HOMEWORK PROBLEMS 01, ANLY 561, FALL 2018
DUE 09/14/18

Readings: Lecture 01 Notes; **Begin** reading Goodfellow and Bengio, Chapter 2; and Chapter 2 from `https://jakevdp.github.io/PythonDataScienceHandbook/`

Exercises:

1. Use what we know from lecture to show that the functions defined in parts (a) through (g) are all convex.
   (a) $f(x) = |x|$
   (b) $f(x) = x^2$
   (c) $f(x) = x^3 - x$ on the interval $[0, 1]$
   (d) $f(x) = |x - \mu|$ for fixed $\mu \in \mathbb{R}$
   (e) $f(x) = \frac{1}{2\sigma^2}(x - \mu)^2$ for fixed $\mu \in \mathbb{R}$ and $\sigma > 0$
   (f) $f(x) = \sum_{i=1}^{n} |x_i - x|$ where $x_1, \ldots, x_n \in \mathbb{R}$ are fixed values
   (g) $f(x) = \frac{1}{2\sigma^2}(x - \mu)^2 + \lambda|x|$ for fixed values $\mu \in \mathbb{R}$, $\sigma > 0$, and $\lambda > 0$.
2. For each of the functions in Exercise 1, find the minimum value and a minimizing solution (and as a function of the fixed parameters μ, σ, x_i, and λ for parts 1(d) through 1(f).
3. We say that $f : I \to \mathbb{R}$ (where I is any subinterval of $\mathbb{R}$) is a *Lipschitz function* if there is a fixed constant $C > 0$ such that
$$|f(x) - f(y)| \leq C|x - y| \text{ for all } x, y \in I.$$
   If C is the minimal constant satisfying this set of inequalities, we say that C is the *Lipschitz constant* of f. One of the main tools for verifying the Lipschitz condition and finding the Lipschitz constant is the following fact:

   Fact: If $f : I \to \mathbb{R}$ is continuous and piecewise differentiable on I (that is, it is differentiable except possibly at a finite set of points in I), and the program
$$\max_{x \in I} |f'(x)| \text{ subject to } f'(x) \text{ existing},$$
   has maximum value $C < \infty$, then f is Lipschitz with Lipschitz constant C. On the other hand, if $|f'(x)|$ is unbounded above on I where it is defined, then f is not Lipschitz on I.

   For each of the functions in Exercise 1, use the above fact to determine if the function is Lipschitz on $I = \mathbb{R}$ (or Lipschitz on $I = [0, 1]$ for part 1(c)). If it is, what is the Lipschitz constant?

   As an aside, if f is a Lipschitz function and we have a point x that is close the a minimum (that is, $|x^\star - x|$, is small where $x^\star$ is the solution), then $f(x)$ is also close to the minimum value of the function on the domain of optimization, $f(x^\star)$. This is why Lipschitz functions are important in the theory of optimization.