How to Morph Tilings Injectively

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Abstract: We describe a method based on convex combinations for morphing corresponding pairs of tilings in $\mathbb{R}^2$. It is shown that the method always yields a valid morph when the boundary polygons are identical, unlike the standard linear morph.

Key words: Morphing, triangulations, tilings, polygons, convex combinations.

1. Introduction

This paper is concerned with how to construct a continuous mapping from one tiling in $\mathbb{R}^2$ to another. A tiling could be a triangulation or a rectangular grid but in general each face can have an arbitrary number of vertices. The continuous evolution from one geometric object into another is generally known as *morphing*, derived from the word 'metamorphosis', and has become important in computer graphics where geometric objects are used to generate computer images.

Several methods are known for morphing two-dimensional images [**2**], planar polygons [**8,14,15,16**], and three-dimensional volume data [**10,11**].

For simple polygons, a complete morph defines both a *correspondence* between the vertices of the two given polygons and a set of *paths* along which the corresponding vertices travel as the first polygon evolves into the second. Some papers have focused on solving the correspondence problem [**3**] and take the paths to be simple linear trajectories between corresponding points. Other papers assume that the correspondence is given and address the problem of finding the vertex paths [**8,15,16**]. Clearly a natural morph of two simple polygons is one which ensures that all the polygons in between are also simple but at the time of writing this appears to be an open problem.

In this paper we study the related problem of morphing two corresponding tilings. To the best of our knowledge the only known morph in this context is the linear morph in which the vertices travel with uniform speed and has been used to morph images as in [**9**]. However we show by way of counterexample that even when the boundaries of the tilings are identical, the intermediate vertex sets of the linear morph may not necessarily induce valid tilings.

We then introduce and study an alternative method for morphing tilings based on averaging convex combinations and we show that at least if the boundaries of the two tilings are identical (and convex) the method always provides a valid morph. In the event that the boundaries differ, the convex combination morph can be generalized by morphing the boundary polygons first. We propose a simple method for morphing the boundaries which, though it does not guarantee to preserve convexity of the intermediate boundaries, yields a valid morph in all our numerical examples.

Recently, Fujimura and Makarov [**7**] have proposed an approach to morphing images by allowing triangulations to change their toplogy, e.g. by swapping edges, in order to prevent foldover.

2. Tilings

We begin our discussion with some notation and definitions. Let the convex hull of a subset A of $\mathbb{R}^2$ be denoted by $[A]$. For points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^2$ let the signed area of the triangle $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ be

$$\operatorname{area}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}. \tag{2.1}$$

By a *polygon* we mean a sequence $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_K)$, $K \geq 3$, of distinct points $\mathbf{p}_i$ in $\mathbb{R}^2$. The *edges* of the polygon $\mathbf{P}$ are the line segments

$$[\mathbf{p}_1, \mathbf{p}_2], [\mathbf{p}_2, \mathbf{p}_3], \ldots, [\mathbf{p}_{K-1}, \mathbf{p}_K], [\mathbf{p}_K, \mathbf{p}_1]$$

and its *vertices* are the points $\mathbf{p}_1, \ldots, \mathbf{p}_K$. The polygon $\mathbf{P}$ is *convex* if

$$\operatorname{area}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) > 0, \qquad 1 \leq i < j < k \leq K.$$

We note that the vertices of a convex polygon are thus ordered anticlockwise.

Next let $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$ be a sequence of distinct points $\mathbf{u}_i = (u_i, v_i)$ in $\mathbb{R}^2$ and further, let $M \geq 1$ and for $j = 1, \ldots, M$, let

$$I_j = (i_{1,j} \ldots, i_{K_j,j})$$

be a sequence of distinct elements of the set $\{1, \ldots, N\}$, $3 \leq K_j \leq N$, and let

$$G = (I_1, \ldots, I_M). \tag{2.2}$$

This leads us to a definition of a tiling.

Definition 2.1. *Let* $\mathbf{U}$ *and* G *be defined as above and for* $j = 1, \ldots, M$ *let*

$$\mathbf{P}_j = (\mathbf{u}_{i_{1,j}}, \ldots, \mathbf{u}_{i_{K_j,j}}). \tag{2.3}$$

We will call $\mathcal{T} = (\mathbf{U}, G)$ *a* `tiling` *if*
(1) $\mathbf{P}_1, \ldots, \mathbf{P}_M$ *are convex planar polygons*
(2) $[\mathbf{P}_i] \cap [\mathbf{P}_j]$ *is either empty or a common vertex or a common edge,* $i \neq j$*,*
(3) $\bigcup_{i=1}^{M} [\mathbf{P}_i] = [\mathbf{U}]$.

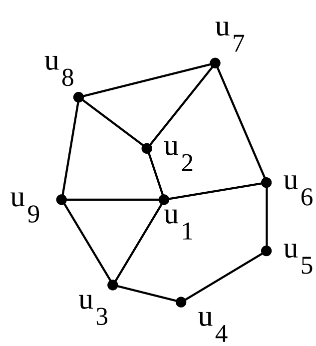

Figure 1: Tiling.

Fig. 1 shows an example of a tiling $\mathcal{T} = (\mathbf{U}, G)$ where

$$G = ((1,9,3),(3,4,5,6,1),(1,6,7,2),(1,2,8,9),(2,7,8)).$$

Note that condition (3) in the definition implies that the tiles $[\mathbf{P}_i]$ must cover a convex region of the plane and that every point $\mathbf{u}_i$ must belong to at least one polygon $\mathbf{P}_j$. We think of G as the 'topology' of the tiling and the point sequence $\mathbf{U}$ as its 'geometry'.

One kind of tiling $\mathcal{T}$ which occurs frequently in applications and which is of special interest to us is when all its polygons have precisely three vertices ($K_j = 3$, $j = 1, \ldots, M$), in which case $\mathcal{T}$ is a *triangulation*. It is well known in computational geometry (see e.g. [**12**]) that every point set admits a *Delaunay* triangulation which has the property that the minimum angle of its triangles is maximized. Thus there exists at least one tiling of every point set. In general, a set of N points will admit an exponential number of tilings.

3. Compatible Point Sequences

Suppose we are given two sequences of N distinct planar points $\mathbf{U}^0$ and $\mathbf{U}^1$. By definition $\mathbf{U}^0$ and $\mathbf{U}^1$ are points *sets* with a given *correspondence*, namely their common ordering. Such a correspondence could be specified in a computer graphics application interactively by a user. Once a correpondence is established, it is easy to extend it to a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ by interpolating between the corresponding points, using one of a variety of scattered data methods, such as radial basis functions. Perhaps the simplest method is a piecewise affine mapping $\phi : [\mathbf{U}^0] \to [\mathbf{U}^1]$ over a triangulation $\mathcal{T}^0 = (\mathbf{U}^0, G)$ of $\mathbf{U}^0$. The triangulation $\mathcal{T}^0$, together with the triangulation $\mathcal{T}^1 = (\mathbf{U}^1, G)$ induced on $\mathbf{U}^1$ by $\mathcal{T}^0$, defines an affine mapping between each of the corresponding pairs of triangles. However the mapping ϕ may not be injective if $\mathcal{T}^1$ is not a (legal) triangulation. As $\mathbf{U}^0$ admits many triangulations, the first problem is to find a triangulation $\mathcal{T}^0$ of $\mathbf{U}^0$ which induces a (legal) triangulation $\mathcal{T}^1$ on $\mathbf{U}^1$, said to be *compatible* with $\mathcal{T}^0$.

This combinatorial problem has been studied for point sets [**1,13**] and polygons [**17**]. In short, we will say that two corresponding polygons, or corresponding point sets, are compatible if they admit a compatible triangulation. To determine whether two given point sets are compatible is believed to be NP-hard. It is possible to formulate many neccesary conditions for compatibility, but a finite set of sufficient conditions has yet to be obtained. Souvaine and Wenger [**17**] show that if *Steiner* points (extraneous points) are allowed, any two sets of N points may be made compatible by adding $O(N^2)$ points, and there exist point sets of size N for whom at least $\Omega(N^2)$ points must be added in order to obtain compatibility. Etzion and Rappoport [**4**] obtained similar results for the star-shaped compatibility of polygons (i.e. when the polygon is decomposed into star-shaped regions).

4. Morphable Tilings

The question now arises, given two compatible triangulations or, more generally, tilings $\mathcal{T}^0 = (\mathbf{U}^0, G)$ and $\mathcal{T}^1 = (\mathbf{U}^1, G)$, is this compatibility preserved when the tilings are morphed.

Definition 4.1. *A* `morph` *of two sequences of N distinct points $\mathbf{U}^0$ and $\mathbf{U}^1$ is any sequence $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$ of continuous mappings $\mathbf{u}_i : [0,1] \to \mathbb{R}^2$, $i = 1, \ldots, N$, such that $\mathbf{u}_i(0) = \mathbf{u}_i^0$ and $\mathbf{u}_i(1) = \mathbf{u}_i^1$.*

A morph $\mathbf{U}$ generates an intermediate point sequence $\mathbf{U}(t) = (\mathbf{u}_1(t), \ldots, \mathbf{u}_N(t))$ for each $t \in (0,1)$. We can think of each mapping $\mathbf{u}_i$ as a parametric curve in $\mathbb{R}^2$ representing the path traced out by the points $\mathbf{u}_i(t)$ in moving from $\mathbf{u}_i^0$ to $\mathbf{u}_i^1$. If furthermore $\mathbf{U}^0$ and $\mathbf{U}^1$ are the the vertex sets of a pair of compatible tilings, we want to know whether the intermediate vertex sets $\mathbf{U}(t)$ induce 'valid' tilings.

Definition 4.2. *We will say that two compatible tilings $\mathcal{T}^0 = (\mathbf{U}^0, G)$ and $\mathcal{T}^1 = (\mathbf{U}^1, G)$ are* `morphable` *if there exists a morph $\mathbf{U}$ of $\mathbf{U}^0$ and $\mathbf{U}^1$ such that $\mathcal{T}(t) = (\mathbf{U}(t), G)$ is a tiling for all $t \in (0,1)$.*

When trying to determine whether two compatible tilings are morphable, a natural candidate is the naive "straight line" or "linear" morph $\mathbf{U}$ defined by taking the weighted average

$$\mathbf{u}_i(t) = (1-t)\mathbf{u}_i^0 + t\mathbf{u}_i^1, \qquad t \in [0,1]. \tag{4.1}$$

If $\mathcal{T}^0$ and $\mathcal{T}^1$ are morphable with respect to this linear morph we say they are *linearly morphable.*

It turns out that not all compatible tilings are linearly morphable. In fact if $\mathcal{T}^0 = (\mathbf{U}^0, G)$ is any tiling with $\mathbf{U}^0 = (\mathbf{u}_1^0, \ldots, \mathbf{u}_N^0)$ we can use the fact, easily deducible from (2.1), that $\text{area}(-\mathbf{u}_i^0, -\mathbf{u}_j^0, -\mathbf{u}_k^0) = \text{area}(\mathbf{u}_i^0, \mathbf{u}_j^0, \mathbf{u}_k^0)$ to show that if $\mathbf{U}^1 = (-\mathbf{u}_1^0, \ldots, -\mathbf{u}_N^0)$ then $\mathcal{T}^1 = (\mathbf{U}^1, G)$ is also a tiling. Yet the morph $\mathbf{U}$ in (4.1) yields $\mathbf{U}(1/2) = (0, 0, \ldots, 0)$ and so $\mathcal{T}(1/2)$ is degenerate. Since $\mathcal{T}^1$ can be viewed as a rotation through π of $\mathcal{T}^0$ in either a clockwise or anticlockwise direction, we could in this case construct a 'rotational' morph by letting $\mathcal{T}^0(t)$ be the rotation of $\mathcal{T}^0$ through an angle of πt.

We will further show that even if the boundaries of the tilings are equal, the tilings may not be linearly morphable.

Lemma 4.3. *Suppose $\mathcal{T}^0 = (\mathbf{U}^0, G)$ and $\mathcal{T}^1 = (\mathbf{U}^1, G)$ are two compatible tilings. If there are two vertices $\mathbf{u}_i^0$ and $\mathbf{u}_j^0$ in $\mathbf{U}^0$, $i \neq j$, such that*

$$\mathbf{u}_i^0 = \mathbf{u}_j^1 \qquad \text{and} \qquad \mathbf{u}_j^0 = \mathbf{u}_i^1, \tag{4.2}$$

then $\mathcal{T}^0$ and $\mathcal{T}^1$ are not linearly morphable.

Proof: From equations (4.1) and (4.2), we find

$$\mathbf{u}_i(1/2) = \frac{\mathbf{u}_i^0 + \mathbf{u}_i^1}{2} = \frac{\mathbf{u}_j^1 + \mathbf{u}_j^0}{2} = \mathbf{u}_j(1/2)$$

which means that the point sequence $\mathbf{U}(1/2) = (\mathbf{u}_1(1/2), \ldots, \mathbf{u}_N(1/2))$ contains two equal points and so $\mathcal{T}(1/2) = (\mathbf{U}(1/2), G)$ cannot be a tiling. ■

We can apply Lemma 4.3 to show that two specific tilings with equal boundaries are not linearly morphable. Let $\mathcal{T}^0 = (\mathbf{U}^0, G)$ be the triangulation given by

$$\mathbf{u}_1^0 = (-1,0), \quad \mathbf{u}_2^0 = (1,0), \quad \mathbf{u}_3^0 = (2,2), \quad \mathbf{u}_4^0 = (-2,-2),$$
$$\mathbf{u}_5^0 = (0,4), \quad \mathbf{u}_6^0 = (0,-4), \quad \mathbf{u}_7^0 = (8,0), \quad \mathbf{u}_8^0 = (-8,0),$$

and

$$G = ((1,4,3),(2,3,4),(1,3,5),(1,5,4),(2,6,3),\\ (2,4,6),(3,6,7),(3,7,5),(4,5,8),(4,8,6)), \tag{4.3}$$

and let $\mathcal{T}^1 = (\mathbf{U}^1, G)$ be the compatible triangulation defined by further letting

$$\mathbf{u}_1^1 = (1,0), \quad \mathbf{u}_2^1 = (-1,0), \quad \mathbf{u}_3^1 = (2,-2), \quad \mathbf{u}_4^1 = (-2,2)$$

and $\mathbf{u}_i^1 = \mathbf{u}_i^0$, $i = 5,\ldots,8$. Since $\mathbf{u}_1^0 = \mathbf{u}_2^1$ and $\mathbf{u}_2^0 = \mathbf{u}_1^1$, Lemma 4.3 implies that $\mathcal{T}(1/2)$ is not a triangulation and so $\mathcal{T}^0$ and $\mathcal{T}^1$ are not linearly morphable, as illustrated in Fig. 2.

In fact even the more general morph $\mathbf{U} = (\mathbf{u}_1,\ldots,\mathbf{u}_N)$ given by

$$\mathbf{u}_i(t) = (1 - w_i(t))\mathbf{u}_i^0 + w_i(t)\mathbf{u}_i^1, \qquad t \in [0,1]$$

where $w_i : [0,1] \to \mathbb{R}$ is a continuous, strictly increasing function satisfying $w_i(0) = 0$ and $w_i(1) = 1$, may not ensure that $\mathcal{T}(t)$ is a tiling. Indeed, since $w_i + w_j$ is an increasing function of t, for any $i \neq j$ there must be a value $t_0 \in (0,1)$ such that $w_i(t_0) + w_j(t_0) = 1$ and so under the hypothesis of Lemma 4.3, we similarly find $\mathbf{u}_i(t_0) = \mathbf{u}_j(t_0)$.

These counterexamples show that a more sophisticated method is needed in order to morph compatible tilings.

5. Morphing by Convex Combinations

In this section we show that all pairs of compatible tilings with identical boundaries are morphable. We do this by explicitly constructing a suitable morph. The basic idea of the morph is to represent the interior vertices of two compatible tilings $\mathcal{T}^0$ and $\mathcal{T}^1$ by convex combinations. We determine the interior vertices of $\mathcal{T}(t)$ by averaging the convex combinations rather than the vertices themselves.

For notational convenience, let I be the set of indices i in $\{1,\ldots,N\}$ such that $\mathbf{u}_i^k$ is an interior vertex of $\mathcal{T}^k$ and B the indices of boundary vertices, so that

$$I \cup B = \{1,\ldots,N\} \qquad \text{and} \qquad I \cap B = \emptyset.$$

For each $i \in I$, let $n(i) \subset \{1,2,\ldots,N\}$ denote the set of indices $j \in \{1,\ldots,N\}$ for which $\mathbf{u}_j^k$ and $\mathbf{u}_i^k$ are neighbours.

We assume in this section that the boundaries of the tilings $\mathcal{T}^0$ and $\mathcal{T}^1$ are identical. Thus we can define a 'constant' morph for boundary points and we let $\mathbf{u}_i(t) = \mathbf{u}_i^0 = \mathbf{u}_i^1$ for $i \in B$.

Next consider each interior vertex $\mathbf{u}_i^k$ of $\mathcal{T}^k$, $k = 0,1$. Since it clearly lies in the interior of the convex hull of its neighbouring vertices it follows that it can be expressed as a convex combination of its neighbours. Thus for each $i \in I$, there exist two sets of strictly positive real values $\lambda_{ij}^k \in \mathbb{R}$, $j \in n(i)$, $k \in \{0,1\}$, such that

$$\sum_{j\in n(i)} \lambda_{ij}^k \mathbf{u}_j^k = \mathbf{u}_i^k \qquad \text{and} \qquad \sum_{j\in n(i)} \lambda_{ij}^k = 1, \qquad k = 0,1. \tag{5.1}$$

Then defining

$$\lambda_{ij}(t) = (1-t)\lambda_{ij}^0 + t\lambda_{ij}^1, \qquad t \in [0,1],$$

we clearly have $\lambda_{ij}(t) > 0$ for each t and

$$\sum_{j \in n(i)} \lambda_{ij}(t) = 1. \tag{5.2}$$

We subsequently define the $|I|$ points $\mathbf{u}_i(t) \in \mathbb{R}^2$, $i \in I$, to be the solutions of the $|I|$ linear equations

$$\sum_{j \in n(i)} \lambda_{ij}(t)\mathbf{u}_j(t) = \mathbf{u}_i(t), \quad i \in I. \tag{5.3}$$

Theorem 5.1. *Two compatible tilings $\mathcal{T}^0 = (\mathbf{U}^0, G)$ and $\mathcal{T}^1 = (\mathbf{U}^1, G)$ with identical boundaries are morphable. Moreover a morph $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$ is given by the constant morph for boundary vertices and the solutions to the linear system (5.3) for interior ones.*

Proof: It was shown in [**5**] and [**6**] that the set of linear equations (5.3) has a unique solution, using the connectedness of the tiling and the weak diagonal dominance of the matrix.

Next we show that $\mathbf{U}$ is a morph. Since the values $\lambda_{ij}(t)$ depend continuously on t, it follows from equation (5.3) that the mappings $\mathbf{u}_i$ are continuous on $[0,1]$. Furthermore, due to the choice of convex combinations in (5.1), the points $\mathbf{u}_i^k$ for $i \in I$, $k = 0,1$, are the unique solutions to the linear system (5.3) in the two cases $t = 0,1$ respectively. Therefore $\mathbf{u}_i(0) = \mathbf{u}_i^0$ and $\mathbf{u}_i(1) = \mathbf{u}_i^1$ for $i \in I$ and indeed all $i = 1, \ldots, N$, which establishes that $\mathbf{U}$ is a morph according to Definition 4.1.

Next we must check that $\mathcal{T}(t) = (\mathbf{U}(t), G)$ is a tiling. It was also observed in [**5**] and [**6**] that the set of linear equations (5.3) is a simple generalization of those arising from the 'barycentric mapping' proposed by Tutte [**19**] for generating straight line drawings of planar graphs. In Tutte's mapping the coefficients $\lambda_{ij}(t)$ for fixed i are constant and equal to $1/d_i$ where d_i is the valency of $\mathbf{u}_i(t)$. Since the theory in [**19**] (see also the prerequisite [**18**]) can be generalized to arbitrary convex combinations and the boundary polygon of $\mathbf{U}(t)$ is convex, it follows, as observed in [**5**], that $\mathcal{T}(t)$ is indeed a tiling according to Definition 2.1. This establishes the condition of Definition 4.2. ■

In order to find such a morph, we need to be able to express the interior vertices of a given tiling $\mathcal{T} = (\mathbf{U}, G)$ as convex combinations of their neighbours. A little thought shows that each interior vertex lies in the interior of the kernel of the star-shaped polygon formed by its neighbours. The following method for finding a suitable convex combination, proposed in [**5**], is based on this observation.

Suppose $\mathbf{p}$ and $\mathbf{p}_1, \ldots, \mathbf{p}_d$ are distinct points in $\mathbb{R}^2$ such that $P = (\mathbf{p}_1, \ldots, \mathbf{p}_d)$ is a star-shaped polygon around the point $\mathbf{p}$ which lies in the interior of the kernel of P. We construct a set of strictly positive values $\lambda_1, \ldots, \lambda_d$ satisfying

$$\sum_{j=1}^{d} \lambda_j \mathbf{p}_j = \mathbf{p} \qquad \text{and} \qquad \sum_{j=1}^{d} \lambda_j = 1, \tag{5.4}$$

by averaging barycentric coordinates of a sequence of triangles covering $\mathbf{p}$. Specifically, for each $k = 1, \ldots, d$, there exists a unique point $\mathbf{q}_k$, not equal to $\mathbf{p}_k$ where the straight line through $\mathbf{p}_k$ and $\mathbf{p}$ intersects an edge of P. Thus there is a unique index i such that $\mathbf{q}_k$ lies on the edge $[\mathbf{p}_i, \mathbf{p}_{i+1}]$ of P (counting cyclically modulo d) and $\mathbf{q}_k \neq \mathbf{p}_{i+1}$. Therefore

$$\mathbf{p} = \tau_1 \mathbf{p}_k + \tau_2 \mathbf{p}_i + \tau_3 \mathbf{p}_{i+1}$$

where $\tau_1, \tau_2 > 0$ and $\tau_3 \geq 0$ and $\tau_1 + \tau_2 + \tau_3 = 1$. The values τ_1, τ_2, τ_3 are the barycentric coordinates of $\mathbf{p}$ with respect to the triangle $[\mathbf{p}_k, \mathbf{p}_i, \mathbf{p}_{i+1}]$. Letting $\mu_{k,k} = \tau_1$, $\mu_{i,k} = \tau_2$ and $\mu_{i+1,k} = \tau_3$ and $\mu_{j,k} = 0$ for all remaining $j \in \{1, \ldots, d\}$, we have then expressed $\mathbf{p}$ as a convex combination

$$\sum_{j=1}^{d} \mu_{jk} \mathbf{p}_j = \mathbf{p} \qquad \text{and} \qquad \sum_{j=1}^{d} \mu_{jk} = 1,$$

though unless $d = 3$, not all the $\mu_{1,k}, \ldots, \mu_{d,k}$ are strictly positive. So we let

$$\lambda_j = \frac{1}{d} \sum_{k=1}^{d} \mu_{jk} \tag{5.5}$$

and since $\lambda_j \geq \mu_{jj}/d > 0$, the values $\lambda_1, \ldots, \lambda_d$ are strictly positive and can be shown to satisfy condition (5.4); see [**5**]. Without strict positivity of the coefficients $\lambda_{ij}(t)$ in equation (5.3) there is no obvious generalization of the theory in [**19**] and so we have no guarantee that $\mathcal{T}(t)$ in Theorem 5.1 is a valid tiling.

6. Morphing Tilings with Different Boundaries

How do we extend the convex combination morph of the previous section to the general case where the boundaries of the two compatible tilings $\mathcal{T}^0 = (\mathbf{U}^0, G)$ and $\mathcal{T}^1 = (\mathbf{U}^1, G)$ are arbitrary? We propose a solution in which we morph the two boundary polygons of $\mathcal{T}^0$ and $\mathcal{T}^1$ first and use the convex combination morph for interior vertices afterwards. Since the theory in [**19**] will only guarantee that $\mathcal{T}(t)$ is a genuine tiling if the boundary polygon of the point sequence $\mathbf{U}(t)$ is convex, it is desirable to morph the two convex boundary polygons in such a way that convexity is preserved. Since we know of no method for doing this, we have implemented the following boundary morph, based on taking convex combinations of polar coordinates. This morph, suggested by Shapira and Rappoport [**16**] preserved convexity in all our numerical examples and it is easy to show that it at least preserves star-shapedness.

For notational convenience let us suppose that the boundary vertices are indexed $\mathbf{u}_1^k, \ldots, \mathbf{u}_n^k$ for $k = 0, 1$ in an anticlockwise direction, where $n = |B|$. Thus $(\mathbf{u}_1^0, \ldots, \mathbf{u}_n^0)$ and $(\mathbf{u}_1^1, \ldots, \mathbf{u}_n^1)$ are the two boundary polygons of $\mathcal{T}^0$ and $\mathcal{T}^1$. Since both boundaries are convex by definition, the two barycentres

$$\mathbf{u}_0^k = \frac{1}{n} \sum_{i=1}^{n} \mathbf{u}_i^k, \qquad k = 0, 1,$$

lie in the interiors of $\mathcal{T}^0$ and $\mathcal{T}^1$. Then we can express $\mathbf{u}_1^k, \ldots, \mathbf{u}_n^k$ in polar coordinates,

$$\mathbf{u}_i^k = \mathbf{u}_0^k + r_i^k(\cos\theta_i^k, \sin\theta_i^k), \qquad k = 0, 1,$$

in such a way that $0 \le \theta_1^k < 2\pi$ and

$$\theta_1^k < \theta_2^k < \ldots < \theta_n^k < \theta_1^k + 2\pi.$$

By taking these ranges of angles we obtain a boundary morph which at least preserves star-shapedness. Indeed, letting

$$\begin{aligned} \mathbf{u}_0(t) &= (1-t)\mathbf{u}_0^0 + t\mathbf{u}_0^1, \\ r_i(t) &= (1-t)r_i^0 + tr_i^1, \qquad i = 1, \ldots, n, \\ \theta_i(t) &= (1-t)\theta_i^0 + t\theta_i^1, \qquad i = 1, \ldots, n, \end{aligned}$$

we define a boundary morph by letting

$$\mathbf{u}_i(t) = \mathbf{u}_0(t) + r_i(t)(\cos\theta_i(t), \sin\theta_i(t)), \qquad i = 1, \ldots, n,$$

and it follows that

$$\theta_1(t) < \theta_2(t) < \ldots < \theta_n(t) < \theta_1(t) + 2\pi.$$

7. Numerical Examples

We have numerically computed both the linear morph and the convex combination morph for several pairs of compatible tilings. In the first example we take $\mathcal{T}^0$ and $\mathcal{T}^1$ to be triangulations with identical boundaries as in example (4.3). Fig. 2a shows $\mathcal{T}^0$ (far left) and $\mathcal{T}^1$ (far right) and triangulations $\mathcal{T}(i/4)$ for $i = 1, 2, 3$ generated by the linear morph (4.1). We can see that two of the triangles in $\mathcal{T}(1/2)$ are degenerate, reflecting the fact that the two vertices $\mathbf{u}_1(1/2)$ and $\mathbf{u}_2(1/2)$ are coincident. Fig. 2b on the other hand shows the result of applying the convex combination morph to $\mathcal{T}^0$ and $\mathcal{T}^1$.

In the second example we let

$$\mathbf{u}_1^0 = (1, 0), \quad \mathbf{u}_2^0 = (-1/2, \sqrt{3}/2), \quad \mathbf{u}_3^0 = (-1/2, -\sqrt{3}/2)$$

and $\mathbf{u}_i^0 = 3\mathbf{u}_{i-3}^0/4$, $i = 4, 5, 6$ and $\mathbf{u}_i^0 = \mathbf{u}_{i-6}^0/2$, $i = 7, 8, 9$. We also let $\mathbf{u}_i^1 = \mathbf{u}_i^0$, $i = 1, 2, 3$, $\mathbf{u}_4^1 = \mathbf{u}_6^0$, $\mathbf{u}_5^1 = \mathbf{u}_4^0$, $\mathbf{u}_6^1 = \mathbf{u}_5^0$, and $\mathbf{u}_7^1 = \mathbf{u}_8^0$, $\mathbf{u}_8^1 = \mathbf{u}_9^0$, $\mathbf{u}_9^1 = \mathbf{u}_7^0$. Then we set

$$\begin{aligned} G = (&(1,5,4), (1,2,5), (2,6,5), (2,3,6), (3,4,6), (3,1,4), \\ &(4,8,7), (4,5,8), (5,9,8), (5,6,9), (6,7,9), (6,4,7), (7,8,9)). \end{aligned}$$

Figs. 3a and 3b show the results of the linear and convex combination morphs respectively. Figs. 4a and 4b show the two morphs of compatible triangulations whose boundaries are different. The boundaries were morphed by taking convex combinations of polar coordinates as explained in Section 6.

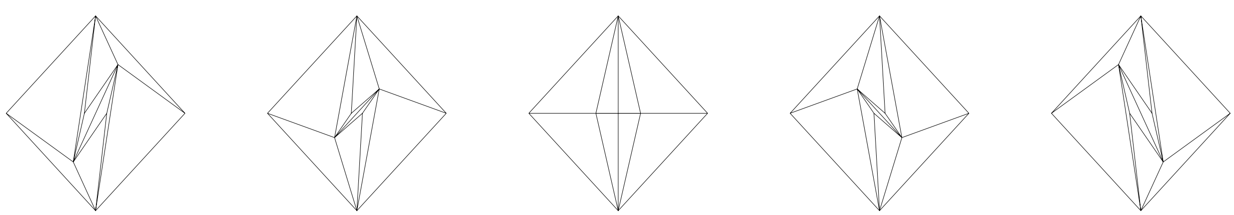

Figure 2a: Linear morph.

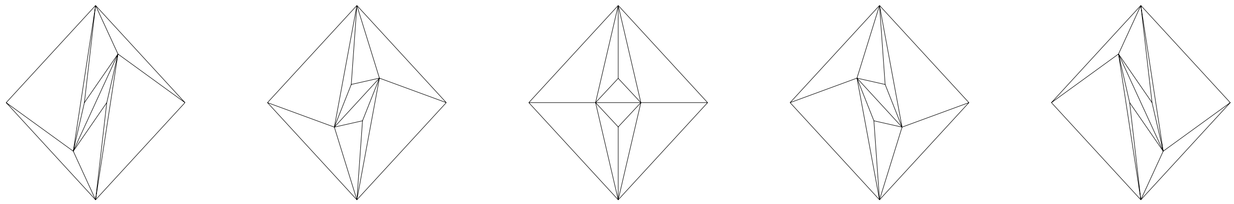

Figure 2b: Convex combination morph.

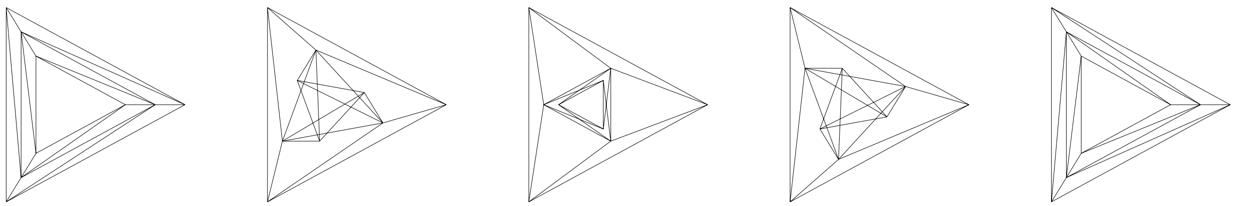

Figure 3a: Linear morph.

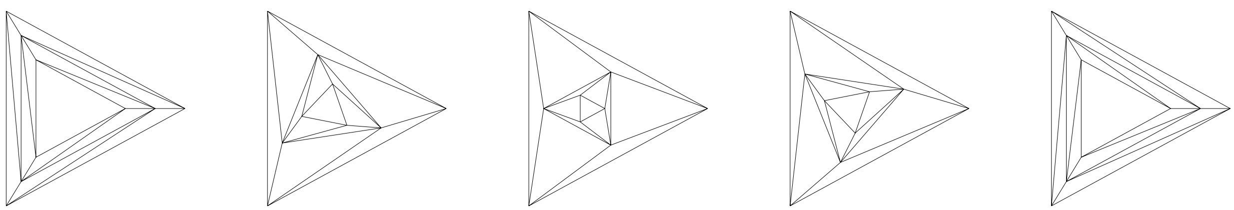

Figure 3b: Convex combination morph.

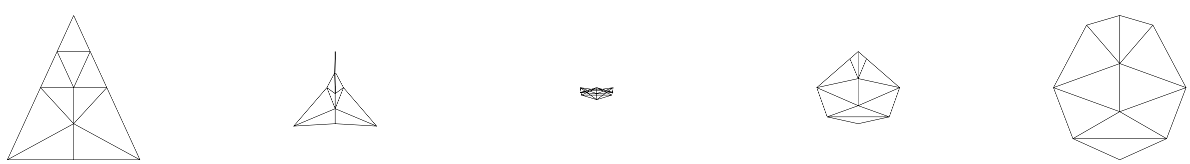

Figure 4a: Linear morph.

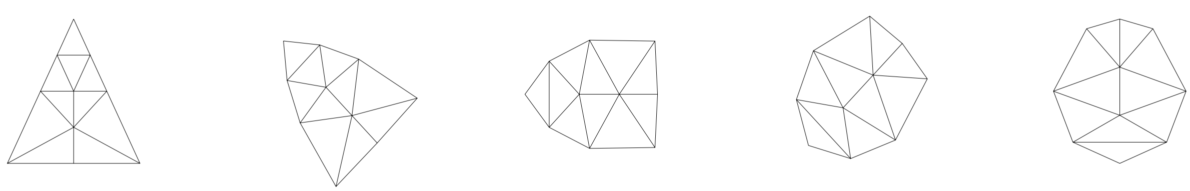

Figure 4b: Convex combination morph.

Fig. 5 shows two more complicated compatible triangulations $\mathcal{T}^0$ and $\mathcal{T}^1$ and their convex combination morph at the values $t = 1/3, 2/3$. For each value of t, the linear system (5.3) was solved by the Bi-CGSTAB iterative method [**20**] applied to each of the two coordinates separately. The number of unknowns in this example is 187 since there are 187 interior vertices in the triangulation. In our implementation, for both $t = 1/3$ and $t = 2/3$, the number of Bi-CGSTAB iterations in each coordinate was between 21 and 24 and the total CPU time for computing each of the triangulations $\mathcal{T}(1/3)$ and $\mathcal{T}(2/3)$ was 0.13 secs. The triangulations $\mathcal{T}^0$ and $\mathcal{T}^1$ were generated by mapping a surface triangulation in $\mathbb{R}^3$ into the unit square using two different 'parametrizations' as in [**5**].

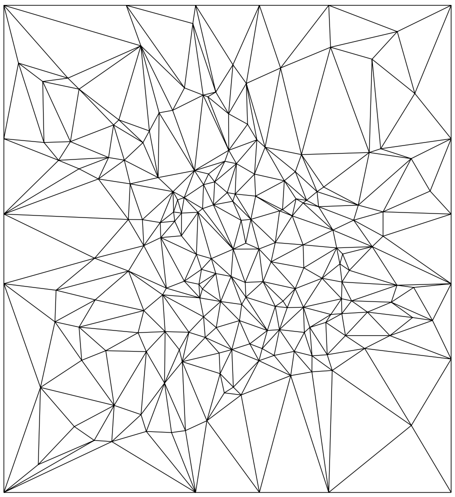

Fig 5a: $t = 0$

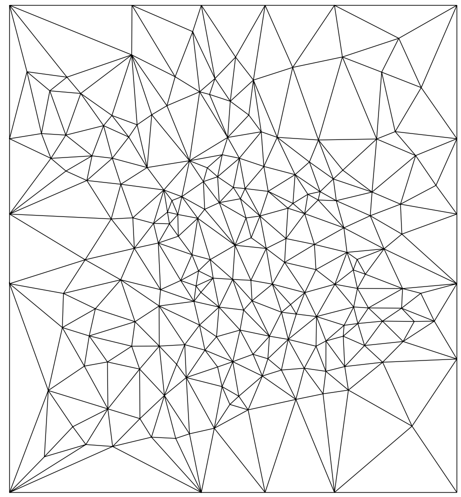

Fig 5b: $t = 1/3$

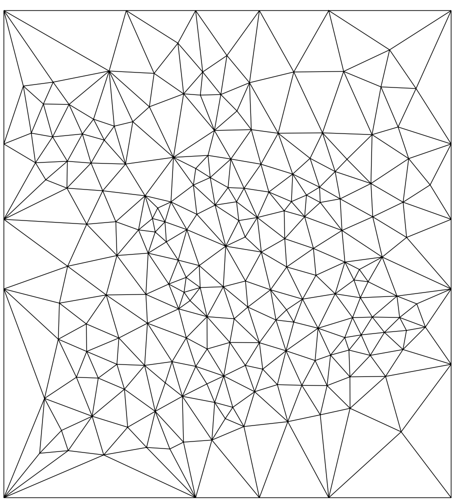

Fig 5c: $t = 2/3$

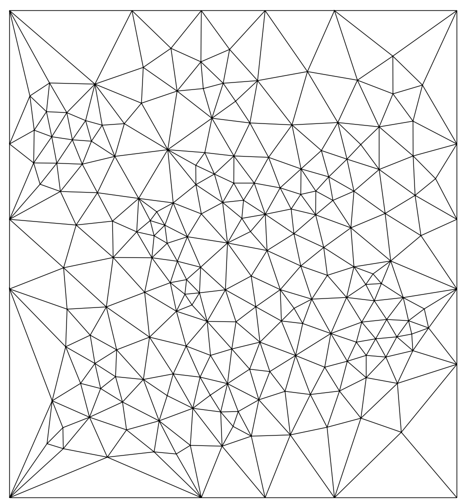

Fig 5d: $t = 1$

Fig. 6 shows the convex combination morph of two rectangular grids where the boundaries are different and the second is not even convex, though star-shaped.

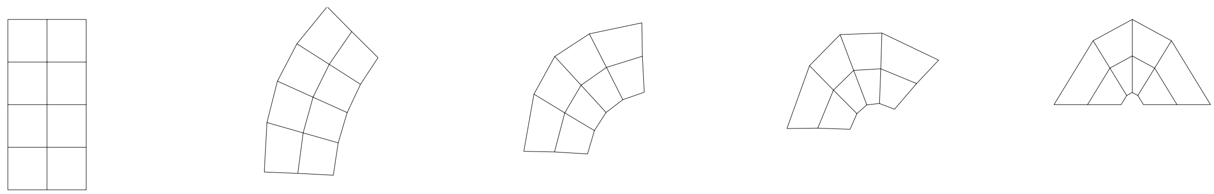

Figure 6: Convex combination morph.

8. Conclusion

We have described a morph based on convex combinations as an alternative to the standard linear one. Though the convex combination morph obviously requires more CPU time, an application of the kind in [**8**] requires only a small number of points, in which case computational speed is not crucial.

One question which naturally arises is whether the choice of the λ_{ij}^k in (5.1) computed from equation (5.5) has a significant effect on the morph and the vertex paths. If the number of edges incident on each interior vertex in the tiling is minimal, i.e. three, then the λ_{ij}^k determined by (5.5) are unique. Otherwise it might be possible to make a choice which yields an optimal morph, for example in the sense that the vertex paths have minimal curvature.

It would also be interesting to characterize the pairs of compatible tilings which can be linearly morphed and which cannot. For example, the linear morph of the example in Fig. 5 also yields valid triangulations.

Finally, we would like a convexity-preserving morph of the two convex boundary polygons. A variety of methods [**8,14,15,16**] have been proposed in the literature for morphing planar polygons, but none of them appear to preserve convexity. We plan to study this problem more closely in a forthcoming paper.

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