Calculus Notes

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Contents

1 Limits and Derivatives

1.1 derivative of a function f at a number a

The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

1.2 derivative of f

The **derivative of** f is f', a new function:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{2}$$

1.3 differentiation

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x) \tag{3}$$

The symbols D and $\frac{d}{dx}$ are called **differentiation operators** because they indicate the operation of **ifferentiation**, which is the process of calculating a derivative. The symbol $\frac{dy}{dx}$, which was introduced by Leibniz, should not be regarded as a ratio.

1.4 second derivative

f'' is called the **second derivative** of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as:

$$\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2} \tag{4}$$

2 Differentiation Rules

2.1 The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{5}$$

2.2 The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{6}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{7}$$

2.3 Linear Approximations and Differentials

$$f(x) \approx f(a) + f'(a)(x - a) \tag{8}$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$F(x) = f(a) + f'(a)(x - a) \tag{9}$$

is called the **linearization** of f at a.

2.4 Differentials

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)dx \tag{10}$$

3 Applications of Differentiation

3.1 The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{11}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{12}$$

3.2 Antiderivative

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

4 Integrals

4.1 Definite Integral

If f is a function defined for $a \geq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, ..., x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{13}$$

4.2 The Fundamental Theory of Calculus, Part 1

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(x)dt \tag{14}$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

4.3 The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{15}$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

4.4 Indefinite Integrals

The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an indefinite integral. Thus

$$\int f(x)dx = F(x) \tag{16}$$

means

$$F'(x) = f(x) \tag{17}$$

5 Partial Derivatives

5.1 Functions of Two Variables

A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on.

5.2 Limit of $f(x, y)$ as (x, y) approaches (a, b)

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{18}$$

indicates that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.

5.3 Partial derivative of f with respect to x at (a, b)

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of** f **with respect to** x **at** (a, b) and denote it by $f_x(a, b)$.

5.4 Partial derivatives

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to0} \frac{f(x + h, y) - f(x, y)}{h} \tag{19}$$

and

$$f_y(x, y) = \lim_{h\to0} \frac{f(x, y + h) - f(x, y)}{h} \tag{20}$$

5.5 Notations for Partial Derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \tag{21}$$

5.6 Second partial derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives of** f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \tag{22}$$

5.7 Directional derivative

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{23}$$

5.8 The Gradient Vector

The directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\begin{aligned} D_u f(x, y) &= f_x(x, y)a + f_y(x, y)b && (24)\\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle && (25)\\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u} && (26) \end{aligned}$$

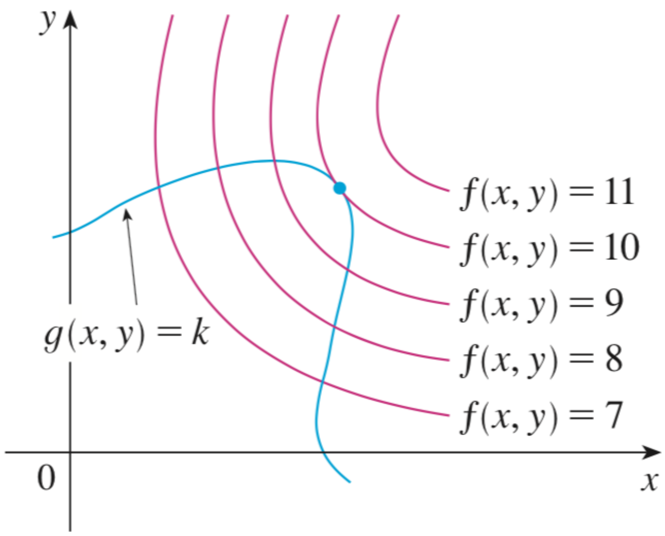
The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read "del f").
If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{27}$$

5.9 Lagrange Multipliers

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f = (x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f = (x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 5.9 shows this curve together with several level curves of f. These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$. To maximize $f = (x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

It appears from Figure 5.9 that this happens when these curves just touch each other, that is, when they have a common tangent line. This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ .



This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.
This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $r(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $r(t_0) = (x_0, y_0, z_0)$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$0 = h'(t_0) \tag{28}$$
$$= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \tag{29}$$
$$= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \tag{30}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $r'(t_0)$ to every such curve C. But we already know that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $r'(t_0)$ for every such curve. This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq 0$, there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \tag{31}$$

The number λ in Equation 33 is called a **Lagrange multiplier**. The procedure based on Equation 33 is as follows.

Theorem 1 (Method of Lagrange Multipliers) *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:*

1. *Find all values of x, y, z, and λ such that*

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \tag{32}$$

   and

$$\nabla g(x, y, z) = k \tag{33}$$

2. *Evaluate f at all the points (x, y, z) that result from step (1). The largest of these values is the maximum value of f; the smallest is the minimum value of f.*

6 Multiple Integrals

6.1 Double Integral

The double integral of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \tag{34}$$

if the limit exits, where $\Delta A = \Delta x \Delta y$

6.2 Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$,then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_x^y f(x,y)dxdy \tag{35}$$