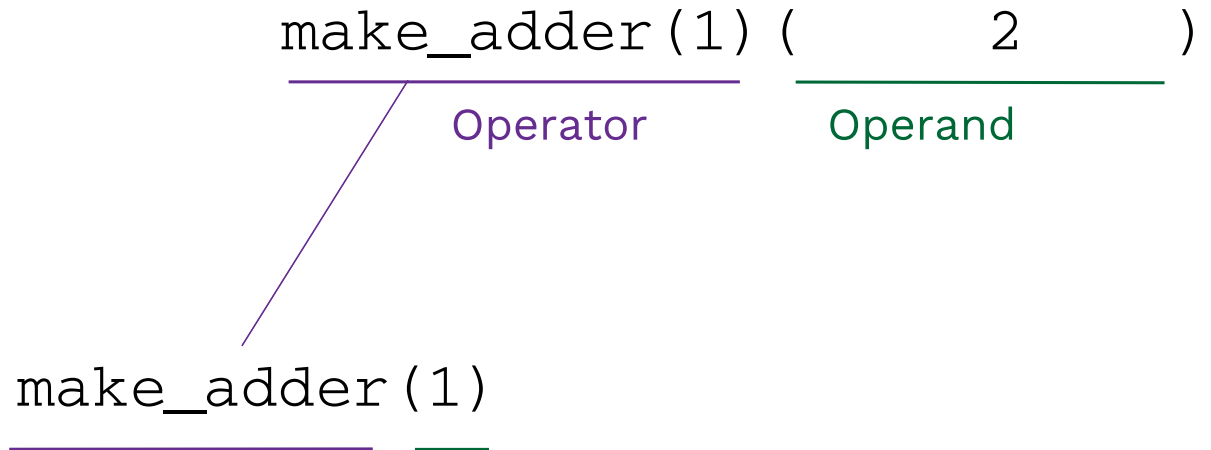
Functions defined within other function bodies are bound to names in a local frame.

```
def make_adder(n):  
    """Return a function that takes one argument k  
       and returns k + n.  
    >>> add_three = make_adder(3)  
    >>> add_three(4)  
    7  
    """  
    def adder(k):  
        return k + n  
    return adder
```

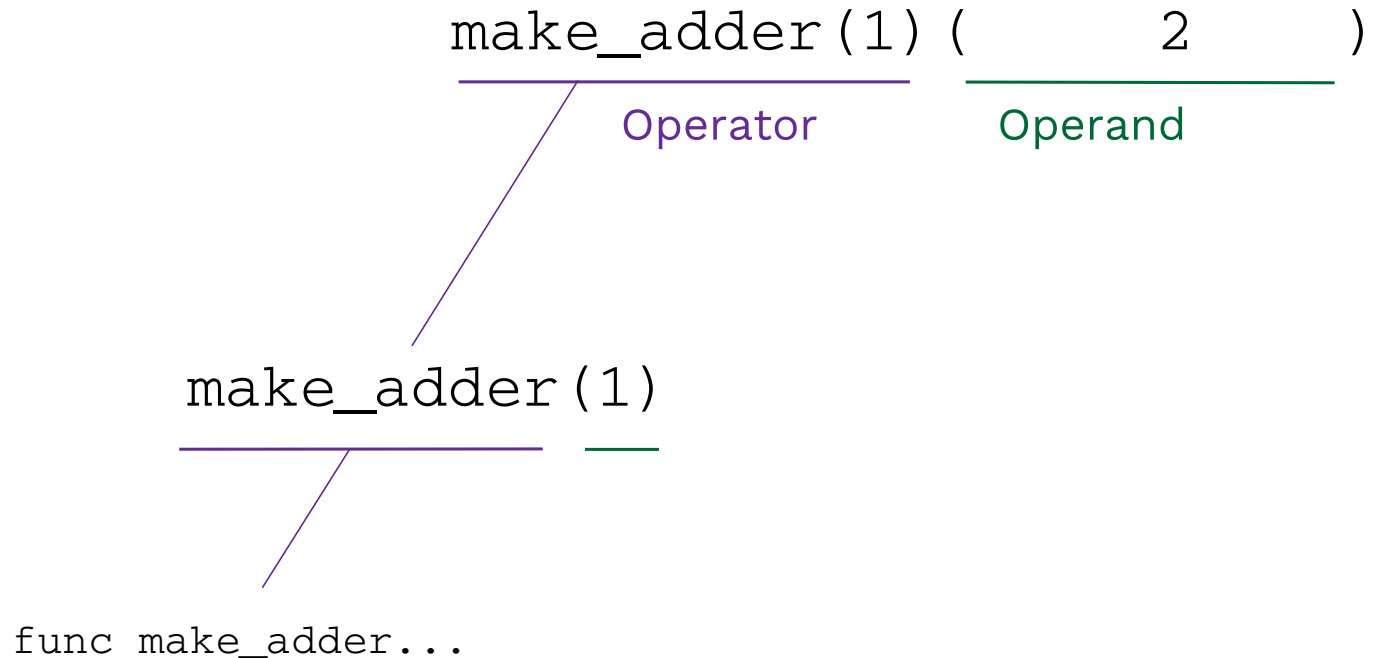
Call expressions as operator expressions

make_adder (1) (2)
Operator Operand

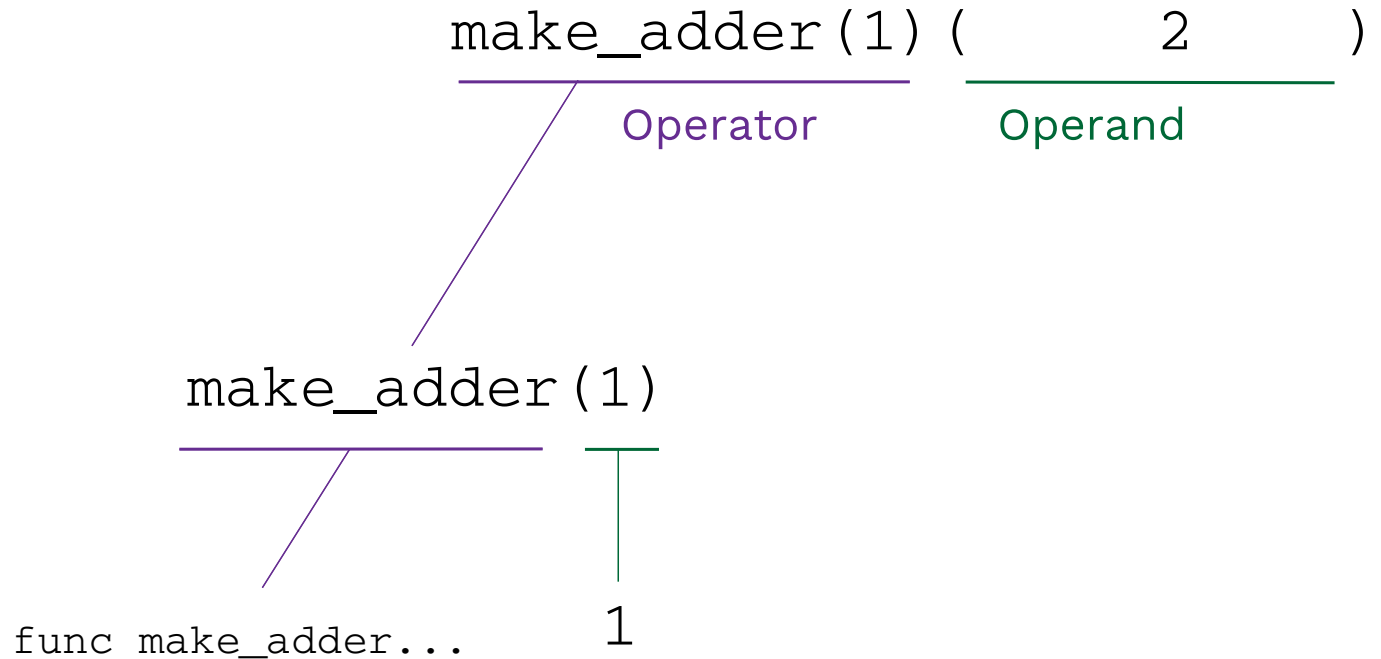
Call expressions as operator expressions



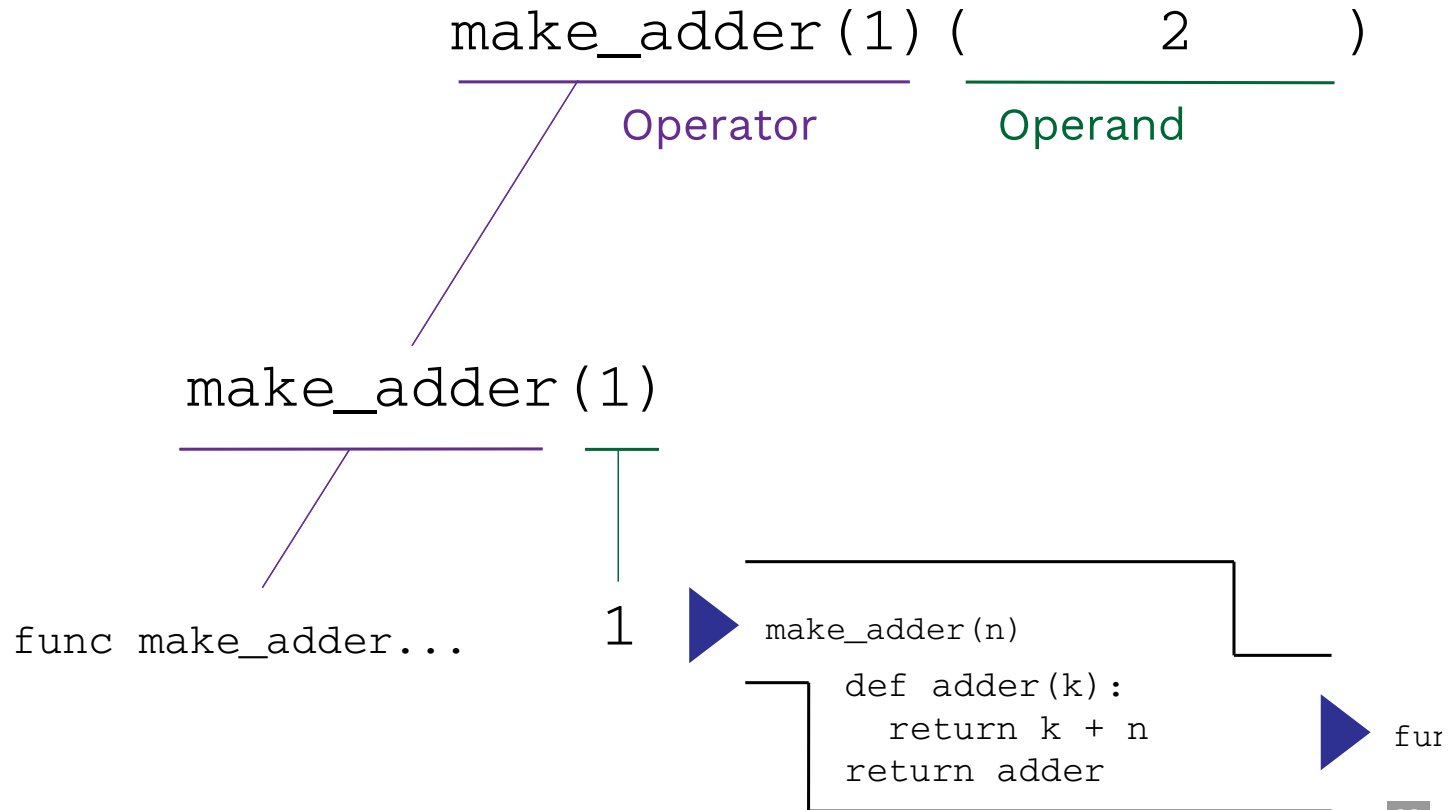
Call expressions as operator expressions



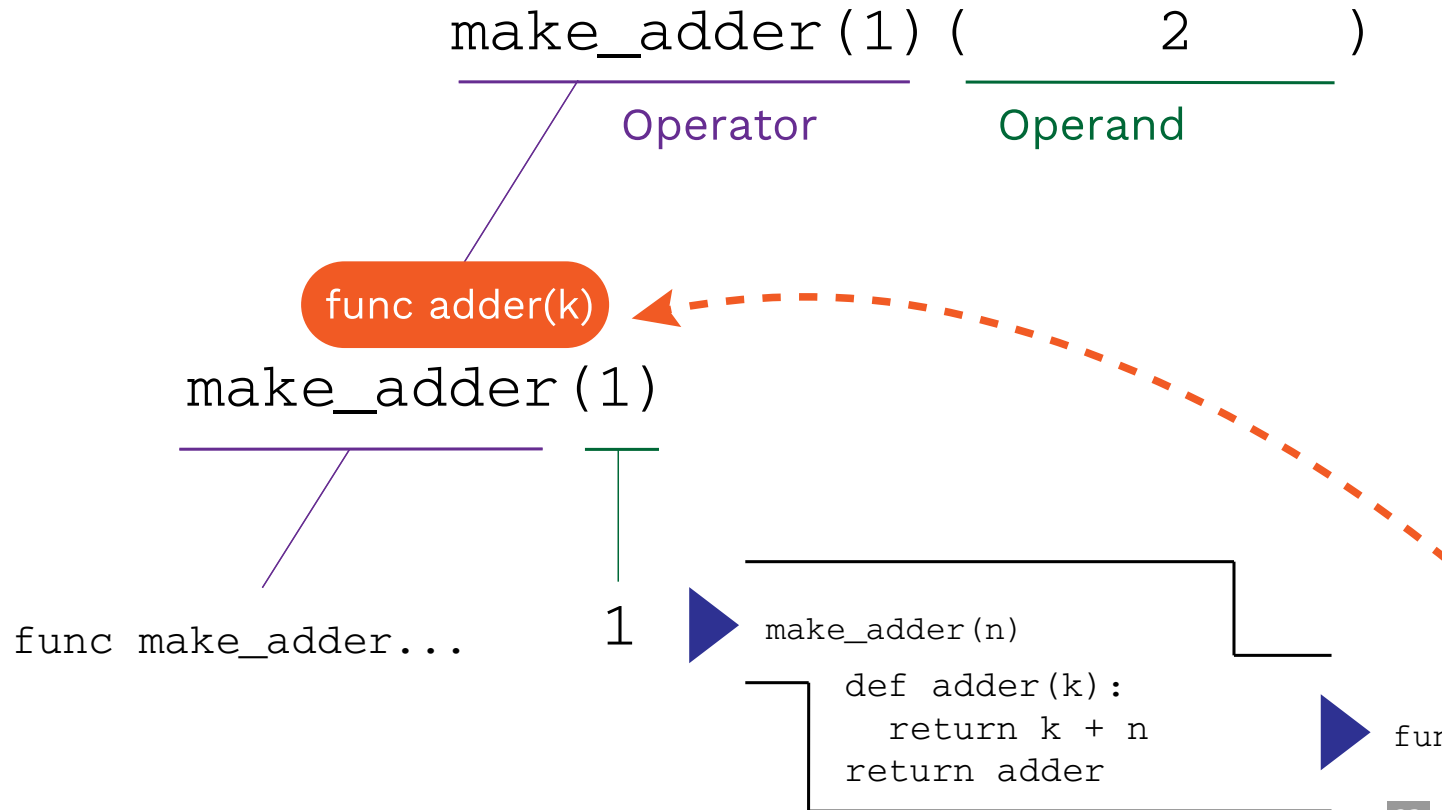
Call expressions as operator expressions



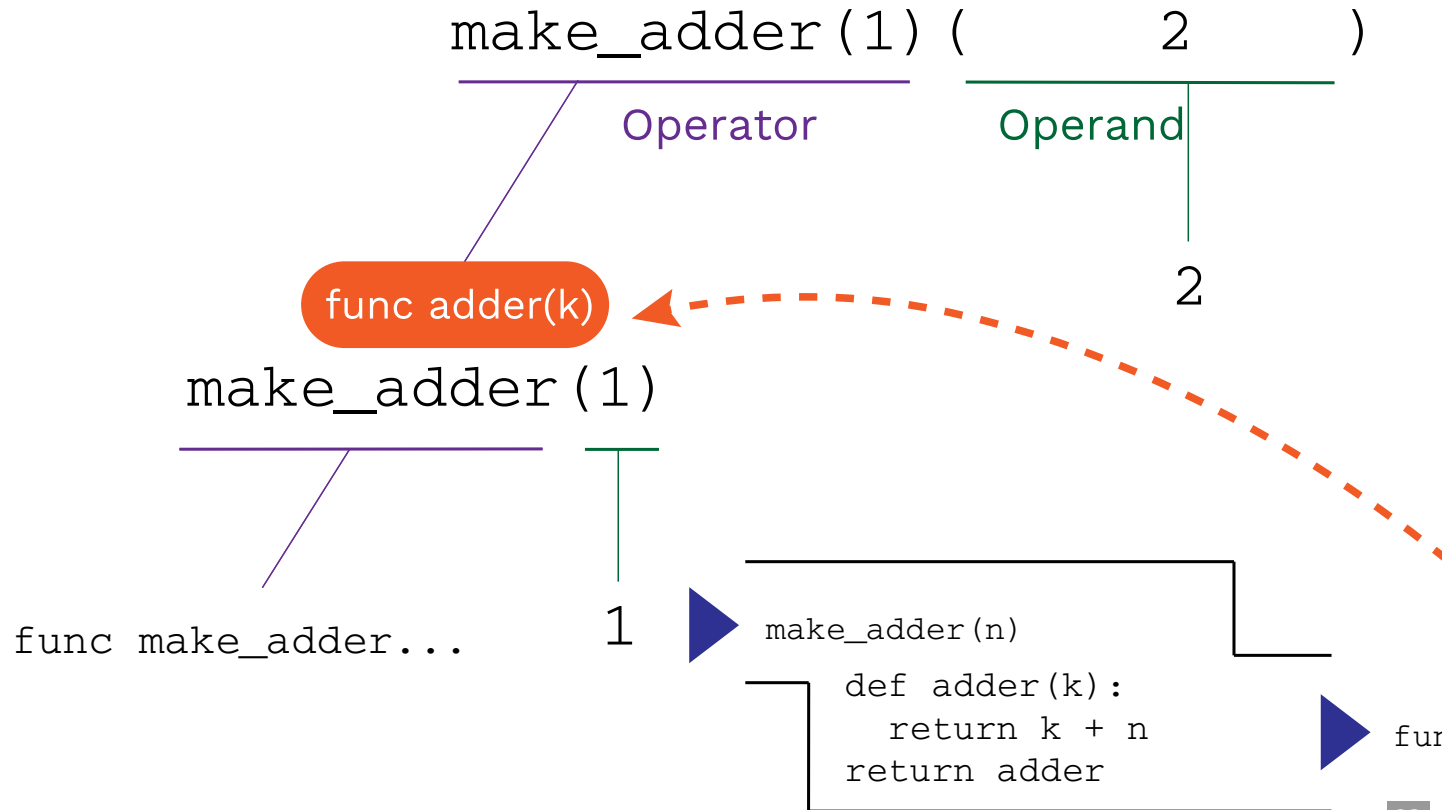
Call expressions as operator expressions



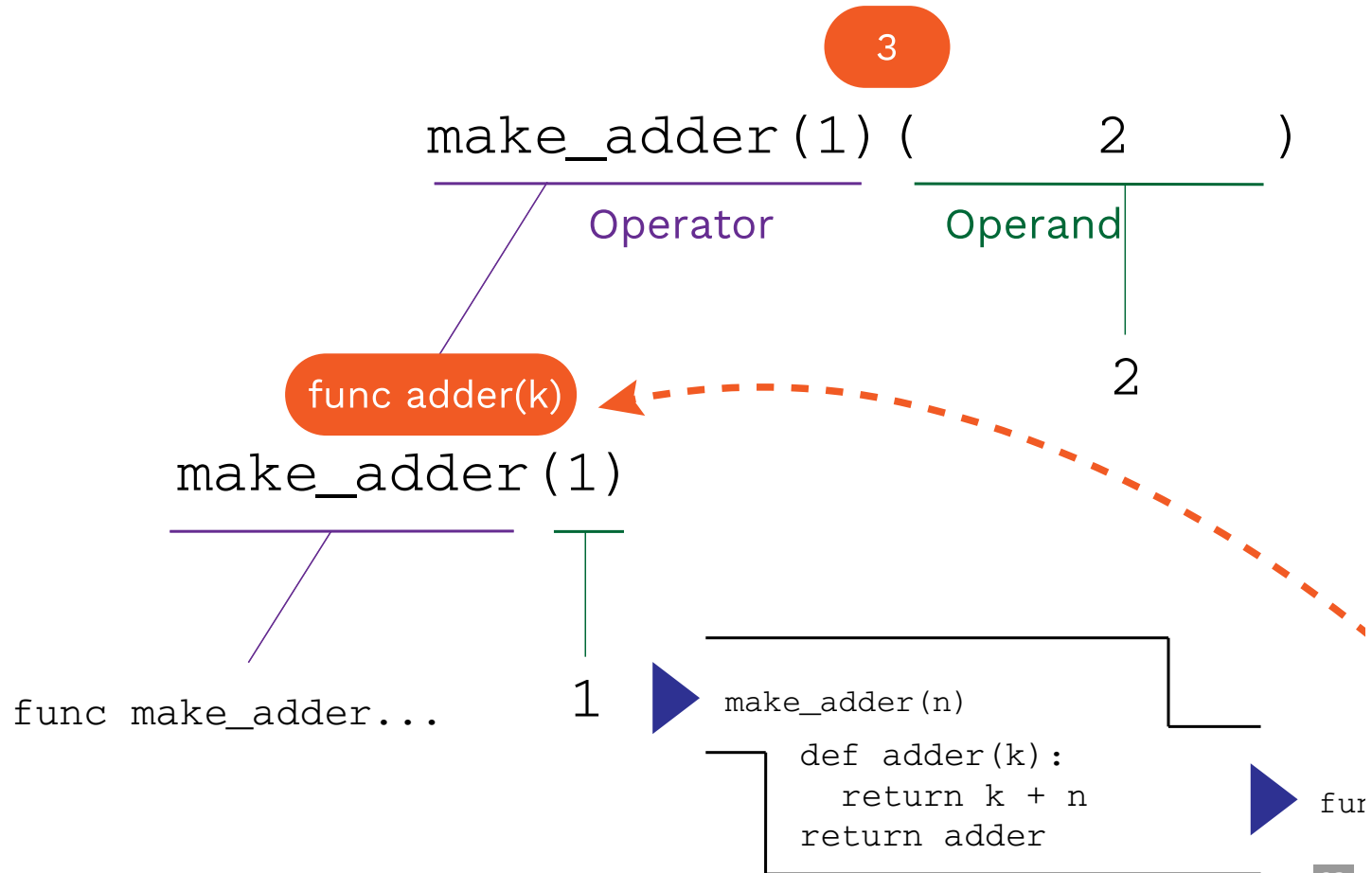
Call expressions as operator expressions



Call expressions as operator expressions



Call expressions as operator expressions



Lambda expressions

Lambda syntax

A **lambda expression** is a simple function definition that evaluates to a function.

The syntax:

```
lambda <parameters>: <expression>
```

A function that takes in **parameters** and returns the result of **expression**.

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A function that takes in **parameters** and returns the result of **expression**.

A lambda version of the **square** function:

```
square = lambda x: x * x
```

A function that takes in parameter **x** and returns the result of **x * x**.

Lambda syntax tips

A lambda expression does **not** contain return statements or any statements at all.

Incorrect:

```
square = lambda x: return x * x
```

Correct:

```
square = lambda x: x * x
```

Def statements vs. Lambda expressions

```
def square(x):  
    return x * x
```

VS

```
square = lambda x: x * x
```

Both create a function with the same domain, range, and behavior.

Both bind that function to the name square.

Only the `def` statement gives the function an **intrinsic name**, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).

Lambda as argument

It's convenient to use a lambda expression when you are passing in a simple function as an argument to another function.

Instead of...

```
def cube(k):  
    return k ** 3  
  
summation(5, cube)
```

We can use a lambda:

```
summation(5, lambda k: k ** 3)
```

Conditional expressions

Conditional expressions

A conditional expression has the form:

```
<consequent> if <predicate> else <alternative>
```

Evaluation rule:

- Evaluate the <predicate> expression.
- If it's a true value, the value of the whole expression is the value of the <consequent>.
- Otherwise, the value of the whole expression is the value of the <alternative>.

Lambdas with conditionals

This is invalid syntax:

```
lambda x: if x > 0: x else: 0
```

Conditional expressions to the rescue!

```
lambda x: x if x > 0 else 0
```