CMSC 250, Fall 2018 **Sections: 020x**

Midterm Exam#1

Date: Wednesday, October 10th

Time: 06:0**5**-08:00pm

*First and last name (***exactly** *as on ELMS):* ______________________________

UID (9 digits): ______________

First name of your professor: ______________

University Honor Pledge:

I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/examination.

Print the text of the University Honor Pledge below:

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Signature: ______________________________

Exam guidelines / rules

- **Turn off all unapproved electronic devices**.
- The exam is **closed book** and **notes**.
- There are **7 (seven)** problems in this exam, with a total grade value equal to **100 (one hundred)** points.
- The exam has been printed **two-sided**, **stapled on the top-left corner** and spans **20 (twenty)** pages across **10 (ten)** sheets. **DO NOT RIP PAGES FROM THE EXAM**. Last page is **scrap paper**. If you need **extra** scrap paper, ask us for it.
- The total time allocated for this exam is **115 (one hundred and fifteen)** minutes.
- You may **not** leave the classroom (e.g to go to the bathroom, or because you're done) during the **last 5 (five) minutes** of the exam.

Provided materials & assumed facts

Logic

Table 1 contains a number of logical equivalences that we have discussed in class.

Commutativity of binary operators	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity of binary operators	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity of binary operators	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation laws	$p \vee (\sim p) \equiv T$	$p \wedge (\sim p) \equiv F$
Double negation	$\sim(\sim p) \equiv p$	
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's axioms	$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$	$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
Universal bound laws	$p \vee T \equiv T$	$p \wedge F \equiv F$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of contradictions / tautologies	$\sim F \equiv T$	$\sim T \equiv F$
Contrapositive	$(a \Rightarrow b) \equiv ((\sim b) \Rightarrow (\sim a))$	
Equivalence between biconditional and implication	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
Equivalence between implication and disjunction	$a \Rightarrow b \equiv \sim a \vee b$	

Table 1: A number of propositional logic axioms you can refer to.

Table 2 contains a number of **valid** rules of inference / natural deduction that we have discussed in class.

Modus Ponens	**Modus Tollens**	**Disjunctive addition**	**Conjunctive addition**
p $p \Rightarrow q$ $\therefore q$	$\sim q$ $p \Rightarrow q$ $\therefore \sim p$	p $\therefore p \vee q$	p, q $\therefore p \wedge q$
Conjunctive Simplification	**Disjunctive syllogism**	**Hypothetical syllogism**	**Resolution**
$p \wedge q$ $\therefore p, q$	$p \vee q$ $\sim p$ $\therefore q$	$p \Rightarrow q$ $q \Rightarrow r$ $\therefore p \Rightarrow r$	$p \vee q$ $(\sim q) \vee z$ $\therefore p \vee z$

Table 2: Some valid rules of inference / natural deduction you can refer to.

Set Theory

The following are Set Theoretic notation and definitions.

Operation	**Symbol**	**Definition**
Membership	$x \in A$	x is a member of set A
Non-membership	$x \notin A$	$\sim(x \in A)$
Union	$A \cup B$	$\{(x \in A) \vee (x \in B)\}$
Intersection	$A \cap B$	$\{(x \in A) \wedge (x \in B)\}$
Relative complement of B given A	$A - B$	$\{(x \in A) \wedge (x \notin B)\}$
Universal (Absolute) complement	$\overline{A}$	$\{x \notin A\}$
Cartesian Product	$A \times B$	$\{(a, b) \mid (a \in A) \wedge (b \in B)\}$
Subset	$A \subseteq B$	$(\forall x \in A)[x \in B]$
Superset	$A \supseteq B$	$B \subseteq A$
Set equality	$A = B$	$(A \subseteq B) \wedge (B \subseteq A)$
Set non-equality	$A \neq B$	$\sim(A = B)$
Proper subset	$A \subset B$	$\{(A \subseteq B) \wedge (A \neq B)\}$
Proper superset	$A \supset B$	$\{(A \supseteq B) \wedge (A \neq B)\}$
Powerset	$\mathcal{P}(A)$	$\{X \mid X \subseteq A\}$

Table 3: Definitions of Set Theory

Others

You may assume/use the following mathematical and logical facts/definitions **without proof**.

- $\mathbb{N} = \{0, 1, 2, ...\}$ (that is, $0 \in \mathbb{N}$)
- $\mathbb{N}$ is closed under **addition** and **multiplication**.
- $\mathbb{Z}$ is closed under **addition**, **subtraction** and **multiplication**.
- $\mathbb{R}$ is closed under **addition**, **subtraction** and **multiplication**.
- An integer can be **either** odd **or** even, but **not both**.

Problem 1: Logic *(10 pts)*

Suppose that we have the following rule of inference:

$$\begin{array}{r} (\sim p) \Rightarrow (\sim q) \\ (\sim p) \vee r \\ \sim r \\ \hline \therefore q \end{array}$$

Question (a): Truth Table (3 pts)

Fill in the following truth table. Write **neatly**; if we can't make out the difference between **T** and **F**, we will be forced to take off points! Also, you **must** write **T** and **F**, **not 1** or **0**.

Row #	p	q	r	$\sim p$	$\sim q$	$\sim r$	$(\sim p) \vee r$	$(\sim p) \Rightarrow (\sim q)$
1	**F**	**F**	**F**					
2	**F**	**F**	**T**					
3	**F**	**T**	**F**					
4	**F**	**T**	**T**					
5	**T**	**F**	**F**					
6	**T**	**F**	**T**					
7	**T**	**T**	**F**					
8	**T**	**T**	**T**					

Question (b): Rule invalidity (2 pts)

Briefly explain on the lines below **why** the truth table you filled tells us that the proof is **invalid**. DO **NOT** MAKE ANY MARKS ON THE TABLE ITSELF! You can refer to a row by its **number**.

Question (c): Deriving a valid conclusion (5 pts)

In questions (a) and (b), we proved that q does **not** logically follow from our given premises. Using **only** valid rules of natural deduction provided for you in table 2, prove that $\sim q$ **does** follow from these premises, establishing the following as a **valid** rule of inference. You **must** label each step of your proof with the name of the rule of natural deduction that you use!

$$\begin{array}{r}(\sim p) \Rightarrow (\sim q)\\ (\sim p) \vee r\\ \sim r\\ \hline \therefore (\sim q)\end{array}$$

BEGIN YOUR ANSWER BELOW THIS LINE

Problem 2: Circuits *(10 pts)*

Observe the truth table of Table 4.

Inputs			Output
p	q	r	z
F	**F**	**F**	**T**
F	**F**	**T**	**F**
F	**T**	**F**	**F**
F	**T**	**T**	**F**
T	**F**	**F**	**T**
T	**F**	**T**	**T**
T	**T**	**F**	**F**
T	**T**	**T**	**F**

Table 4: A truth table for a formula that maps inputs p, q, r to an output z.

Question (a): Boolean Formula (5 pts)

Based on the truth table shown to you in Table 4, provide us with a boolean (propositional) formula that computes the output z based on the inputs p, q, r. Your formula should be in *Disjunctive Normal Form* (**DNF**), i.e it must be a **disjunction of conjunctions**, like in your homework.

BEGIN YOUR ANSWER BELOW THIS LINE

Question (b): Circuit (5 pts)

Draw a **circuit** that computes the boolean formula

$$(p \vee (\sim q)) \wedge (r \vee q) \wedge ((\sim r) \vee p\ )$$

You should use **only AND, OR** and **NOT** gates!

BEGIN YOUR ANSWER BELOW THIS LINE

Problem 3: Set Theory *(10 pts)*

For every one of the following statements, fill in the circle corresponding to the appropriate choice (**True** or **False**). For example, if any given statement is **True**, you should fill in the **first** circle, such that ○ becomes ● . **PLEASE DO NOT USE CHECKMARKS (✓), CROSSES (×), ETC: FILL-IN THE CIRCLES AS INDICATED ABOVE.** You **do not** need to justify your answers.

	Statement	**True**	**False**
(a)	$0 \in \mathbb{Q}$	○	○
(b)	$50 \in \mathbb{Z}$	○	○
(c)	$\emptyset \in \emptyset$	○	○
(d)	$\emptyset \subseteq \{0\}$	○	○
(e)	$(A - B) = A \cup B^c$	○	○
(f)	$\mathbb{N} \cup \mathbb{Z} = \mathbb{Q}$	○	○
(g)	$\lvert\{1,6\} - \{\{1,6\}\}\rvert = 2$	○	○
(h)	$\emptyset \in \mathcal{P}(\{\emptyset\})$	○	○
(i)	$\{\sqrt{2}, \sqrt{4}\} \in \mathcal{P}(\mathbb{R} - \mathbb{Q})$	○	○
(j)	$(\forall x, y \in \mathbb{Q}^{>0})[x^y \in \mathbb{Q}^{>0}]$	○	○

Problem 4: Quantifiers *(15 pts)*

In the following quantified expressions, push the negation operator (~) **as far inside the expression as possible**. Your final expression should **not have any instances of ~!** Do **not** concern yourselves with the **truth values** of the statements **or** their negations; just concentrate on the task at hand. You do **not** need to show us intermediate steps; just the **final expression** is sufficient.

(a) $\sim(\forall n \in \mathbb{Z})[n^2 \geq 0]$

BEGIN YOUR ANSWER BELOW THIS LINE

(b) $\sim(\forall n_1 \in \mathbb{Z})(\exists n_2 \in \mathbb{Z})[(n_1 - n_2) < 0]$

BEGIN YOUR ANSWER BELOW THIS LINE

(c) $\sim(\forall n_1 \in \mathbb{Z})(\forall n_2 \in \mathbb{Z})\big[(n_1 < n_2) \Rightarrow (\sim\exists q \in \mathbb{Q})[n_1 < q < n_2]\big]$

BEGIN YOUR ANSWER BELOW THIS LINE

Problem 5: Direct proofs *(20 pts)*

Question (a): Rationals (5 pts)

Prove **directly** that, if $q \in \mathbb{Q}$, then $\frac{q}{3} \in \mathbb{Q}$.

BEGIN YOUR ANSWER BELOW THIS LINE

Question (b): Divisiblity (15 pts)

Suppose $x, y \in \mathbb{Z}$. When divided by 7, x leaves a remainder of 2. When divided by 14, y leaves a remainder of 3. Prove **directly** that $x \cdot y$ leaves a remainder of 6 when divided by 7.

BEGIN YOUR ANSWER BELOW THIS LINE

Problem 6: Indirect proofs *(30 pts)*

Question (a): Lemma (5 pts)

Suppose n is an integer. Prove **indirectly** that, if n^3 is a multiple of 5, then so is n.

BEGIN YOUR ANSWER BELOW THIS LINE

CONTINUE YOUR ANSWER BELOW THIS LINE

Question (b): Euclidean proof I (10 pts)

Prove **indirectly** that $\sqrt[3]{5} \notin \mathbb{Q}$ using the **Euclidean argument**.

BEGIN YOUR ANSWER BELOW THIS LINE

CONTINUE YOUR ANSWER BELOW THIS LINE

Question (c): Proof via UPFT (15 pts)

Prove **indirectly** that $\sqrt[3]{5} \notin \mathbb{Q}$ using the **Unique Prime Factorization Theorem**.

BEGIN YOUR ANSWER BELOW THIS LINE

CONTINUE YOUR ANSWER BELOW THIS LINE

Problem 7: Show me what you got *(5 pts)*

Suppose that $n \in \mathbb{Z}$. Using **any** methodology that you have learned so far in the class, prove that $8n^2 + 5n$ is even **if, and only if**, $n^4 + 6$ is even. You can use the page in the back if you run out of space.

BEGIN YOUR ANSWER BELOW THIS LINE

CONTINUE YOUR ANSWER BELOW THIS LINE

SCRAP PAPER