Constructing Pairwise Disjoint Paths with Few Links

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Abstract. Let P be a simple polygon and let $\{(u_i, u_i')\}$ be m pairs of distinct vertices of P where for every distinct $i, j \leq m$, there exist pairwise disjoint paths connecting u_i to u_i' and u_j to u_j'. We wish to construct m pairwise disjoint paths in the interior of P connecting u_i to u_i' for $i = 1, \ldots, m$, with minimal total number of line segments. We give an approximation algorithm which in $O(n \log m + M \log m)$ time constructs such a set of paths using $O(M)$ line segments where M is the number of line segments in the optimal solution.

1 Introduction

Let P be a simple polygon and let u and u' be two distinct vertices of P. The *(interior) link distance* from u to u' is the minimum number of line segments (also called *links*) required to connect u to u' by a polygonal path lying in (the interior of) P. The interior link distance from u to u' may differ greatly from the link distance between the two points. (See Figure 1.) A polygonal path which uses the minimum number of required line segments is called a *minimum link (interior) path*. Suri in [11] gave a linear time algorithm for determining the link distance and a minimal link path between two vertices.

Let u_1, u_1', u_2, u_2' be four vertices lying in the given order around P. By virtue of the relative locations of these four vertices, there are nonintersecting paths, ζ_1 and ζ_2, connecting u_1 to u_1' and u_2 to u_2', respectively. However, it is possible that every minimum interior link path connecting u_1 to u_1' intersects every minimum interior link path connecting u_2 to u_2'. (See Figure 1.) To simultaneously connect u_1 to u_1' and u_2 to u_2' by nonintersecting interior paths requires more line segments. In general, two additional line segments suffice to construct two such nonintersecting interior paths. (See [7].)

A set $\Pi = \{(u_i, u_i')\}$, $i \leq m$, of m pairs of distinct vertices of P is *untangled* if some set of pairwise disjoint paths connects each u_i to u_i'. Let $\Pi = \{(u_i, u_i')\}$, $i \leq m$, be an untangled set of m pairs of distinct vertices of P. Let $l(u_i, u_i')$ be the interior link distance from u to u' and let $L = \sum_{i=1..m} l(u_i, u_i')$ be the sum of those distances. Clearly, L line segments are required to construct a set of pairwise disjoint interior paths connecting u_i to u_i', for $i = 1, \ldots, m$. How many additional line segments are required? In [7] we proved that $O(m \log m)$

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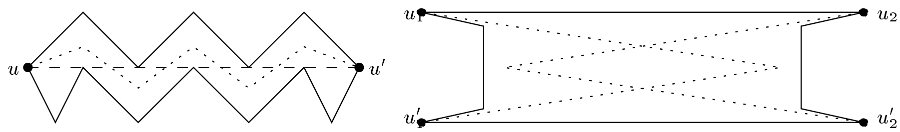


Fig. 1. Minimum link paths, minimum link interior paths and intersecting link paths.

additional line segments suffice and claimed without proof that $\Omega(m \log m)$ additional line segments may be required. A proof of the lower bound is provided in [6].

We define the *pairwise disjoint link paths problem* as: given an untangled set, $\{(u_i, u_i')\}$, of m pairs of distinct vertices of P, find the minimum total number of line segments required by a set of pairwise disjoint interior paths connecting u_i to u_i'. We were unable to give a polynomial time algorithm for this problem or to determine if the problem is NP-complete. Instead we present an algorithm which finds a solution within a constant factor of the optimal solution. Related problems are shown to be NP-complete in [2] and [5], but we do not know if those results can be applied to our problem.

A triangulation T_P of P (possibly with interior vertices) is *isomorphic* to a triangulation T_Q of Q if there is a one-to-one, onto mapping f between the vertices of T_P and the vertices of T_Q such that p, p', p'' are vertices of a triangle in T_P if and only if $f(p), f(p'), f(p'')$ are vertices of a triangle in T_Q. An isomorphic triangulation of P and Q defines a piecewise linear homeomorphism between P and Q. The size of a triangulation is the total number of vertices, edges and triangles in the triangulation.

Algorithms for constructing isomorphic triangulations and piecewise linear homeomorphisms between simple polygons are also given in [1,8,7]. Algorithms for constructing isomorphic triangulations between labelled point sets are described in [9] and [10]. The main result in this paper improves the output size and running time of the approximation algorithm in [7] from $O(M_1 \log n + n \log^2 n)$ to $O(M_1 \log n)$ where n is the input size and M_1 is the size of the optimal solution. The improvement is described in [6].

2 Approximation Algorithm

We first give a an approximation algorithm for connecting a set of vertices $\mathcal{U}$ by pairwise disjoint interior paths to a distinguished edge e^* of P. We start with some definitions.

Point $p \in P$ is *visible* from point $p' \in P$ if P contains the open line segment (p, p'). Point p is *clearly visible* from point $p' \in P$ if the interior of P contains the open line segment (p, p').

Point $p \in P$ is *(clearly) visible* from edge $e \in P$ if there is some point $p' \in e$ such that p is (clearly) visible from p'. (This definition of visibility is sometimes called *weak visibility* as opposed to *strong visibility* where p must be visible from

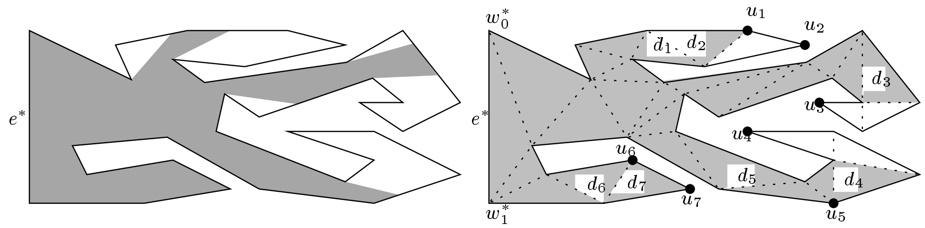


Fig. 2. $\hat{\mathrm{V}}\mathrm{is}(e)$ and Γ_{e^*}.

every point $p' \in e$. Throughout this paper, visibility refers to weak visibility.) Edge e or triangle t is *(clearly) visible* from edge e' or triangle t' if there are points $p \in e$ or $p \in t$ and $p' \in e'$ or $p' \in t'$ such that p is (clearly) visible from p'.

We let $\hat{\mathrm{V}}\mathrm{is}(p)$ and $\hat{\mathrm{V}}\mathrm{is}(e)$ denote the points clearly visible from point p and edge e. (See Figure 2.) Note that $\hat{\mathrm{V}}\mathrm{is}(p)$ and $\hat{\mathrm{V}}\mathrm{is}(e)$ are not necessarily closed sets.

Let u and e be a vertex and an edge of P, respectively. Edge d of triangulation T_P *separates* u from e if every interior path from u to the interior of e must intersect the interior of d. Triangle t of triangulation T_P *separates* u from e if every interior path from u to the interior of e must intersect the interior of t.

To construct pairwise disjoint paths connecting the vertices $\mathcal{U}$ to edge e^* of P, we construct a triangulated region Γ_{e^*} which contains and approximates $\hat{\mathrm{V}}\mathrm{is}(e^*)$, the set of points clearly visible from e^*. For each $u_i \in \mathcal{U}$, let d_i be the diagonal of Γ_{e^*} farthest from e^* which separates e^* from u_i. Let s_i be the portion of e^* visible from d_i. Note that s_i is a line segment. (See Figures 2 and 3.)

For each u_i we wish to choose a point p_i on e^* to be the endpoint of the path from u_i to e^*. Obviously, a point in s_i is a good candidate since it can reach d_i with a single line segment. However, we also need to choose the p_i such that their order on e^* is consistent with the order of $\mathcal{U}$ around P. In other words, u_i, u_j, p_j, p_i should lie clockwise or counter-clockwise around P in the given order.

We partition the set of line segments $\{s_i\}$ into groups and associate each such group with a point g_j on e^* which is in the "middle" of the line segments in the groups. If many of the line segments in the group contain g_j, then the corresponding diagonals can be connected to g_j by pairwise disjoint line segments. If few line segments contain g_j, then g_j partitions the line segments into roughly two equals subgroups in e^* with the property that many line segments connecting d_i to s_i from one subgroup intersect many line segments connecting d_j to s_j from the other subgroup. In addition, the order that the points g_j lie on e^* is consistent with the order that the associated vertices of $\mathcal{U}$ lie on the boundary of P. Partitioning the line segments is conceptually and technically the most difficult part of the algorithm.

From all the $\hat{\mathrm{V}}\mathrm{is}(g_j)$, we construct another triangulated region $\Gamma \subseteq \Gamma_{e*}$. We recursively connect $\mathcal{U}$ by pairwise disjoint paths to the edges on the boundary of

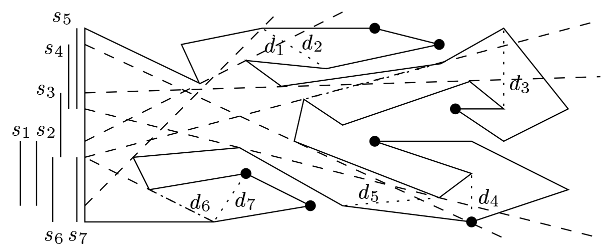


Fig. 3. Line segments $s_i = \text{int}(e^*) \cap \hat{\text{V}}\text{is}(d_i)$.

Γ and then connect those boundary edges by pairwise disjoint line segments to Γ. A careful analysis shows that Γ must contain many line segments in any set of pairwise disjoint paths connecting $\mathcal{U}$ to e^*. Thus the number of line segments in our solution is proportional to the number in the optimal solution.

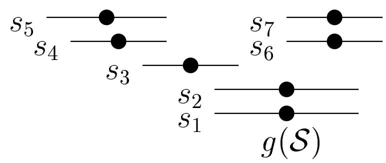
Lemma 1. *Let P be a simple polygon on n vertices with distinguished edge $e^* = \{w_0^*, w_1^*\}$ and let $\mathcal{U}$ be a subset of $\text{Vert}(P) \setminus \{w_0^*, w_1^*\}$ of size m. A set of m pairwise disjoint interior paths connecting the vertices in $\mathcal{U}$ to the interior of e^* can be constructed in $O(n \log m + M \log m)$ time using a total of at most $240M$ line segments where M is the minimum total number of line segments necessary to connect $\mathcal{U}$ to e^* by m pairwise disjoint paths.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the points in $\mathcal{U}$ labeled in clockwise order around P starting at e^*. Construct a triangulation T_P of P. Let Γ_{e^*} be union of the triangles of T_P which are clearly visible from edge e^*. The region Γ_{e^*} is a simple polygon in P. (See Figure 2.)

For each $u_i \in \mathcal{U}$, let d_i be the diagonal of Γ_{e^*} farthest from e^* which separates e^* from u_i. Let s_i be $\text{int}(e^*) \cap \hat{\text{V}}\text{is}(d_i)$, the interior of e^* which is clearly visible from s_i. Set s_i is an open line segment lying on e^*. Note that s_i may equal s_j (and d_i may equal d_j) for many distinct points $u_i, u_j \in \mathcal{U}$. (See Figure 3.)

Let $\mathcal{S}$ be any set of open line segments in $\mathbf{R}^1$, not necessarily distinct. For each point $q \in \mathbf{R}^1$, let $f(q, \mathcal{S})$ be the number of line segments of $\mathcal{S}$ which contain the point q. The line segments of $\mathcal{S}$ are open and do not contain their endpoints. Let $f^-(q, \mathcal{S})$ and $f^+(q, \mathcal{S})$ be the number of line segments of $\mathcal{S}$ contained in the open intervals $(-\infty, q)$ and (q, ∞), respectively. Note that $f(q, \mathcal{S}) + f^-(q, \mathcal{S}) + f^+(q, \mathcal{S})$ equals $|\mathcal{S}|$.

Let $\mathcal{R}$ be the set of midpoints of line segments of $\mathcal{S}$, again not necessarily distinct. The median point of $\mathcal{R}$ is the $\lceil |\mathcal{R}|/2 \rceil$'th point in $\mathcal{R}$, ordered from $-\infty$ to ∞. Let $g(\mathcal{S})$ be this median point of $\mathcal{R}$. At least $|\mathcal{R}|/2 = |\mathcal{S}|/2$ points of $\mathcal{R}$ lie in each of the closed intervals $(-\infty, g(\mathcal{S})]$ and $[g(\mathcal{S}), \infty)$. If the midpoint of segment $s \in \mathcal{S}$ lies in $(-\infty, g(\mathcal{S})]$, then segment s either contains $g(\mathcal{S})$ or lies in the open interval $(-\infty, g(\mathcal{S}))$. Thus $f(g(\mathcal{S}), \mathcal{S}) + f^-(g(\mathcal{S}), \mathcal{S})$ is greater than or equal to $\lceil |\mathcal{S}|/2 \rceil$. Similarly $f(g(\mathcal{S}), \mathcal{S}) + f^+(g(\mathcal{S}), \mathcal{S})$ is greater than or equal to $\lceil |\mathcal{S}|/2 \rceil$. (See Figure 4.)



$f(g(\mathcal{S}), \mathcal{S}) = 2,\ f^-(g(\mathcal{S}), \mathcal{S}) = 3,\ f^+(g(\mathcal{S}), \mathcal{S}) = 2.$

Fig. 4. $\mathcal{S}$, $g(\mathcal{S})$ and $f(g(\mathcal{S}), \mathcal{S})$.

Now consider two sets of line segments $\mathcal{S}_0$ and $\mathcal{S}_1$ on $\mathbf{R}^1$ and let $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. Define

$$\mathcal{F}(\mathcal{S}_0, \mathcal{S}_1) = f(g(\mathcal{S}), \mathcal{S}) + f^+(g(\mathcal{S}), \mathcal{S}_0) + f^-(g(\mathcal{S}), \mathcal{S}_1).$$

Without loss of generality, assume that w_0^*, e^*, w_1^* appear in counter-clockwise order around P. Let $\mathcal{S}_{\mathcal{U}}$ be the sequence $(s_1, s_2, \ldots, s_m)$. Embed e^* and the line segments $s_i \in \mathcal{S}$ in the real line $\mathbf{R}^1$, mapping w_0^* to zero and w_1^* to one. In the next section, we describe an algorithm to partition $\mathcal{S}_{\mathcal{U}}$ into contiguous subsequences $\sigma_1 = (s_1, s_2, \ldots, s_{i_1})$, $\sigma_2 = (s_{i_1+1}, s_{i_1+2}, \ldots, s_{i_2}), \ldots$, $\sigma_{2h} = (s_{i_{2h-1}+1}, s_{i_{2h-1}+2}, \ldots, s_m)$, such that:

1. $g(\sigma_1 \cup \sigma_2) \leq g(\sigma_3 \cup \sigma_4) \leq \cdots \ldots \leq g(\sigma_{2h-1} \cup \sigma_{2h})$;
2. $|\sigma_{2j-1}| = |\sigma_{2j}|\ \ (+1)$ for $1 \leq j \leq h$;
3. $\sum_{j=1..h} \mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \geq m/40$.

(One possible partition of the segments in Figure 3 is $\sigma_1 = \{s_1, s_2, s_3\}$, $\sigma_2 = \{s_4, s_5\}$, $\sigma_3 = \{s_6\}$, $\sigma_4 = \{s_7\}$.)

Let g_j equal $g(\sigma_{2j-1} \cup \sigma_{2j})$ for $j = 1, \ldots, h$. Note that $g_1, g_2, \ldots, g_h$ lie in counter-clockwise order around P. Let $\mathcal{U}_j = \{u_i : s_i \in \sigma_j\}$ be the points in $\mathcal{U}$ corresponding to the line segments in σ_j for $j = 1, \ldots, h$. For each g_j, let Γ_j be the union of the triangles of T_P which intersect $\hat{\mathrm{V}}\mathrm{is}(g_j)$ and separate some $u \in \mathcal{U}_{2j-1} \cup \mathcal{U}_{2j}$ from e^*. (See Figure 5.) Let Γ be the union of all the Γ_j. Similar to Γ_{e^*}, the region Γ is also a simple polygon in P, its boundary is composed of edges and chords of P, and it has a triangulation $T_\star$ induced by the triangulation T_P of P.

Let $\mathcal{C}$ be the set of chords of P bounding Γ. Each chord $c \in \mathcal{C}$ separates P into two subpolygons. Let P_c be the subpolygon not containing Γ. Let w_0^c and w_1^c be the endpoints of c. For each chord $c \in \mathcal{C}$, let $\mathcal{U}_c$ be the points of $\mathcal{U} \setminus \{w_0^c, w_1^c\}$ in P_c. Recursively, construct pairwise disjoint paths connecting the points in $\mathcal{U}_c$ to c. (See Figure 6.)

For each $u_i \in \mathcal{U}$, let $\tilde{d}_i$ be the diagonal of Γ farthest from e^* which separates e^* from u_i. Choose the minimum j such that $\tilde{d}_i$ is a diagonal of Γ_j. Connect $\tilde{d}_i$ to g_j by a line segment λ_i in the interior of P. (See Figure 7.) Diagonal $\tilde{d}_i$ may also separate other vertices of $\mathcal{U}$ from e^* and there may be many line segments which intersect $\tilde{d}_i$. The line segments λ_i should be chosen so that their order along $\tilde{d}_i$ corresponds to the order of the vertices around P. The choice of $\tilde{d}_i$

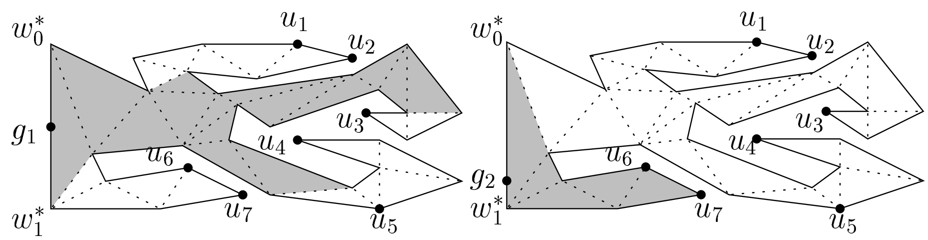


Fig. 5. Γ_1 and Γ_2.

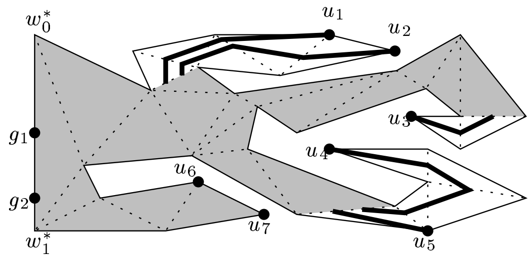


Fig. 6. Γ, triangulation $T_\star$ and paths to the boundary of Γ.

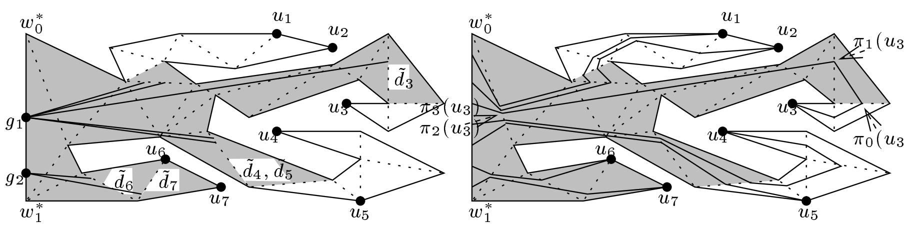


Fig. 7. Diagonals $\tilde{d}_i$, line segments λ_i and paths connecting $\mathcal{U}$ to e^*.

and the associated point g_j ensures that line segments λ_i intersect only at their endpoints. (See [6].)

For each point $u_i \in \mathcal{U}(c)$, let $\pi_0(u_i)$ be the endpoint on c of the path connecting u_i to c. For each point $u_i \in \mathcal{U}$ which lies in Γ, let $\pi_0(u_i)$ equal u_i. Let $\pi_1(u_i)$ be the endpoint of λ_i on $\tilde{d}_i$. Let $\pi_2(u_i)$ be the first intersection point of λ_i and the triangle containing e^*. Place m points equally spaced on e^*. Let $\pi_3(u_i)$ be the i'th point, ordered counter-clockwise from w_0^*. Connect $\pi_0(u_i)$ to e^* with a polygonal line through $\pi_0(u_i), \pi_1(u_i), \pi_2(u_i), \pi_3(u_i)$. (See Figure 7.)

We claim that this algorithm connects $\mathcal{U}$ to e^* using $O(M)$ links where M is the number of links in some optimal solution. For each $u_i \in \mathcal{U}$, let ζ_i be the path constructed from u_i to e^* by our algorithm while η_i is the path from u_i to e^* in the optimal solution. Path ζ_i has at most three line segments in Γ. Line segment s_i is in $\sigma_{2j-1} \cup \sigma_{2j}$ for some j. If s_i contains g_j, then some point on diagonal d_i is clearly visible from g_j and d_i is a diagonal of $\Gamma_j \subseteq \Gamma$. Since d_i is the farthest diagonal visible from e^* which separates u_i from e^*, any path from u_i to e^* must have at least one line segment contained in $\Gamma_j \subseteq \Gamma$. Thus if s_i contains g_j, then we can charge the three links of ζ_i in Γ to a line segment of η_i in Γ. However, s_i may not contain g_j.

Consider the case where $s_i \in \sigma_{2j-1}$ lies between g_j and w_1^* while $s_{i'} \in \sigma_{2j}$ lies between w_0^* and g_j. Any two paths from u_i to s_i and $u_{i'}$ to $s_{i'}$ must intersect. Since paths η_i and $\eta_{i'}$ are pairwise disjoint, either the endpoint of η_i must lie between w_0^* and g_j or the endpoint of $\eta_{i'}$ must lie between g_j and w_1^*. Without loss of generality, assume that the endpoint ρ of η_i lies between w_0^* and g_j. In that case, g_j lies between ρ and s_i. Let d be the farthest diagonal of P visible from ρ and separating ρ from u_i. By the construction of d_i and s_i, diagonal d separates d_i from e^* and hence is visible to s_i. Since g_j lies between ρ and s_i, diagonal d is also visible to g_j and is contained in $\Gamma_j \subseteq \Gamma$. Thus if ρ lies between w_0^* and g_j, path η_i must have at least one line segment contained in $\Gamma_j \subseteq \Gamma$. Similarly, if the endpoint of $\eta_{i'}$ lies between g_j and w_1^*, path $\eta_{i'}$ must have at least one line segment contained in $\Gamma_j \subseteq \Gamma$. It follows that either η_i or $\eta_{i'}$ must have a line segment contained in $\Gamma_j \subseteq \Gamma$.

Let m_0, m_-, m_+ equal $f(g_j, \sigma_{2j-1})$, $f^-(g_j, \sigma_{2j-1})$ and $f^+(g_j, \sigma_{2j-1})$, respectively, while m'_0, m'_-, m'_+ equal $f(g_j, \sigma_{2j})$, $f^-(g_j, \sigma_{2j})$ and $f^+(g_j, \sigma_{2j})$, respectively. By the arguments above, the paths connecting the points in $\mathcal{U}_{2j-1} \cup \mathcal{U}_{2j}$ to e^* in the optimal solution must have at least $m_0 + m'_0 + \min(m_+, m'_-)$ line segments contained in Γ. By the choice of point g_j, $m_0 + m_- + m'_0 + m'_- \geq |\sigma_{2j-1} \cup \sigma_{2j}|/2$. Since $|\sigma_{2j-1}|$ equals $|\sigma_{2j}|$ or $|\sigma_{2j-1}|+1$, $m_0 + m_- + m'_0 + m'_- \geq |\sigma_{2j-1}|$. On the other hand, $m_0 + m_- + m_+ = |\sigma_{2j-1}|$. Subtracting the second equation from the first gives $m'_0 + m'_- \geq m_+$. Thus

$$
\begin{aligned}
m_0 + m'_0 + \min(m_+, m'_-) &= \min(m_0 + m'_0 + m_+, m_0 + m'_0 + m'_-) \\
&\geq \min(m_0 + m'_0 + m_+, m_0 + m_+) \\
&= m_0 + m_+
\end{aligned}
$$

Similarly, $m_0 + m_+ \geq m'_-$ and $m_0 + m'_0 + \min(m_+, m'_-) \geq m'_0 + m'_-$. Thus

$$
m_0 + m'_0 + \min(m_+, m'_-) \geq \max(m_0 + m_+, m'_0 + m'_-) \geq \mathcal{F}(g_j, \sigma_{2j-1}, \sigma_{2j})/2.
$$

The paths connecting the points in $\mathcal{U}_{2j-1} \cup \mathcal{U}_{2j}$ to e^* in the optimal solution must have at least $\mathcal{F}(g_j, \sigma_{2j-1}, \sigma_{2j})/2$ line segments in Γ. Since $\sum_{j=1..h} \mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \geq m/40$, any pairwise disjoint paths connecting the points in $\mathcal{U}$ to e^* must have at least $m/80$ line segments contained in Γ. The construction produces at most $3m$ line segments in Γ, so the solution is at most 240 times the optimal.

Finally, we discuss the running time of our algorithm. Constructing the initial triangulation T_P takes $O(n)$ time [3]. As discussed in the next section, partitioning $\mathcal{S}_{\mathcal{U}}$ into the subsequences σ_j takes $O(m \log m)$ time. Constructing Γ_{e^*} takes $O(n^*)$ time where n^* is the number of triangles of T_P intersected by $\hat{\mathrm{V}}\mathrm{is}(e^*)$ [4]. All the other steps in the algorithm can be done in $O(n^* + m)$ time. Thus the

PARTITION($\mathcal{S}$)
/* $\mathcal{S}$ = a sequence of line segments $(s_1, s_2, \ldots, s_m)$ */
/* Returns a linked list of contiguous subsequences of $\mathcal{S}$ */
1. Initialize linked list $\mathcal{A}$ to $\emptyset$;
2. FOR $i = 1$ TO m DO
3.   Create new node a where $a.seq = (s_i)$ and $a.size = 1$;
4.   Add a to the end of linked list $\mathcal{A}$;
5. WHILE $\exists a \in \mathcal{A}$ such that $g(a.seq) > g(a.next.seq)$ DO
6.   Merge a and $a.next$ to form a new node a' in $\mathcal{A}$;
7.   BALANCE-NEXT(a');
8.   BALANCE-PREV(a');
9. Return($\mathcal{A}$).

Fig. 8. Algorithm PARTITION.

non-recursive steps in this algorithm take $O(n^* + m \log m)$ time. A careful accounting for the recursive steps gives the desired $O(n \log m + M \log m)$ bound. Details appear in [6]. □

Using arguments similar to those given in [7], the previous algorithm can be turned into an algorithm for connecting an untangled set of m pairs of vertices of P. The algorithm and its analysis is provided in [6].

Theorem 2. *Let P be a simple polygon on n vertices let $\Pi = \{(u, u')\}$ be an untangled set of m pairs of distinct vertices of P. A set of m pairwise disjoint interior paths connecting u to u' for each $(u, u') \in \Pi$ can be constructed in $O(n \log m + M \log m)$ time using $O(M)$ line segments where M is the minimum total number of line segments necessary to connect all pairs $(s, s') \in \Pi$ by pairwise disjoint paths.*

3 Partition Algorithm

In this section, we describe and analyze the algorithm for partitioning a sequence of line segments. The functions f, f^+, f^- and $\mathcal{F}$ were defined in the previous section.

Lemma 3. *Let $\mathcal{S}$ be a sequence of line segments$(s_1, s_2, \ldots, s_m)$ on the real line $\mathbf{R}^1$. In $O(m \log m)$ time, $\mathcal{S}$ can be partitioned into contiguous subsequences $\sigma_1 = (s_1, s_2, \ldots, s_{i_1})$, $\sigma_2 = (s_{i_1+1}, s_{i_1+2}, \ldots, s_{i_2}), \ldots$, $\sigma_{2h} = (s_{i_{2h-1}+1}, s_{i_{2h-1}+2}, \ldots, s_m)$, such that:*

1. $g(\sigma_1 \cup \sigma_2) \le g(\sigma_3 \cup \sigma_4) \le \cdots \ldots \le g(\sigma_{2h-1} \cup \sigma_{2h})$;
2. $|\sigma_{2j-1}| = |\sigma_{2j}|$ $(+1)$ *for* $1 \le j \le h$;
3. $\sum_{j=1..h} \mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \ge m/40$.

Proof (outline). Split $\mathcal{S}$ into m distinct subsequences, (s_i), consisting of one element each. Store the m subsequences in a linked list $\mathcal{A}$ in the order they appear in $\mathcal{S}$. Each node $a \in \mathcal{A}$ contains a subsequence $a.seq$. Call $\mathcal{A}$ *balanced* if the size of each subsequence is at most three times the size of any adjacent subsequence in $\mathcal{A}$.

While $\mathcal{A}$ contains two adjacent subsequences, $a.seq$ followed by $a.next.seq$, such that $g(a.seq) > g(a.next.seq)$, merge the subsequences $a.seq$ and $a'.seq$. After each merge of two such subsequences, rebalance list $\mathcal{A}$ by merging adjacent subsequences, as necessary. Figure 8 contains the main algorithm. A complete description of the subroutines BALANCE-NEXT and BALANCE-PREV is provided in [6].

Let a_j be the j'th node in $\mathcal{A}$ when the algorithm is completed. Partition $a_j.seq = (s_i, \ldots, s_{i'})$ into two approximately equal sized sequences $\sigma_{2j-1} = (s_i, \ldots, s_{\lceil (i+i')/2 \rceil})$ and $\sigma_{2j} = (s_{\lceil (i+i')/2 \rceil + 1}, \ldots, s_{i'})$. We claim that this is a partitioning of $\mathcal{S}$ with the desired properties. Initially, the s_j are stored in $\mathcal{A}$ in sorted order. The merging and splitting steps in the main algorithm and in the subroutines BALANCE-NEXT and BALANCE-PREV preserve the order of the s_i, so the σ_j properly partition $\mathcal{S}$ into contiguous subsequences.

Let g_j be $g(a_j.seq) = g(\sigma_{2j-1} \cup \sigma_{2j})$. The while loop only terminates when $g_1 \leq g_2 \leq \cdots \leq g_h$, so property 1 is clearly satisfied. Sets σ_{2j-1} and σ_{2j} are created by partitioning $a_j.seq$ into two equal sized sequences, so property 2 is satisfied.

To show property 3 holds, note that a_j could be an initial node or it could be created when $g(a.seq) > g(a.next.seq)$ or it could be created in the rebalancing procedure.

If a_j is an initial node, then $a_j.seq = \{s\}$ for some $s \in \mathcal{S}$ and $\sigma_{2j-1} = \{s\}$ and $\sigma_{2j} = \emptyset$. Point g_j is the midpoint of s and $\mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \geq 1 \geq (1/8)|a_j|$.

Assume a_j is created when $g(a.seq)$ is greater than $g(a.next.seq)$ and that $|a.next| \geq |a|$. The sequence $a.seq$ is a subsequence of σ_{2j-1}, so

$$f(g_j, \sigma_{2j-1}) \geq f(g_j, a.seq) \text{ and}$$
$$f^+(g_j, \sigma_{2j-1}) \geq f^+(g_j, a.seq).$$

The point $g_j = g(a_j.seq)$ must lie between $a.g$ and $a.next.g$, so

$$\begin{aligned} f(g_j, a.seq) + f^+(g_j, a.seq) &\geq f(g(a.seq), a.seq) + f^+(g(a.seq), a.seq) \\ &\geq |a|/2. \end{aligned}$$

Since $|a.next| \leq 3|a|$, we have $|a_j| \leq 4|a|$. Thus,

$$\mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \geq f(g_j, \sigma_{2j-1}) + f^+(g_j, \sigma_{2j-1}) \geq (1/8)|a_j|.$$

In the case that $|a.next| < |a|$, similar reasoning gives

$$\mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \geq f(g_j, \sigma_{2j}) + f^-(g_j, \sigma_{2j}) \geq (1/8)|a_j|.$$

Finally, if a_j is created in the rebalancing step, $\mathcal{F}(\sigma_{2j-1}, \sigma_{2j})$ may not have the desired lower bound. However, at most $(4/5)m$ line segments lie in nodes created in the rebalancing step. By counting the $m/5$ line segments which are not in a rebalanced node, we find $\sum_{i=1..k} \mathcal{F}(\sigma_{2j-1}, \sigma_{2j}) \geq m/40$.

A complete description and analysis of the algorithm, its correctness and $O(m \log m)$ running time appears in [6]. $\square$

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