HOMEWORK PROBLEMS 04, ANLY 561, FALL 2018
DUE 10/06/18

Exercises:

1. Using the `Mathbox2` code from class, display the graphs of the functions
   - $f(x, y) = x^2 + y^2$,
   - $f(x, y) = x^2$,
   - $f(x, y) = x^2 - y^2$,
   - $f(x, y) = -x^2$,
   - and $f(x, y) = -x^2 - y^2$.

   By visual inspection, which of these functions is convex? Which of these functions are strictly convex?

2. For each of the functions in Exercise 1, either show that the first order conditions for (strict) convexity hold or fail.

3. Consider the function defined by $f(x, y) = \frac{y^2}{\sqrt{x^2+y^2}}$ when $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.
   (a) Explain why f is continuous at $(0, 0)$.
   (b) Show that, for any $a, b \in \mathbb{R}$, the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = f(at, bt)$ is convex over $\mathbb{R}$.
   (c) Show that f is not convex over $\mathbb{R}^2$.

4. Compute the second-order multivariate Taylor approximation for $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1,\ x_2) = -\log\left(\det\begin{pmatrix} 1 + x_1^2 & x_1 x_2 \\ x_1 x_2 & 1 + x_2^2 \end{pmatrix}\right),$$

   and at the point $(x_1^*,\ x_2^*) = (1,\ 1)$, and where

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

5. We let $M_{2,2}$ denote the set of all 2 by 2 matrices. That is,

$$M_{2,2} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}\right\}$$

   A matrix $A \in M_{2,2}$ is said to be **symmetric** if equals its **transpose**. That is,

$$A^T = A \Leftrightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   Clearly, this is equivalent to $b = c$. We say that a symmetric matrix $A \in M_{2,2}$ is **positive semidefinite** if

$$\mathbf{x}^T A \mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a x_1^2 + 2b x_1 x_2 + d x_2^2 \geq 0$$

   for all

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

   We let SPD(2) denote the set of all 2 by 2 symmetric positive semidefinite matrices.

(a) Show that if $A, B \in \mathrm{SPD}(2)$, then $A + B \in \mathrm{SPD}(2)$, where the sum is defined entry-wise. *Hint*: Use the distributive property of matrix-vector multiplication: $(A + B)\mathbf{v} = A\mathbf{v} + B\mathbf{v}$.

(b) A **convex subset** $\mathcal{X}$ of a vector space $\mathcal{V}$ (i.e. $\mathcal{V}$ has vector addition and scalar multiplication operations) is a set such that if $v, w \in \mathcal{X}$ and $t \in [0, 1]$, then $tv + (1 - t)w \in \mathcal{X}$. Geometrically, this means that the set $\mathcal{X}$ contains all line segments connecting points inside the set. Show that $\mathrm{SPD}(2)$ is a convex subset of $M_{2,2}$.

(c) If $X \in M_{2,2}$, explain why $X^T X \in M_{2,2}$, and also why $X^T X \in \mathrm{SPD}(2)$.

(d) We say that a matrix $B \in M_{2,2}$ is **positive definite** if $\mathbf{x}^T B\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. If $A \in M_{2,2}$ is positive semidefinite and $B \in M_{2,2}$ is positive definite, explain why $A + B$ is positive definite.

(e) If $A \in M_{2,2}$ is positive definite, explain why A is always invertible (that is, A^{-1} exists).