TOLOSA-SW v0.1 : User and Scientific Documentation

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1 Model

The complete model resolved is,

$$\left\{\begin{array}{l}\dfrac{\partial h}{\partial t}+\operatorname{div}\left(h\mathbf{u}\right)=0\\[2ex]\dfrac{\partial\left(h\mathbf{u}\right)}{\partial t}+\operatorname{div}\left(h\mathbf{u}\otimes\mathbf{u}\right)=-gh\nabla\left(h+b\right)-fh\boldsymbol{u}^{\perp}-S_F\left(h,\boldsymbol{u}\right)\end{array}\right.\tag{1}$$

where b is the bottom elevation, $fh\boldsymbol{u}^{\perp}$ is the Coriolis force and $S_F\left(h,\boldsymbol{u}\right)$ the friction source term which can have multiple expression,

$$\left\{\begin{array}{l}S_F\left(h,\boldsymbol{u}\right)=C_f\,\dfrac{\|\mathbf{u}\|\,\mathbf{u}}{h^{1/3}},\quad\text{with } C_f=gn^2\\[2ex]S_F\left(h,\boldsymbol{u}\right)=C_l\,\mathbf{u}+C_b\left(h\right)\,\|\mathbf{u}\|\,\mathbf{u}\end{array}\right.\tag{2}$$

the first expression is called the Manning-Strickler friction source term and is adapted to hydraulic simulations like rivers calibrating the C_f constant and the second expression is adapted to oceanic simulation.

2 Numerical resolution

2.1 The hyperbolic part with topography

The global numerical scheme is a combination of two schemes. The first one is the low Froude number asymptotic preserving and entropy dissipative scheme of Vila and al. and is used for the vast majority of the computational domain. The second one is a variant of the Rusanov scheme and is only used to stabilize solutions close to the boundary conditions and also to wet-dry cells. In order to switch smoothly from the first scheme to the second one, a function of interpolation δ varying from 0 to 1 is a priori computed for all cells before applying the time step. In practice, δ_K is set to 1 for cells with an edge at the physical boundary as for cells for which there is a neighbouring one with a dry state, and then going from cell to cell with common edges from a given number of (typically from 3 to 5), the value goes to 0 linearly. Then for all edges is computed $\delta_e=\max\left(\delta_{\mathrm{K}},\delta_{\mathrm{K}_e}\right)$ giving the final interpolation value to compute the antisymmetric fluxes to preserve the convervation property of a Finite Volume scheme,

$$\left\{\begin{array}{ll}h_K^{n+1}=h_K^n & -\dfrac{\Delta t}{m_K}\displaystyle\sum_{e\in\partial K}\left(\left(1-\delta_e\right)\mathcal{F}_e^{n,*}+\delta_e\,\mathcal{F}_e^{n,\sharp}\right)m_e\\[3ex]h_K^{n+1}\boldsymbol{u}_K^{n+1}=h_K^n\boldsymbol{u}_K^n & -\dfrac{\Delta t}{m_K}\displaystyle\sum_{e\in\partial K}\left(\left(1-\delta_e\right)\mathcal{G}_e^{n,*}+\delta_e\,\mathcal{G}_e^{n,\sharp}\right)m_e\\[3ex] & -\dfrac{\Delta t}{m_K}h_K^n\displaystyle\sum_{e\in\partial K}\left(\left(1-\delta_e\right)\Phi_e^{n,*}+\delta_e\,\Phi_e^{n,\sharp}\right)\boldsymbol{n}_{e,K}\,m_e\end{array}\right.\tag{3}$$

where $\left(\mathcal{F}_e^{n,*},\mathcal{G}_e^{n,*},\Phi_e^{n,*}\right)$ is the low Froude number asymptotic preserving and entropy dissipative scheme of Vila and al.,

$$\left\{\begin{array}{l}\mathcal{F}_e^{n,*}=\dfrac{1}{2}\left(h_K^n\boldsymbol{u}_K^n+h_{K_e}^n\boldsymbol{u}_{K_e}^n\right).\boldsymbol{n}_{e,K}-\dfrac{1}{8}\gamma\left(\dfrac{m_{\partial K}}{m_K}+\dfrac{m_{\partial K_e}}{m_{K_e}}\right)\left(\Phi_{K_e}^n-\Phi_K^n\right)\\[3ex]\mathcal{G}_e^{n,*}=\boldsymbol{u}_K^n\left(\mathcal{F}_e^{n,*}\right)^{+}+\boldsymbol{u}_{K_e}^n\left(\mathcal{F}_e^{n,*}\right)^{-}\\[3ex]\Phi_e^{n,*}=\dfrac{1}{2}\left(\Phi_K^n+\Phi_{K_e}^n\right)-\dfrac{1}{4}\alpha\,g\left(\dfrac{m_{\partial K}}{m_K}+\dfrac{m_{\partial K_e}}{m_{K_e}}\right)\left(h_{K_e}^n\boldsymbol{u}_{K_e}^n-h_K^n\boldsymbol{u}_K^n\right).\boldsymbol{n}_{e,K}\end{array}\right.\tag{4}$$

and where $\left(\mathcal{F}_e^{n,\sharp},\mathcal{G}_e^{n,\sharp},\Phi_e^{n,\sharp}\right)$ is a Rusanov like scheme,

$$
\left\{
\begin{aligned}
&\mathcal{F}_e^{n,\sharp} = \frac{1}{2}\left(h_K^{n,R}\boldsymbol{u}_K^n + h_{K_e}^{n,R}\boldsymbol{u}_{K_e}^n\right).\boldsymbol{n}_{e,K} - \lambda\left(h_{K_e}^{n,R} - h_K^{n,R}\right)\\
&\mathcal{G}_e^{n,\sharp} = \frac{1}{2}\left(h_K^{n,R}\boldsymbol{u}_K^n \otimes \boldsymbol{u}_K^n + h_{K_e}^{n,R}\boldsymbol{u}_{K_e}^n \otimes \boldsymbol{u}_{K_e}^n\right).\boldsymbol{n}_{e,K} - \lambda\left(h_{K_e}^{n,R}\boldsymbol{u}_{K_e}^n - h_K^{n,R}\boldsymbol{u}_K^n\right)\\
&\Phi_e^{n,\sharp} = \frac{1}{2}\left(\Phi_K^{n,R} + \Phi_{K_e}^{n,R}\right)\\
&\lambda = \max\left(\left|\boldsymbol{u}_K^n.\boldsymbol{n}_{e,K}\right| + \sqrt{g h_K^{n,R}}, \left|\boldsymbol{u}_{K_e}^n.\boldsymbol{n}_{e,K}\right| + \sqrt{g h_{K_e}^{n,R}}\right)
\end{aligned}
\right.
\tag{5}
$$

using the famous hydrostatic reconstruction of Audusse and al.,

$$
\left\{
\begin{aligned}
&h_K^{n,R} = \max\left(0, h_K^n + b_K - \max\left(b_K, b_{K_e}\right)\right)\\
&h_{K_e}^{n,R} = \max\left(0, h_{K_e}^n + b_{K_e} - \max\left(b_K, b_{K_e}\right)\right)
\end{aligned}
\right.
\tag{6}
$$

with as well a reconstruction of the pressure potential to find the balance when water is below the bathymetry of a dry neighbouring cell,

$$
\left\{
\begin{aligned}
&\text{if} \quad h_K^n + b_K < b_{K_e} \text{ and } h_{K_e} < h_\epsilon \quad \text{, then} \quad \Phi_{K_e}^{n,R} = \Phi_K^n \quad \text{, else } \Phi_{K_e}^{n,R} = \Phi_{K_e}^n\\
&\text{if} \quad h_{K_e}^n + b_{K_e} < b_K \text{ and } h_K < h_\epsilon \quad \text{, then} \quad \Phi_K^{n,R} = \Phi_{K_e}^n \quad \text{, else } \Phi_K^{n,R} = \Phi_K^n
\end{aligned}
\right.
\tag{7}
$$

2.2 The friction source term

A time-splliting scheme is used to treat the friction source term, then we focus here on the resolution of,

$$
\left\{
\begin{aligned}
&\frac{\partial h}{\partial t} = 0\\
&\frac{\partial\left(h\mathbf{u}\right)}{\partial t} = -S_F\left(h, \boldsymbol{u}\right)
\end{aligned}
\right.
\tag{8}
$$

2.2.1 Oceanic friction source term

$$
\left\{
\begin{aligned}
&\frac{\partial h}{\partial t} = 0\\
&\frac{\partial\left(h\mathbf{u}\right)}{\partial t} = -C_l\,\mathbf{u} - C_b\left(h\right)\,\|\mathbf{u}\|\,\mathbf{u}
\end{aligned}
\right.
\tag{9}
$$

Semi-implicit resolution The semi-implicit time step resolution writes,

$$
h^n\boldsymbol{u}^{n+1} = h^n\boldsymbol{u}^n - \Delta t\left(C_l\,\mathbf{u}^{n+1} + C_b\left(h^n\right)\,\|\mathbf{u}^n\|\,\mathbf{u}^{n+1}\right)
\tag{10}
$$

which easely gives the final expression of the velocity,

$$
\boldsymbol{u}^{n+1} = \frac{h^n\boldsymbol{u}^n}{h^n + \Delta t\left(C_l + C_b\left(h^n\right)\,\|\mathbf{u}^n\|\right)}
\tag{11}
$$

Full implicit resolution The full implicit time step resolution writes,

$$
h^n\boldsymbol{u}^{n+1} = h^n\boldsymbol{u}^n - \Delta t\left(C_l\,\mathbf{u}^{n+1} + C_b\left(h^n\right)\,\left\|\mathbf{u}^{n+1}\right\|\,\mathbf{u}^{n+1}\right)
\tag{12}
$$

$$
\Delta t\,C_b\left(h^n\right)\,\left\|\mathbf{u}^{n+1}\right\|\,\mathbf{u}^{n+1} + \boldsymbol{u}^{n+1}\left(h^n + \Delta t\,C_l\right) = h^n\boldsymbol{u}^n
\tag{13}
$$

Remarking that $\boldsymbol{u}^{n+1} = \alpha\boldsymbol{u}^n$, the solution is given by a quadratic equation,

$$
\Delta t\,C_b\left(h^n\right)\,\|\mathbf{u}^n\|\,\alpha^2 + \left(h^n + \Delta t\,C_l\right)\,\alpha - h^n = 0
\tag{14}
$$

with the final solution,

$$
\boldsymbol{u}^{n+1} = \frac{2\,h^n\,\boldsymbol{u}^n}{h^n + \Delta t\,C_l + \sqrt{\left(h^n + \Delta t\,C_l\right)^2 + 4\,\Delta t\,h^n\,C_b\left(h^n\right)\,\|\mathbf{u}^n\|}}
\tag{15}
$$

2.2.2 Manning-Strickler source term

$$\begin{cases} \dfrac{\partial h}{\partial t} = 0 \\ \dfrac{\partial \left(h\mathbf{u}\right)}{\partial t} = C_f \, \dfrac{\|\mathbf{u}\| \, \mathbf{u}}{h^{1/3}} \end{cases} \tag{16}$$

Semi-implicit resolution The semi-implicit time step resolution writes,

$$h^n \boldsymbol{u}^{n+1} = h^n \boldsymbol{u}^n - \Delta t \, C_f \, \frac{\|\mathbf{u}^n\| \, \mathbf{u}^{n+1}}{\left(h^n\right)^{1/3}} \tag{17}$$

which easely gives the final expression of the velocity,

$$\boldsymbol{u}^{n+1} = \frac{\left(h^n\right)^{4/3} \, \boldsymbol{u}^n}{\left(h^n\right)^{4/3} + C_f \, \Delta t \, \|\mathbf{u}^n\|} \tag{18}$$

Full implicit resolution The full implicit time step resolution writes,

$$h^n \boldsymbol{u}^{n+1} = h^n \boldsymbol{u}^n - \Delta t \, C_f \, \frac{\|\mathbf{u}^n\| \, \mathbf{u}^{n+1}}{\left(h^n\right)^{1/3}} \tag{19}$$

$$\Delta t \, C_f \, \left\|\mathbf{u}^{n+1}\right\| \mathbf{u}^{n+1} + \left(h^n\right)^{4/3} \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) = 0 \tag{20}$$

Again, remarking that $\boldsymbol{u}^{n+1} = \alpha \boldsymbol{u}^n$, the solution is given by a quadratic equation,

$$\Delta t \, C_f \, \|\mathbf{u}^n\| \, \alpha^2 + \left(h^n\right)^{4/3} \alpha - \left(h^n\right)^{4/3} = 0 \tag{21}$$

with the final solution,

$$\boldsymbol{u}^{n+1} = \frac{2 \, \left(h^n\right)^{2/3} \, \boldsymbol{u}^n}{\left(h^n\right)^{2/3} + \sqrt{\left(h^n\right)^{4/3} + 4 \, \Delta t \, C_f \, \|\mathbf{u}^n\|}} \tag{22}$$

2.3 The Coriolis force