

# Midterm Exam#1

**Date:** Wednesday, October 10th

**Time:** 06:05-08:00pm

First and last name (**exactly** as on ELMS): \_\_\_\_\_

UID (9 digits): \_\_\_\_\_

## University Honor Pledge:

*I pledge on my honor that I have not given or received  
any unauthorized assistance on this assignment/examination.*

**Print the text of the University Honor Pledge below:**

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**Signature:** \_\_\_\_\_

## Exam guidelines / rules

- Turn off all unapproved electronic devices.
- The exam is **closed book** and **notes**.
- There are **7 (seven)** problems in this exam, with a total grade value equal to **100 (one hundred)** points.
- The exam has been printed **two-sided, stapled on the top-left corner** and spans **20 (twenty)** pages across **10 (ten)** sheets. **DO NOT RIP PAGES FROM THE EXAM**. Last page is **scrap paper**. If you need **extra** scrap paper, ask us for it.
- The total time allocated for this exam is **115 (one hundred and fifteen)** minutes.
- You may **not** leave the classroom (e.g to go to the bathroom, or because you're done) during the **last 5 (five) minutes** of the exam.

# Provided materials & assumed facts

## Logic

Table 1 contains a number of logical equivalences that we have discussed in class.

<b>Commutativity of binary operators</b>	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
<b>Associativity of binary operators</b>	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
<b>Distributivity of binary operators</b>	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
<b>Identity laws</b>	$p \wedge T \equiv p$	$p \vee F \equiv p$
<b>Negation laws</b>	$p \vee (\sim p) \equiv T$	$p \wedge (\sim p) \equiv F$
<b>Double negation</b>	$\sim(\sim p) \equiv p$	
<b>Idempotence</b>	$p \wedge p \equiv p$	$p \vee p \equiv p$
<b>De Morgan's axioms</b>	$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$	$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
<b>Universal bound laws</b>	$p \vee T \equiv T$	$p \wedge F \equiv F$
<b>Absorption laws</b>	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
<b>Negations of contradictions / tautologies</b>	$\sim F \equiv T$	$\sim T \equiv F$
<b>Contrapositive</b>	$(a \Rightarrow b) \equiv ((\sim b) \Rightarrow (\sim a))$	
<b>Equivalence between biconditional and implication</b>	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
<b>Equivalence between implication and disjunction</b>	$a \Rightarrow b \equiv \sim a \vee b$	

Table 1: A number of propositional logic axioms you can refer to.

Table 2 contains a number of **valid** rules of inference / natural deduction that we have discussed in class.

<b>Modus Ponens</b>	<b>Modus Tollens</b>	<b>Disjunctive addition</b>	<b>Conjunctive addition</b>
$p$ $p \Rightarrow q$ $\therefore q$	$\sim q$ $p \Rightarrow q$ $\therefore \sim p$	$p$ $\therefore p \vee q$	$p, q$ $\therefore p \wedge q$
<b>Conjunctive Simplification</b>	<b>Disjunctive syllogism</b>	<b>Hypothetical syllogism</b>	<b>Resolution</b>
$p \wedge q$ $\therefore p, q$	$p \vee q$ $\sim p$ $\therefore q$	$p \Rightarrow q$ $q \Rightarrow r$ $\therefore p \Rightarrow r$	$p \vee q$ $(\sim q) \vee z$ $\therefore p \vee z$

Table 2: Some valid rules of inference / natural deduction you can refer to.

## Set Theory

The following are Set Theoretic notation and definitions.

Operation	Symbol	Definition
Membership	$x \in A$	$x$ is a member of set $A$
Non-membership	$x \notin A$	$\sim(x \in A)$
Union	$A \cup B$	$\{(x \in A) \vee (x \in B)\}$
Intersection	$A \cap B$	$\{(x \in A) \wedge (x \in B)\}$
Relative complement of B given A	$A - B$	$\{(x \in A) \wedge (x \notin B)\}$
Universal (Absolute) complement	$\overline{A}$	$\{x \notin A\}$
Cartesian Product	$A \times B$	$\{(a, b) \mid (a \in A) \wedge (b \in B)\}$
Subset	$A \subseteq B$	$(\forall x \in A)[x \in B]$
Superset	$A \supseteq B$	$B \subseteq A$
Set equality	$A = B$	$(A \subseteq B) \wedge (B \subseteq A)$
Set non-equality	$A \neq B$	$\sim(A = B)$
Proper subset	$A \subset B$	$\{(A \subseteq B) \wedge (A \neq B)\}$
Proper superset	$A \supset B$	$\{(A \supseteq B) \wedge (A \neq B)\}$
Powerset	$\mathcal{P}(A)$	$\{X \mid X \subseteq A\}$

Table 3: Definitions of Set Theory

## Others

You may assume/use the following mathematical and logical facts/definitions **without proof**.

- $\mathbb{N} = \{0, 1, 2, \dots\}$  (that is,  $0 \in \mathbb{N}$ )
- $\mathbb{N}$  is closed under **addition** and **multiplication**.
- $\mathbb{Z}$  is closed under **addition**, **subtraction** and **multiplication**.
- $\mathbb{R}$  is closed under **addition**, **subtraction** and **multiplication**.
- An integer can be **either** odd **or** even, but **not both**.

## Problem 1: Logic (10 pts)

Suppose that we have the following rule of inference:

$$\begin{array}{c} (\sim p) \Rightarrow (\sim q) \\ (\sim p) \vee r \\ \sim r \\ \hline \therefore q \end{array}$$

*Question (a): Truth Table (3 pts)*

Fill in the following truth table. Write **neatly**; if we can't make out the difference between **T** and **F**, we will be forced to take off points! Also, you **must** write **T** and **F**, **not** 1 or 0.

Row #	$p$	$q$	$r$	$\sim p$	$\sim q$	$\sim r$	$(\sim p) \vee r$	$(\sim p) \Rightarrow (\sim q)$
1	<b>F</b>	<b>F</b>	<b>F</b>					
2	<b>F</b>	<b>F</b>	<b>T</b>					
3	<b>F</b>	<b>T</b>	<b>F</b>					
4	<b>F</b>	<b>T</b>	<b>T</b>					
5	<b>T</b>	<b>F</b>	<b>F</b>					
6	<b>T</b>	<b>F</b>	<b>T</b>					
7	<b>T</b>	<b>T</b>	<b>F</b>					
8	<b>T</b>	<b>T</b>	<b>T</b>					

*Question (b): Rule invalidity (2 pts)*

**Briefly** explain on the lines below **why** the truth table you filled tells us that the proof is **invalid**. **DO NOT** MAKE ANY MARKS ON THE TABLE ITSELF! You can refer to a row by its **number**.

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*Question (c): Deriving a valid conclusion (5 pts)*

In questions (a) and (b), we proved that  $q$  does **not** logically follow from our given premises. Using **only** valid rules of natural deduction provided for you in table 2, prove that  $\sim q$  **does** follow from these premises, establishing the following as a **valid** rule of inference. You **must** label each step of your proof with the name of the rule of natural deduction that you use!

$$\begin{array}{c} (\sim p) \Rightarrow (\sim q) \\ (\sim p) \vee r \\ \sim r \\ \hline \therefore (\sim q) \end{array}$$

**BEGIN YOUR ANSWER BELOW THIS LINE**

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## Problem 2: Circuits (10 pts)

Observe the truth table of Table 4.

Inputs			Output
$p$	$q$	$r$	$z$
<b>F</b>	<b>F</b>	<b>F</b>	<b>T</b>
<b>F</b>	<b>F</b>	<b>T</b>	<b>F</b>
<b>F</b>	<b>T</b>	<b>F</b>	<b>F</b>
<b>F</b>	<b>T</b>	<b>T</b>	<b>F</b>
<b>T</b>	<b>F</b>	<b>F</b>	<b>T</b>
<b>T</b>	<b>F</b>	<b>T</b>	<b>T</b>
<b>T</b>	<b>T</b>	<b>F</b>	<b>F</b>
<b>T</b>	<b>T</b>	<b>T</b>	<b>F</b>

Table 4: A truth table for a formula that maps inputs  $p, q, r$  to an output  $z$ .

### Question (a): Boolean Formula (5 pts)

Based on the truth table shown to you in Table 4, provide us with a boolean (propositional) formula that computes the output  $z$  based on the inputs  $p, q, r$ . Your formula should be in *Disjunctive Normal Form* (**DNF**), i.e it must be a **disjunction of conjunctions**, like in your homework.

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*Question (b): Circuit (5 pts)*

Draw a **circuit** that computes the boolean formula

$$(p \vee (\sim q)) \wedge (r \vee q) \wedge ((\sim r) \vee p)$$

You should use **only AND, OR and NOT** gates!

**BEGIN YOUR ANSWER BELOW THIS LINE**

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### Problem 3: Set Theory (10 pts)

For every one of the following statements, fill in the circle corresponding to the appropriate choice (**True** or **False**). For example, if any given statement is **True**, you should fill in the **first** circle, such that  $\circ$  becomes  $\bullet$ . **PLEASE DO NOT USE CHECKMARKS ( $\checkmark$ ), CROSSES ( $\times$ ), ETC: FILL-IN THE CIRCLES AS INDICATED ABOVE.** You **do not** need to justify your answers.

	Statement	True	False
(a)	$0 \in \mathbb{Q}$	<input type="radio"/>	<input type="radio"/>
(b)	$50 \in \mathbb{Z}$	<input type="radio"/>	<input type="radio"/>
(c)	$\emptyset \in \emptyset$	<input type="radio"/>	<input type="radio"/>
(d)	$\emptyset \subseteq \{0\}$	<input type="radio"/>	<input type="radio"/>
(e)	$(A - B) = A \cup B^c$	<input type="radio"/>	<input type="radio"/>
(f)	$\mathbb{N} \cup \mathbb{Z} = \mathbb{Q}$	<input type="radio"/>	<input type="radio"/>
(g)	$ \{1, 6\} - \{\{1, 6\}\}  = 2$	<input type="radio"/>	<input type="radio"/>
(h)	$\emptyset \in \mathcal{P}(\{\emptyset\})$	<input type="radio"/>	<input type="radio"/>
(i)	$\{\sqrt{2}, \sqrt{4}\} \in \mathcal{P}(\mathbb{R} - \mathbb{Q})$	<input type="radio"/>	<input type="radio"/>
(j)	$(\forall x, y \in \mathbb{Q}^{>0})[x^y \in \mathbb{Q}^{>0}]$	<input type="radio"/>	<input type="radio"/>



## Problem 4: Quantifiers (15 pts)

In the following quantified expressions, push the negation operator ( $\sim$ ) **as far inside the expression as possible**. Your final expression should **not have any instances of  $\sim$ !** Do **not** concern yourselves with the **truth values** of the statements **or** their negations; just concentrate on the task at hand. You do **not** need to show us intermediate steps; just the **final expression** is sufficient.

(a)  $\sim(\forall n \in \mathbb{Z})[n^2 \geq 0]$

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(b)  $\sim(\forall n_1 \in \mathbb{Z})(\exists n_2 \in \mathbb{Z})[(n_1 - n_2) < 0]$

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(c)  $\sim(\forall n_1 \in \mathbb{Z})(\forall n_2 \in \mathbb{Z})[(n_1 < n_2) \Rightarrow (\sim \exists q \in \mathbb{Q})[n_1 < q < n_2]]$

**BEGIN YOUR ANSWER BELOW THIS LINE**

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**Problem 5: Direct proofs (20 pts)**

*Question (a): Rationals (5 pts)*

Prove **directly** that, if  $q \in \mathbb{Q}$ , then  $\frac{q}{3} \in \mathbb{Q}$ .

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*Question (b): Divisiblity (15 pts)*

Suppose  $x, y \in \mathbb{Z}$ . When divided by 7,  $x$  leaves a remainder of 2. When divided by 14,  $y$  leaves a remainder of 3. Prove **directly** that  $x \cdot y$  leaves a remainder of 6 when divided by 7.

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**Problem 6: Indirect proofs (30 pts)**

*Question (a): Lemma (5 pts)*

Suppose  $n$  is an integer. Prove **indirectly** that, if  $n^3$  is a multiple of 5, then so is  $n$ .

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**CONTINUE YOUR ANSWER BELOW THIS LINE**

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*Question (b): Euclidean proof I (10 pts)*

Prove **indirectly** that  $\sqrt[3]{5} \notin \mathbb{Q}$  using the **Euclidean argument**.

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*Question (c): Proof via UPFT (15 pts)*

Prove **indirectly** that  $\sqrt[3]{5} \notin \mathbb{Q}$  using the **Unique Prime Factorization Theorem**.

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### Problem 7: Show me what you got (*5 pts*)

Suppose that  $n \in \mathbb{Z}$ . Using **any** methodology that you have learned so far in the class, prove that  $8n^2 + 5n$  is even **if, and only if**,  $n^4 + 6$  is even. You can use the page in the back if you run out of space.

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## SCRAP PAPER

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