ANLY561 Homework 6

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Question 1.

In [11]:

```
# function for backtracking
def backtracking(xi,yi,x0,dx,f,df0,alpha=0.2,beta=0.8,verbose=False):

    delta=alpha*np.sum(dx*df0)
    t=1
    f0= f(xi,yi,x0)
    x=x0+dx
    fx=f(xi,yi,x)
    #print(fx)
    while (not np.isfinite(fx)) or fx>f0+delta*t:
        #print(fx)
        t=beta*t
        x=x0+t*dx
        fx= f(xi,yi,x)
        #print(x)

    return x
```

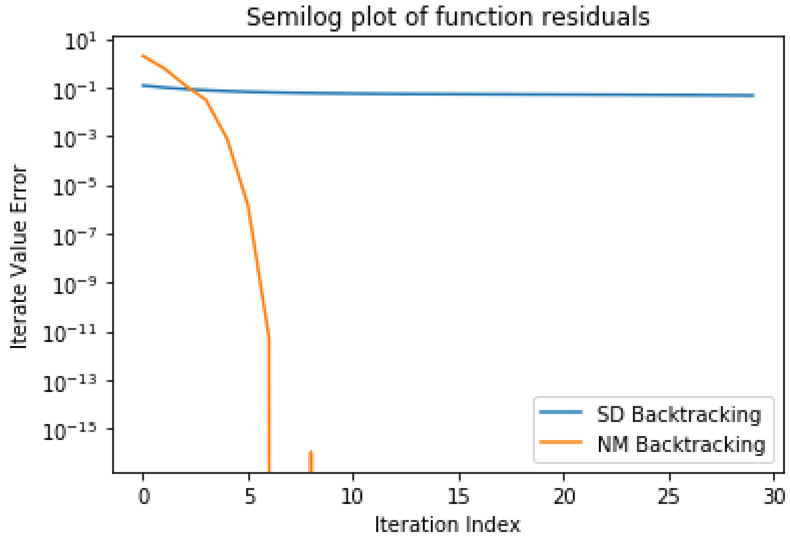
In [12]:

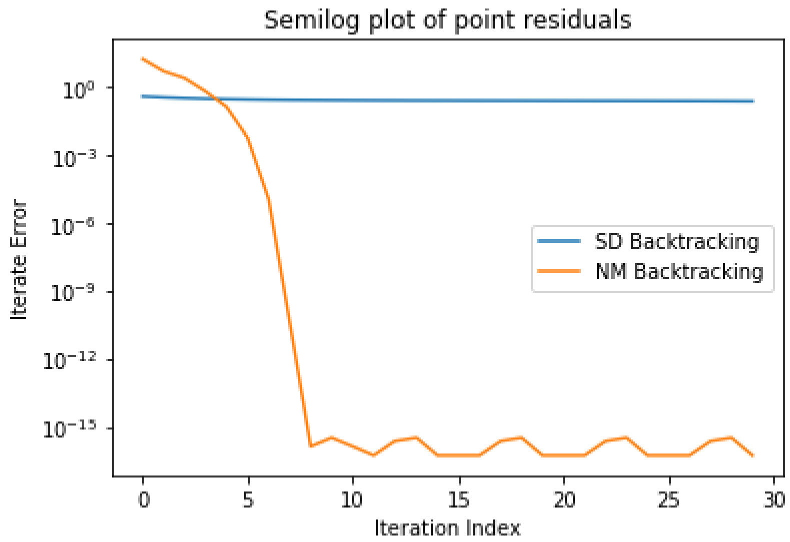
```
import numpy as np
import matplotlib.pyplot as plt


# function logistic_objective
def logistic_objective(x,y,b):
        L0=0
        N=len(x)
        i=0
        while i < N:
            L0=L0+np.log(1+np.exp(-y[i]*(b[1]*x[i] + b[0])))
            i=i+1
        return L0/10

# function dlogistic_objective
def dlogistic_objective(x,y,b):
        G0=0
        G1=0
        N=len(x)
        i=0
        while i < N:
            s=np.exp(-y[i]*(b[1]*x[i]+b[0]))
            G0=G0+(s*-y[i])/(1+s)
            G1=G1+(s*-y[i]*x[i])/(1+s)
            i=i+1
        return np.array([G0/10,G1/10])

# function d2logistic_objective
def d2logistic_objective(x,y,b):
        df11=0
        df12=0
        df21=0
        df22=0
        N=len(x)
        i=0
        while i < N:
            s=np.exp(-y[i]*(b[1]*x[i]+b[0]))
            df11=df11+(s*(y[i])**2)/((1+s)**2)
            df12=df12+(s*x[i]*(y[i])**2)/((1+s)**2)
            df21=df21+(s*x[i]*(y[i])**2)/((1+s)**2)
            df22=df22+(s*(x[i]**2)*(y[i])**2)/((1+s)**2)
            i=i+1
        return np.array([[df11/10,df12/10],[df21/10,df22/10]])

# load data
X=[-1,-1,-1,0,0,0,1,1,1,1]
Y=[-1,-1,-1,1,-1,1,1,1,1,-1]
# initial point b0
b0 = np.array([10, 10])

iter = 30
alpha = 0.2
beta = 0.8


#### backtracking of gradient
x_sd_bt = [b0]
fb0 = logistic_objective(X,Y,b0)
f_sd_bt = [fb0]
```

```python
x = b0

for i in range(iter):
    dfb = dlogistic_objective(X,Y,x)
    fb =  logistic_objective(X,Y,x)
    x = backtracking(X,Y,x, -dfb, logistic_objective, dfb)
    x_sd_bt.append(x)
    fb =  logistic_objective(X,Y,x)
    f_sd_bt.append(fb)

# calculate difference
dx_sd_bt = []
df_sd_bt=[]
for i in range(0,30):
    d1 = f_sd_bt[i+1]-f_sd_bt[i]

    d2 = x_sd_bt[i+1]-x_sd_bt[i]
    d2 = (d2[0]**2+d2[1]**2)**0.5

    df_sd_bt.append(d1)
    dx_sd_bt.append(d2)

#### backtracking of newton
x_nm_bt = [b0]
fb0 =   logistic_objective(X,Y,b0)
f_nm_bt = [fb0]
x = b0

for i in range(iter):
    d2fb =  d2logistic_objective(X,Y,x)
    dfb =  dlogistic_objective(X,Y,x)
    dx_newt = - np.linalg.solve(d2fb, dfb)
    fb =  logistic_objective(X,Y,x)

    x = backtracking(X,Y,x,dx_newt, logistic_objective, dfb)
    x_nm_bt.append(x)
    fb =  logistic_objective(X,Y,x)
    f_nm_bt.append(fb)

# calculate difference
dx_nm_bt = []
df_nm_bt=[]
for i in range(0,30):
    d3 = f_nm_bt[i+1]-f_nm_bt[i]

    d4 = x_nm_bt[i+1]-x_nm_bt[i]
    d4 = (d4[0]**2+d4[1]**2)**0.5

    df_nm_bt.append(d3)
    dx_nm_bt.append(d4)

## function residuals
sd_bt, = plt.semilogy(np.abs(df_sd_bt), label='SD Backtracking')
nm_bt, = plt.semilogy(np.abs(df_nm_bt), label='NM Backtracking')

plt.xlabel('Iteration Index')
plt.ylabel('Iterate Value Error')
plt.legend(handles=[sd_bt, nm_bt])
plt.legend(handles=[sd_bt, nm_bt])
plt.title('Semilog plot of function residuals')
plt.show()
```

```python
## point residuals
sd_bt, = plt.semilogy(np.abs(dx_sd_bt), label='SD Backtracking')
nm_bt, = plt.semilogy(np.abs(dx_nm_bt), label='NM Backtracking')

plt.xlabel('Iteration Index')
plt.ylabel('Iterate Error')
plt.legend(handles=[sd_bt, nm_bt,])
plt.legend(handles=[sd_bt, nm_bt,])
plt.title('Semilog plot of point residuals')
plt.show()
```

```
/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:11: Run
timeWarning: overflow encountered in exp
  # This is added back by InteractiveShellApp.init_path()
```





Question 2

a.

$f(x, y) = 2x + 3y, h_1 = x - 1, h_2 = -x - 1, h_3 = y - 1, h_4 = -y - 1$

1.(Stationarity) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2.(Primal Feasibility) $h_1 = x^* - 1 \leqslant 0, h_2 = -x^* - 1 \leqslant 0, h_3 = y^* - 1 \leqslant 0, h_4 = -y^* - 1 \leqslant 0$

3.(Dual Feasibility) $\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_3 \geqslant 0, \lambda_4 \geqslant 0$

4.(Complementary Slackness) $\lambda_1(x^* - 1) = \lambda_2(-x^* - 1) = \lambda_3(y^* - 1) = \lambda_4(-y^* - 1) = 0$

b.

To satisfy stationarity, $\lambda_1 - \lambda_2 = -2$ and $\lambda_3 - \lambda_4 = -3$

i.

When $x = -1, y = -1, \lambda_1 = \lambda_3 = 0$ and therefore, $\lambda_2 = 2, \lambda_4 = 3$

1.(Stationarity) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2.(Primal Feasibility) $h_1 = x^* - 1 = -2 \leqslant 0, h_2 = -x^* - 1 = 0 \leqslant 0, h_3 = y^* - 1 = -2 \leqslant 0,$
$h_4 = -y^* - 1 = 0 \leqslant 0$

3.(Dual Feasibility) $\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_3 \geqslant 0, \lambda_4 \geqslant 0$

4.(Complementary Slackness) $\lambda_1(x^* - 1) = \lambda_2(-x^* - 1) = \lambda_3(y^* - 1) = \lambda_4(-y^* - 1) = 0$

Therefore point $(-1, -1)$ satisfy all KKT conditions.

ii.

When $x = -1, y = 1, \lambda_1 = \lambda_4 = 0$ and therefore, $\lambda_2 = 2, \lambda_3 = -3$

Dual Feasibility does not satisfy since $\lambda_3 = -3 < 0$.

iii.

When $x = 1, y = -1, \lambda_2 = \lambda_3 = 0$ and therefore, $\lambda_1 = -2, \lambda_4 = 3$

Dual Feasibility does not satisfy since $\lambda_1 = -2 < 0$.

iv.

When $x = 1, y = 1, \lambda_2 = \lambda_4 = 0$ and therefore, $\lambda_1 = -2, \lambda_3 = -3$

Dual Feasibility does not satisfy since $\lambda_1 = -2 < 0, \lambda_3 = -3 < 0$.

1.(Stationarity) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + -2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + -3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2.(Primal Feasibility) $h_1 = x^* - 1 = 0 \leqslant 0, h_2 = -x^* - 1 = -2 \leqslant 0, h_3 = y^* - 1 = 0 \leqslant 0,$
$h_4 = -y^* - 1 = -2 \leqslant 0$

4.(Complementary Slackness) $\lambda_1(x^* - 1) = \lambda_2(-x^* - 1) = \lambda_3(y^* - 1) = \lambda_4(-y^* - 1) = 0$

Therefore, $(1, 1)$ satisfies all the KKT conditions except dual feasibility.

v.

When points are in the interior area, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

Stationarity does not satisfy.

vi.

When x=1, $\lambda_2 = \lambda_3 = \lambda_4 = 0.$

Stationarity does not satisfy.

vii.

When x=-1, $\lambda_1 = \lambda_3 = \lambda_4 = 0.$

Stationarity does not satisfy.

viii.

When y=1, $\lambda_1 = \lambda_2 = \lambda_4 = 0.$

Stationarity does not satisfy.

ix.

When y=-1, $\lambda_1 = \lambda_2 = \lambda_3 = 0.$

Stationarity does not satisfy.

Overall, $(-1, -1)$ is the only point which satisfies the KKT conditions, and $(1, 1)$ satisfies all KKT conditions except dual feasibility.

c.

When $x = 0, -1 \leqslant 0 \leqslant 1$, when $y = 0, -1 \leqslant 0 \leqslant 1$. Therefore, point $(0, 0)$ is an interior point.

The python program is:

In [13]:

```
import numpy as np


def lb1D(x, a, b):
    return -np.log(x[0]-a)-np.log(b-x[0])-np.log(x[1]-a)-np.log(b-x[1])

def dlb1D(x, a, b):
    return np.array([ 1/(b-x[0]) - 1/(x[0]-a),1/(b-x[1]) - 1/(x[1]-a)])

def d2lb1D(x, a, b):
    return np.array([[1/((b-x[0])**2) + 1/((x[0]-a)**2),0],[0,1/((b-x[1])**2) +
1/((x[1]-a)**2)]])


def backtracking2(x0,dx,f,df0,alpha=0.2,beta=0.8,verbose=False):
    if verbose:
        n=0
        xs=[x0+dx]*3


    delta=alpha*np.sum(dx*df0)
    t=1
    f0=f(x0)
    x=x0+dx
    fx=f(x)

    while (not np.isfinite(fx)) or fx>f0+delta*t:

        t=beta*t
        x=x0+t*dx
        fx=f(x)


        if verbose:
            n=n+1
            xs.append(x)
            xs.pop(0)

    return x

def log_barrier_opt_1D(a, b, x0, f, d1f, d2f, al=0.2, be=0.8, M=10, init_iter=5,
 out_iter=5, in_iter=3, verbose=False):

    # First, approximate the solution with t=1
    x = x0
    if verbose:
        pts = [x0]

    # Centering step
    for i in range(init_iter):
        flb = lambda z: f(z) + lb1D(z, a, b)
        d1fb0 = d1f(x) + dlb1D(x, a, b)
        d2fb0 = d2f(x) + d2lb1D(x, a, b)
        dx_newt = - np.linalg.solve(d2fb0, d1fb0)

        x = backtracking2(x, dx_newt, flb, d1fb0, alpha=al, beta=be)

        if verbose:
            pts.append(x)
```

```python
    # Now begin the outer iterations
    t=1
    for i in range(out_iter):
        t = M * t
        if verbose:
            pts = [x]
        for j in range(in_iter):
            flb = lambda z: f(z) + lb1D(z, a, b)/t
            d1fb0 = d1f(x) + d1b1D(x, a, b)/t
            d2fb0 = d2f(x) + d2lb1D(x, a, b)/t
            dx_newt = - np.linalg.solve(d2fb0, d1fb0)

            x = backtracking2(x, dx_newt, flb, d1fb0, alpha=al, beta=be)

    return x

a= -1
b= 1

x0=np.array([0,0])
f=lambda x: 2*x[0]+3*x[1]
d1f=lambda x: np.array([2,3])
d2f= lambda x: np.array([[0,0],[0,0]])

x = log_barrier_opt_1D(a, b, x0, f, d1f,d2f, verbose=True, out_iter=5)
print(x)
```

```
[-0.99999501 -0.99999683]

/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:5: Runt
imeWarning: divide by zero encountered in log
  """
/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:5: Runt
imeWarning: invalid value encountered in log
  """
```