CMSC250, Spring 2018 **Sections: all**

Final Exam

Date: Tuesday, 05-15-2018

Time: 04:05pm-06:00pm

P1	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**TOTAL**

*First and last name (**exactly** as on ELMS):* ______________________________

UID (9 digits): ______________

Section number (4 digits): __________

University Honor Pledge:

I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/examination.

Print the text of the University Honor Pledge below:

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Signature: ____________________________

Exam guidelines / rules

- **TURN OFF ALL ELECTRONIC DEVICES** (e.g phones, tablets, laptops, **calculators**). Setting a device on "silent" or "sleep" mode does **not** constitute it being turned **off**: Your device is turned off if and only if it requires **pushing a power button to begin the execution of a bootloader**. **Proctors reserve the right to confiscate an electronic device if it is not turned off according to the definition above.**
- The exam is **CLOSED BOOK AND NOTES**.
- **DO NOT RIP PAGES FROM THE EXAM**. You can ask us for **extra scrap paper** if you need it.

- **WRITE NEATLY**. If we can't read your response, you will receive **no credit** for it.
- There are **7 (seven)** problems in this exam, with a total grade value that adds to **100 (one hundred)**.
- The exam has been printed **two-sided**, **stapled on the top-left corner** and spans **24 (twenty-four)** pages across **12 (twelve)** sheets.
- The total time allocated for this exam is **115 (one hundred and fifteen)** minutes.
- You may **not** leave the classroom (e.g to go to the bathroom, or because you're done) during the **last 5 (five) minutes** of the exam.

Provided materials & assumed facts

Logic

Table 1 contains a number of logical equivalences that we have discussed in class. Recall that, in Logic, the symbol $\equiv$ means "logically equivalent to".

Commutativity of binary operators	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity of binary operators	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity of binary operators	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation laws	$p \vee (\sim p) \equiv T$	$p \wedge (\sim p) \equiv F$
Double negation	$\sim(\sim p) \equiv p$	
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's axioms	$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$	$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
Universal bound laws	$p \vee T \equiv T$	$p \wedge F \equiv F$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of contradictions / tautologies	$\sim F \equiv T$	$\sim T \equiv F$
Equivalence between biconditional and implication	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
Equivalence between implication and disjunction	$a \Rightarrow b \equiv \sim a \vee b$	

Table 1: A number of propositional logic axioms you can refer to.

Set Theory

The following are Set Theoretic notation and definitions.

Operation	Symbol	Definition
Membership	$x \in A$	x is a member of set A
Non-membership	$x \notin A$	$\sim(x \in A)$
Union	$A \cup B$	$\{(x \in A) \vee (x \in B)\}$
Intersection	$A \cap B$	$\{(x \in A) \wedge (x \in B)\}$
Relative complement of B given A	$A - B$	$\{(x \in A) \wedge (x \notin B)\}$
Universal (Absolute) complement	$\overline{A}$	$\{x \notin A\}$
Cartesian Product	$A \times B$	$\{(a, b) \mid (a \in A) \wedge (b \in B)\}$
Subset	$A \subseteq B$	$(\forall x \in A)[x \in B]$
Superset	$A \supseteq B$	$B \subseteq A$
Set equality	$A = B$	$(A \subseteq B) \wedge (B \subseteq A)$
Set non-equality	$A \neq B$	$\sim(A = B)$
Proper subset	$A \subset B$	$\{(A \subseteq B) \wedge (A \neq B)\}$
Proper superset	$A \supset B$	$\{(A \supseteq B) \wedge (A \neq B)\}$
Powerset	$\mathcal{P}(A)$	$\{X \mid X \subseteq A\}$

Table 2: Definitions of Set Theory

Number Theory

- The set of naturals $\mathbb{N}$ is **closed** under **addition** and **multiplication**.
- The set of integers $\mathbb{Z}$ is **closed** under **addition**, **subtraction** and **multiplication**.
- $0 \in \mathbb{N}$.

Problem 1: Various *(10 pts)*

For every one of the following statements, fill in the circle corresponding to the appropriate choice (**True** or **False**). For example, if any given statement is **True**, you should fill in the **first** circle, such that ○ becomes ● . **PLEASE DO NOT USE CHECKMARKS (✓), CROSSES (×), ETC: FILL-IN THE CIRCLES AS INDICATED ABOVE.** You do **not** need to justify your answers. You may refer to Table 2 for a list of all axioms of Set Theory that we have learned in the class.

	Statement	True	False
(a)	$\{2\} \in \{2, \{\{2\}\}\}$	○	○
(b)	$\mathbb{Q}^{\leq 0}$ is **countable**.	○	○
(c)	$f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}, f(x) = x^{16}$ is a **bijection**.	○	○
(d)	$f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}, f(x) = \frac{1}{\|x\|}$ is a **bijection**.	○	○
(e)	$(\exists A)[\emptyset \in (A - A)]$	○	○
(f)	$(\forall A)[\|\mathcal{P}(A)\| = \|A\|]$	○	○
(g)	$(\forall A)[\|\mathcal{P}(\{A\})\|] = 2$	○	○
(h)	$\|\mathcal{P}(\mathcal{P}(\{1, 2, 3\}))\| = 8$	○	○
(i)	If $a, b \in \mathbb{Q}^{>0}$, then $a^b \in \mathbb{Q}$	○	○
(j)	If $a, b \notin \mathbb{Q}$, then $\frac{a}{b} \notin \mathbb{Q}$	○	○

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Problem 2: Logic *(10 pts)*

Question (a): Truth Tables (5 pts)

Complete the following **truth table** for the logical expression

$$(p \wedge q) \vee (\sim(z \wedge q))$$

To start you off, we are giving you the first three columns. Write **neatly**; if we can't make out the difference between a **T** and an **F**, we will be forced to take off points! Also, you should write **T** and **F**, **not 1** or **0**. You can use the scrap space available if you want to first fill in the table with **0**s and **1**s.

p	q	z	$p \wedge q$	$z \wedge q$	$\sim(z \wedge q)$	$(p \wedge q) \vee (\sim(z \wedge q))$
F	**F**	**F**				
F	**F**	**T**				
F	**T**	**F**				
F	**T**	**T**				
T	**F**	**F**				
T	**F**	**T**				
T	**T**	**F**				
T	**T**	**T**				

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Question (b): Logical equivalence (5 pts)

Prove that the following compound logical expression:

$$\big((z \vee q) \vee (z \wedge {\sim}q)\big) \wedge \big(z \vee ({\sim}({\sim}p \wedge {\sim}z))\big)$$

is logically equivalent to the expression:

$$z \vee (q \wedge p)$$

For every derivation you make, write the **name** of the propositional logic axiom that you are using **on the right of the derivation.** Refer to Table 1 for a list of all propositional logic axioms we have learned in the class.

BEGIN YOUR ANSWER FOR QUESTION (b) BELOW THIS LINE

Problem 3: Relations *(10 pts)*

For every one of the following relations, fill in the square corresponding to the choices you believe are appropriate for the given relation A. For example, if you believe that A **is** reflexive but **neither** symmetric **nor** transitive, you should turn the first square from □ into ■ with your pen or pencil, yet leave the other two ones empty, like so: □. If you believe that the relation doesn't have **any** of the three properties, you should fill in **the last** box. If you believe that the relation has **all** properties, you should fill in **the first three** boxes. **PLEASE DO NOT USE CHECKMARKS (✓), CROSSES, (×), ETC: FILL IN THE SQUARES AS INDICATED ABOVE.** Note that for every relation, you are **given** the sets for which the relation is defined; **those change from relation to relation**. You do **not** need to justify your answers.

	Relation	Reflexive	Symmetric	Transitive	None
(a)	$A \subseteq \mathbb{R} \times \mathbb{R}$, $A = \{(x, y) \mid x \geq y\}$	□	□	□	□
(b)	$A \subseteq \mathbb{N}^{\geq 1} \times \mathbb{N}^{\geq 1}$, $A = \{(x, y) \mid x \leq y^2\}$	□	□	□	□
(c)	$A \subseteq \mathbb{R} \times \mathbb{R}$, $A = \{(x, y) \mid \lvert x - y \rvert \geq 1\}$	□	□	□	□
(d)	$A \subseteq \mathbb{N}^{\geq 2} \times \mathbb{N}^{\geq 2}$, $A = \{(x, y) \mid y \equiv 0 \pmod{x}\}$	□	□	□	□
(e)	$A \subseteq \mathbb{R} \times \mathbb{R}, A = \{(x, y) \mid x \geq \lfloor y \rfloor\}$	□	□	□	□

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Problem 4: Number Theory *(20 pts)*

Question (a): Direct Proof (5 pts)

Prove **directly** that, if $q \in \mathbb{Q}$, $q - 1 \in \mathbb{Q}$.

BEGIN YOUR ANSWER FOR QUESTION (a) BELOW THIS LINE

Question (b): Indirect Proof I (5 pts)

Suppose that $a \in \mathbb{Z}$. Using an *indirect* proof methodology, prove that, if $a^3 \equiv 0 \pmod 5$, then $a \equiv 0 \pmod 5$.

BEGIN YOUR ANSWER FOR QUESTION (b) BELOW THIS LINE

Question (c): Indirect Proof II (10 pts)

Using an *indirect* proof methodology as well as the theorem of question (b), prove that $\sqrt[3]{5} \notin \mathbb{Q}$. You should take the theorem of question (b) **as a given, whether you were able to prove it or not.**

BEGIN YOUR ANSWER FOR QUESTION (c) BELOW THIS LINE

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Problem 5: Induction *(20 pts)*

Question (a): Strong (5 pts)

Let a_n be a sequence recursively defined as follows:

$$a_n = \begin{cases} 6, & n = 0 \\ 8, & n = 1 \\ -2n + a_{n-1} + a_{n-2}, & n \geq 2 \end{cases}$$

Using **strong induction**, show that

$$(\forall n \geq 0)[a_n = 2n + 6]$$

WRITE YOUR INDUCTIVE BASE BELOW THIS LINE

WRITE YOUR INDUCTIVE HYPOTHESIS BELOW THIS LINE

WRITE YOUR INDUCTIVE STEP BELOW THIS LINE

Question (b): Structural on binary trees (7 pts)

We recursively define a **perfect binary tree** of height h, as follows:

- A single node called the tree's **root**, if $h = 0$, or
- A root node pointing to **two perfect** binary trees of height $h - 1$, if $h > 0$.

Let V be the number of nodes and E the number of edges in a perfect binary tree. Use structural induction to prove that $E = V - 1$. *Hint: Induct on the height of the tree.*

WRITE YOUR INDUCTIVE BASE BELOW THIS LINE

WRITE YOUR INDUCTIVE HYPOTHESIS BELOW THIS LINE

WRITE YOUR INDUCTIVE STEP BELOW THIS LINE

Question (c): Structural on k-ary trees (8 pts)

A **perfect tree** (not perfect **binary** tree!) of height h is either

- A single node called the tree's **root**, if $h = 0$, or
- A root node that points to the roots of **one** or **more** perfect trees, each of height $h - 1$, if $h > 0$.

Let V be the number of nodes and E the number of edges in a perfect tree. Use structural induction to prove that $E = V - 1$. *Hint: Induct on the height of the tree.*

WRITE YOUR INDUCTIVE BASE BELOW THIS LINE

WRITE YOUR INDUCTIVE HYPOTHESIS BELOW THIS LINE

WRITE YOUR INDUCTIVE STEP BELOW THIS LINE

Problem 6: Combinatorics / Probability *(20 pts)*

Answer the following questions on the line available to you after each and every one of them. Your answer should be in terms of **factorials**, **permutation / combination symbols** or **ratios** of the aforementioned quantities. Do **not** simplify any ratios: for example, if you end up with a result of form $^{4!}/_{2!}$, leave it **exactly as is**; do **not** simplify the ratio to 6.

(a) The **Greek alphabet** has **24 (twenty-four)** characters, of which **7 (seven)** are vowels, and **17 (seventeen)** are consonants. How many strings of length **13 (thirteen)** can we construct from this alphabet...

(i) If we **CAN reuse** the same characters (WITH replacement)? ____________

(ii) If we **CANNOT reuse** the same characters (WITHOUT replacement)? ____________

(iii) If we want **exactly** three vowels, and we **CAN** re-use characters? ____________

(iv) If we want **exactly** three vowels, and we **CANNOT** re-use characters? ____________

(b) The **octal** numbering system uses the digits 0, 1, 2, 3, 4, 5, 6, 7 to build "octal strings". How many octal strings of length 10 (ten) contain:

(i) **Exactly** three **6**s? ____________

(ii) **At most** three **6**s? ____________

(iii) **Exactly** three **6**s and **exactly** four **5**s? ____________

(c) A standard deck of cards has **52 (fifty-two)** cards, which are divided into **4 (four) suits**: **clubs, diamonds, spades and hearts**. Every suit is subdivided into **13 (thirteen) ranks**: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King. The Jack, Queen and King of **all four suits** are also called *face* cards. A "hand" is just a set of $n \geq 1$ cards. Order **never matters** in hands. For example, A♠ 7♠ 4♣ and 7♠ A♠ 4♣ are the **same 3 (three) - card hand**.

(i) How many **6 (SIX)** - card hands are there? ____________

(ii) How many **6 (SIX)** - card hands of **only face cards** are there? ____________

(iii) How many **6 (SIX)** - card hands contain **all 4 (four) Aces**? __________

(iv) How many **4 (FOUR)** - card hands contain **all 4 (four) Aces**? __________

(v) How many **6 (SIX)** - card hands **do not** contain **any** face cards? __________

(d) Suppose that we have a group of people from **4 (four)** different countries: **8 (eight)** from the USA, **2 (two)** from Egypt, **4 (four)** from Greece and **6 (six)** from Israel. We are interesting in creating committees from all of these people. In committees, order of the people in the committee does **NOT** matter.

(i) How many **4 (four)**-person committees can we make from this group of people? __________

(ii) How many **4 (four)**-person committees can we make from this group of people if we want every person to be **from a different country?** __________

(iii) What is the **probability** that a **4 (four)**-person committee, chosen at random from this group of people consists of people from **4 (four)** different countries? __________

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Problem 7: Show me what you got *(10 pts)*

Suppose that T_n is a sequence recursively defined as follows:

$$T_n = \begin{cases} 0, & n = 0 \\ 5, & n = 1 \\ T_{\lfloor n/6 \rfloor} + T_{\lfloor 2n/3 \rfloor} + 3n, & n \geq 2 \end{cases}$$

Using **Constructive STRONG induction**, find the **smallest possible** constant $c \in \mathbb{R}^{>0}$ such that:

$$(\forall n \in \mathbb{N})[T_n \leq cn]$$

WRITE YOUR INDUCTIVE BASE BELOW THIS LINE

WRITE YOUR INDUCTIVE HYPOTHESIS BELOW THIS LINE

WRITE YOUR INDUCTIVE STEP BELOW THIS LINE

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