# On Compatible Triangulations of Simple Polygons*


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**Abstract**

It is well known that, given two simple $n$-sided polygons, it may not be possible to triangulate the two polygons in a compatible fashion, if one's choice of triangulation vertices is restricted to polygon corners. Is it always possible to produce compatible triangulations if additional vertices inside the polygon are allowed? We give a positive answer and construct a pair of such triangulations with $O(n^2)$ new triangulation vertices. Moreover, we show that there exists a "universal" way of triangulating an $n$-sided polygon with $O(n^2)$ extra triangulation vertices. Finally, we also show that creating compatible triangulations requires a quadratic number of extra vertices in the worst case.


## 1 The Problem

Given two simple polygons $\mathcal{P}_1$ and $\mathcal{P}_2$, each with $n$ vertices, it is not always possible to find compatible triangulations of the two polygons. In other words, there may not exist a circular labeling of the corners of each polygon by consecutive numbers 1 through $n$ and a set of $n-3$ non-crossing interior chords in $\mathcal{P}_1$, so that the open chords joining corresponding vertices in $\mathcal{P}_2$ are disjoint and lie completely inside $\mathcal{P}_2$. Consider figure 1—here each hexagon admits a unique triangulation, but the two triangulations are not compatible in the above sense, since the three interior chords "fan out" from a common vertex in $\mathcal{P}_1$ and form a triangle in $\mathcal{P}_2$. The following question was asked by Goodman and Pollack [G]:

> Is it possible to triangulate any two polygons with the same number of vertices compatibly if additional vertices (*Steiner points*) are allowed in the interior? If yes, how many such points are required?


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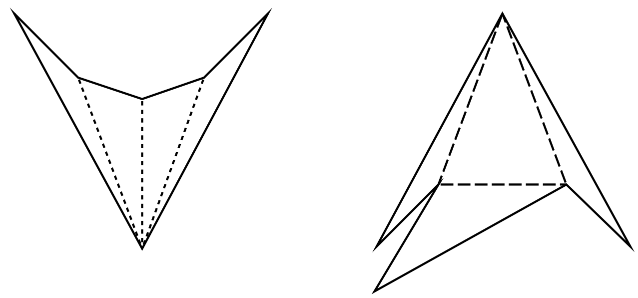

Figure 1: $\mathcal{P}_1$ and $\mathcal{P}_2$ cannot be triangulated compatibly without extra points

More formally, is there a choice of $2k$ points, $k$ in the interior of each polygon, and a numbering of the vertices of each polygon and the newly added points, such that polygon vertices are numbered consecutively 1 through $n$, the remaining points are labeled by numbers $n+1$ through $n+k$, and there exist triangulations of the two polygon interiors (1) which use polygon corners and the newly added points as vertices and (2) in which two points are connected by an edge in one triangulation if and only if there is an edge between the corresponding points in the other triangulation (see figure 2)? If yes, how large must $k$ be? We give a positive answer to the first question and provide a simple

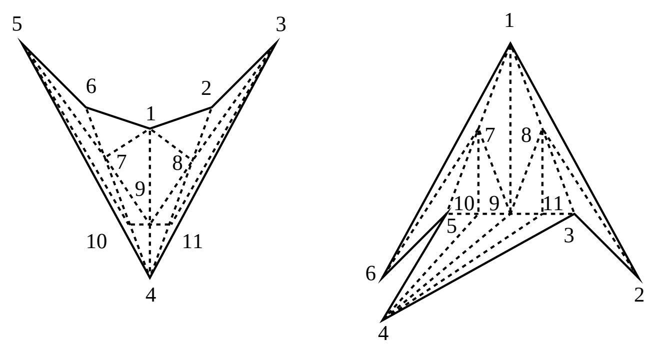


Figure 2: Compatible triangulations of $\mathcal{P}_1$ and $\mathcal{P}_2$

construction of a pair of compatible triangulations with $k = O(n^2)$ Steiner points. Moreover, we show that there is a "universal" triangulation with $\Theta(n^2)$ Steiner points that serves all $n$-sided polygons simultaneously. This note concludes with an argument that our constructions are asymptotically worst-case optimal in the sense that in general a quadratic number Steiner points is necessary for creating compatible triangulations.

## 2 A Simple Construction

We start by describing the construction of compatible triangulations for a pair of arbitrary polygons. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two simple polygons, each with $n$ corners, and number the vertices of $\mathcal{P}_1$ ($\mathcal{P}_2$) consecutively, starting from an arbitrary corner. Let $\mathcal{P}$ be an arbitrary convex $n$-gon with its vertices, just as those of $\mathcal{P}_1$ and $\mathcal{P}_2$, numbered consecutively 1 through $n$. Consider an arbitrary triangulation (without additional vertices) $T_1$ of $\mathcal{P}_1$ and map its chords to the corresponding chords of $\mathcal{P}$. Refer to figure 3. This induces a triangulation $T_1'$ of $\mathcal{P}$ and a piecewise-linear homeomorphism $\mathcal{L}_1 : \mathcal{P} \to \mathcal{P}_1$ that matches vertices, edges, and triangles of $T_1'$ to those of $T_1$; a point in some triangle $\Delta$ of $T_1$ is mapped to the point of the corresponding triangle $\Delta'$ of $T_1'$ given by the unique linear map that carries $\Delta$ onto $\Delta'$ and preserves vertex labeling. Repeat this procedure for a triangulation $T_2$ of $\mathcal{P}_2$, obtaining $\mathcal{L}_2 : \mathcal{P} \to \mathcal{P}_2$. Consider the convex subdivision obtained by overlaying $T_1'$ and $T_2'$ on $\mathcal{P}$. It has $O(n^2)$ vertices. Let $\mathcal{S}$ be an arbitrary triangulation of this subdivision that does not introduce additional vertices—it is sufficient to triangulate every face from one of its vertices. By Euler's formula, $\mathcal{S}$ consists of $O(n^2)$ vertices, edges, and faces. We argue that $\mathcal{S}_1 = \mathcal{L}_1(\mathcal{S})$ and $\mathcal{S}_2 = \mathcal{L}_2(\mathcal{S})$ are compatible triangulations of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Since each region $R$ of $\mathcal{S}$ is a triangle fully contained in some triangle $\Delta$ of $T_1'$ and $\mathcal{L}_1$ is linear when restricted to $\Delta$, the corresponding region $\mathcal{L}_1(R)$ in $\mathcal{S}_1$ is a triangle. Thus $\mathcal{S}_1$ and, similarly, $\mathcal{S}_2$ is a triangulation, as claimed. Compatibility follows by construction.

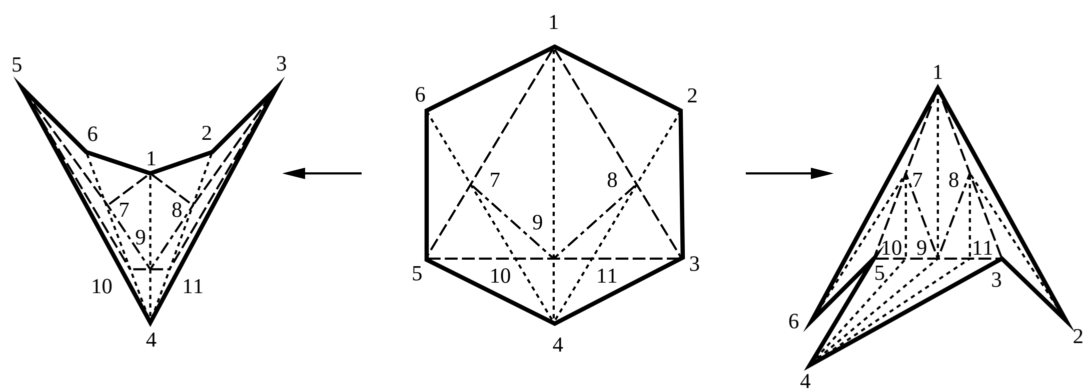


Figure 3: Construction of compatible triangulations

Algorithmically the construction can be carried out by first triangulating the given polygons (see, for example, Garey et al. [GJPT] for an $O(n \log n)$ algorithm, or Chazelle [C] for an $O(n)$ algorithm) and then computing their triangulated overlay by tracing each edge of $T_1'$ through $T_2'$ in overall time $O(n + k)$ where $k$ is the number of Steiner points introduced in the construction.

## 3 A Universal Triangulation

Observe that our construction of compatible triangulations yields a "universal triangulation". More formally, in any $n$-gon there exists a set of $\Theta(n^4)$ Steiner points and a way of triangulating the polygon using its corners and the added points as vertices so that the combinatorial structure of the resulting triangulation is independent of the initial polygon. These points can be obtained by considering *all* intersection points of *all* chords in a convex $n$-gon $\mathcal{P}$—the universality of the resulting triangulation follows from the same arguments as the construction of a pair of compatible triangulations described at the beginning of this note. We now use a different construction to show that there exists a universal triangulation that only contains $\Theta(n^2)$ Steiner points.

In its simplest form the construction is as follows: the canonical triangulation has the form of a "spiderweb" of $n-2$ concentric layers of regular $n$-polygons plus one point in the common center (see figure 4); all corners of the innermost polygon are connected to the centerpoint; corresponding corners of polygons are connected by spokes; the resulting quadrilaterals are triangulated canonically (not shown in the figure).

An arbitrary $n$-vertex polygon can be triangulated in such a spiderweb pattern as follows: Assume $P$ is an $n$-vertex polygon with $m$ corners (i.e. the remaining $n-m$ vertices lie on the sides of $\mathcal{P}$). Place a slightly smaller copy $\mathcal{P}'$ of $\mathcal{P}$ inside $\mathcal{P}$. The difference between $\mathcal{P}$ and $\mathcal{P}'$ is a "circular" corridor. Connecting corresponding vertices of $\mathcal{P}$ and $\mathcal{P}'$ with edges decomposes this corridor into $n$ convex quadrilaterals (just as in one layer of a spiderweb). $\mathcal{P}'$ must have some corner $c$ that induces an ear (i.e. the two neighboring corners of $c$ can be connected by a chord $d$ interior to $\mathcal{P}'$). Replace the two sides of $\mathcal{P}'$ that are incident to $c$ by that chord $d$ and place the appropriate number of vertices on $d$ (see figure 4) to obtain $\mathcal{P}''$. This new polygon still has $n$ vertices but only $m-1$ corners.

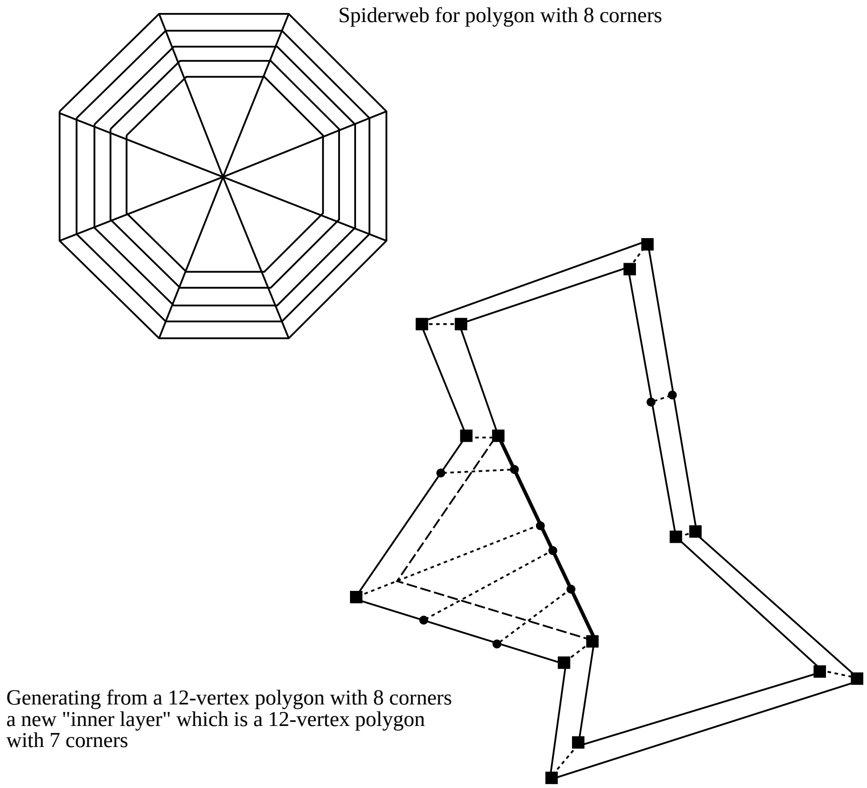


Figure 4: A spiderweb and one iteration

Starting with a polygon with $n$ vertices, all of which are corners, this whole process can be repeated $n-3$ times after which the innermost polygon has only 3 corners. Now connect all vertices of that polygon to the center point of the spiderweb.

This construction uses $n(n-3)+1$ Steiner points. This can be somewhat improved by observing that every polygon has at least two ears (i.e. two corners can be eliminated in every iteration) and that every polygon with 5 corners is star-shaped (i.e. no more iterations are necessary). This modification yields a universal triangulation of $n$-vertex polygons that uses $n\lceil(n-5)/2)\rceil+1$ Steiner points (when $n > 4$). Slight further improvements are possible by considering pairs or triples of successive iterations. Finally note that it is not very difficult to perform this construction on any $n$-sided polygon in $O(n^2)$ time (since constructing $\mathcal{P}'$ from $\mathcal{P}$ is easy if a "normal" triangulation of $\mathcal{P}$ is available).

# 4 A Lower Bound

We now argue that it is sometimes necessary to introduce $\Omega(n^2)$ new vertices in order to triangulate two simple $n$-gons compatibly. Initially, we show that, if the corners of each polygon are already numbered 1 through $n$, one sometimes needs $\Omega(n^2)$ additional vertices to produce compatible triangulations consistent with this numbering. The argument is then extended to the case that the numbering is not fixed *a priori*. For convenience, we will assume that $n$ is a multiple of 4.

Let $\mathcal{P}_1$ be a polygon composed of two convex "domes" each with $\frac{n}{4}$ vertices connected by a "snake" of length $\frac{n}{4}$. $\mathcal{P}_2$ is a copy of $\mathcal{P}_1$ with vertex labels shifted by $\frac{n}{4}$. Refer to figure 5.

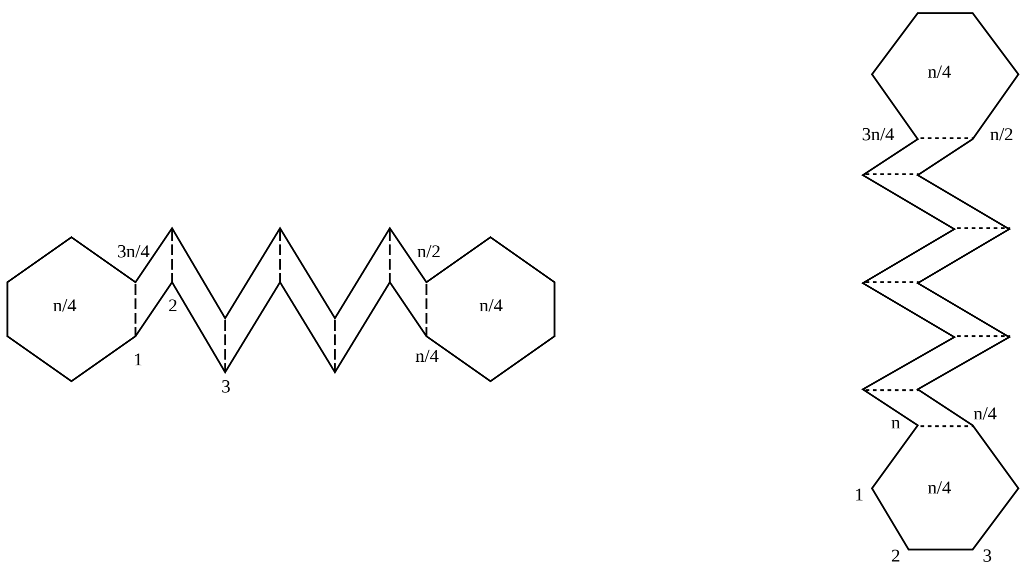


Figure 5: Lower bound construction: polygons

Consider two compatible triangulations, $T_1$ of $\mathcal{P}_1$ and $T_2$ of $\mathcal{P}_2$. Observe that, by Euler's formula, it is sufficient to show that $T_1$ and $T_2$ have $\Omega(n^2)$ triangles. As before, the correspondence between $T_1$ and $T_2$ induces a 1-1 piecewise-linear mapping between $\mathcal{P}_1$ and $\mathcal{P}_2$ that carries faces, edges, and vertices of $T_1$ into like-numbered faces, edges, and vertices of $T_2$. In particular, this map transforms any straight-line segment in $\mathcal{P}_1$ into a simple polygonal path in $\mathcal{P}_2$.

Consider the open chords of $\mathcal{P}_1$ connecting pairs of vertices on opposite sides of the "snake," drawn dashed in figure 5. Each such chord $c$ is mapped to a simple polygonal path in the interior of $\mathcal{P}_2$ whose vertices lie on edges of $T_2$. We thus obtain a family of $\frac{n}{4}$ disjoint simple "vertical"

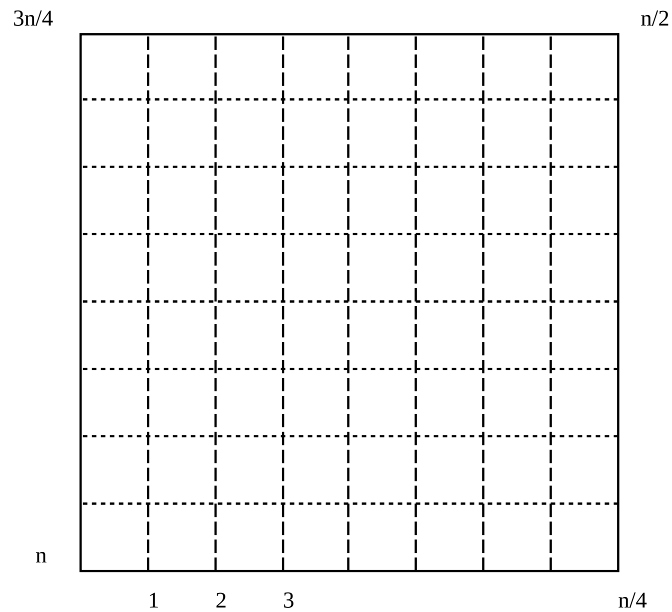


Figure 6: Lower bound construction: paths and chords

paths in $\mathcal{P}_2$, schematically shown dashed in figure 6, connecting the top dome of $\mathcal{P}_2$ with its bottom dome. Similarly, there is a collection of $\frac{n}{4}$ "horizontal" chords in $\mathcal{P}_2$, shown dotted. Every vertical path intersects every horizontal chord, possibly in more than one point, yielding $(\frac{n}{4})^2$ intersecting *chord-path pairs.*

We will now show that, for a triangle $\Delta$ of $T_2$, there are no more than 9 chord-path pairs that intersect in $\Delta$. First of all, notice that, by construction, no edge $e$ of $T_2$ cuts more than two chords of the horizontal family. Similarly, $e$ cannot meet more than two vertical paths, for otherwise the edge of $T_1$ corresponding to $e$ would meet three or more dashed chords, which is impossible. In particular, as any vertical path meeting $\Delta$ has to meet at least two of its edges and every edge meets at most two vertical paths, only $(2 \times 3)/2 = 3$ vertical paths can enter $\Delta$. Similarly, only 3 horizontal chords can meet $\Delta$. Thus at most 9 chord-path pairs can have points of intersection in $\Delta$, implying that it would require at least $\frac{n^2}{16}/9 = \frac{n^2}{144}$ triangles to cover all intersections. This completes our argument.

What if the labeling of polygon corners is not given, so that any matching between corners of $\mathcal{P}_1$ and $\mathcal{P}_2$ that preserves their circular ordering is allowed? One is tempted to conjecture that the number of new vertices required would always be reduced, as it is for the polygons of figure 5. However, in the worst case relabeling does not help by more than a constant factor, for the following reason. The above proof relies crucially on the fact that, in a collection of "vertical" and "horizontal" paths, arranged in a manner schematically depicted in figure 6, a quadratic number of path pairs meet. Let $\mathcal{P}_1$ be a snake of length $\frac{n}{2}$ and $\mathcal{P}_2$ consist of three snakes each of length $\frac{n}{6}$ pasted together in a star-like fashion. Refer to figure 7. Joining vertices lying across the snake from each other, as in the previous argument, we arrive at the situation schematically pictured in figure 8. Notice that overlaying the two families always produces $\Omega(n^2)$ intersections, no matter how polygon corners are matched, and, as above, the number of path pairs intersecting in a single triangle of a common triangulation is at most 9. Similar reasoning as above yields an $\Omega(n^2)$ lower bound on the number

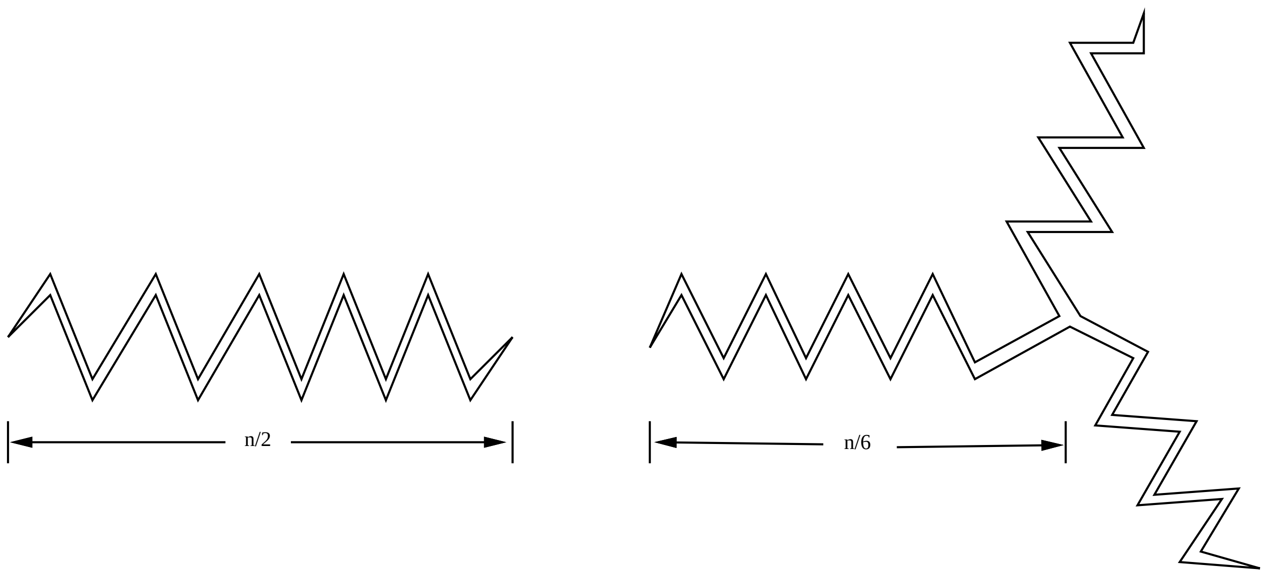


Figure 7: Stronger lower bound: polygons

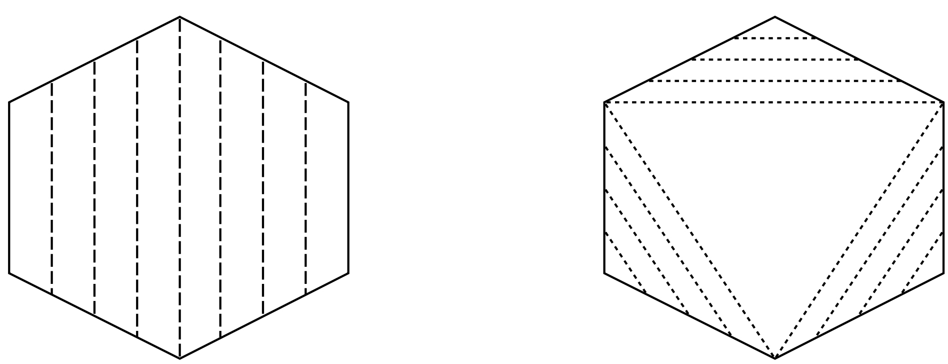

Figure 8: Stronger lower bounds: families of paths

of vertices in any pair of compatible triangulations, as asserted.

## 5 Remarks

The above arguments give a $\Theta(n^2)$ bound on the number of Steiner points required in the worst case for constructing compatible triangulations for two polygons, which leaves open the question of determining the exact value of the coefficient of $n^2$ in this bound. Note that the coefficient may be different depending on whether a numbering of polygon vertices is fixed from the start or one is free to choose a numbering. For example, in the fixed vertex-numbering case, our lower bound construction requires at least $\frac{n^2}{144}$ triangles in the common triangulation which, by Euler's formula, means that at least $\frac{n^2}{288}$ vertices are needed, including the vertices of the input polygons. An easy modification of this argument implies a better bound of $\frac{n^2-4n}{64}$, which is still quite far from the upper bound.

Another obvious question remains open: How does one find the least number of Steiner points required for a given pair of simple polygons? One may ask this question both in the fixed vertex-numbering setting and in the more general case. Note that the question whether two $n$-gons admit a compatible triangulation with no Steiner points can be decided via relatively straightforward dynamic programming techniques in $O(n^3)$ time in the fixed vertex-numbering setting, and hence in $O(n^4)$ time in the general setting.

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