ANLY561 Homework 8

Hongyang Zheng

Question1

a)

$$\nabla f(\mathbf{x}^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \nabla g_1(\mathbf{x}^*) = \begin{pmatrix} 3(x_1 - 1)^2 \\ 1 \end{pmatrix}, \nabla h_1(\mathbf{x}^*) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla h_2(\mathbf{x}^*) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

When $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have:

$$\nabla g_1(\mathbf{x}^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nabla h_1(\mathbf{x}^*) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla h_2(\mathbf{x}^*) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Therefore we have $\nabla g_1(\mathbf{x}^*) = -\nabla h_2(\mathbf{x}^*)$, implying they are not linear independent.

As a result, Linear Independence Constraint Qualificaiton fails at $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) ¶

$$\nabla f(\mathbf{x}^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \nabla g_1(\mathbf{x}^*) = \begin{pmatrix} 3(x_1 - 1)^2 \\ 1 \end{pmatrix}, \nabla h_1(\mathbf{x}^*) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla h_2(\mathbf{x}^*) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

1. (Stationarity) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3(x_1 - 1)^2 \\ 1 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. (Primal Feasibility) $g_1(\mathbf{x}) = 0; h_1(\mathbf{x}) \le 0, h_2(\mathbf{x}) \le 0$
3. (Dual Feasibility) $\mu_1 \ge 0, \mu_2 \ge 0$
4. (Complementary Slackness) $\mu_1 h_1(\mathbf{x}) = 0$ and $\mu_2 h_2(\mathbf{x}) = 0$

c)

When $x_1 = 1$ and $x_2 = 0$, $h_1 = -1$, and $h_2 = 0$

According to complementary slackness, we have $\mu_1 = 0$

Therefore, to satisfy stationarity, we need $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Therefore, $1 + \lambda_1 * 0 + \mu_2 * 0 = 0$ and there is no such λ_1, μ_2

As a result, $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ does not satisfy the KKT conditions.

d)

From the constraints we know that $x_2 + (x_1 - 1)^3 \leqslant 0$, since $x_2 \geqslant 0$, $(x_1 - 1)^3 \leqslant 0$.

Therefore, $(x_1 - 1) \leqslant 0$ and $x_1 \leqslant 1$.

Since $x_1 \geqslant 0, x_1 \in [0, 1]$

To maximum the program x_1, we have $x_1^* = 1$.

When $x_1^* = 1$, plug it into the constraint $x_2 + (x_1 - 1)^3 \leqslant 0$, we get $x_2^* = 0$

Therefore, the solution to this program is $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Question2

a)

The constraints are:

$$\begin{aligned} h_1 &= x_1 - 1 \leqslant 0, h_2 = -x_1 \leqslant 0 \\ h_3 &= x_2 - 1 \leqslant 0, h_4 = -x_2 \leqslant 0 \\ h_5 &= x_3 - 1 \leqslant 0, h_6 = -x_3 \leqslant 0 \end{aligned}$$

Since they are all in the form $ax_1 + bx_2 + cx_3 + d \leqslant 0$, they are all affine functions.

When the constraints are all affine, Linear Independence Constraint Qualificaiton holds.

Therefore, the solution to this program must satisfy the KKT conditions.

b)

$$\nabla f(\mathbf{x}^*) = \begin{pmatrix} 6x_1 - 2x_2 \\ -2x_1 + 6x_2 - 2x_3 \\ -2x_2 + 6x_3 \end{pmatrix}, \nabla h_1(\mathbf{x}^*) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \nabla h_2(\mathbf{x}^*) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \nabla h_3(\mathbf{x}^*) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$
$$\nabla h_4(\mathbf{x}^*) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \nabla h_5(\mathbf{x}^*) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \nabla h_6(\mathbf{x}^*) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

1. (Stationarity)
$$\begin{pmatrix} 6x_1 - 2x_2 \\ -2x_1 + 6x_2 - 2x_3 \\ -2x_2 + 6x_3 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \mu_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu_6 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
2. (Primal Feasibility) $h_1(\mathbf{x}) \leq 0, h_2(\mathbf{x}) \leq 0, h_3(\mathbf{x}) \leq 0, h_4(\mathbf{x}) \leq 0, h_5(\mathbf{x}) \leq 0, h_6(\mathbf{x}) \leq 0$
3. (Dual Feasibility) $\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0$
4. (Complementary Slackness) $\mu_i h_j(\mathbf{x}) = 0$ for $i, j \in 1, 2, 3, 4, 5, 6$

c)

Since the constraints are all affine, we only need to check the points on the boundary.

(1) When $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mu_1 = \mu_3 = \mu_5 = 0$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \mu_6 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_2 = \mu_4 = \mu_6 = 0$.

At this point, $h_1 = h_3 = h_5 = -1 \leqslant 0$, and $h_2 = h_4 = h_6 = 0 \leqslant 0$

Therefore, KKT is satisfied.

(2) When $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mu_2 = \mu_3 = \mu_5 = 0$

$$\begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \mu_6 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_1 = -6, \mu_4 = -2, \mu_6 = 0$.

Since $\mu_1 = -6 < 0, \mu_4 = -2 < 0$, KKT is not satisfied.

(3) When $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mu_1 = \mu_4 = \mu_5 = 0$

$$\begin{pmatrix} -2 \\ 6 \\ -2 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_6 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_2 = 2, \mu_3 = -6, \mu_6 = -2$.

Since $\mu_3 = -6 < 0, \mu_6 = -2 < 0$, KKT is not satisfied.

(4) When $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mu_1 = \mu_3 = \mu_6 = 0$

$$\begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \mu_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_2 = 0, \mu_4 = -2, \mu_5 = -6$.

Since $\mu_4 = -2 < 0, \mu_5 = -6 < 0$, KKT is not satisfied.

(5) When $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mu_2 = \mu_4 = \mu_5 = 0$

$$\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_6 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_1 = -4, \mu_3 = -4, \mu_6 = -2.$

Since μ_1, μ_3, μ_6 are less than zero, KKT is not satisfied.

(6) When $x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mu_2 = \mu_3 = \mu_6 = 0$

$$\begin{pmatrix} 6 \\ -4 \\ 6 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \mu_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_1 = -6, \mu_4 = -4, \mu_5 = -6.$

Since μ_1, μ_4, μ_5 are less than zero, KKT is not satisfied.

(7) When $x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mu_1 = \mu_4 = \mu_6 = 0$

$$\begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_2 = -2, \mu_3 = -4, \mu_5 = -4.$

Since μ_2, μ_3, μ_5 are less than zero, KKT is not satisfied.

(8) When $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mu_2 = \mu_4 = \mu_6 = 0$

$$\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $\mu_1 = -4, \mu_3 = -2, \mu_5 = -6.$

Since μ_1, μ_3, μ_5 are less than zero, KKT is not satisfied.

As a result, $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies KKT conditions.

d)

When $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, f(x) = 0$

Therefore, the minimum value is 0 and the minimizer is $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.