

COMP9334

# Capacity Planning for Computer Systems and Networks

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## Week 3: Queues with Poisson arrivals

## Pre-lecture exercise 1:

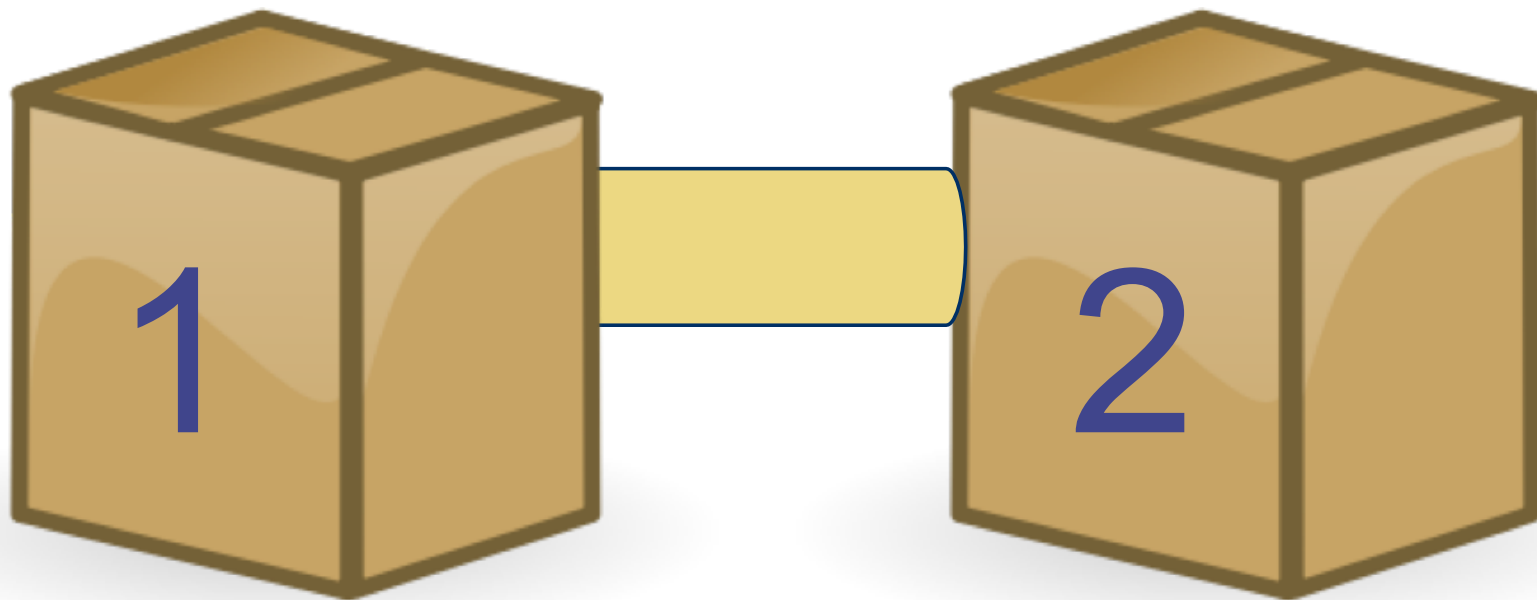
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- Let  $X$  and  $Y$  be two events
- Let  $\text{Prob}[X]$  = Probability that event  $X$  occurs
- Let  $\text{Prob}[Y]$  = Probability that event  $Y$  occurs
- Question: Under what condition will the following equality hold?
  - $\text{Prob}[X \text{ or } Y] = \text{Prob}[X] + \text{Prob}[Y]$

## Pre-lecture exercise 2: Where is Felix? (Page 1)

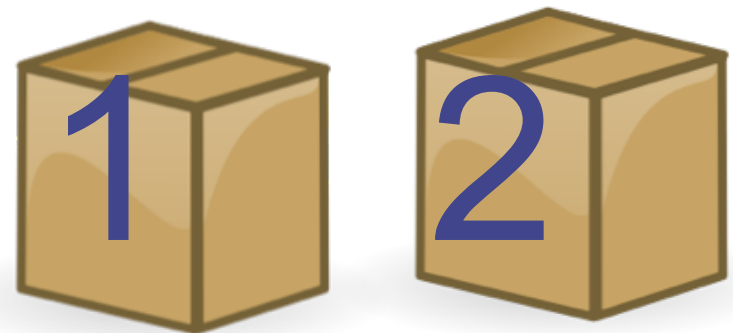
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- You have two boxes: Box 1 and Box 2, as well as a cat called Felix
- The two boxes are connected by a tunnel
- Felix likes to hide inside these boxes and travels between them using the tunnel.
- Felix is a very fast cat so the probability of finding him in the tunnel is zero
- You know Felix is in one of the boxes but you don't know which one



## Pre-lecture exercise 2: Where is Felix? (Page 2)

- Notation:
  - $\text{Prob}[A]$  = probability that event A occurs
  - $\text{Prob}[A \mid B]$  = probability that event A occurs given event B
- You do know
  - Felix is in one of the boxes at times 0 and 1
  - $\text{Prob}[\text{Felix is in Box 1 at time 0}] = 0.3$
  - $\text{Prob}[\text{Felix will be in Box 2 at time 1} \mid \text{Felix is in Box 1 at time 0}] = 0.4$
  - $\text{Prob}[\text{Felix will be in Box 1 at time 1} \mid \text{Felix is in Box 2 at time 0}] = 0.2$
- Calculate
  - $\text{Prob}[\text{Felix is in Box 1 at time 1}]$
  - $\text{Prob}[\text{Felix is in Box 2 at time 1}]$



## Pre-lecture exercise 3

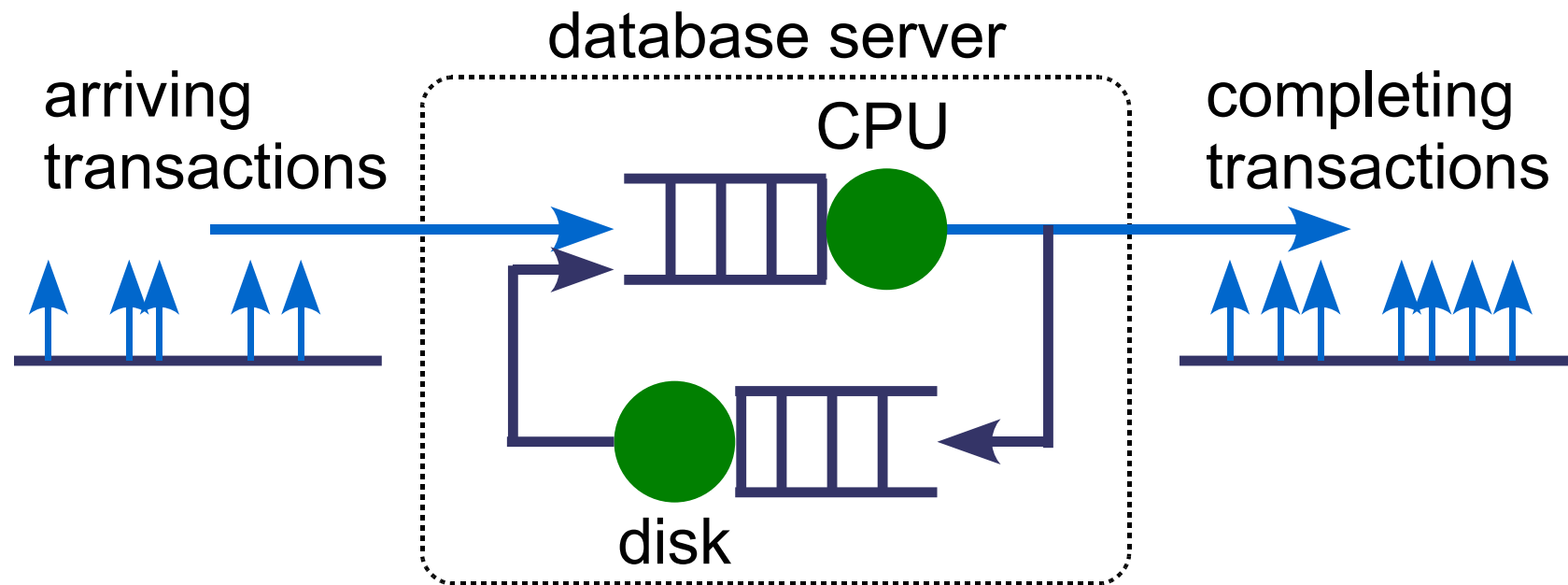
- You have a loaded die with 6 faces with values 1, 2, 3, 4, 5 and 6
- The probability that you can get each face is given in the table below
- What is the mean value that you can get?

Value	Probability
1	0.1
2	0.1
3	0.2
4	0.1
5	0.3
6	0.2



## Week 1:

- Modelling a computer system as a network of queues
- Example: Open queueing network consisting of two queues



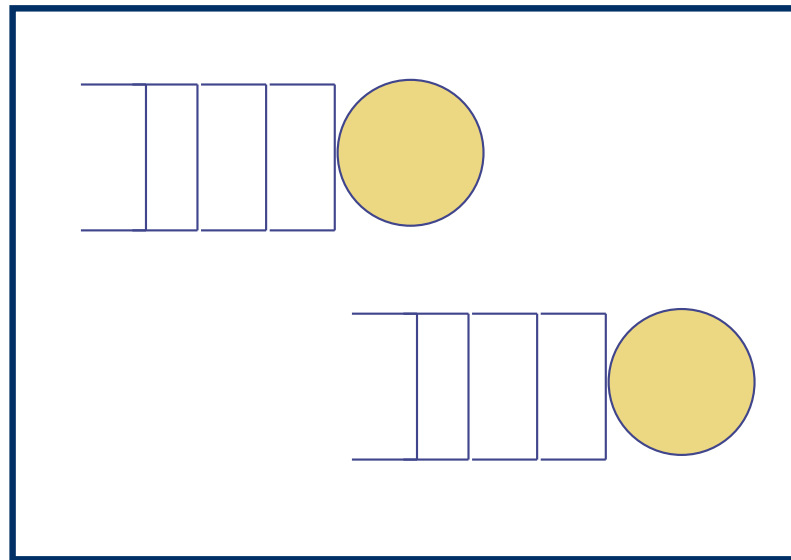
## Week 2:

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- Operational analysis
  - Measure #completed jobs, busy time etc
  - Operational quantities: utilisation, response time, throughput etc.
  - Operational laws relate the operational quantities
- Bottleneck analysis

# Little's Law

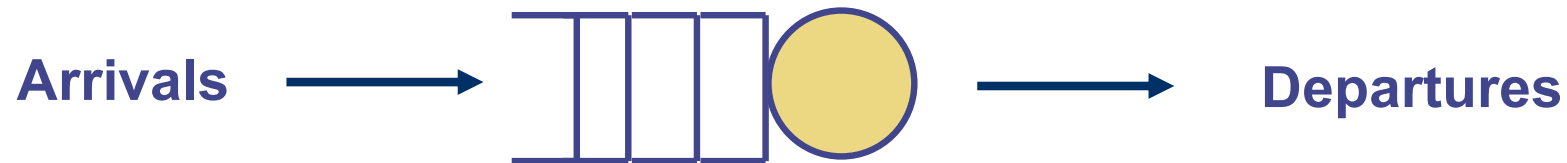
- Applicable to any “box” that contains some queues or servers
- Mean number of jobs in the “box” =  
Mean response time x Throughput
- We will use Little's Law in this lecture to derive the mean response time
  - We first compute the mean number of jobs in the “box” and throughput





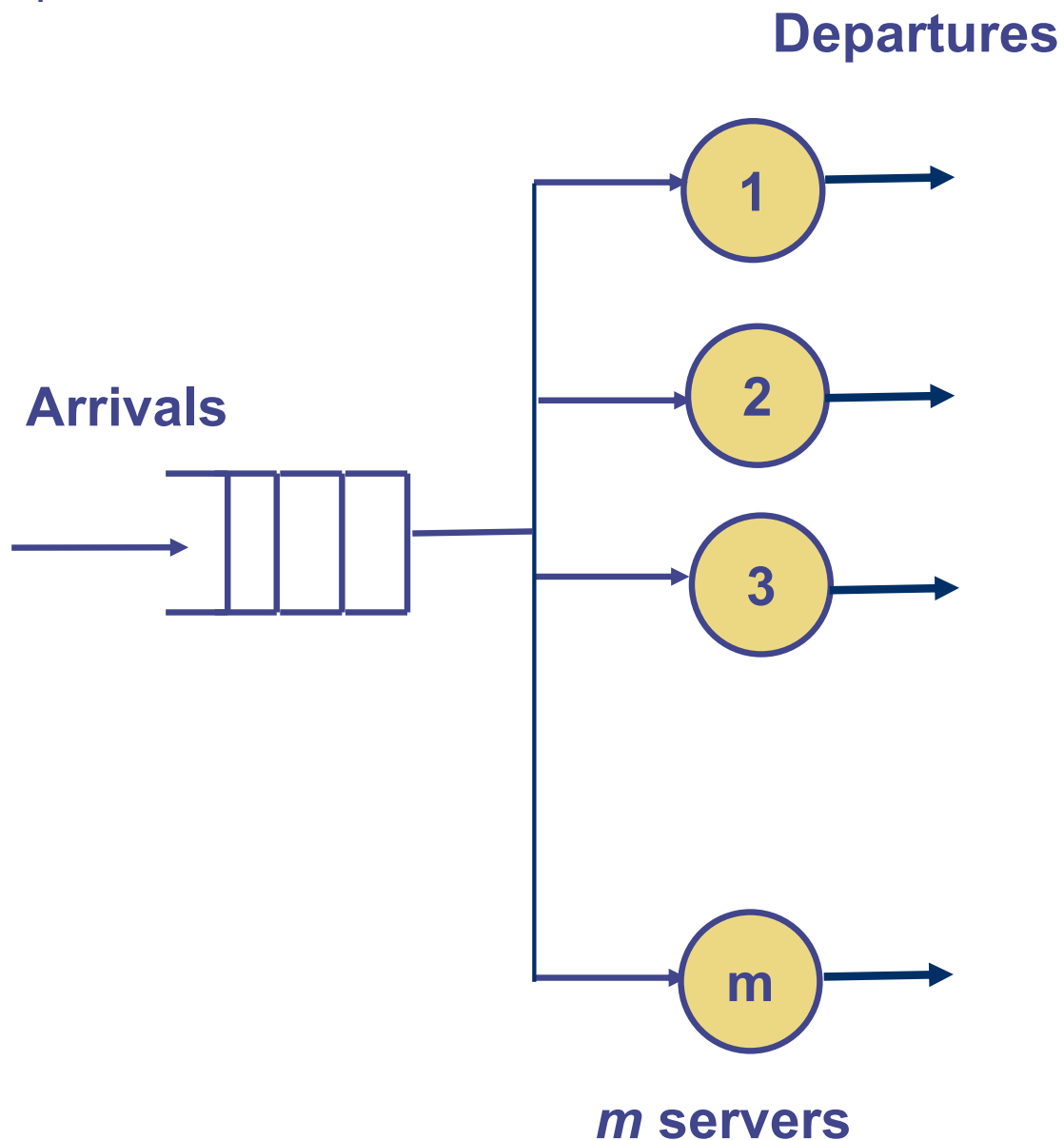
## This week (1)

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- Open, single server queues and
- How to find:
  - Waiting time
  - Response time
  - Mean queue length etc.
- The technique to find waiting time etc. is called *Queueing Theory*

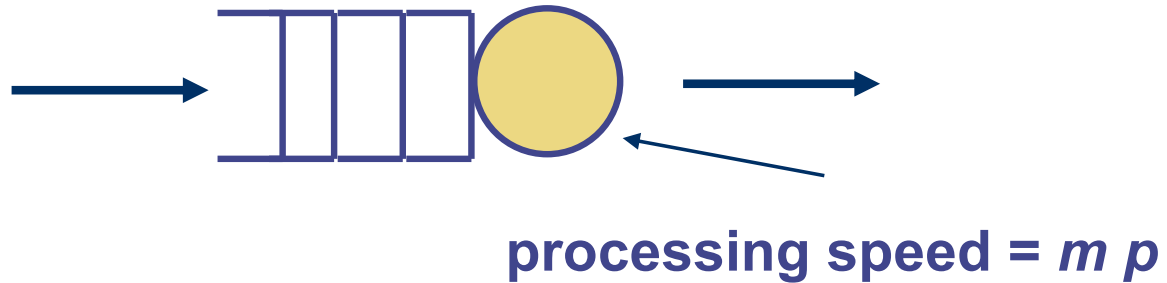
## This week (2)



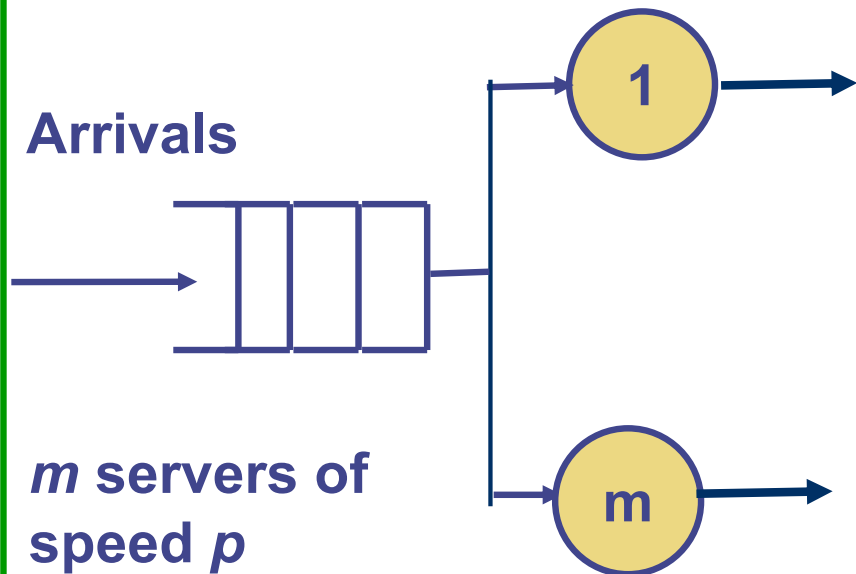
- Open, multi-server queue
- How to find:
  - Waiting time
  - Response time
  - Mean queue length etc.

# What will you be able to do with the results?

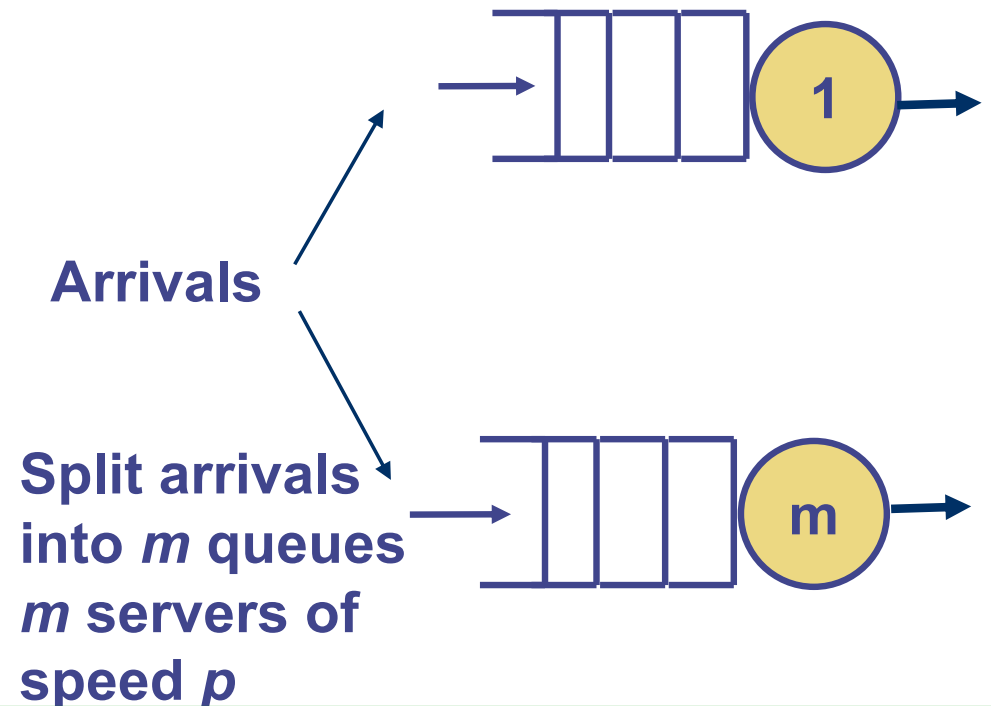
Configuration 1:



Configuration 2:



Configuration 3:



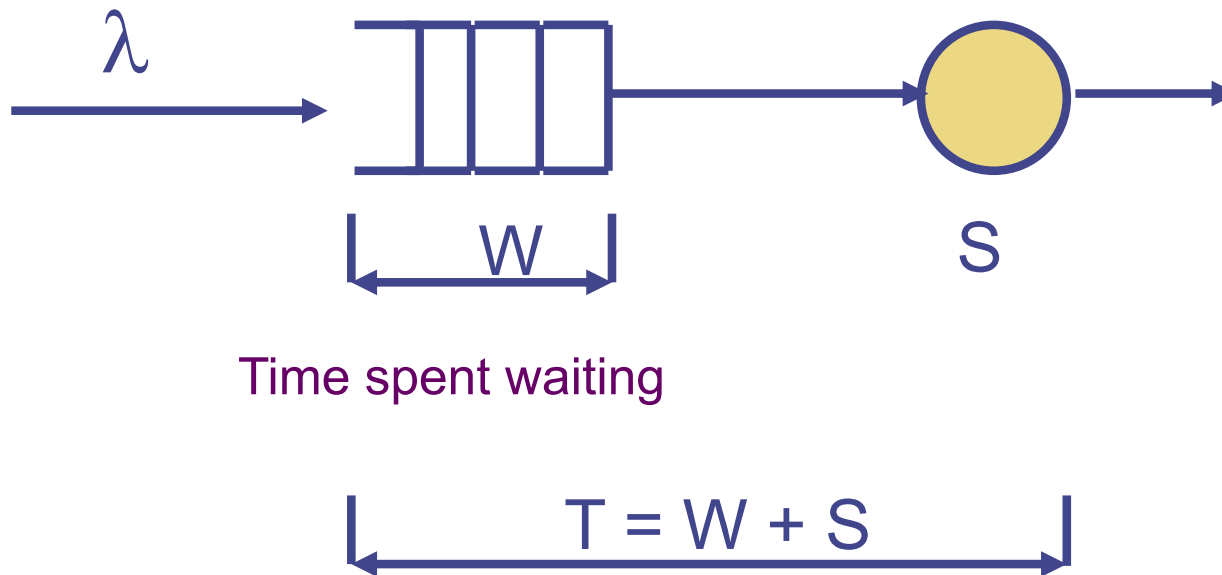
Which configuration has the best response time?

# Be patient

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- We will show how we can obtain the response time
  - It takes a number of steps to obtain the answer
- It takes time to stand in a queue, it also takes time to derive results in queuing theory!

# Single Server Queue: Terminology



Response Time  $T$   
= Waiting time  $W$  + Service time  $S$

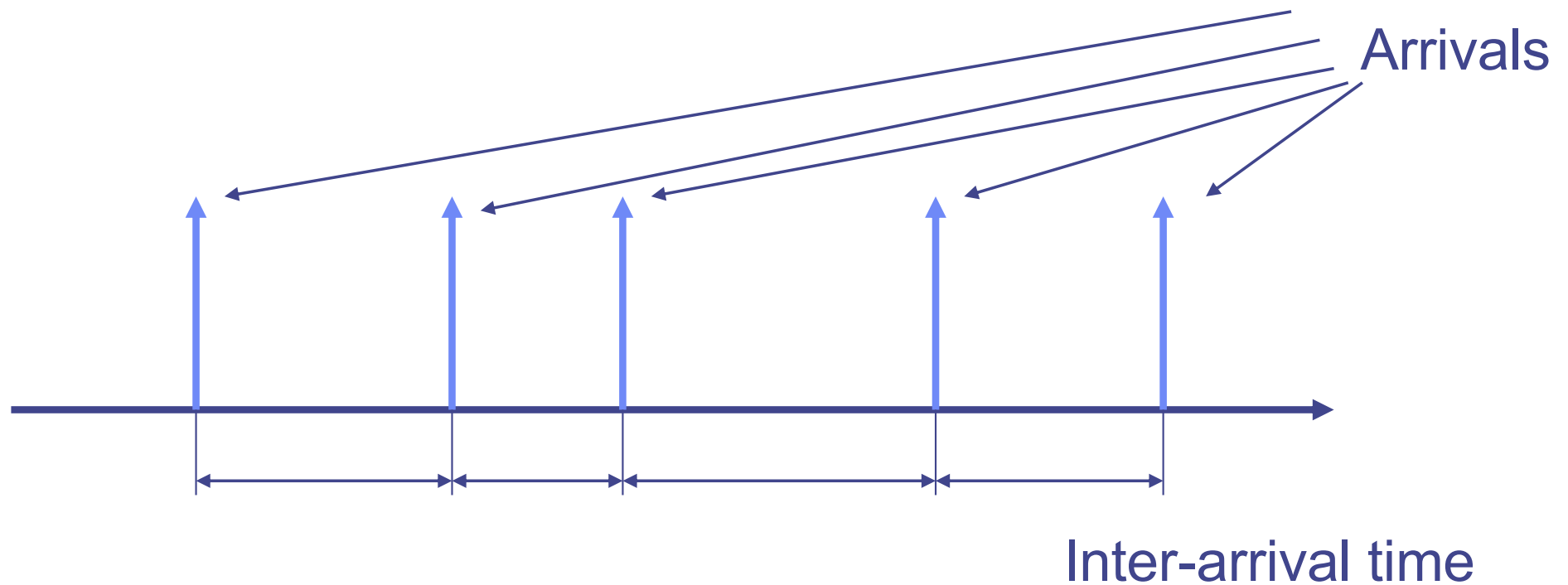
Note: We use  $T$  for response time because this is the notation in many queueing theory books. For a similar reason, we will use  $\rho$  for utilisation rather than  $U$ .

# Single server system

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- In order to determine the response time, you need to know
  - The inter-arrival time probability distribution
  - The service time probability distribution
- Possible distributions
  - Deterministic
    - Constant inter-arrival time
    - Constant service time
  - Exponential distribution
- We will focus on exponential distribution

## Exponential inter-arrival with rate $\lambda$



We assume that successive arrivals are independent

Probability that inter-arrival time is between  $x$  and  $x + \delta x$   
 $= \lambda \exp(-\lambda x) \delta x$

# Poisson distribution (1)

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- The following are equivalent
  - The inter-arrival time is independent and exponentially distributed with parameter  $\lambda$
  - The number of arrivals in an interval  $T$  is a Poisson distribution with parameter  $\lambda$

$$Pr[k \text{ arrivals in a time interval } T] = \frac{(\lambda T)^k \exp(-\lambda T)}{k!}$$

- Mean inter-arrival time =  $1 / \lambda$
- Mean number of arrivals in time interval  $T = \lambda T$
- Mean arrival rate =  $\lambda$



## Poisson distribution (2)


- Poisson distribution arises from a large number of independent sources
  - An example from Week 2:
    - $N$  customers, each with a probability of  $p$  per unit time to make a request.
    - This creates a Poisson arrival with  $\lambda = Np$
- Another interpretation of Poisson arrival:
  - Consider a small time interval  $\delta$ 
    - This means  $\delta^n$  (for  $n \geq 2$ ) is negligible
  - Probability [ no arrival in  $\delta$  ] =  $1 - \lambda \delta$
  - Probability [ 1 arrival in  $\delta$  ] =  $\lambda \delta$
  - Probability [ 2 or more arrivals in  $\delta$  ]  $\approx 0$
- This interpretation can be derived from:

$$Pr[k \text{ arrivals in a time interval } T] = \frac{(\lambda T)^k \exp(-\lambda T)}{k!}$$

# Service time distribution

- Service time = the amount of processing time a job requires from the server
- We assume that the service time distribution is exponential with parameter  $\mu$ 
  - The probability that the service time is between  $t$  and  $t + \delta t$  is:

$$\mu \exp(-\mu t) \delta t$$

- Here:  $\mu$  = service rate =  $1 / \text{mean service time}$
- Another interpretation of exponential service time:
  - Consider a small time interval  $\delta$
  - Probability [ a job will finish its service in next  $\delta$  seconds ] =  $\mu \delta$
  - Probability [ a job will **not** finish its service in next  $\delta$  seconds ] = 

# Sample queueing problems

- Consider a call centre
  - Calls are arriving according to Poisson distribution with rate  $\lambda$
  - The length of each call is exponentially distributed with parameter  $\mu$ 
    - Mean length of a call is  $1/\mu$  (in, e.g. seconds)

**Call centre:**

**Arrivals**



**$m$  operators**

**If all operators are busy, the centre can put at most  $n$  additional calls on hold.**

**If a call arrives when all operators and holding slots are used, the call is rejected.**



- Queueing theory will be able to answer these questions:
  - What is the mean waiting time for a call?
  - What is the probability that a call is rejected?

# Road map

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- We will start by looking at a call centre with one operator and no holding slot
  - This may sound unrealistic but we want to show how we can solve a typical queueing network problem
  - After that we go into queues that are more complicated

# Call centre with 1 operator and no holding slots

- Let us see how we can solve the queuing problem for a very simple call centre with 1 operator and no holding slots
- What happens to a call that arrives when the operator is busy?
  - 
- What happens to a call that arrives when the operator is idle?
  - 
- We are interested to find the probability that an arriving call is rejected.

**Arrivals**



**Call centre:**

**1 operator. No holding slot.**

## Solution (1)

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- There are two possibilities for the operator:
  - Busy or
  - Idle
- Let
  - State 0 = Operator is idle (i.e. #calls in the call centre = ?)
  - State 1 = Operator is busy (i.e. #calls in the call centre = ?)


$P_0(t)$  = Prob. 0 call in the call centre at time  $t$

$P_1(t)$  = Prob. 1 call in the call centre at time  $t$

## Solution (2)

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We try to express  $P_0(t + \Delta t)$  in terms of  $P_0(t)$  and  $P_1(t)$

- No call at call centre at  $t + \Delta t$  can be caused by
  - **No call at time  $t$**  and **no call arrives in  $[t, t + \Delta t]$** , or
  - 

$$P_0(t + \Delta t) = \img alt="Yellow rectangular box" data-bbox="347 558 652 708"/> + \img alt="Yellow rectangular box" data-bbox="698 558 928 708"/>$$

**Question: Why do we NOT have to consider the following possibility:  
No customer at time  $t$  & 1 customer arrives in  $[t, t + \Delta t]$  &  
the call finishes within  $[t, t + \Delta t]$ .**

## Solution (3)

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- Similarly, we can show that

$$P_1(t + \Delta t) = P_0(t)\lambda\Delta t + P_1(t)(1 - \mu\Delta t)$$

- If we let  $\Delta t \rightarrow 0$ , we have

$$\frac{dP_0(t)}{dt} = -P_0(t)\lambda + P_1(t)\mu$$

$$\frac{dP_1(t)}{dt} = P_0(t)\lambda - P_1(t)\mu$$



## Solution (4)

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- We can solve these equations to get

$$P_0(t) = \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\mu + \lambda)t}$$

$$P_1(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\mu + \lambda)t}$$

- This is too complicated, let us look at **steady state** solution

$$P_0 = P_0(\infty) = \frac{\mu}{\lambda + \mu}$$

$$P_1 = P_1(\infty) = \frac{\lambda}{\lambda + \mu}$$

## Solution (5)

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- From the steady state solution, we have
  - The probability that an arriving call is rejected
  - = The probability that the operator is busy

- =
$$P_1 = \frac{\lambda}{\lambda + \mu}$$

- Let us check whether it makes sense
  - For a constant  $\mu$ , if the arrival rate  $\lambda$  increases, will the probability that the operator is busy go up or down?
  - Does the formula give the same prediction?

## An alternative interpretation

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- We have derived the following equation:

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda\Delta t) + P_1(t)\mu\Delta t$$

- Which can be rewritten as:

$$P_0(t + \Delta t) - P_0(t) = -P_0(t)\lambda\Delta t + P_1(t)\mu\Delta t$$

- At steady state:

Change in Prob in State 0 = 0

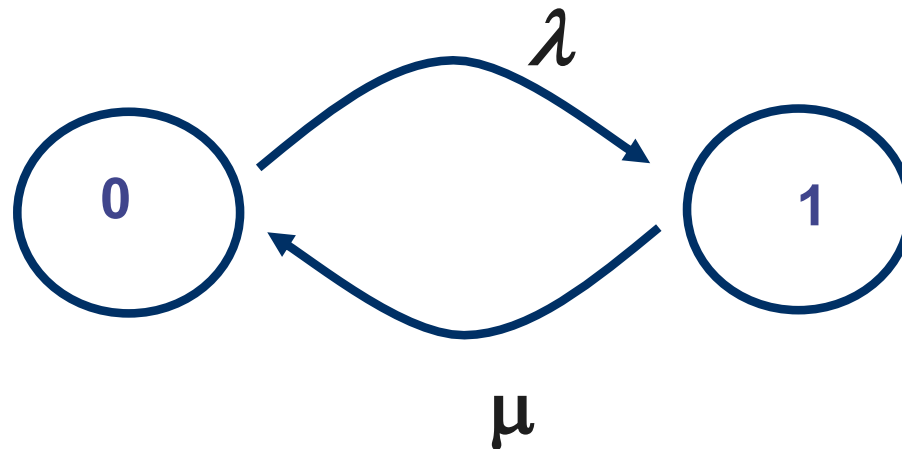
$$\Rightarrow 0 = -\boxed{P_0\lambda}\Delta t + \boxed{P_1\mu}\Delta t$$

**Rate of leaving state 0**

**Rate of entering state 0**

# Faster way to obtain steady state solution (1)

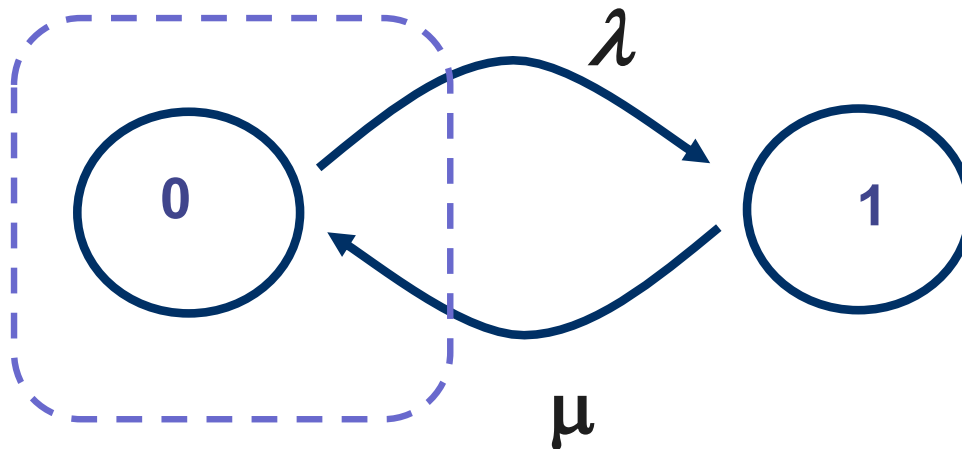
- Transition from State 0 to State 1
  - Caused by an arrival, the rate is  $\lambda$
- Transition from State 1 to State 0
  - Caused by a completed service, the rate is  $\mu$
- State diagram representation
  - *Each circle is a state*
  - *Label the arc between the states with transition rate*



## Faster way to obtain steady state solution (2)

- Steady state means
  - **rate of transition out of a state = Rate of transition into a state**
- We have for state 0:

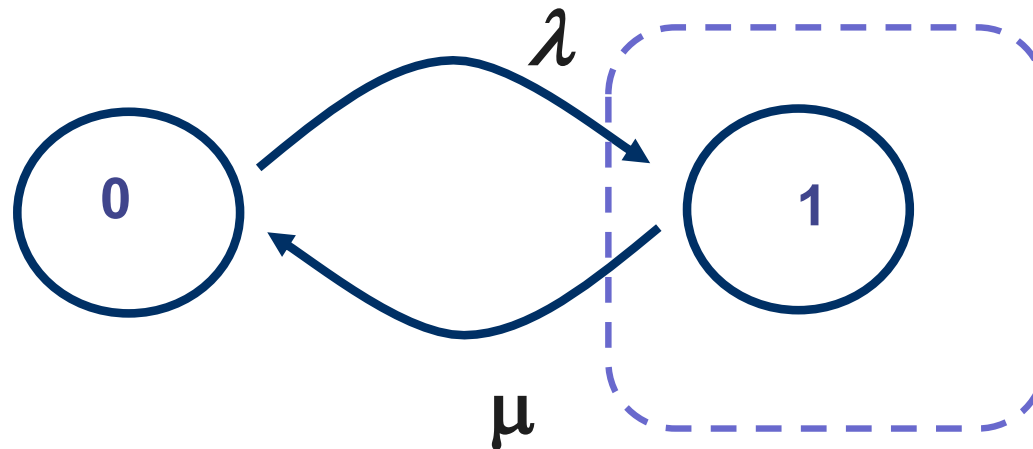
$$\underline{\lambda P_0} = \underline{\mu P_1}$$



## Faster way to obtain steady state solution (3)

- We can do the same for State 1:
- Steady state means
  - **Rate of transition into a state** = **rate of transition out of a state**
- We have for state 1:

$$\underline{\lambda P_0} = \underline{\mu P_1}$$



## Faster way to obtain steady state solution (4)

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- We have one equation  $\lambda P_0 = \mu P_1$
- We have 2 unknowns and we need one more equation.
- Since we must be either one of the two states:

$$P_0 + P_1 = 1$$

- Solving these two equations, we get the same steady state solution as before

$$P_0 = \frac{\mu}{\lambda + \mu} \quad P_1 = \frac{\lambda}{\lambda + \mu}$$

# Summary

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- Solving a queueing problem is not simple
- It is harder to find how a queue evolves with time
- It is simpler to find how a queue behaves at steady state
  - Procedure:
    - Draw a diagram with the states
    - Add arcs between states with transition rates
    - Derive flow balance equation for each state, i.e.
      - Rate of entering a state = Rate of leaving a state
    - Solve the equation for steady state probability



# Let us have a look at our call centre problem again

- Consider a call centre
  - Calls are arriving according to Poisson distribution with rate  $\lambda$
  - The length of each call is exponentially distributed with parameter  $\mu$ 
    - Mean length of a call is  $1/\mu$

## Call centre:

Arrivals



**$m$  operators**

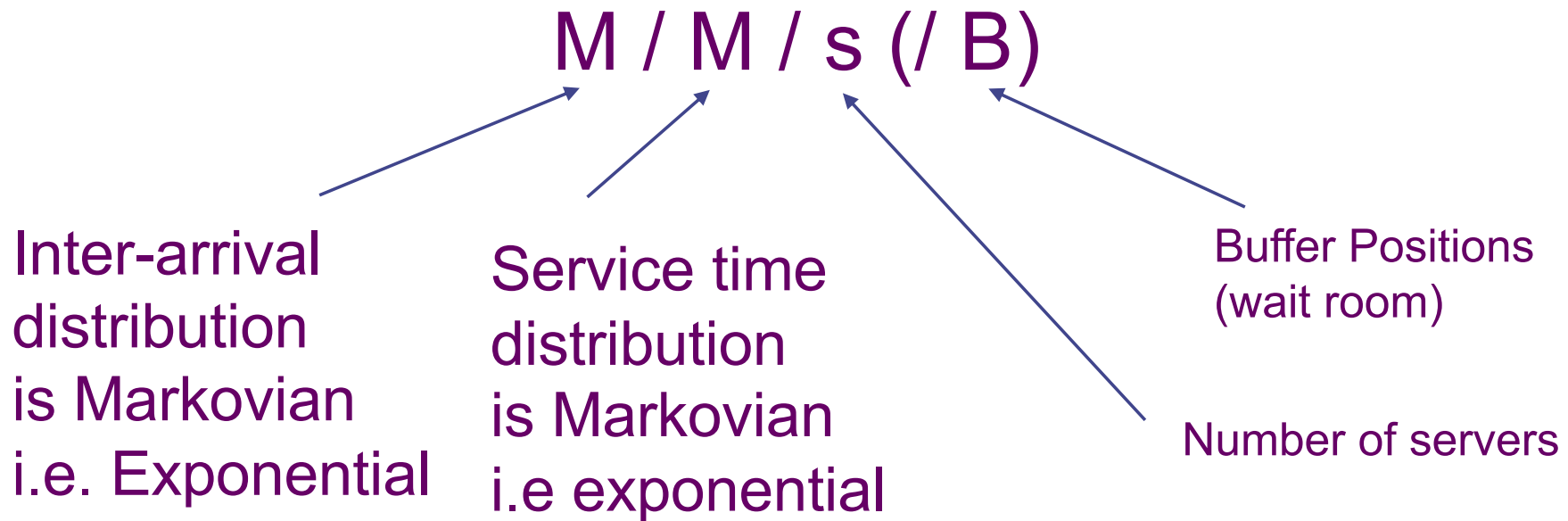
**If all operators are busy, the centre can put at most  $n$  additional calls on hold.**

**If a call arrives when all operators and holding slots are used, the call is rejected.**

- We solve the problem for  $m = 1$  and  $n = 0$ 
  - We call this a M/M/1/1 queue (explanation on the next page)
- How about other values of  $m$  and  $n$

# Kendall's notation

- To represent different types of queues, queueing theorists use the Kendall's notation
- The call centre example on the previous page can be represented as:

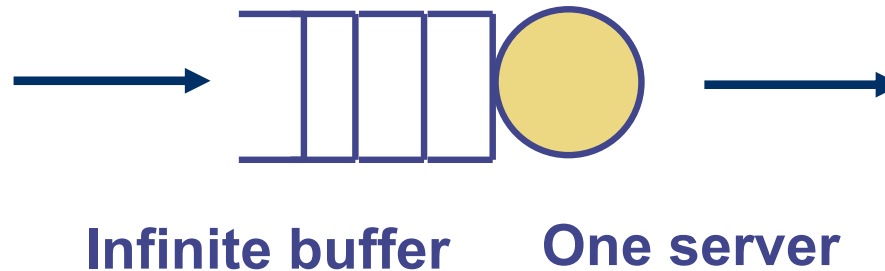


The call centre example on the last page is a  $M/M/m/(m+n)$  queue  
If  $n = \infty$ , we simply write  $M/M/m$

# M/M/1 queue

Exponential  
Inter-arrivals ( $\lambda$ )

Exponential  
Service time ( $\mu$ )



- Consider a call centre analogy
  - Calls are arriving according to Poisson distribution with rate  $\lambda$
  - The length of each call is exponentially distributed with parameter  $\mu$ 
    - Mean length of a call is  $1/\mu$

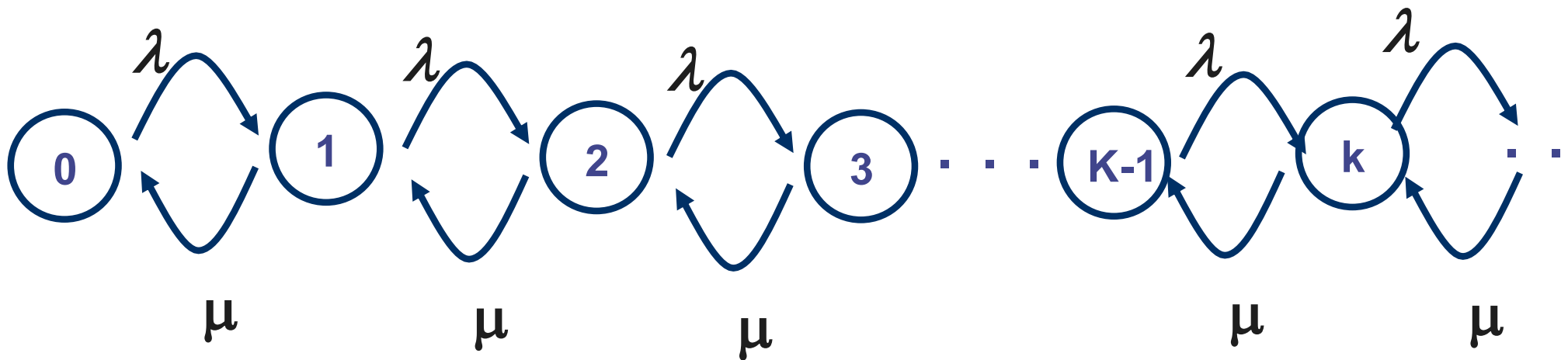
Arrivals  
→

**Call centre with 1 operator**  
**If the operator is busy, the centre will put the call on hold.**  
**A customer will wait until his call is answered.**

- Queueing theory will be able to answer these questions:
  - What is the mean waiting time for a call?

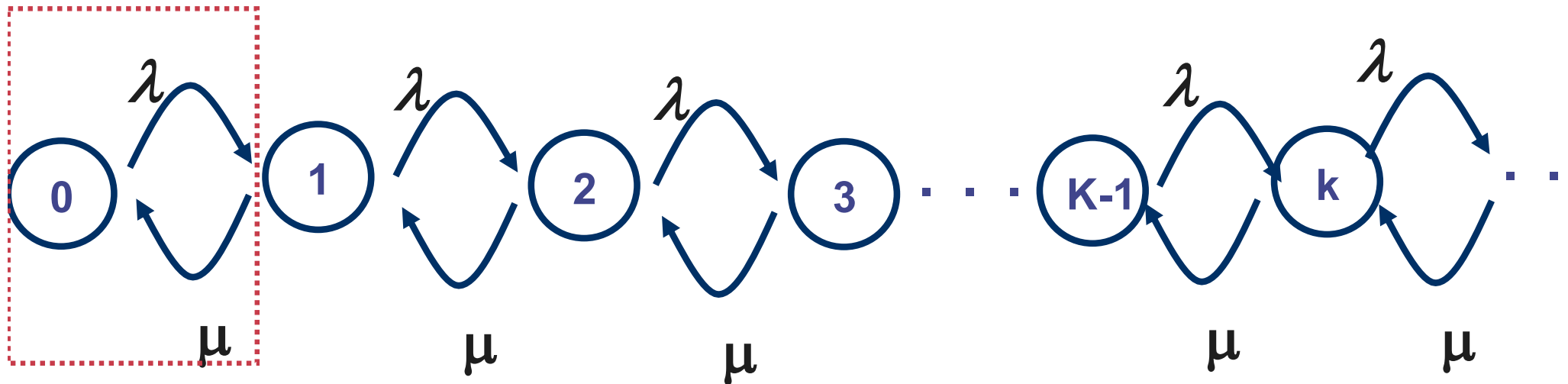
# Solving M/M/1 queue (1)

- We will solve for the steady state response
- Define the states of the queue
  - State 0 = There is zero job in the system (= The server is idle)
  - State 1 = There is 1 job in the system (= 1 job at the server, no job queueing)
  - State 2 = There are 2 jobs in the system (= 1 job at the server, 1 job queueing)
  - State  $k$  = There are  $k$  jobs in the system (= 1 job at the server,  $k-1$  job queueing)
- The state transition diagram



## Solving M/M/1 queue (2)

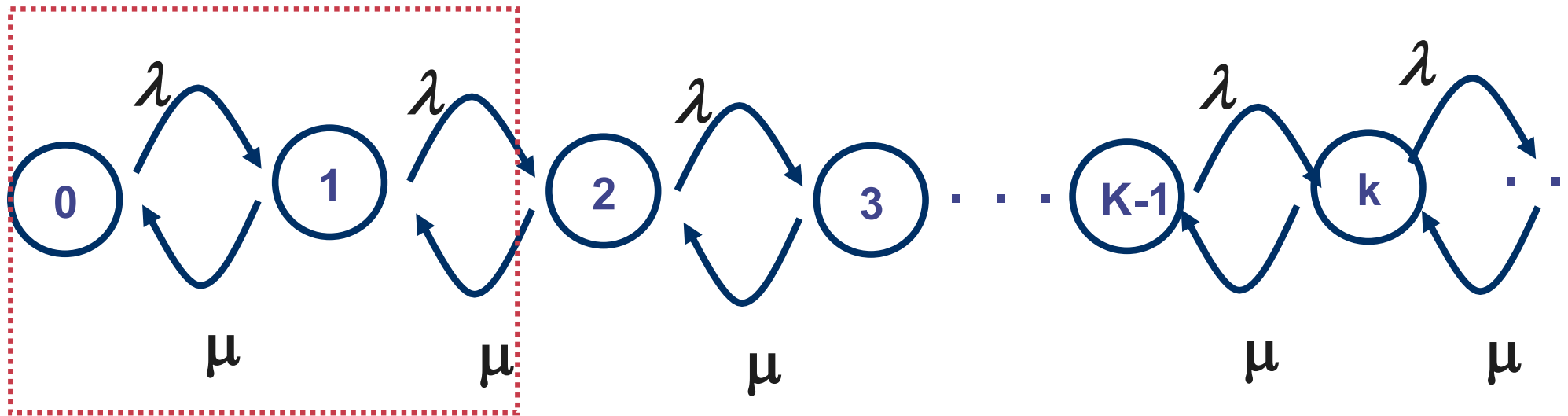
$P_k$  = Prob.  $k$  jobs in system



$$\lambda P_0 = \mu P_1$$

$$\Rightarrow P_1 = \frac{\lambda}{\mu} P_0$$

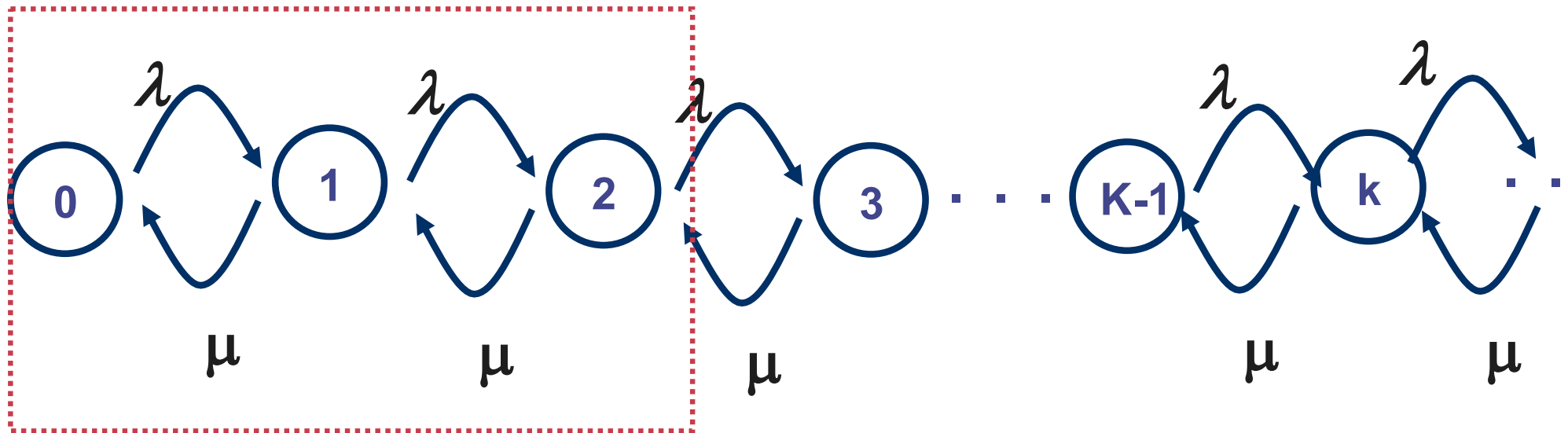
## Solving M/M/1 queue (3)



$$\lambda P_1 = \mu P_2$$

$$\Rightarrow P_2 = \frac{\lambda}{\mu} P_1 \Rightarrow P_2 = \left( \frac{\lambda}{\mu} \right)^2 P_0$$

## Solving M/M/1 queue (4)



$$\lambda P_2 = \mu P_3$$

$$\Rightarrow P_3 = \frac{\lambda}{\mu} P_2 \Rightarrow P_3 = \left( \frac{\lambda}{\mu} \right)^3 P_0$$

## Solving M/M/1 queue (5)

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**In general** 
$$P_k = \left( \frac{\lambda}{\mu} \right)^k P_0$$

Let 
$$\rho = \frac{\lambda}{\mu}$$

**We have** 
$$P_k = \rho^k P_0$$



## Solving M/M/1 queue (6)

With  $P_k = \rho^k P_0$  and

$$P_0 + P_1 + P_2 + P_3 + \dots = 1$$

$$\Rightarrow (1 + \rho + \rho^2 + \dots)P_0 = 1$$

$$\Rightarrow P_0 = 1 - \rho \text{ if } \rho < 1$$

$$\Rightarrow P_k = (1 - \rho)\rho^k$$

Since  $\rho = \frac{\lambda}{\mu}$ ,  $\rho < 1 \Rightarrow \lambda < \mu$

$\rho$  = utilisation  
= Prob server is busy  
=  $1 - P_0$   
= 1 - Prob server is idle

Arrival rate < service rate

## Solving M/M/1 queue (7)

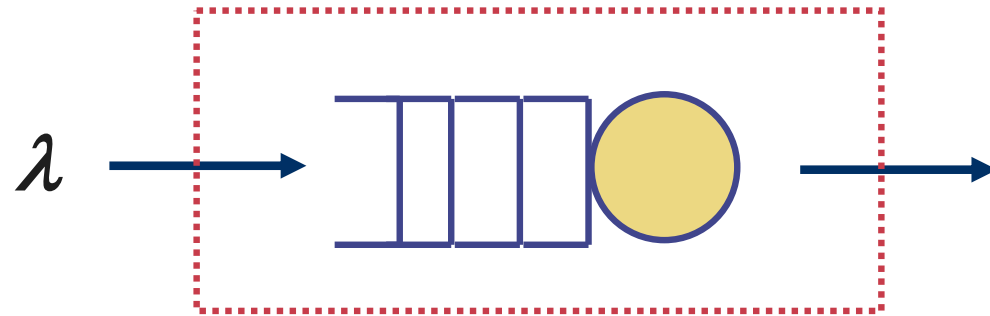
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With  $P_k = (1 - \rho)\rho^k$

This is the probability that there are  $k$  jobs in the system.  
To find the response time, we will make use of Little's law.  
First we need to find the mean number of customers =

$$\begin{aligned}\sum_{k=0}^{\infty} k P_k &= \sum_{k=0}^{\infty} k (1 - \rho) \rho^k \\ &= \frac{\rho}{1 - \rho}\end{aligned}$$

## Solving M/M/1 queue (8)



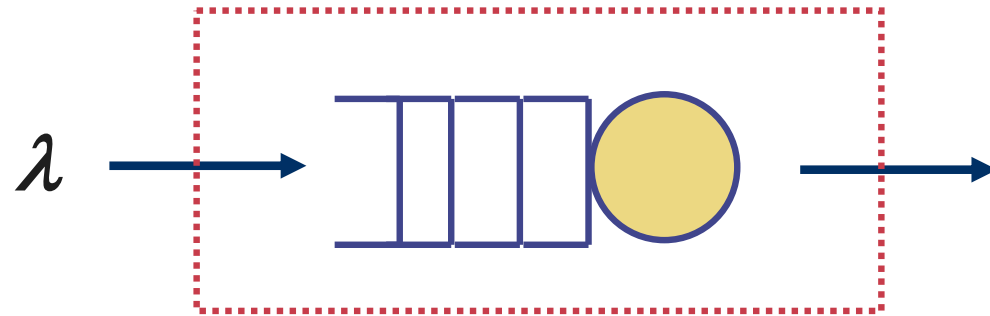
Little's law:

mean number of customers = throughput x response time

Throughput is  $\lambda$  (*why?*)

$$\text{Response time } T = \frac{\rho}{\lambda(1 - \rho)} = \frac{1}{\mu - \lambda}$$

## Solving M/M/1 queue (9)



**What is the mean waiting time at the queue?**

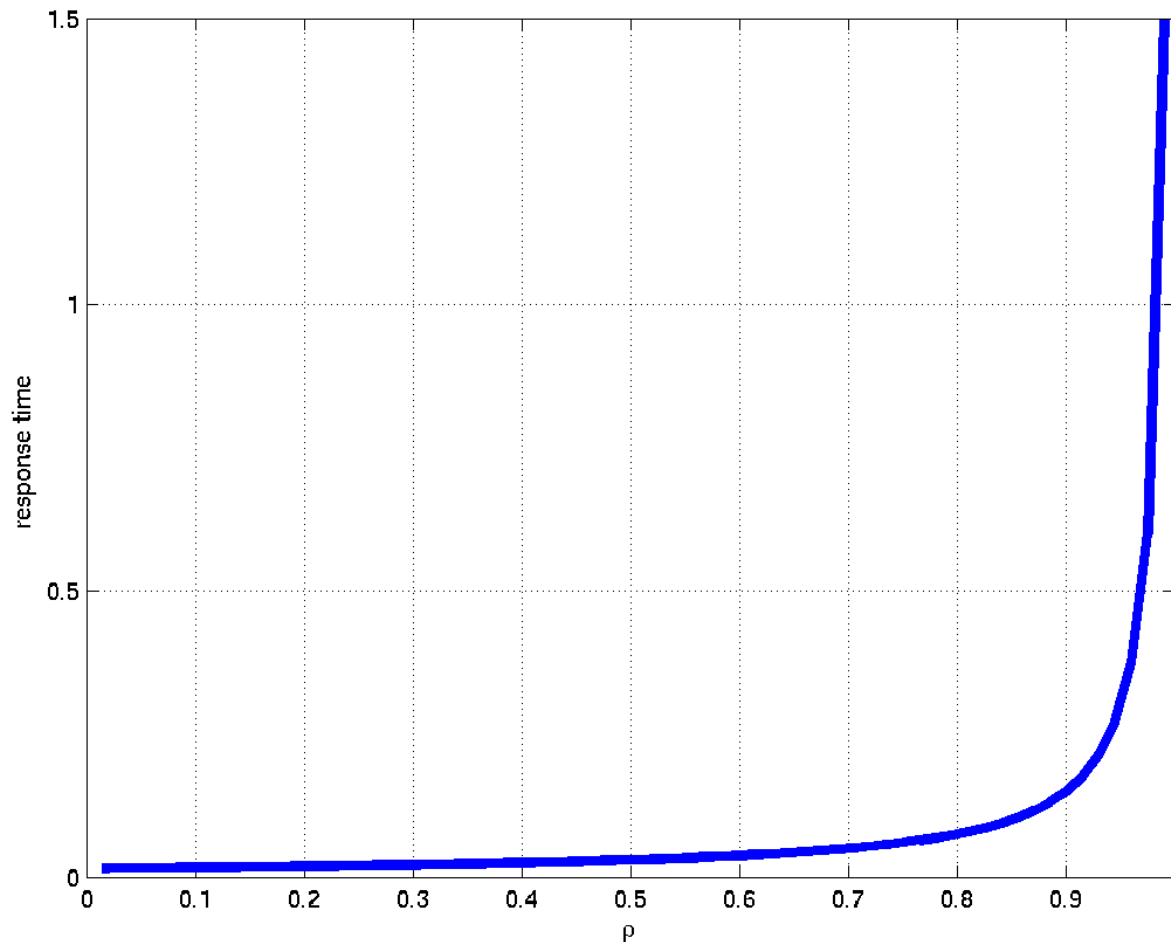
**Mean waiting time = mean response time - mean service time**

**We know mean response time (from last slide)**

**Mean service time is  $= 1 / \mu$**

Using the service time parameter ( $1/\mu = 15\text{ms}$ ) in the example, let us see how response time  $T$  varies with  $\lambda$

$$T = \frac{1}{\mu(1 - \rho)}$$



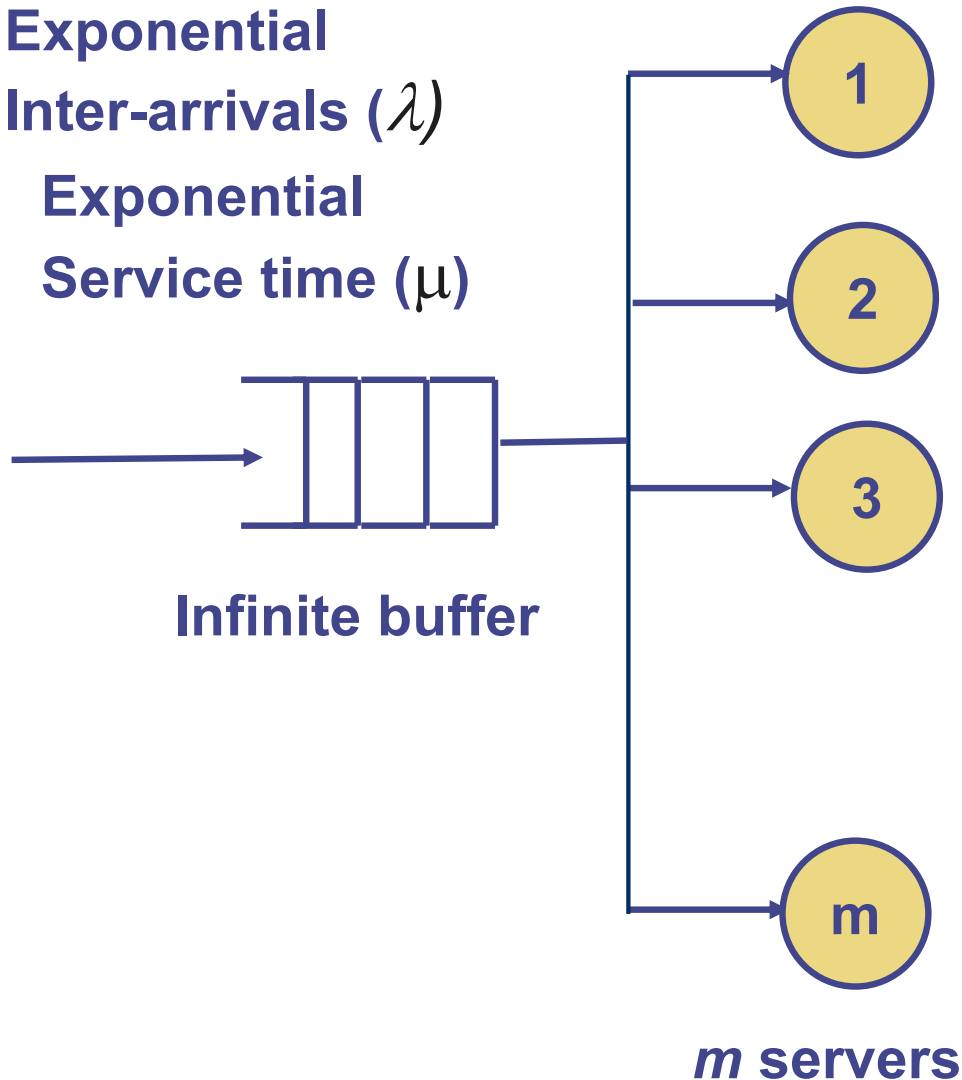
Observation:  
Response time  
increases  
sharply when  
 $\rho$  gets close  
to 1

Infinite queue  
assumption  
means  $\rho \rightarrow 1$ ,  
 $T \rightarrow \infty$

# Multi-server queues M/M/m

Exponential  
Inter-arrivals ( $\lambda$ )

Exponential  
Service time ( $\mu$ )



All arrivals go into one queue.

Customers can be served by any one of the  $m$  servers.

When a customer arrives

- If all servers are busy, it will join the queue
- Otherwise, it will be served by one of the available servers

# A call centre analogy of M/M/m queue

- Consider a call centre analogy
  - Calls are arriving according to Poisson distribution with rate  $\lambda$
  - The length of each call is exponentially distributed with parameter  $\mu$ 
    - Mean length of a call is  $1/\mu$

Arrivals

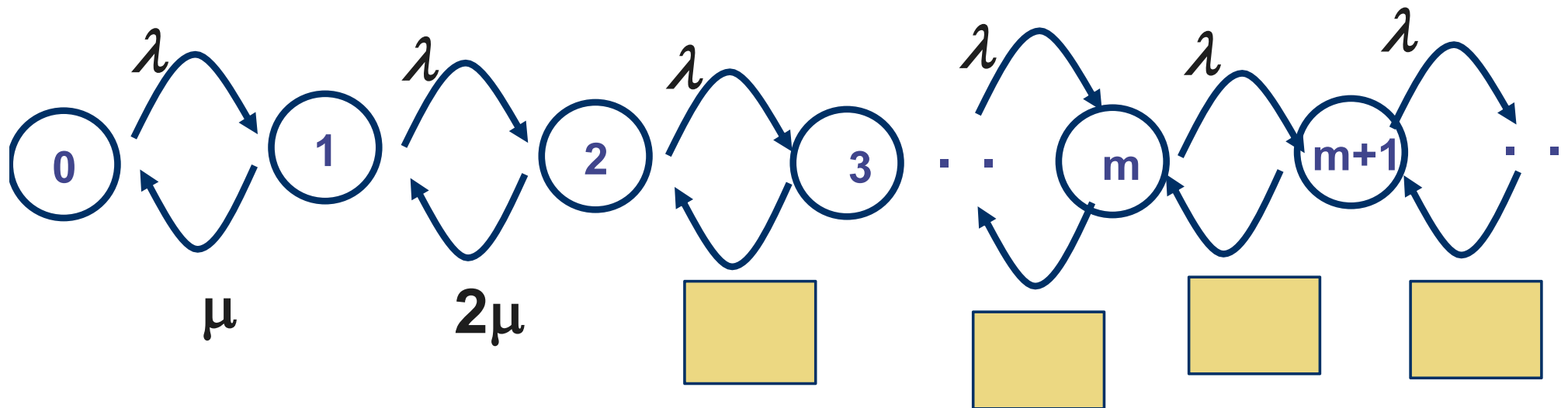


**Call centre with  $m$  operators**

**If all  $m$  operators are busy, the centre will put the call on hold.**

**A customer will wait until his call is answered.**

## State transition for M/M/m





- Following the same method, we have mean response time  $T$  is

$$T = \frac{C(\rho, m)}{m\mu(1 - \rho)} + \frac{1}{\mu}$$

where

$$\rho = \frac{\lambda}{m\mu}$$

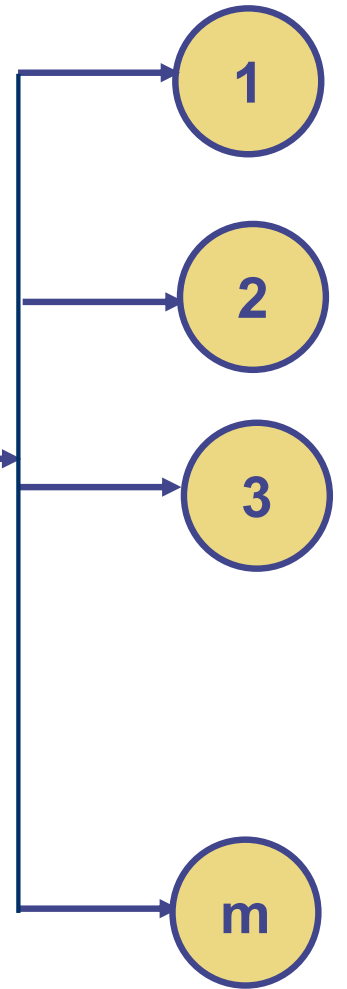
$$C(\rho, m) = \frac{\frac{(m\rho)^m}{m!}}{(1 - \rho) \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!}}$$

# Multi-server queues M/M/m/m with no waiting room

Exponential  
Inter-arrivals ( $\lambda$ )

Exponential  
Service time ( $\mu$ )

No waiting  
Room or  
No buffer



***m* servers**

An arrival can be served  
by any one of the  $m$   
servers.

When a customer arrives  
• If all servers are busy, it  
will *depart* from the  
system

• Otherwise, it will be  
served by one of the  
available servers

# A call centre analogy of M/M/m/m queue

---

- Consider a call centre analogy
  - Calls are arriving according to Poisson distribution with rate  $\lambda$
  - The length of each call is exponentially distributed with parameter  $\mu$ 
    - Mean length of a call is  $1/\mu$

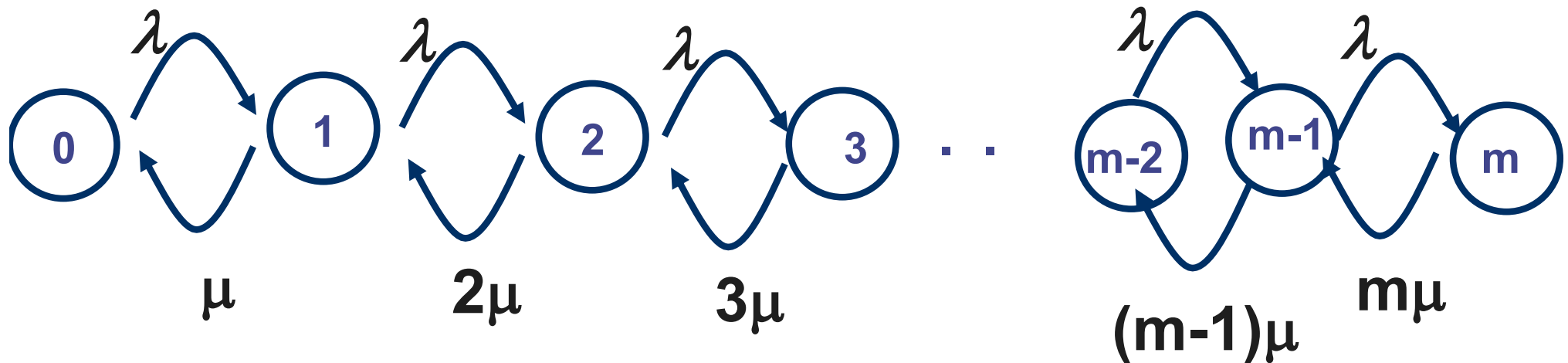
Arrivals



**Call centre with  $m$  operators**

**If all  $m$  operators are busy, the call is dropped.**

## State transition for M/M/m/m



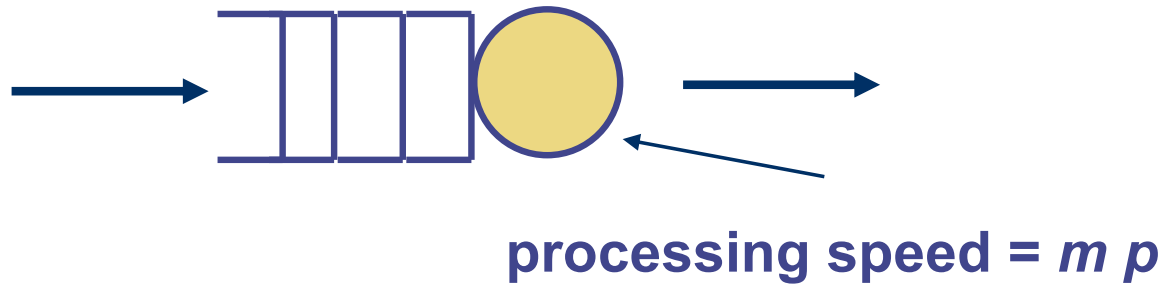
**Probability that an arrival is blocked**  
**= Probability that there are m customers in the system**

$$P_m = \frac{\frac{\rho^m}{m!}}{\sum_{k=0}^m \frac{\rho^k}{k!}} \quad \text{where} \quad \rho = \frac{\lambda}{\mu}$$

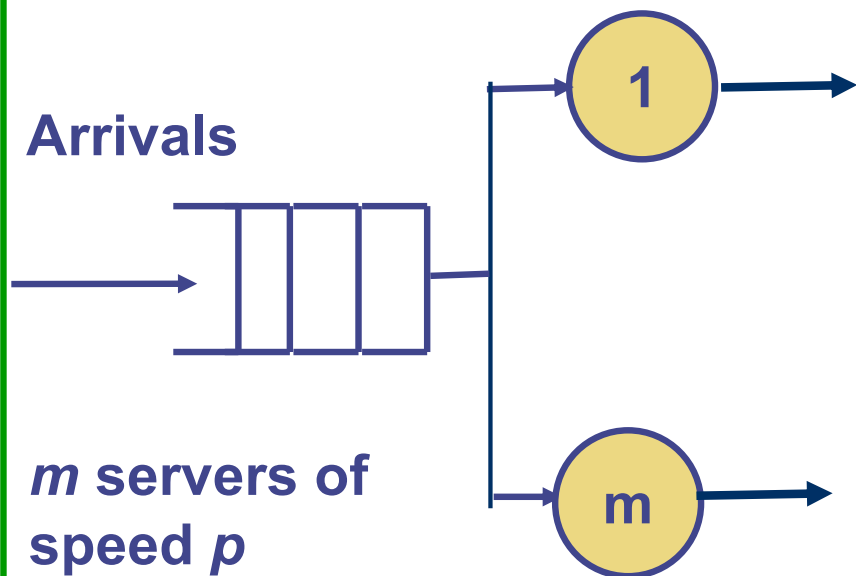
“Erlang B formula”

# What configuration has the best response time?

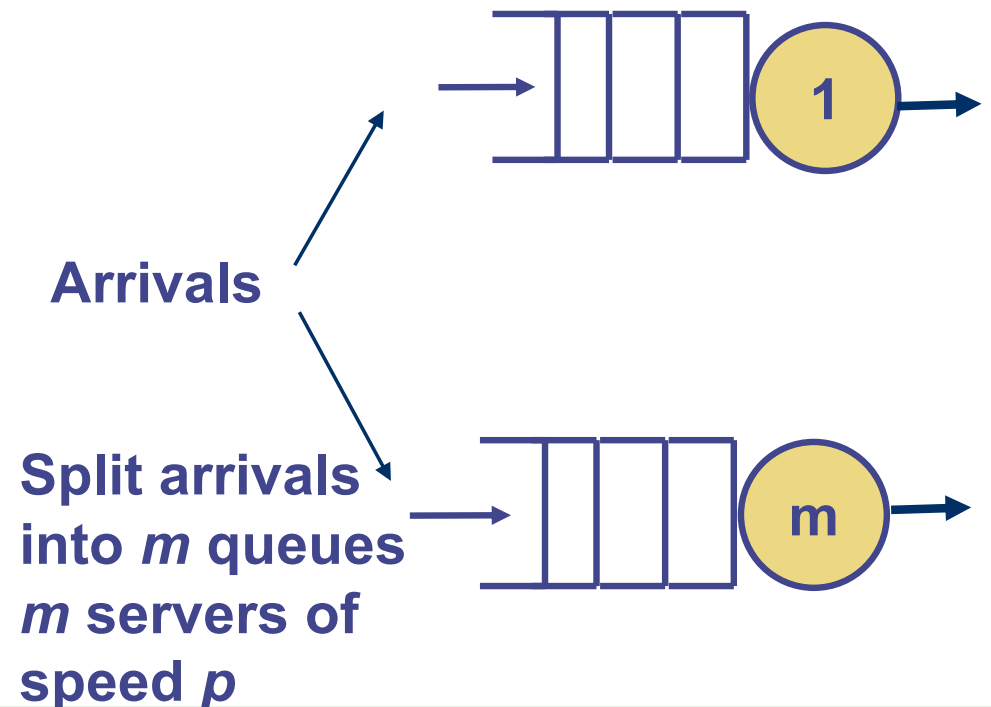
Configuration 1:



Configuration 2:



Configuration 3:



Try out the tutorial question!

# References

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- Recommended reading
  - Queues with Poisson arrival are discussed in
  - Bertsekas and Gallager, *Data Networks*, Sections 3.3 to 3.4.3
  - Note: I derived the formulas here using continuous Markov chain but Bertsekas and Gallager used discrete Markov chain
  - Mor Harchal-Balter. Chapters 13 and 14