ANLY561 Assignment 1

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1.

a).

Let $x, y \in \mathbb{R}$ and $t \in [0, 1]$, then we have

$$\begin{aligned}
&f((1-t)x + ty) \\
&= |(1-t)x + ty| \\
&\leqslant |(1-t)x| + |ty| \\
&= (1-t)|x| + t|y| \\
&= (1-t)f(x) + tf(y)
\end{aligned}$$

Therefore, $f(x) = |x|$ is convex according to the definition of convex.

b).

$f'(x) = 2x$

$f''(x) = 2$

Since $f''(x) > 0$, $f(x)$ is convex and it is strictly convex.

c).

$f'(x) = 3x^2 - 1$

$f''(x) = 6x$

Since $x \in [0, 1]$, $f''(x)$ is monotonically increasing, $f''(x) \in [0, 6]$.

Since $f''(x) \geqslant 0$, $f(x) = x^3 - x$ is convex.

d).

Let $m, n \in \mathbb{R}$ and $r \in [0, 1]$, then we have

$$\begin{aligned}
&(1-r)f(m) + rf(n) \\
&= (1-r)|m-\mu| + r|n-\mu| \\
&= |(1-r)(m-\mu)| + |r(n-\mu)| \\
&\geqslant |(1-r)(m-\mu) + r(n-\mu)| \\
&= |(1-r)m + rn - u| \\
&= f((1-r)m + rn)
\end{aligned}$$

Therefore, $f(x) = |x - \mu|$ is convex according to the definition of convex.

e).

$f'(x) = \frac{1}{\sigma^2}(x - \mu)$

$f''(x) = \frac{1}{\sigma^2}$

Since σ is greater than $0, f''(x) > 0$.

Therefore, $f(x)$ is strictly convex.

f).

Since each component of $f(x)$ is $|x_i - x|$ and x_i is a fixed value, let $x_i = a$.

Then each component of $f(x)$ is $|a - x|$, which is equal to $|x - a|$.

According to part d), $|x - a|$is a convex function. Thus $|x_i - x|$is a convex function

Since the sum of convex functions is also convex, $f(x) = \sum_{i=1}^{n} |x_i - x|$ is convex.

g).

According to part a) and part e), $f(x_1) = \lambda|x|$ with $\lambda > 0$ and $f(x_2) = \frac{1}{2\sigma^2}(x - \mu)^2$ are convex.

Thus the sum of these two functions $f(x)$ is also convex.

2.

a).

Since $f(x) = |x| \geqslant 0$ for all $x \in \mathbb{R}, f(x) = 0$ is the minimum value.

When $f(x) = 0$, we have $x = 0$ as the minimizing solution.

b).

Since $f(x) = x^2 \geqslant 0$ for all $x \in \mathbb{R}, f(x) = 0$ is the minimum value.

When $f(x) = 0$, we have $x = 0$ as the minimizing solution.

c).

$f'(x) = 3x^2 - 1$, let $f'(x) = 0$, then we have $x = \pm\frac{\sqrt{3}}{3}$.

Since $x \in [0,1], f'(0) < 0$ and $f'(1) > 0$. When $x = \frac{\sqrt{3}}{3}, f(x)$ has the minimum value $-\frac{2\sqrt{3}}{9}$.

d).

Since $f(x) = |x - \mu| \geqslant 0$ for all $x \in \mathbb{R}, f(x) = 0$ is the minimum value.

When $f(x) = 0$, we have $x = \mu$ as the minimizing solution.

e).

Since $f(x) = \frac{1}{2\sigma^2}(x-\mu)^2 \geqslant 0$ for all $x \in \mathbb{R}, f(x) = 0$ is the minimum value.

When $f(x) = 0$, we have $x = \mu$ as the minimizing solution.

f).

We have $f(x) = \sum_{i=1}^{n} |x - x_i|$.

This function means the sum of distance between x and x_i, and we want the minimum.

Thus, when x is the midpoint in the line $[x_1, x_n], f(x)$ has the minimum value.

we have $x = \frac{(x_1 + x_n)}{2}$ as the minimizing solution.

Case A: when x is even.

The minimum value of $f(x)$ is $(x_n - x_1) + (x_{n-1} - x_2) + \ldots + (x_{\frac{n+2}{2}} - x_{\frac{n}{2}})$

Case B. when n is odd.

The minimum value of $f(x)$ is $(x_n - x_1) + (x_{n-1} - x_2) + \ldots + (x_{\frac{n+3}{2}} - x_{\frac{n-1}{2}})$

g).

We have

$$f(x) = \begin{cases} \frac{1}{2\sigma^2}(x-\mu)^2 - \lambda(x), & x \in (-\infty, 0) \\ \frac{1}{2\sigma^2}(x-\mu)^2 + \lambda(x), & x \in (0, \infty) \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{\sigma^2}(x-\mu) - \lambda(x), & x \in (-\infty, 0) \\ \frac{1}{\sigma^2}(x-\mu) + \lambda(x), & x \in (0, \infty) \end{cases}$$

1) When $x < 0$, let $f'(x) = 0$, we have $x^{*1} = \sigma^2\lambda + \mu$.

since $x^{*1} < 0, f(x)$ is decreasing on $(-\infty, \sigma^2\lambda + \mu)$ and increasing on $[\sigma^2\lambda + \mu, 0)$

Therefore, $x^{*1} = \sigma^2\lambda + \mu$ is a minimizing solution and a minimum value is $f(x) = -\mu - \frac{\lambda^2\sigma^2}{2}$

2) When $x > 0$, let $f'(x) = 0$, we have $x^{*2} = -\sigma^2\lambda + \mu$.

since $x^{*2} > 0, f(x)$ is decreasing on $(0, -\sigma^2\lambda + \mu)$ and increasing on $[-\sigma^2\lambda + \mu, \infty)$

Therefore, $x^{*2} = -\sigma^2\lambda + \mu$ is a minimizing solution and a minimum value is $f(x) = \mu - \frac{\lambda^2\sigma^2}{2}$

3.

a).

$f(x)$ is continuous and differentiable on $\mathbb{R}$ except $x = 0$.

We have $|f'(x)| = 1$, for $x > 0, x < 0$

Thus, there is a maximum value $C = 1$ for $|f'(x)|$ and $f(x)$ is Lipschitz.

b).

$f(x)$ is continuous and differentiable on $\mathbb{R}$ and f'(x)=2x.

Since $|f'(x)| = 2|x|$ is decreasing on $(-\infty, 0]$ and inceasing on $[0, \infty)$, $|f'(x)|$ doesn't have maximum value $C < \infty$.

Therefore, $f(x)$ is not Lipschitz.

c).

$f(x)$ is continuous and differentiable on $[0, 1]$ and $f'(x) = 3x^2 - 1$.

$$|f'(x)| = \begin{cases} 1 - 3x^2, & x \in [0, \frac{\sqrt{3}}{3}) \\ 3x^2 - 1, & x \in [\frac{\sqrt{3}}{3}, 1] \end{cases}$$

$|f'(0)| = 1, |f'(\frac{\sqrt{3}}{3})| = 0, |f'(1)| = 2.$

Thus, there is a maximum value $C = 2$ for $|f'(x)|$ and $f(x)$ is Lipschitz.

d).

$f(x)$ is continuous and differentiable on $\mathbb{R}$ except $x = \mu$.

We have $|f'(x)| = 1$ for$x > \mu, x < \mu$

Thus, there is a maximum value $C = 1$ for $|f'(x)|$ and $f(x)$ is Lipschitz.

e).

$f(x)$ is continuous and differentiable on $\mathbb{R}$ and $f'(x) = \frac{1}{\sigma^2}(x-\mu)$.

$|f'(x)| = |\frac{1}{\sigma^2}(x-\mu)| = \frac{1}{\sigma^2}|(x-\mu)|$

Since $|f'(x)| = \frac{1}{\sigma^2}|(x-\mu)|$is decreasing on $(-\infty, \mu]$ and inceasing on $[\mu, \infty)$, $|f'(x)|$ doesn't have maximum value $C < \infty$.

Therefore, $f(x)$ is not Lipschitz.

f).

$f(x)$ is continuous and differentiable on $\mathbb{R}$ except several points.

We have

$$|f'(x)| = \begin{cases} n, & x \in (-\infty, x_1], [x_n, \infty) \\ n-1, & x \in (x_1, x_2], [x_{n-1}, x_n) \\ n-2, & x \in (x_2, x_3], [x_{n-2}, x_{n-1}) \\ \dots \end{cases}$$

Thus, there is a maximum value $C = n$ for $|f'(x)|$ and $f(x)$ is Lipschitz.

g).

$f(x)$ is continuous and differentiable on $\mathbb{R}$ except $x = 0$.

We have

$$f'(x) = \begin{cases} \frac{1}{\sigma^2}(x-\mu) - \lambda(x), & x \in (-\infty, 0) \\ \frac{1}{\sigma^2}(x-\mu) + \lambda(x), & x \in (0, \infty) \end{cases}$$

Since $\lambda > 0$ and $\sigma > 0$, $\frac{1}{\sigma^2}(x-\mu) + \lambda(x)$is increasing, and $f'(x)$ cannot have a maximum value less than ∞.

Therefore, $|f'(x)|$ cannot have a maximum value $C < \infty$. $f(x)$ is not Lipschitz.