NOTES ON HYPOTROCHOIDS AND EPITROCHOIDS

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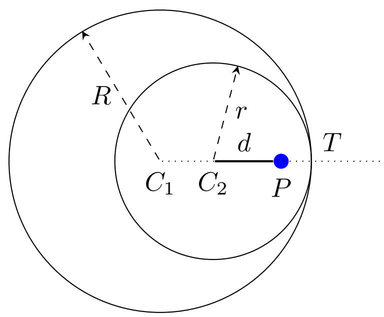
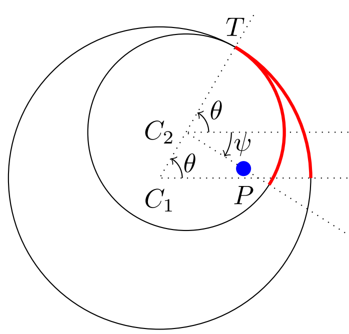
1. Hypotrochoids

A *hypotrochoid* is the curve obtained by tracing the positions taken by a point P rigidly attached to a circle $\mathcal{C}_2$ of centre C_2 and radius r, P being at a distance d from C_2, with $\mathcal{C}_2$ rolling around the inside of another circle $\mathcal{C}_1$ of centre C_1 and radius R. To compute the equation of the curve, one assumes that C_1 is located at the origin of the plane, so has coordinates $(0,0)$, and C_1, C_2 and P are horizontally aligned, in that order from left to right, as shown in the following picture.



As $\mathcal{C}_2$ rotates clockwise and moves anticlockwise around the inside of $\mathcal{C}_1$, when $\overrightarrow{C_1C_2}$ has gone from an angle of 0 to a positive angle of θ, and $\overrightarrow{C_2P}$ from an angle of 0 to a negative angle of ψ, the point of contact T between both circles has travelled the same distance along both circles—represented in red in the picture below—, namely, θR on $\mathcal{C}_1$, and $(\theta-\psi)r$ on $\mathcal{C}_2$. Hence:

$$\psi = -\frac{R-r}{r}\theta$$



At this stage, since $\overrightarrow{C_1P} = \overrightarrow{C_1C_2} + \overrightarrow{C_2P}$, the point P has coordinates:

$$x = (R-r)\cos(\theta) + d\cos(-\psi)$$
$$y = (R-r)\sin(\theta) + d\sin(-\psi)$$

that is:

$$x = (R-r)\cos(\theta) + d\cos\Big(\frac{R-r}{r}\theta\Big)$$
$$y = (R-r)\sin(\theta) - d\sin\Big(\frac{R-r}{r}\theta\Big)$$

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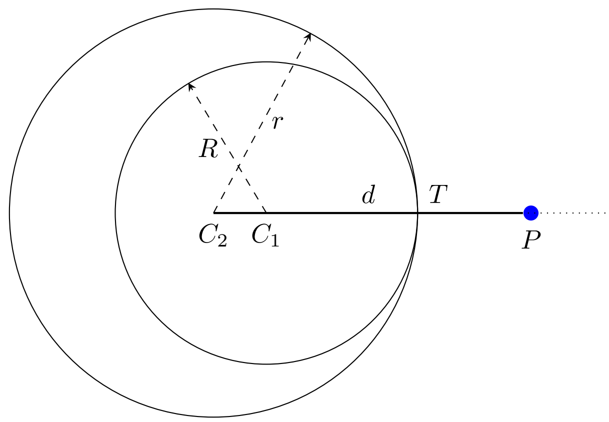
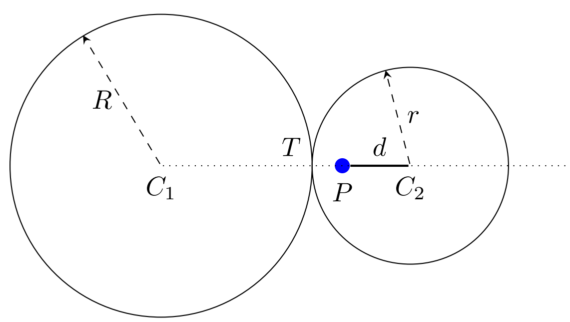
Note that P can "stick out" of $\mathcal{C}_2$, that is, d can be larger than r, as shown in the following picture, which also illustrates that $\mathcal{C}_2$ can be larger than $\mathcal{C}_1$, that is, r can be greater than R; that does not change the above reasoning and the equations still hold.



The *period* of a hypotrochoid is the number of rolls of $\mathcal{C}_2$ needed for P to get back to its original position. It is equal to the least strictly positive integer ρ such that $\rho \times 2\pi r$ is a multiple of $2\pi R$; hence it is equal to $\frac{r}{\gcd(r,R)}$.

2. Epitrochoids

If we let $\mathcal{C}_2$ roll around the outside rather than the inside of $\mathcal{C}_1$, then the curve obtained by tracing the positions taken by P is called an *epitrochoid*. To compute the equation of the curve, one assumes that C_1, C_2 and P are horizontally aligned, with C_2 to the right of C_1 and with P to the left of C_2, and also to the left of C_1 in case d is greater than $R + r$; the following picture illustrates the case where $r < R$ and $d < r$.



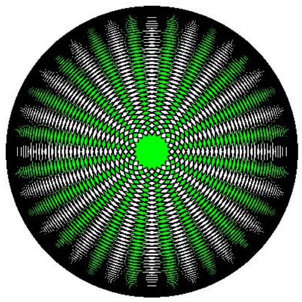
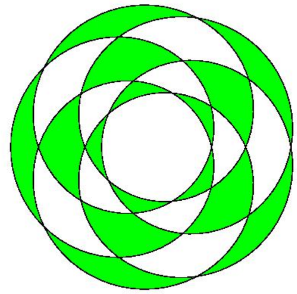
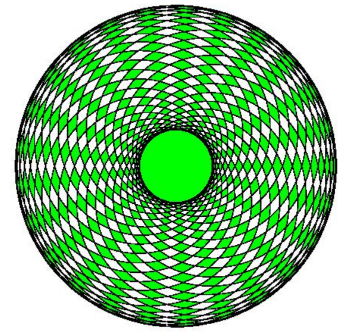
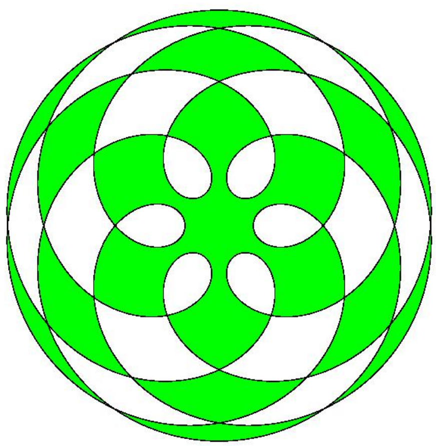
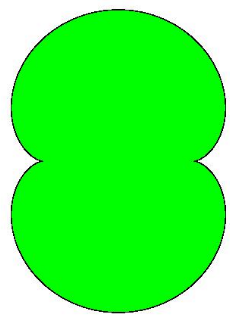
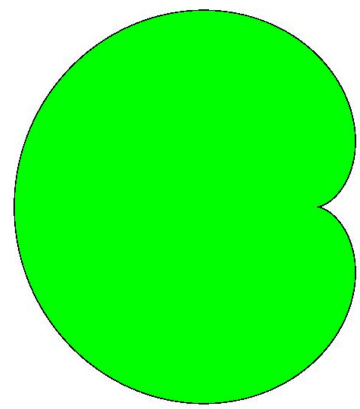
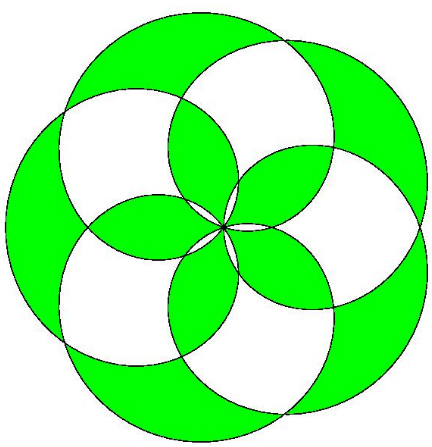
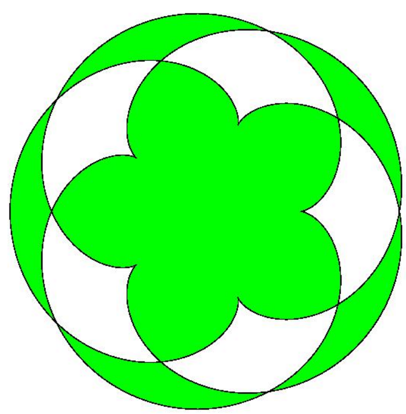
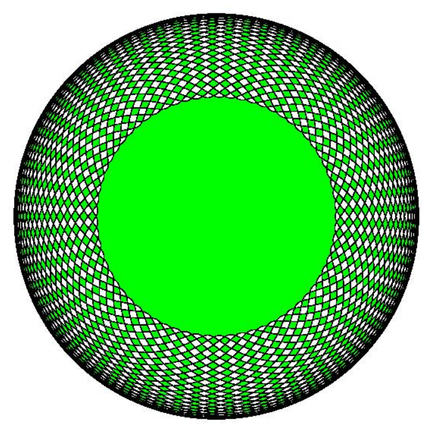
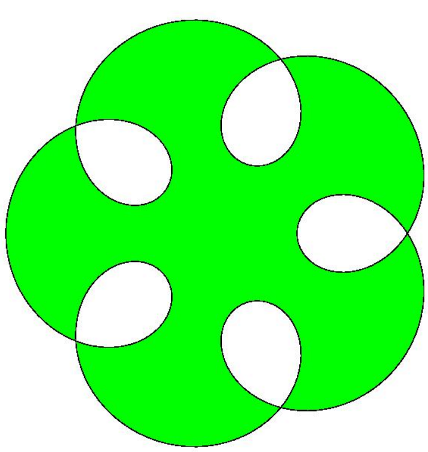
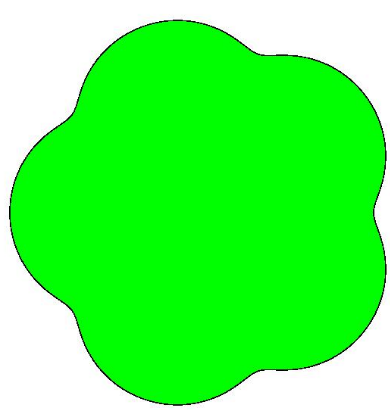
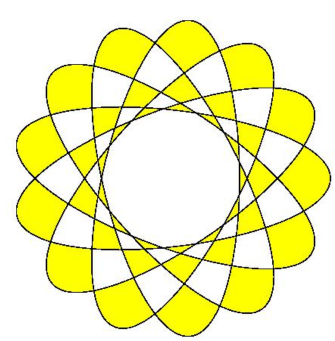
The reasoning that yields the equations for hypotrochoids can be immediately adapted to epitrochoids and result in the following equations:

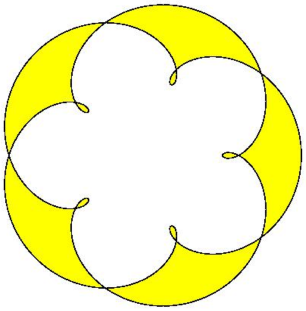
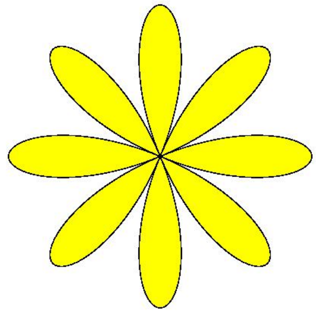
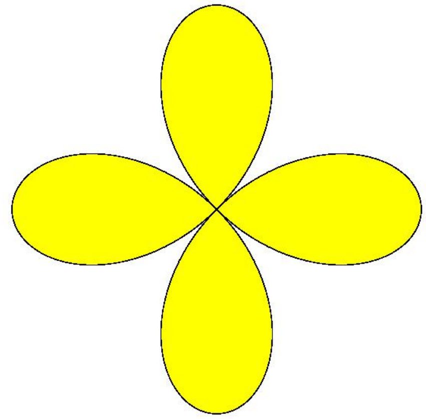
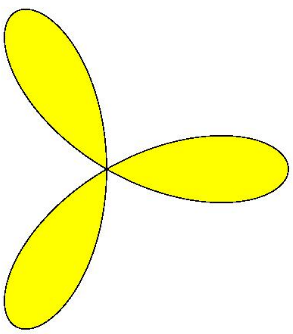
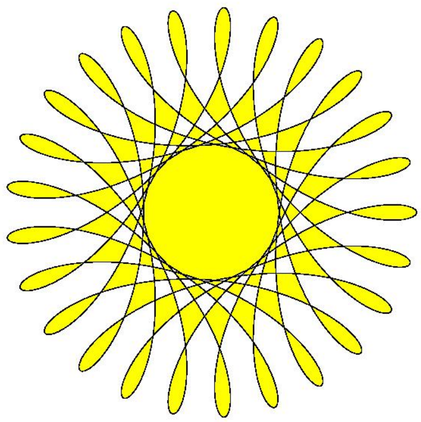
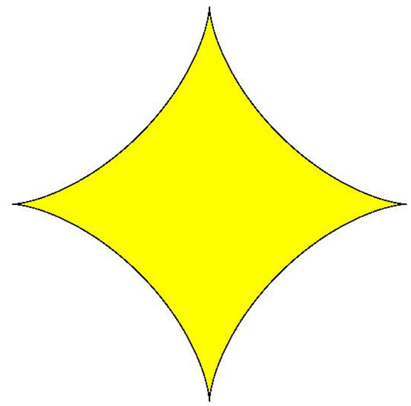
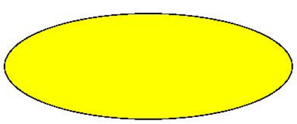
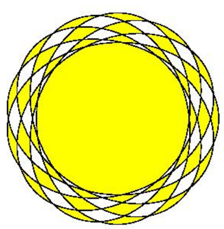
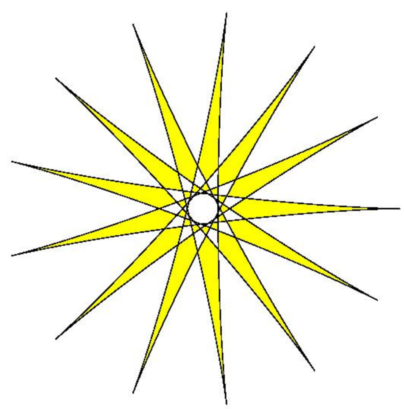
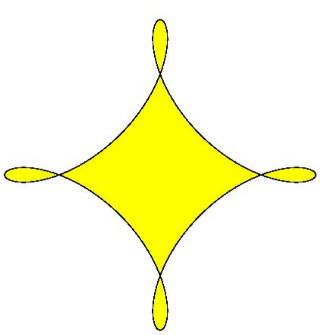
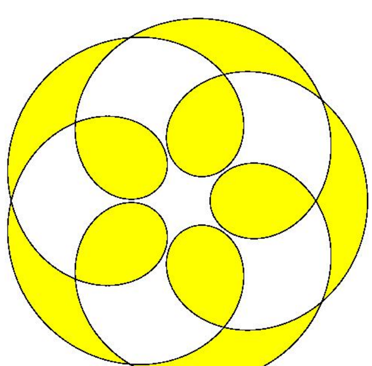
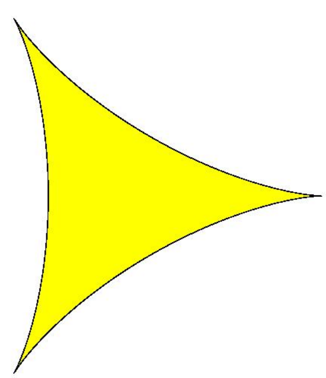
$$x = (R + r)\cos(\theta) - d\cos\Big(\frac{R + r}{r}\theta\Big)$$
$$y = (R + r)\sin(\theta) - d\sin\Big(\frac{R + r}{r}\theta\Big)$$

The period of an epitrochoid is also equal to $\frac{r}{\gcd(r,R)}$.

3. Particular cases

Ellipse, *deltoid*, *astroid*, *nephroid*, *cardioid* and *roses* are amongst the following pictures of epitrochoids (with a green filling) and hypotrochoids (with a yellow filling).

The following table shows how ellipse, deltoid, astroid, nephroid and a few other particular cases are obtained. When d is equal to r, hypotrochoids are also called *hypocycloids*, and epitrochoids are also called *epicycloids*.

	Hypotrochoids					Epitrochoids	
	$r = \frac{R}{2}$	$r \in \{\frac{R}{3}, \frac{2R}{3}\}$	$r \in \{\frac{R}{4}, \frac{3R}{4}\}$	$r = \frac{3R}{2}$	$r = 2R$	$r = \frac{R}{2}$	$r = R$
$d = r$	ellipse	deltoid	astroid	nephroid	cardioid	nephroid	cardioid
$d = 0$	segment	circle					
Any d							Pascal limaçon

To be complete, one should let R be ∞; then $\mathcal{C}_1$ is a line and the associated curves are called *trochoids*, with *cycloids* as a particular case when $d = r\ldots$