MA 515: Introduction to Algorithms & MA353 : Design and Analysis of Algorithms [3-0-0-6]

Lecture 3

http://www.iitg.ernet.in/psm/indexing_ma353/y09/index.html

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Mon 10:00-10:55 Tue 11:00-11:55 Fri 9:00-9:55
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Big-Oh Notation

- To simplify the running time estimation, for a function *f(n),* we ignore the constants and lower order terms.

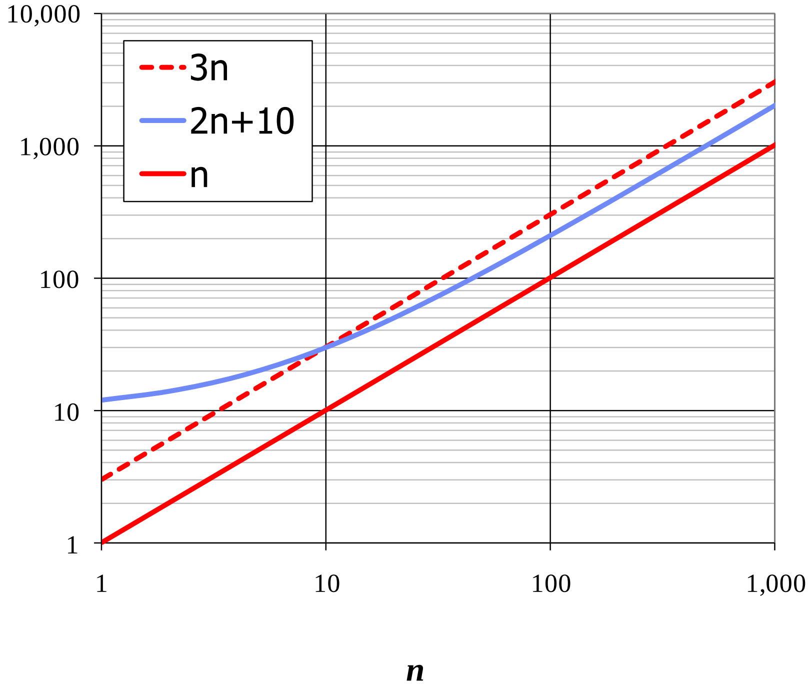
Example: $10n^3+4n^2-4n+5$ is $O(n^3)$.

Big-Oh Notation (Formal Definition)

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

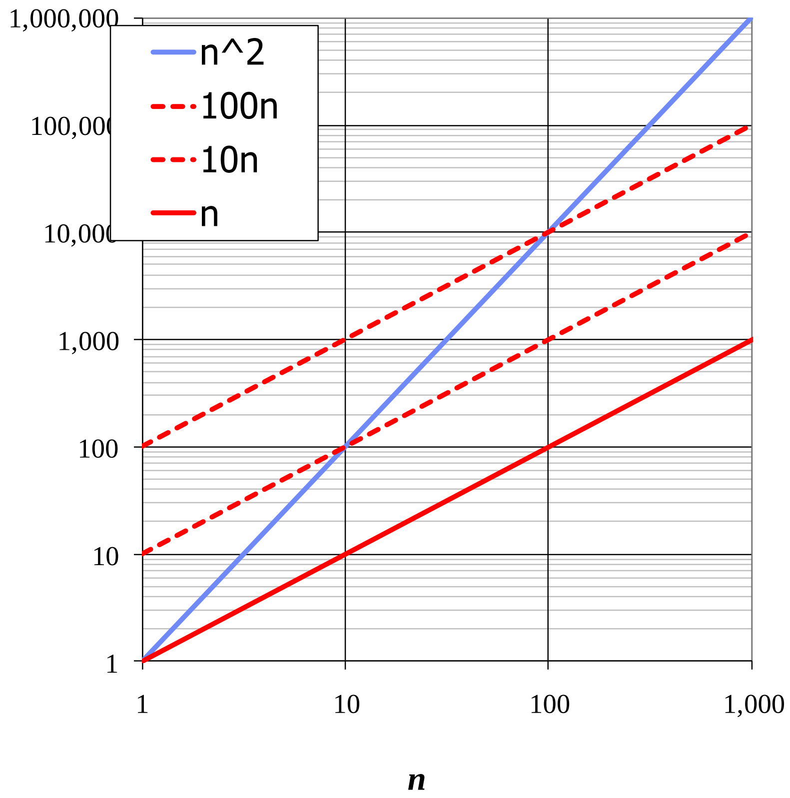
$0 \leq f(n) \leq cg(n)$ for $n \geq n_0$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant
  - n^2 is $O(n^2)$.



More Big-Oh Examples

- $7n-2$

  $7n-2$ is $O(n)$

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq cn^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- $3 \log n + 5$

  $3 \log n + 5$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

Big-Oh Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  - Drop lower-order terms
  - Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Growth Rate of Running Time

- Consider a program with time complexity $O(n^2)$.
  - for the input of size n, it takes 5 seconds.
  - If the input size is doubled (2n).
  - then it takes 20 seconds.

- Consider a program with time complexity O(n).
  - for the input of size n, it takes 5 seconds.
  - If the input size is doubled (2n).
  - then it takes 10 seconds.

- Consider a program with time complexity $O(n^3)$.
  - for the input of size n, it takes 5 seconds.
  - If the input size is doubled (2n) .
  - then it takes 40 seconds.

Asymptotic Algorithm Analysis

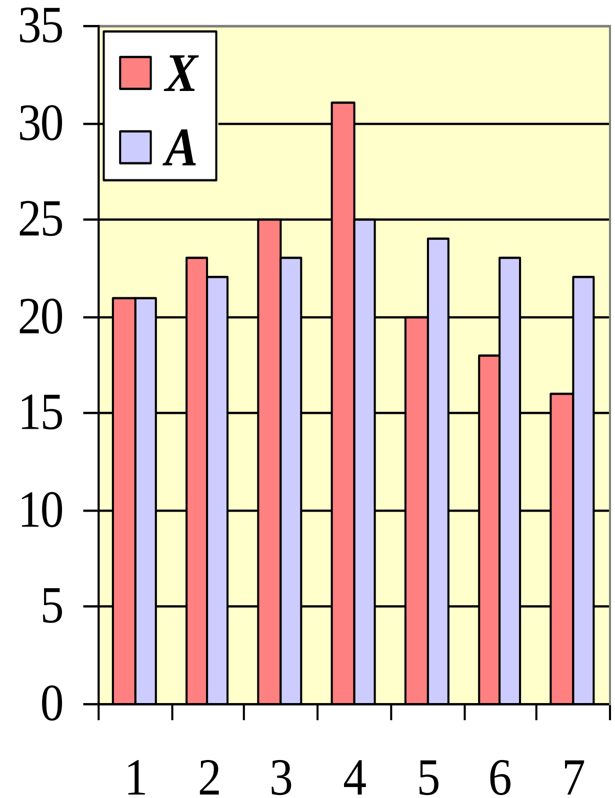
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm $\boldsymbol{arrayMax}$ executes at most $6\boldsymbol{n} - 1$ primitive operations
  - We say that algorithm $\boldsymbol{arrayMax}$ "runs in $\boldsymbol{O}(\boldsymbol{n})$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can **disregard** them when counting primitive operations.

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The $\boldsymbol{i}$-th prefix average of an array $\boldsymbol{X}$ is average of the first $(\boldsymbol{i} + 1)$ elements of $\boldsymbol{X}$:

$$\boldsymbol{A}[\boldsymbol{i}] = (\boldsymbol{X}[0] + \boldsymbol{X}[1] + \ldots + \boldsymbol{X}[\boldsymbol{i}])/(i+1)$$

- Computing the array $\boldsymbol{A}$ of prefix averages of another array $\boldsymbol{X}$ has applications to financial analysis



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition.

Algorithm ***prefixAverages1***(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\{ s \leftarrow X[0]$	n
for $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$A[i] \leftarrow s / (i + 1) \}$	n
return A	1

Arithmetic Progression

- The running time of ***prefixAverages1*** is $\boldsymbol{O}(1 + 2 + \ldots + \boldsymbol{n})$
- The sum of the first $\boldsymbol{n}$ integers is $\boldsymbol{n}(\boldsymbol{n} + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in T(n)=$\boldsymbol{O}(\boldsymbol{n}^2)$ time

Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum.

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\{s \leftarrow s + X[i]$	n
$A[i] \leftarrow s / (i + 1) \}$	n
return A	1

- Algorithm ***prefixAverages2*** runs in $T(n)=O(n)$ time

Exercise: Give a big-Oh characterization

Algorithm Ex1(A, ***n***)

Input an array ***X*** of ***n*** integers

Output the sum of the elements in A

s ← A[0]

for ***i*** ← 0 **to** ***n*** – 1 **do**

  s ← ***s*** + A[***i***]

return s

Exercise: Give a big-Oh characterization

Algorithm Ex2(A, n)
  Input an array X of n integers
  Output the sum of the elements at even cells in A

  $s \leftarrow$ A[0]
  for $i \leftarrow 2$ **to** $n - 1$ by increments of 2 **do**
    $s \leftarrow s +$ A[i]
  return s

Exercise: Give a big-Oh characterization

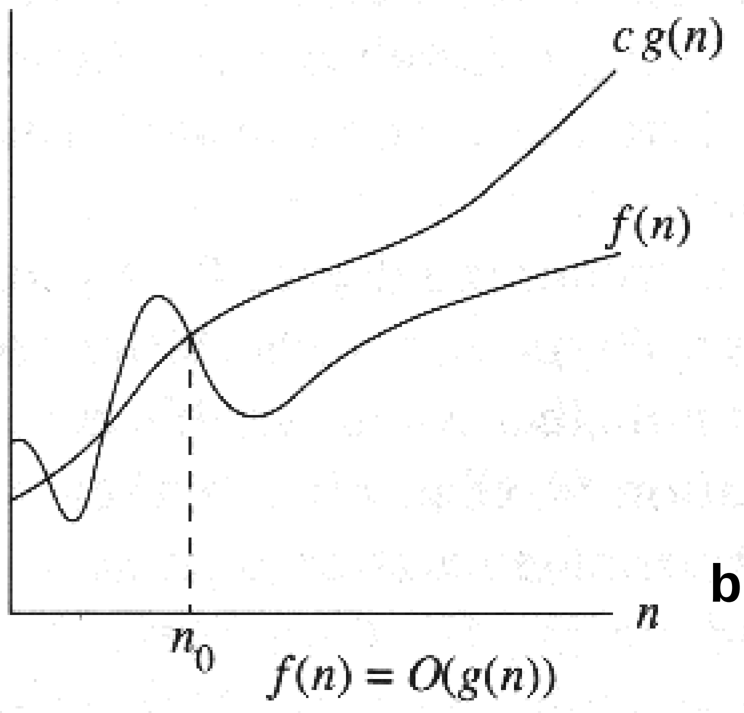
```
Algorithm Ex3(A, n)
    Input an array X of n integers
    Output the sum of the prefix sums  A
    s ← 0
    for i ← 0 to n − 1 do
        s ← s + A[0]
        for j← 1 to i do
            s ← s +A[j]
    return s
```

Asymptotic Notation

- Convenient for describing worst-case running time function T(n)

- Big-Oh
  - f(n) is O(g(n)) if f(n) is asymptotically less than or equal to g(n)
- big-Omega
  - f(n) is Ω(g(n)) if f(n) is asymptotically greater than or equal to g(n)
- big-Theta
  - f(n) is Θ(g(n)) if f(n) is asymptotically equal to g(n)
- little-oh
  - f(n) is o(g(n)) if f(n) is asymptotically strictly less than g(n)
- little-omega
  - f(n) is ω(g(n)) if is asymptotically strictly greater than g(n)

big-oh (O)-notation



big-Oh (O)-notation:

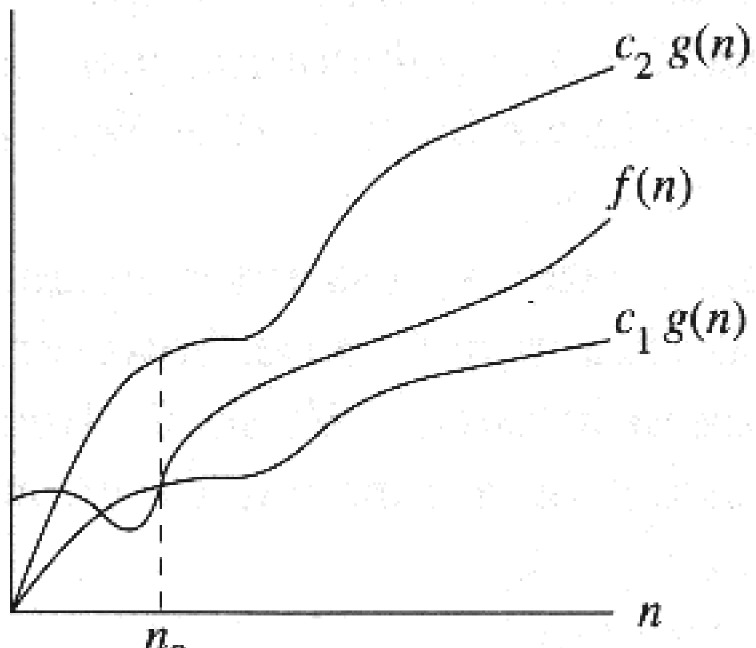
f(n) is O(g(n)) if there is a constant c > 0 and an integer constant n_0>=1 such that 0=<f(n) =< c g(n) for n >= n_0

- a upper bound that is asymptotically tight.

big-Theta (Θ)-notation

Θ-notation

- asymptotic tight bound
- $\Theta(g(n))$ = { f(n) : there exists positive constants c_1, c_2,
    - and n_0 ,s.t. $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$
- $\Theta(g(n))$ is a set
- $f(n) = \Theta(g(n))$
    - is an abuse of '='
    - really
        - f(n) is a member of $\Theta(g(n))$
        - $f(n) \in \Theta(g(n))$
- asymptotically nonnegative
    - f(n) is nonnegative for large enough n



$f(n) = \Theta(g(n))$

Example

Show that:

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

- determine c1, c2 and n0 s.t.

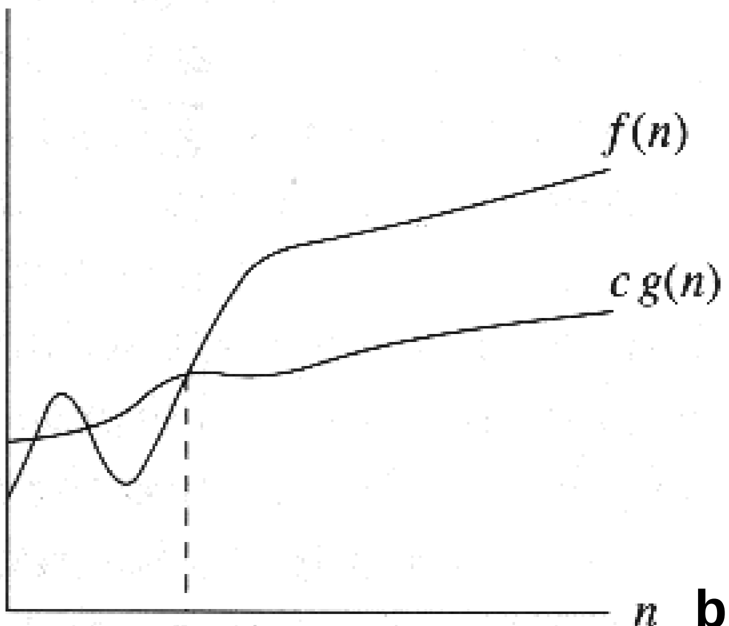
$$c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2$$

- for an n$\geq$n0.
- dividing all by n^2:

$$c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2$$

- this holds for
  - c2 $\geq$ 1/2, so choose c2 = 1/2
  - n $\geq$ 7, so choose n0 = 7
  - c1 $\leq$ 1/14, so choose c1 = 1/14
  - i.e., some choice exists.

big-Omega (Ω)-notation



$f(n) = \Omega(g(n))$

big-Omega (Ω)-notation:

f(n) is Ω(g(n)) if there is a constant c > 0 and an integer constant n_0>=1 such that f(n) >= c g(n) >=0 for n >= n_0

- a lower bound and that is asymptotically tight.

Little-oh & Little-omeha

- **little-oh (o)-notation**
  - f(n) is o(g(n)) if, for any constan c > 0, there is an integer constant n_0 >= 0 such that 0=<f(n) < c g(n) for n >= n_0
  - a upper bound that is not asymptotically tight.

- similar to O() but
  - $2n^2 = O(n^2)$ is tight
  - $2n = O(n^2)$ is not tight
- main difference
  - O() - some constant c
  - o() - for all constants c
- note
  - $2n = o(n^2)$
  - $2n^2 \neq o(n^2)$

- **little-omega (ω)-notation**
  - f(n) is ω(g(n)) if, for any constant c > 0, there is an integer constant n_0 >= 0 such that f(n) > c g(n) >=0 for n >= n_0
  - a lower bound that is not asymptotically tight.

- similar to Ω() but
- $f(n) \in \boldsymbol{\omega}(g(n))$ iff $g(n) \in o(f(n))$
- note
  - $n^2/2 = \boldsymbol{\omega}(n)$
  - $n^2/2 \neq \boldsymbol{\omega}(n^2)$

The family of Bachmann–Landau notations

Notation	Name	Intuition	As $n \to \infty$, eventually...	Definition
$f(n) \in O(g(n))$	Big Omicron; Big O; Big Oh	f is bounded above by g (up to constant factor) asymptotically	$\|f(n)\| \leq g(n) \cdot k$ for some k	$\exists k > 0, n_0 \; \forall n > n_0 \; \|f(n)\| \leq \|g(n) \cdot k\|$ or $\exists k > 0, n_0 \; \forall n > n_0 \; f(n) \leq g(n) \cdot k$
$f(n) \in \Omega(g(n))$ (Older math papers sometimes use this in the weaker sense that f=o(g) is false)	Big Omega	f is bounded below by g (up to constant factor) asymptotically	$\|f(n)\| \geq g(n) \cdot k$ for some k	$\exists k > 0, n_0 \; \forall n > n_0 \; g(n) \cdot k \leq \|f(n)\|$
$f(n) \in \Theta(g(n))$	Big Theta	f is bounded both above and below by g asymptotically	$\|g(n)\| \cdot k_1 \leq \|f(n)\| \leq \|g(n)\| \cdot k_2$ for some k_1, k_2	$\exists k_1, k_2 > 0, n_0 \; \forall n > n_0 \; \|g(n) \cdot k_1\| < \|f(n)\| < \|g(n) \cdot k_2\|$
$f(n) \in o(g(n))$	Small Omicron; Small O; Small Oh	f is dominated by g asymptotically	$\|f(n)\| \leq \|g(n)\| \cdot \varepsilon$ for every ε	$\forall \varepsilon > 0 \; \exists n_0 \; \forall n > n_0 \; \|f(n)\| \leq \|g(n) \cdot \varepsilon\|$
$f(n) \in \omega(g(n))$	Small Omega	f dominates g asymptotically	$\|f(n)\| \geq \|g(n)\| \cdot k$ for every k	$\forall k > 0 \; \exists n_0 \; \forall n > n_0 \; \|g(n) \cdot k\| \leq \|f(n)\|$
$f(n) \sim g(n)$	on the order of; "twiddles"	f is equal to g asymptotically	$\|f(n)/g(n) - 1\| < \varepsilon$ for every ε	$\forall \varepsilon > 0 \; \exists n_0 \; \forall n > n_0 \; \|f(n)/g(n) - 1\| < \varepsilon$

Comparison of Functions

- Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$
  $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$
  $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$
  $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$
  $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$
- Reflexivity: $f(n) = \Theta(f(n))$
  $f(n) = O(f(n))$
  $f(n) = \Omega(f(n))$
- Symmetry: $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose Symmetry:
  $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$
  $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$