Robust Control
Inverted Pendulum on a Cart

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Abstract

In this report will be modelled and controlled the problem of the (single) inverted pendulum on a cart. The control will be developed with three different techniques tought by Prof. *Patrizio Colaneri* and Prof. *Gian Paolo Incremona* during the course *Robust Control*.

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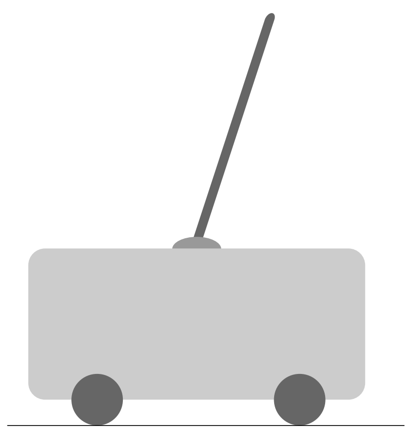
Chapter 1

System Modeling

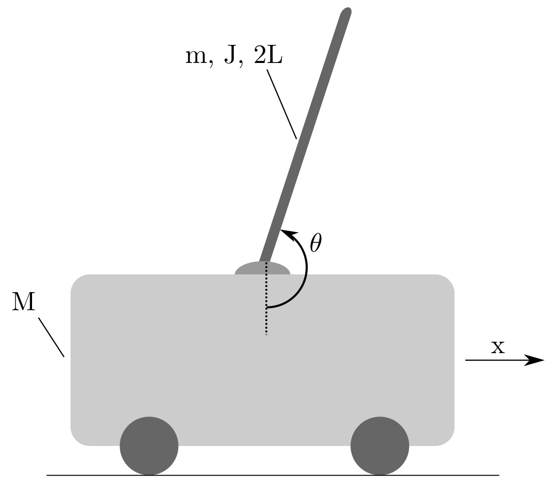
1.1 Problem Definition

This chapter makes reference to the modelling developed by Prof. Bill Messner of the Department of Mechanical Engineering at Tufts University (formerly of Carnegie Mellon University) and Prof. Dawn Tilbury of the Department of Mechanical Engineering and Applied Mechanics at the University of Michigan. For more information about their work, click here.

The system under analysis consists of an inverted pendulum mounted on a motorized cart. The system is composed of a cart, a joint and a bar as in the following image:



The system has two degrees of freedom: x and θ. Below the image is showing the system in more detail:



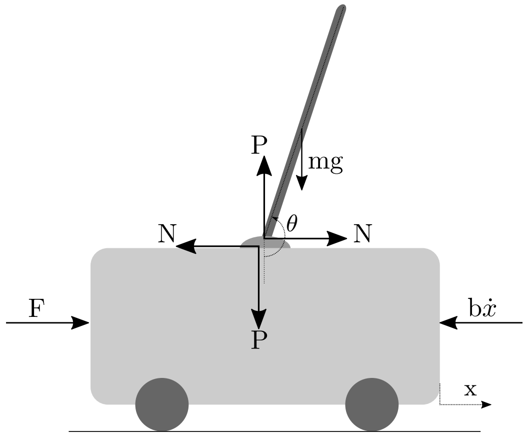
where:

- M = 5 kg, Mass of the cart;
- m = 3 kg, Mass of the bar;
- L = 0.4 m, Distance of the c.o.m. of the bar from the joint;
- J = 0.16 $\mathrm{kg\,m^2}$, Mass moment of inertia of the bar;
- b_n = 0.5 $\mathrm{N\,s\,m^{-1}}$, Coefficient of friction between cart and track.

where it will be supposed an uncertainty of 30% over the value of b_n resulting in a coefficient of friction with values $b \in [0.35, 0.65]$.
The goal is to control the system so to set it in the equilibrium position $x = 0$ and $\theta = \pi$. We will assume that the only accessible sensors are the two that represent the degrees of freedom.

1.2 Force Analysis and Equations of Motion

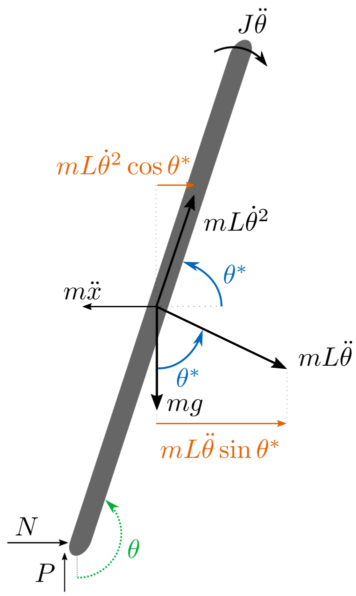
The forces applied to the two elements of the system are showed in the following image:

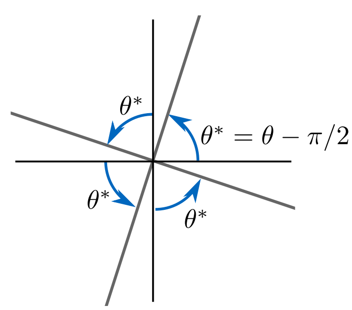


summing the forces of the cart in the horizontal direction, we get the following equation of motion:

$$M\ddot{x} + b\dot{x} + N = F \tag{1.1}$$

Now, considering only the the free-body diagram of the pendulum, the forces applied to the body are:





Therefore, summing the horizontal forces:

$$\sum F_x = 0 : N = m\ddot{x} - mL\ddot{\theta}\sin\theta^* - mL\dot{\theta}^2\cos\theta^*$$

and observing that

$$-mL\ddot{\theta}\sin\theta^* = -mL\ddot{\theta}\sin(\theta - \pi/2) = mL\ddot{\theta}\sin(\pi/2 - \theta) = mL\ddot{\theta}\cos\theta$$

$$-mL\dot{\theta}^2\cos\theta^* = -mL\dot{\theta}^2\cos(\theta-\pi/2) = -mL\dot{\theta}^2\cos(\pi/2-\theta) = -mL\dot{\theta}^2\sin\theta$$

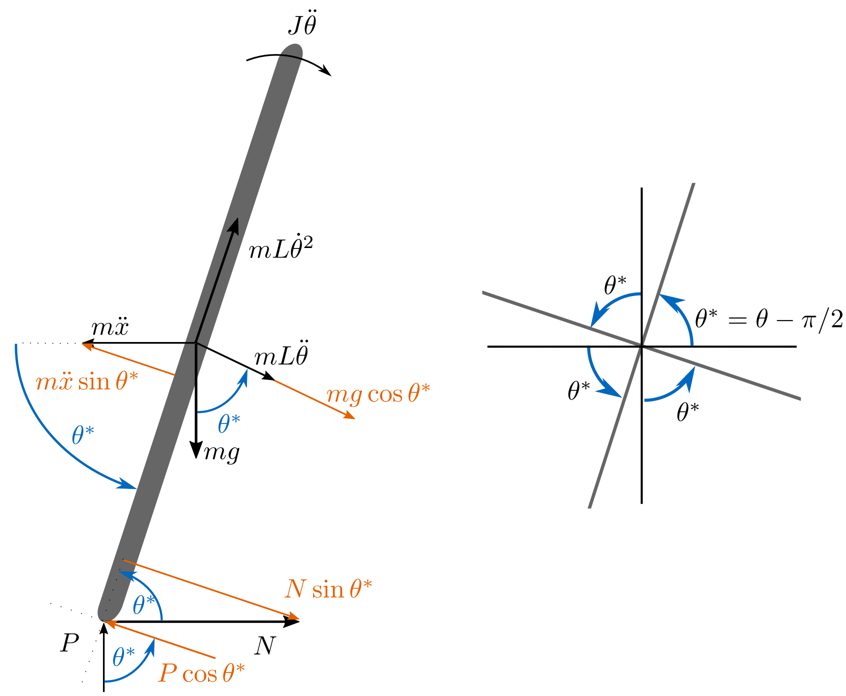
it is possible to rewrite F_x as follows:

$$N = m\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta$$

Now, substituting the latter result into 1.1, we obtain the first equation of motion:

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{1.2}$$

In order to obtain the second equation, we consider the forces perpendicular to the pendulum (to facilitate the computations). The free body diagram is:



Similarly to what has been done before, summing the perpendicular forces we obtain:

$$\sum F_\perp = 0 : N\sin\theta^* - P\cos\theta^* - m\ddot{x}\sin\theta^* + mL\ddot{\theta} + mg\cos\theta^* = 0$$

and observing that

$$N\sin\theta^* = N\sin(\theta-\pi/2) = -N\sin(\pi/2-\theta) = -N\cos\theta$$

$$-P\cos\theta^* = -P\cos(\theta-\pi/2) = -P\cos(\pi/2-\theta) = -P\sin\theta$$

$$-m\ddot{x}\sin\theta^* = -m\ddot{x}\sin(\theta-\pi/2) = m\ddot{x}\sin(\pi/2-\theta) = m\ddot{x}\cos\theta$$

$$mg\cos\theta^* = mg\cos(\theta - \pi/2) = mg\cos(\pi/2 - \theta) = mg\sin\theta$$

we can substitute into the previous equation and obtain $F_\perp$ written as:

$$P\sin\theta + N\cos\theta - mg\sin\theta = mL\ddot{\theta} + m\ddot{x}\cos\theta$$

To discard the terms P and N from the previous equation, we sum the moments about the centroid of the pendulum to get the following equation:

$$-PL\sin\theta - NL\cos\theta = J\ddot{\theta}$$

Substituting, we obtain the second equation of motion:

$$(J + mL^2)\ddot{\theta} + mgL\sin\theta = -mL\ddot{x}\cos\theta \tag{1.3}$$

Thus, combining the equations 1.2 and 1.3, we get the overall system of equations of motion:

$$\begin{cases} (M+m)\ddot{x} + b\dot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = F \\ (J + mL^2)\ddot{\theta} + mgL\sin\theta = -mL\ddot{x}\cos\theta \end{cases} \tag{1.4}$$

1.3 Linearised Equations of Motion

The objective of this project is to stabilise the system around equilibrium position $x = 0$ and $\theta = \pi$, assuming that it stays within a small neighbourhood of this equilibrium.
We can represent the angular deviation (error) from the equilibrium position by setting:

$$\phi : \theta = \pi + \phi$$

Assuming small deviation ϕ from the equilibrium position (within 20°), we can use the following approximations for the nonlinear equations of 1.4:

$$\cos\theta = \cos(\pi + \phi) \approx -1$$

$$\sin\theta = \sin(\pi + \phi) \approx -\phi$$

$$\dot{\theta}^2 = \dot{\phi}^2 \approx 0$$

Now, substituting the above expressions into 1.4 and substituting input u with the external force F, we obtain the nominal linearised equations of motion:

$$\Sigma_n : \begin{cases} (M+m)\ddot{x} + b\dot{x} - mL\ddot{\phi} = u \\ (J + mL^2)\ddot{\phi} - mgL\phi = mL\ddot{x} \end{cases} \tag{1.5}$$

1.4 Transfer Function

In order to obtain the transfer function of Σ_n we assume zero initial conditions and then we apply the Laplace transform obtaining the following system of equations:

$$\Sigma_n : \begin{cases} (M+m)X(s)s^2 + bX(s)s - mL\Phi(s)s^2 = U(s) \\ (J+mL^2)\Phi(s)s^2 - mgL\Phi(s) = mLX(s)s^2 \end{cases} \tag{1.6}$$

From the second equation of the previous system, we get:

$$\Phi(s) = \left[\frac{mL}{(J+mL^2) - mgL} s^2 \right] X(s)$$

Replacing $\Phi(s)$ into the first equation of the system above, we can develop the following algebraic computations:

$$(M+m)X(s)s^2 + bX(s)s - mLs^2 \frac{mL}{(J+mL^2)s^2 - mgL} s^2 X(s) = U(s)$$

Thus, we can perform the following simple algebraic computations:

$$\Big[(J+mL^2) - mgL\Big]\Big\{(M+m)X(s)s^2 + bX(s)s\Big\} - (mL)^2 s^4 X(s) = \Big[(J+mL^2) - mgL\Big]U(s)$$

$$\left[qs^4 + b(J+mL^2)s^3 - mgL(M+m)s^2 - mgLbs\right] X(s) = \Big[(J+mL^2) - mgL\Big]U(s)$$

where $q = (M+m)(J+mL^2) - (mL)^2$. Therefore, the cart transfer function is:

$$G_{cart_n} = \frac{X(s)}{U(s)} = \frac{\frac{(J+mL^2)s^2 - mgL}{q}}{s^4 + \frac{b(J+mL^2)}{q}s^3 - \frac{mgL(M+m)}{q}s^2 - \frac{bgmL}{q}s} \quad \left[\frac{\mathrm{m}}{N}\right] \tag{1.7}$$

Similarly from what has been done before, from the second equation of 1.6, we get:

$$X(s) = \left[\frac{J+mL^2}{mL} - \frac{g}{s^2} \right] \Phi(s)$$

Now, substituting the latter result into the first equation of 1.6, we obtain the following intermediate results:

$$(M+m)\left[\frac{J+mL^2}{mL} - \frac{g}{s^2} \right]\Phi(s)s^2 + b\left[\frac{J+mL^2}{mL} - \frac{g}{s^2} \right]\Phi(s)s - mL\Phi(s)s^2 = U(s)$$

$$\Phi(s)\left\{ \Big[\frac{(M+m)(J+mL^2)}{mL} - mL\Big]s^3 + \frac{b(J+mL^2)}{mL}s^2 - g(M+m)s - bg \right\} = sU(s)$$

$$\Phi(s)\left\{\left[(M+m)(J+mL^2)-(mL)^2\right]s^3+b(J+mL^2)s^2-mgL(M+m)s-mLbg\right\}=mLsU(s)$$

and, setting $q=(M+m)(J+mL^2)-(mL)^2$, we obtain the pendulum transfer function:

$$G_{pend_n}=\frac{\Phi(s)}{U(s)}=\frac{\frac{mL}{q}s}{s^3+\frac{b(J+mL^2)}{q}s^2-\frac{(M+m)mgL}{q}s-\frac{mLbg}{q}} \qquad \left[\frac{\text{rad}}{N}\right] \tag{1.8}$$

Hence, combining 1.7 and 1.8, we obtain the nominal transfer function of system Σ_n:

$$G_n=\begin{bmatrix} G_{cart_n} \\ \\ G_{pend_n}\end{bmatrix}=\begin{bmatrix} \frac{\frac{(J+mL^2)s^2-mgL}{q}}{s^4+\frac{b_n(J+mL^2)}{q}s^3-\frac{(M+m)mgL}{q}s^2-\frac{b_n gmL}{q}s} \\ \\ \frac{\frac{mL}{q}s}{s^3+\frac{b_n(J+mL^2)}{q}s^2-\frac{(M+m)mgL}{q}s-\frac{b_n gmL}{q}}\end{bmatrix} \tag{1.9}$$

where b_n is the nominal value of the coefficient of friction b. It should be noticed that G_{cart_n} has one pole in the origin.

1.5 State-Space in Controllable Canonical Form

State space can be written in many different ways. Here in the following will be used the canonical controllable form for system adaptation reasons that will be clearer later.
Given the equations 1.9, it is possible to rewrite them making use of the controllable canonical form:

Proposition 1 (Controllable Canonical Form)**.** Given a transfer function G(s) :

$$G(s)=\frac{\beta_n s^{n-1}+\beta_{n-1}s^{n-2}+\cdots+\beta_2 s+\beta_1}{s^n+\alpha_n s^{n-1}+\alpha_{n-1}s^{n-2}+\cdots+\alpha_2 s+\alpha_1}+\gamma$$

Its Controllable Canonical Form (in the SISO case) is:

$$A=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_n \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n \end{bmatrix} \qquad D = \gamma$$

In order to exploit previous realization for a $S.I.M.O.$ system it is needed to modify G_{pend_n} so to make it have the same characteristic polynomial of G_{cart_n} and thus to have the same least common denominator (of the system). To do so, the following has to be done:

$$G_{pend_n} \frac{s}{s}$$

Now it is possible to notice that G_{pend_n} and G_{cart_n} share the same denominator whereas the numerator of G_{pend_n} can be rewritten as:

$$\frac{mL}{q} s^2$$

Therefore, it is possible to write:

$$\Sigma_n : \begin{cases} \underline{\dot{x}} = A\underline{x} + Bu \\ \underline{y} = C\underline{x} + Du \end{cases} \tag{1.10}$$

where:

$$\underline{y} = \begin{bmatrix} x \\ \phi \end{bmatrix}$$

In accordance to proposition 1 and with transfer function 1.9, the nominal state space is:

$$\Sigma_n : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_n gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_n(J+mL^2)}{q} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} x \\ \phi \end{bmatrix} = \begin{bmatrix} -\frac{mgL}{q} & 0 & \frac{(J+mL^2)}{q} & 0 \\ 0 & 0 & \frac{mL}{q} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{cases} \tag{1.11}$$

where b_n is the value of coefficient of friction between cart and track under nominal conditions.

1.6 State-Space from Differential Equations

Making reference to the linearised equations of motions 1.5, the system can be represented in state-space by rearranging them into a series of first order differential equations. From the first equation of 1.5, by doing some computation we get:

$$\ddot{x} = \frac{mL}{M+m}\ddot{\phi} - \frac{b}{M+m}\dot{x} + \frac{1}{M+m}u$$

From the second equation, instead:

$$\ddot{\phi} = \frac{mgL}{J+mL^2}\phi + \frac{mL}{J+mL^2}\ddot{x}$$

Thus, substituting $\ddot{\phi}$ into $\ddot{x}$ we obtain:

$$\ddot{x} = \frac{g(mL)^2}{(J+mL^2)(M+m)}\phi + \frac{(mL)^2}{(J+mL^2)(M+m)}\ddot{x} - \frac{b}{M+m}\dot{x} + \frac{1}{M+m}u$$

$$\frac{p}{(J+mL^2)(M+m)}\ddot{x} = \frac{g(mL)^2}{(J+mL^2)(M+m)}\phi - \frac{b}{M+m} + \frac{1}{M+m}u$$

where $p = (J+mL^2)(M+m) - (mL)^2$. Hence, we can write:

$$\ddot{x} = \frac{-b(J+mL^2)}{p}\dot{x} + \frac{g(mL)^2}{p}\phi + \frac{J+mL^2}{p}u \tag{1.12}$$

Now,in order to obtain $\ddot{\phi}$, we proceed by substituting 1.12 into $\ddot{\phi}$:

$$\ddot{\phi} = \Big[\frac{mgL}{J+mL^2} + \frac{g(mL)^3}{p(J+mL^2)}\Big]\phi - \frac{mLb}{p}\dot{x} + \frac{mL}{p}u$$

but, remembering that $p = (J+mL^2)(M+m) - (mL)^2$ and observing that:

$$\begin{aligned}
\frac{mgL}{J+mL^2} + \frac{g(mL)^3}{p(J+mL^2)} &= \\
&= \frac{mgL}{J+mL^2} + \frac{g(mL)^3}{\big[(J+mL^2)(M+m)-(mL)^2\big](J+mL^2)} \\
&= \frac{mgL}{J+mL^2}[1 + \frac{(mL)^2}{p}] \\
&= \frac{mgL(M+m)}{p}
\end{aligned}$$

it is possible to write the following relationship:

$$\ddot{\phi} = \frac{mgL(M+m)}{p}\phi - \frac{mLb}{p}\dot{x} + \frac{mL}{p}u \tag{1.13}$$

Being in a $2-d.o.f.$ system, we need 4 state variables. Accordingly to the definition of the linearised equation of motion 1.5, we can set the vector of state variables with an immediate physical meaning:

$$\underline{x_{ph}} = \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

and thus, from relationships 1.12 and 1.13, and in accordance to previous definition of $\underline{x}$, we can write the state-space realization of the system:

$$\Sigma_n^{ph} : \begin{cases} \underline{\dot{x}_{ph}} = A_{ph} \cdot \underline{x_{ph}} + B_{ph} \cdot u \\ \underline{y} = C_{ph} \cdot \underline{x_{ph}} + D_{ph} \cdot u \end{cases} \tag{1.14}$$

$$\Sigma_n^{ph} : \begin{cases} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(J+mL^2)b_n}{p} & \frac{g(mL)^2}{p} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mLb_n}{p} & \frac{mgL(M+m)}{p} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{J+mL^2}{p} \\ 0 \\ \frac{mL}{p} \end{bmatrix} u \\ \begin{bmatrix} x \\ \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{cases} \tag{1.15}$$

1.7 Similarity Transformation Matrix

In the previous two sections have been obtained two different state spaces: $\Sigma_n : (A, B, C, D)$ and $\Sigma_n^{ph} : (A_{ph}, B_{ph}, C_{ph}, D_{ph})$. It may be useful to find:

$$T : \Sigma_n^{ph} \longleftrightarrow \Sigma_n$$

Therefore, considering the following relationship:

$$\underline{\dot{x}} = T \cdot \underline{\dot{x}_{ph}} \longrightarrow \underline{\dot{x}_{ph}} = T^{-1} \underline{\dot{x}}$$

Substituting we have:

$$\begin{cases} \underline{\dot{x}_{ph}} = A_{ph} \cdot \underline{x_{ph}} + B_{ph} \cdot u \\ \underline{y} = C_{ph} \cdot \underline{x_{ph}} + D_{ph} \cdot u \end{cases} \implies \begin{cases} T^{-1}\underline{\dot{x}} = A_{ph} \cdot T^{-1}\underline{x} + B_{ph} \cdot u \\ \underline{y} = C_{ph} \cdot T^{-1}\underline{x} + D_{ph} \cdot u \end{cases}$$

Multiplying both the members of first equation by T we obtain:

$$\begin{cases} \underline{\dot{x}} = T \cdot A_{ph} \cdot T^{-1}\underline{x} + T \cdot B_{ph} \cdot u \\ \underline{y} = C_{ph} \cdot T^{-1}\underline{x} + D_{ph} \cdot u \end{cases}$$

And, defining:

$$A_n = T \cdot A_{ph} \cdot T^{-1} \qquad B_n = T \cdot B_{ph} \qquad C_n = C_{ph} \cdot T^{-1} \qquad D_n = D_{ph}$$

we obtain the same form of equations 1.10.
Now, considering the general controllability matrices:

$$M_{ph} = \begin{bmatrix} B_{ph} & A_{ph} \cdot B_{ph} & A_{ph}^2 \cdot B_{ph} & \dots A_{ph}^{n-1} \cdot B_{ph} \end{bmatrix}$$

$$
\begin{aligned}
M_n &= \begin{bmatrix} B_n & A_n B & A_n^2 B_n & \dots A_n^{n-1} B_n \end{bmatrix} \\
&= \begin{bmatrix} TB_{ph} & TA_{ph}T^{-1}TB_{ph} & (TA_{ph}T^{-1})^2 TB_{ph} & \dots & (TA_{ph}T^{-1})^{n-1} TB_{ph} \end{bmatrix} \\
&= T \cdot M_{ph}
\end{aligned}
$$

Therefore, similarity transformation matrix is obtained by the following equation:

$$T = M_n \cdot M_{ph}^{-1} \tag{1.16}$$

1.8 Numerical Results

Given the numerical values proposed in 1.1, substituting them in 1.9 we obtain the following transfer function:

$$G_n = \begin{bmatrix} G_{cart_n} \\ \\ G_{pend_n} \end{bmatrix} = \begin{bmatrix} \frac{8.667s^2 - 159.4}{49.84s^4 + 0.4334s^3 - 1275s^2 - 7.971s} \\ \\ \frac{16.25s}{49.84s^3 + 0.4334s^2 - 1275s - 7.971} \end{bmatrix} \tag{1.17}$$

Substituting the numerical values into 1.14:

$$\Sigma_n^{ph} : \begin{cases} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0,0870 & 3,8387 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0,1630 & 25,5913 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0,1739 \\ 0 \\ 0,3261 \end{bmatrix} u \\ \begin{bmatrix} x \\ \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{cases} \tag{1.18}$$

Instead, substituting the numerical values into the canonical realization 1.11:

$$\Sigma_n : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1,5995 & 25,5913 & -0.0870 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} x \\ \phi \end{bmatrix} = \begin{bmatrix} -3,1989 & 0 & 0,1739 & 0 \\ 0 & 0 & 0,3261 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{cases} \tag{1.19}$$

where the poles of the system are:

$$p_1 = 0.0 \qquad p_2 = 5,0467 \qquad p_3 = -5,0712 \qquad p_4 = -0,0625$$

Similarity transformation matrix T, whose relationship with Σ_n^{ph} and Σ_n has been described previously in section 1.7, is:

$$T = \begin{bmatrix} -0,3126 & 0 & 0,1667 & 0 \\ 0 & -0,3126 & 0 & 0,1667 \\ 0 & 0 & 3,0667 & 0 \\ 0 & 0 & 0 & 3,0667 \end{bmatrix} \tag{1.20}$$

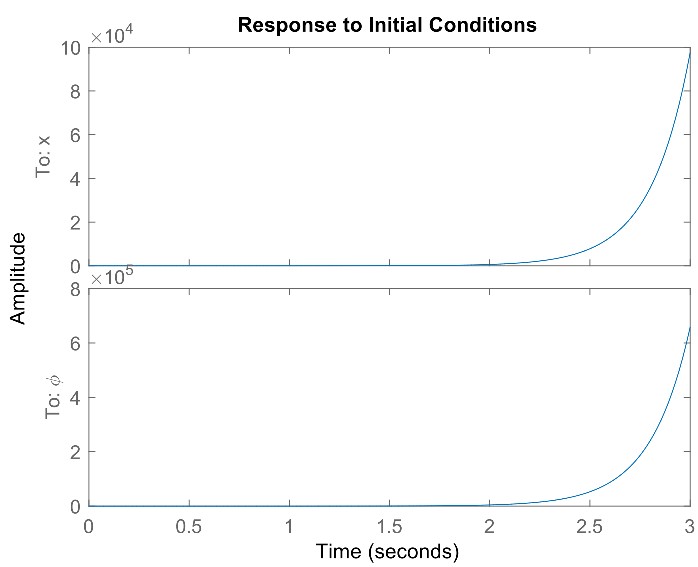
1.9 Open Loop Simulation

Here will be shown the open loop system response of the system. The plot is obtained with Matlab R2018a.

Before proceeding, it is necessary to define an initial state. Supposing the system to have an error on the position of 1m and an error on the angle of 20° (thus 0,349066 rad) and no initial linear nor angular speed, we can write:

$$\underline{x_{ph}}(0) = \begin{bmatrix} 1 \\ 0 \\ 0,349066 \\ 0 \end{bmatrix} \xrightarrow{\underline{x}(0)=T\cdot\underline{x_{ph}}(0)} \underline{x}(0) = \begin{bmatrix} -0,2544 \\ 0 \\ 1,0705 \\ 0 \end{bmatrix} \tag{1.21}$$

Thus, the open loop response of the system Σ_n with initial state $\underline{x}(0)$ is the following:



As shown in the plot above and as predicted by the unstable pole, the open loop system is unstable. Matlab code is reported in last chapter.

Chapter 2

System Robust Control

2.1 Introductory Theory

The goal of this chapter is to show several methods to efficiently control the system in the desired equilibrium position:

$$\underline{x_{ph_{eq}}} = \begin{bmatrix} x_{eq} \\ \dot{x}_{eq} \\ \phi_{eq} \\ \dot{\phi}_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The design will rely on *Robust Control Theory*, especially on the methods taught from *Prof. Patrizio Colaneri* and from *Prof. Gian Paolo Incremona*.

Hereunder will be presented three different projects. Before showing them, it's necessary to remember some theory on robust control and to verify the assumptions:

Theorem 1 (Space $L_2(\tau$,T)). *The L_2 space is defined by all (real valued, matrix) functions of time defined in (τ, T) and zero elsewhere such that:*

$$\int_{\tau}^{T} trace[v(t)'v(t)]d\omega = \int_{\tau}^{T} trace[v(t)v(t)']d\omega < \infty$$

The L_2 norm of $v : v \in L_2(0, \infty)$ is:

$$||v||_2 = \sqrt{\int_{\tau}^{T} trace[v(t)'v(t)]dt}$$

Spaces $L_2(0, \infty)$ and $L_2(-\infty, 0)$ are really important in engineering applications. Here below the frequency interpretation of previous theorem.

Theorem 2 (L_2 Space - Frequency Interpretation)**.** *The L_2 space is defined as the set G(s) of rational and strictly proper transfer functions with no poles on the imaginary axis such that:*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} trace[G(-j\omega)'G(j\omega)]d\omega < \infty$$

The L_2 norm of a function G(s) is:

$$||G(s)||_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} trace[G(-j\omega)'G(j\omega)]d\omega}$$

Given theorems 1 and 2 we can now further classify two useful subspaces:

Theorem 3 (Subspaces H_2 and $H_2^{\perp}$)**.** *The H_2 subspace is constituted by the functions of L_2 which are analytic in the right half plane (strictly proper and stable transfer functions).*
The $H_2^{\perp}$ is constituted by the functions of L_2 which are analytic in the left half plane (strictly proper and anti-stable transfer functions).

Therefore, a function in L_2 can be written in an unique way as a sum of a function in H_2 and a function in $H_2^{\perp}$ such that, given $G_1(s) \in H_2$ and $G_2(s) \in H_2^{\perp}$: $G_1(s) \perp G_2(s)$, $||G(s)||_2^2 = ||G_1(s)||_2^2 + ||G_2(s)||_2^2$.

From the previous theorems, it is possible to derive the following two propositions:

Proposition 2. Given a G(s)$\in L_2(0,\infty) : y(t) = Ce^{At}B$, given P observability gramian : $P = \int_0^{\infty} e^{A't}C'Ce^{At}dt$, the following relation holds:

$$||G(s)||_2^2 = \int_0^{\infty} trace\big(y(t)'y(t)\big)dt = \int_0^{\infty} trace\big(B'e^{A't}C'Ce^{At}B\big)dt$$

Thus:

$$||G(s)||_2^2 = ... = trace\Big(B'\Big[\int_0^{\infty} e^{A't}C'Ce^{At}dt\Big]B\Big)$$

If the system is observable we have that $P > 0$ thus, considering the control-type Lyapunov equation:

$$A'P + PA = \frac{d}{dt}\int_0^{\infty} e^{A't}C'Ce^{At}dt = -C'C$$

therefore, we obtain:

$$A'P + PA + C'C = 0$$

Finally, the H_2 norm of G(s) is obtained from the following system:

$$\begin{cases} ||G(s)||_2^2 = trace(B'PB) \\ A'P + PA + C'C = 0 \end{cases}$$

The dual derivation is:

Proposition 3. Given a G(s)$\in L_2(0, \infty) : y(t) = Ce^{At}B$, given S controllability gramian : $S = \int_0^\infty e^{A't}BB'e^{At}dt$, the following relation holds:

$$||G(s)||_2^2 = \int_0^\infty trace\big(y(t)y(t)'\big)dt = \int_0^\infty trace\big(Ce^{At}BB'e^{A't}C'\big)dt$$

Thus:

$$||G(s)||_2^2 = ... = trace\Big(C\Big[\int_0^\infty e^{A't}BB'e^{At}dt\Big]C'\Big)$$

If the system is observable we have that $S > 0$ thus, considering the Lyapunov equation:

$$AS + SA' = \frac{d}{dt}\int_0^\infty e^{A't}BB'e^{At}dt = -BB'$$

therefore, we obtain:

$$A'P + PA + BB' = 0$$

Finally, the H_2 norm of G(s) is obtained from the following system:

$$\begin{cases} ||G(s)||_2^2 = trace(CSC') \\ AS + SA' + BB' = 0 \end{cases}$$

In the following will be also make use of the H_∞ norm, thus of the L_∞ norm.

Proposition 4 (L_∞ Norm)**.** Given a system Σ with transfer function $G(s)$, the L_∞ space is the space of all $G(s)$ such that:

$$\sup_\omega |G(s)| < +\infty$$

So G(s) must be a proper (rational) function without poles on the imaginary axis.

And, similarly to some previous considerations:

Proposition 5 (H_∞ Space and H_∞ Norm)**.** The H_∞ space is composed by all the G(s) that belong to the L_∞ space and with poles with strictly negative real parts. In addition, the norm is given by:

$$||G(s)||_\infty = \sup_{\omega} |G(j\omega)|$$

It is sometime necessary to use Riccati equations in order to design a controller that satisfies some constraints with respect to the H_∞ norm. Below some theorems and intuitive proofs related to H_∞ control:

Theorem 4. *Given a function $G(s) = C(sI - A)^{-1}B + D$ that belongs to the H_∞ space, G(s) in minimal form, and given $\gamma > 0$, then:*

$$||G(s)||_\infty < \gamma \iff \begin{cases} \det(I - G(-j\omega)'G(j\omega) \neq 0 & \forall\omega \in [0, \infty) \\ ||G(j\infty)|| < \gamma \end{cases}$$

Or, equivalently:

$$||G(s)||_\infty < \gamma \iff \begin{cases} ||D|| < \gamma \\ H \textit{ is unmixed (no eigenvalues on the imaginary axis)} \end{cases}$$

where the Hamiltonian matrix is:

$$H = \begin{bmatrix} A + B(I - \frac{D'D}{\gamma^2})^{-1}\frac{D'C}{\gamma^2} & B(I - \frac{D'D}{\gamma^2})^{-1}\frac{B'}{\gamma^2} \\ \\ -C'(I - \frac{D'D}{\gamma^2})^{-1}C & -A' - \frac{C'D}{\gamma^2}(I - \frac{D'D}{\gamma^2})^{-1}B' \end{bmatrix}$$

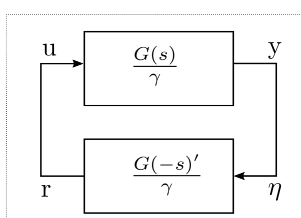
Proof. We want to test if $||G(s)||_\infty < \gamma$, thus it must be verified:

$$||G(s)||^2_\infty < \gamma^2 \qquad \forall\omega$$

Translating previous relation into a matrix inequality:

$$G(-j\omega)G(j\omega) < \gamma^2 I \qquad \forall\omega \implies I - \frac{G(-j\omega)'}{\gamma}\frac{G(j\omega)}{\gamma} > 0 \qquad \forall\omega$$

The system can now be represented as:



$\Gamma(s)$

where $\Gamma(s) \triangleq I - \frac{G(-s)'}{\gamma}\frac{G(s)}{\gamma}$. Now, considering that for $s = j\omega \to +\infty$ we have:

$$G(s) = C(sI - A)^{-1}B + D \overset{s\to\infty}{\longrightarrow} D \implies G(s) \simeq D \quad \forall s = j\omega \to +\infty$$

Therefore:

$$\Gamma(s) \to I - \frac{D'D}{\gamma^2} > 0 \iff ||D|| < \gamma$$

Now, assuming that $||D|| < \gamma$ and assuming that $\det\left(I - \frac{G(-j\omega)'}{\gamma}\frac{G(j\omega)}{\gamma}\right) > 0$, we consider the following realization:

$$\frac{G(s)}{\gamma} : \begin{cases} \dot{x} = Ax + \frac{B}{\gamma}u \\ y = Cx + \frac{D}{\gamma}u \end{cases}$$

And, noticing that $G(-s)' = -B'(sI + A')^{-1}C' + D'$, we obtain also the adjoin system

$$\frac{G(-s)'}{\gamma} : \begin{cases} \dot{\lambda} = -A'\lambda - C'\eta \\ r = \frac{B'}{\gamma}\lambda + \frac{D'}{\gamma}\eta \end{cases}$$

Therefore, from the equations we obtain:

$$u = \frac{B'}{\gamma}\lambda + \frac{D'}{\gamma}\left(Cx + \frac{D}{\gamma}u\right)$$
$$\left(I - \frac{D'D}{\gamma^2}\right)u = \frac{B'}{\gamma}\lambda + \frac{D'C}{\gamma}x$$

hence:

$$u = \left(I - \frac{D'D}{\gamma^2}\right)^{-1}\left(\frac{B'}{\gamma}\lambda + \frac{D'C}{\gamma}x\right)$$

Rearranging the state vector as:

$$\begin{bmatrix} x \\ \lambda \end{bmatrix}$$

and managing the equations of $\Gamma(s)$ is possible to obtain the following relation:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = H \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

where H takes the form described in the theorem above. Therefore, to conclude the second part of the demonstration, H must not have any pole on the imaginary axis. □

As appendix to this proof, it should also be noticed that the poles of the feedback system (the eigenvalues of H) are also the zeros of $\Gamma(s)$.

Theorem 5 (Bounded Real Lemma)**.** *Given a function $G(s) = C(sI - A)^{-1}B + D$ with A stable and in minimal form, that belongs to the H_∞ space and given $\gamma > 0$, then the three following conditions are equivalent to each other:*

1. $||G(s)||_\infty < \gamma$

2. $||D|| < \gamma$ *and* $\exists P \geq 0$ *solution of the Riccati equation:*

$$A'P + PA + (PB + C'D)(\gamma^2 I - D'D)^{-1}(B'P + DC') + C'C = 0$$

And, in addition:

$$A + B(\gamma^2 I - D'D)^{-1}(B'P + DC') \quad \textit{stable}$$

3. $||D|| < \gamma$ *and* $\exists P > 0$ *solution of the Riccati inequality:*

$$A'P + PA + (PB + C'D)(\gamma^2 - D'D)^{-1}(B'P + DC') + C'C < 0$$

Intuitive/Non-Formal Proof: Defining matrix $\tilde{A}$ stable, known the Hamiltonian matrix (supposing, for simplicity, D=0) defined in theorem 4, from the following relation:

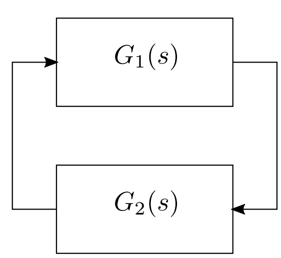
$$H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \tilde{A}$$

Therefore:

$$\begin{cases} A + \frac{BB'}{\gamma^2}P = \tilde{A} \\ -C'C - A'P = P(A + \frac{BB'}{\gamma^2}P) \end{cases}$$

we obtain $A'P + PA + \frac{PBB'P}{\gamma^2} + C'C = 0$ that is the Algebrai Riccati Equation. In addition, if $\gamma \to \infty$ we obtain same equation of 2 (equation for the H_2 case).

Theorem 6 (Small Gain Theorem)**.** *Given two functions* $G_1(s)$ *and* $G_2(s)$*, with* $G_1(s) \in H_2$*, related as in following scheme:*



Then:

1. The interconnected system is stable $\forall G_2(s)$*, with* $||G_2(s)||_\infty < \alpha$*, if* $||G_1(s)||_\infty < \alpha^{-1}$*;*

2. If $||G_1(s)||_\infty > \alpha^{-1}$ *then* $\exists G_2(s)$ *stable, with* $||G_2(s)||_\infty < \alpha$ *that destabilizes the interconnected system.*

2.2 System Adaptation

Given the theory written so far, one may notice that the L_2 norm of the nominal system described from transfer function 1.9 or, equivalently, from state space 1.15 do not exists. The reason why the L_2 norm is not defined lies in the assumptions of theorem 2.
One of the poles of the nominal system is in the origin and this makes the integral of theorem 1 to diverge so that the L_2 norm for this system does not exist. So, considering the frequency interpretation of the L_2 space, written in theorem 2, it is possible to check if the assumption are satisfied:

- G(s) rational and strictly proper: ✓
- G(s) with no poles on the imaginary axis: ✗

In order to control the system making use of robust control theory based on L_2 space, it is required to modify the transfer function (so the state space) of the original system. The small modification that will solve the problem consists in finding the sub-optimal solution for the control problem by moving the pole from the origin to the left of a small infinitesimal quantity ϵ (in the following will be used $\epsilon = 1 \cdot 10^{-4}$) .

Given the cart transfer function 1.7, its denominator

$$\sigma(s) = s^4 + \frac{b(J+mL^2)}{q}s^3 - \frac{mgL(M+m)}{q}s^2 - \frac{bgmL}{q}s \tag{2.1}$$

can be rewritten as:

$$\sigma(s) = s\left[s^3 + \frac{b(J+mL^2)}{q}s^2 - \frac{mgL(M+m)}{q}s - \frac{bgmL}{q}\right]$$

where $q = (M+m)(J+mL^2) - (mL)^2$. Now, adding the small adjustment to the pole we obtain:

$$\sigma^{\epsilon}(s) = (s+\epsilon)\left[s^3 + \frac{b(J+mL^2)}{q}s^2 - \frac{mgL(M+m)}{q}s - \frac{bgmL}{q}\right] \tag{2.2}$$

And, distributing the terms the equivalent form is:

$$s^4 + \frac{b(J+mL^2)+q\epsilon}{q}s^3 - \frac{mgL(M+m)+q\epsilon}{q}s^2 - \frac{bgmL+q\epsilon}{q}s - \frac{bgmL\epsilon}{q}$$

Therefore, 1.7 can be rewritten as:

$$G^{\epsilon}_{cart_n} = \frac{\frac{(J+mL^2)s^2 - mgL}{q}}{s^4 + \frac{b(J+mL^2)+q\epsilon}{q}s^3 - \frac{mgL(M+m)+q\epsilon}{q}s^2 - \frac{bgmL+q\epsilon}{q}s - \frac{bgmL\epsilon}{q}} \tag{2.3}$$

And, by observing that

$$\frac{q\epsilon}{b(J+mL^2)} \approx 0 \qquad \frac{q\epsilon}{mgL(M+m)} \approx 0 \qquad \frac{q\epsilon}{bgmL} \approx 0$$

equation 2.3 can be rewritten as:

$$G^{\epsilon}_{cart_n} \simeq \frac{\frac{(J+mL^2)s^2-mgL}{q}}{s^4+\frac{b(J+mL^2)}{q}s^3-\frac{mgL(M+m)}{q}s^2-\frac{bgmL}{q}s-\frac{bgmL\epsilon}{q}} \tag{2.4}$$

Thanks to the previous modification, we can rewrite also 1.9 as:

$$G^{\epsilon}_n = \begin{bmatrix} G^{\epsilon}_{cart_n} \\ \\ G_{pend_n} \end{bmatrix} = \begin{bmatrix} \frac{\frac{(J+mL^2)s^2-mgL}{q}}{s^4+\frac{b_n(J+mL^2)}{q}s^3-\frac{mgL(M+m)}{q}s^2-\frac{b_ngmL}{q}s-\frac{b_ngmL\epsilon}{q}} \\ \\ \frac{\frac{mL}{q}s}{s^3+\frac{b_n(J+mL^2)}{q}s^2-\frac{mgL(M+m)}{q}s-\frac{b_ngmL}{q}} \end{bmatrix} \tag{2.5}$$

In order to obtain the Canonical form, it has to be rewritten as follows:

$$G^{\epsilon} \simeq \begin{bmatrix} \frac{\frac{(J+mL^2)s^2-mgL}{q}}{s^4+\frac{b(J+mL^2)}{q}s^3-\frac{mgL(M+m)}{q}s^2-\frac{bgmL}{q}s-\frac{bgmL\epsilon}{q}} \\ \\ \frac{\frac{mL}{q}s(s+\epsilon)}{s^4+\frac{b(J+mL^2)}{q}s^3-\frac{mgL(M+m)}{q}s^2-\frac{bgmL}{q}s-\frac{bgmL\epsilon}{q}} \end{bmatrix}$$

Where the numerator of G_{pend_n} can be also rewritten as:

$$\frac{mL}{q}s^2 + \frac{mL\epsilon}{q}s$$

Therefore, the state space of 2.5 is $\Sigma^{\epsilon} : (A^{\epsilon}, B^{\epsilon}, C^{\epsilon}, D^{\epsilon})$:

$$\Sigma^{\epsilon} : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{bgmL\epsilon}{q} & \frac{bgmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b(J+mL^2)}{q} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} x \\ \phi \end{bmatrix} = \begin{bmatrix} -\frac{mgL}{q} & 0 & \frac{(J+mL^2)}{q} & 0 \\ 0 & \frac{mL\epsilon}{q} & \frac{mL}{q} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{cases} \tag{2.6}$$

Substituting in state space 2.6 the numerical values proposed in section 1.1 we obtain:

$$\Sigma^\epsilon : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0,0002 & 1,5995 & 25,5913 & -0,0870 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} x \\ \phi \end{bmatrix} = \begin{bmatrix} -3,1989 & 0 & 0,1739 & 0 \\ 0 & 3,2609 \cdot 10^{-5} & 0,3261 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{cases}$$

The system Σ^ϵ eigenvalues are:

$$p_1 = -0,0001 \qquad p_2 = 5,0467 \qquad p_3 = -5,0712 \qquad p_4 = -0,0624$$

And with this adaptation the assumption of theorems 1 and 2 are satisfied.

2.3 Project 1 - H_2 Control with Polytopic Uncertainties

In this first project, the system will be controlled by state feedback taking into account polytopic uncertainties.

Theorem 7 (Stability of Polytopically Uncertain System). *Given*

$$A = \sum_{i=1}^{M} A_i \delta_i \qquad \delta_i \geq 0, \; \sum_{i=1}^{M} \delta_i = 1$$

the system is quadratically stable if and only if $\exists P$>0 such that:

$$A_i' P + P A_i < 0 \quad \forall i$$

Proof.

$$A_i' P + P A_i < 0 \;\; \forall i$$

$$\implies \big(\sum_{i=1}^{M} A_i \delta_i\big)' P + P\big(\sum_{i=1}^{M} A_i \delta_i\big) < 0 \;\; \forall \delta_i \;\text{ in the simplex}$$

Viceversa, if $\exists P > 0$:

$$\big(\sum_{i=1}^{M} A_i \delta_i\big)' P + P\big(\sum_{i=1}^{M} A_i \delta_i\big) < 0 \;\; \forall \delta_i \;\text{ in the simplex}$$

$$\implies A_i' P + P A_i < 0 \;\; \forall i$$

□

2.3.1 Design

In the following will be exploited the second part of the proof above. Given:

$$b \in [b_{min}, b_{max}]$$

in order to compute the controller, we have to substitute the value of b with b_{min} and b_{max} into stability matrix of system Σ^ϵ and to test quadratic stability property we will need also matrix $A_{n,1} = A_n(b_{min}),\ A_{n,2} = A_n(b_{max})$ of Σ_n, thus:

$$A_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_{min} gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_{min}(J+mL^2)}{q} \end{bmatrix}$$

$$A_{n,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_{max} gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_{max}(J+mL^2)}{q} \end{bmatrix}$$

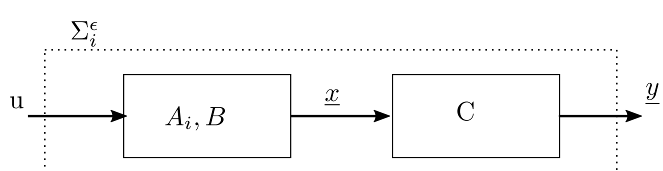
Considering matrix A^ϵ of the adapted system Σ^ϵ, we have:

$$A_1^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{b_{min} gmL\epsilon}{q} & \frac{b_{min} gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_{min}(J+mL^2)}{q} \end{bmatrix} \qquad \delta_1 = 0.5$$

$$A_2^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{b_{max} gmL\epsilon}{q} & \frac{b_{max} gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_{max}(J+mL^2)}{q} \end{bmatrix} \qquad \delta_2 = 0.5$$

Thus obtaining the following equivalent systems:

$$\Sigma_i^\epsilon : \begin{cases} \underline{\dot{x}} = A_i^\epsilon \underline{x} + B^\epsilon \bar{u} \\ \underline{y} = C^\epsilon \underline{x} \end{cases} \qquad i = 1, 2$$



In order to proceed with the design, I set matrix $\tilde{A}^\epsilon = \frac{A_1^\epsilon + A_2^\epsilon}{2}$. The objective is to effectively control the system by state feedback considering the following metric:

$$\begin{aligned}\bar{u} : J^o &= \min \int_0^\infty (\underline{x}' Q \underline{x} + \bar{u}' R \bar{u}) dt \qquad Q \geq 0, R > 0 \\ &= \min \int_0^\infty (z'z) dt \\ &= \min ||z||_2^2\end{aligned}$$

Thus, with $w(t) = imp(t)$, generalizing to the usual formulation for the augmented plant:

$$\begin{cases} \dot{\underline{x}} = A\underline{x} + B_1 w + B_2 u \\ \underline{z} = C_1 \underline{x} + D_{12} u \\ \underline{y} = C_2 \underline{x} + D_{21} u \\ \underline{x}(0) = \underline{x_0} \end{cases}$$

where $u = R^{1/2}\bar{u}$ and:

$$\begin{gathered} A = \tilde{A}^\epsilon \qquad B_1 = x_0 \qquad B_2 = B^\epsilon R^{-1/2} \\ C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \end{gathered} \tag{2.7}$$

Therefore, from proposition 2 it is possible to compute the solution:

$$\begin{cases} ||T_{zw}(s)||_2^2 = trace(B'PB) \\ A'P + PA + C'C = 0 \end{cases}$$

And substituting the matrices into second equation, remembering that with $\bar{u} = K\underline{x}$ previous equations become $\dot{\underline{x}} = (A + B_2 K)\underline{x} + B_1 w$, $z = (C_1 + D_{12}K)\underline{x}$, it is possible to write:

$$A'P + PA' + C'C = 0$$

$$(A + B_2K)'P + P(A + B_2K) + (C_1 + D_{12}K)'(C_1 + D_{12}K) = 0$$

$$A'P + PA + C_1'C_1 + K'(B_2'P + D_{12}'C_1) + (PB_2 + C_1'D_{12})K + K'D_{12}'D_{12}K = 0$$

Now, managing again last equation:

$$\begin{aligned} &A'P + PA + C_1'C_1 + \\ &\Big[K' + (C_1'D_{12} + PB_2)(D_{12}'D_{12})^{-1}\Big]\Big(D_{12}'D_{12}\Big)\Big[K + (D_{12}'D_{12})^{-1}(D_{12}'C_1 + B_2'P)\Big] \\ &\qquad - \Big(C_1'D_{12} + PB_2\Big)\Big(D_{12}'D_{12}\Big)^{-1}\Big(C_1'D_{12} + PB_2\Big)' = 0 \end{aligned}$$

Observing that $C_1'D_{12} = 0$, last equation becomes:

$$\begin{aligned} &A'P + PA + C_1'C_1 + \\ &\qquad \Big[K' + PB_2(D_{12}'D_{12})^{-1}\Big]\Big(D_{12}'D_{12}\Big)\Big[K + (D_{12}'D_{12})^{-1}B_2'P\Big] \\ &\qquad\qquad - PB_2(D_{12}'D_{12})^{-1}B_2'P = 0 \end{aligned}$$

Now, remembering that $A = \tilde{A}^\epsilon$, $B_2 = B^\epsilon R^{-1/2}$ and noticing that $C_1'C_1 = Q$, $D_{12}'D_{12} = R$, we get with respect to u:

$$\tilde{A}^{\epsilon\prime} P + P\tilde{A}^\epsilon + Q + \Big[K' + PB^\epsilon R^{-1/2}R^{-1}\Big]R\Big[K + R^{-1}R^{-1/2\prime}B^{\epsilon\prime}P\Big] \\ - PB^\epsilon R^{-1/2}R^{-1}R^{-1/2\prime}B^{\epsilon\prime}P = 0$$

So, the minimizing K with respect to u is:

$$K^{opt,u} = -R^{-1}R^{-1/2\prime}B^{\epsilon\prime}P$$

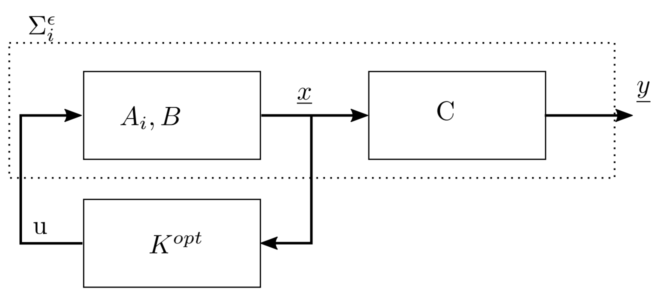
and remembering that $u = R^{1/2}\bar{u}$, since we are designing to have $\bar{u} = K\underline{x}$, the following result holds:

$$K^{opt} = K^{opt,\bar{u}} = -R^{1/2}R^{-1}R^{-1/2\prime}B^{\epsilon\prime}P = -R^{-1}B^{\epsilon\prime}P$$

Setting $Q = C^{\epsilon\prime}C^\epsilon$ and $R = \epsilon^2$, the equivalent solution of the problem is:

$$\begin{cases} ||T_{zw}(s)||_2^2 = trace(B^{\epsilon\prime}PB^\epsilon) \\ \tilde{A}^{\epsilon\prime}P + P\tilde{A}^\epsilon - PB^\epsilon R^{-1}B^{\epsilon\prime}P + Q = 0 \\ \bar{u} = K^{opt}\underline{x} \\ K^{opt} = -R^{-1}B^{\epsilon\prime}P \end{cases} \tag{2.8}$$

With the state feedback control action, system scheme is:



In order to test the controller performances we need to plug K^{opt} to the real system (so to Σ_n), obtaining the following state space:

$$\Sigma_n : (A_n + B_nK^{opt}, B_n, C_n, D_n)$$

And, with the solution previously found, to check if the controlled system is quadratically stable we have to find a $\bar{P} > 0$:

$$A_{new,i}'\bar{P} + \bar{P}A_{new,i} < 0 \;\; i = 1, 2$$

where $A_{new,i} = A_{n,i} + B_nK^{opt}$

2.3.2 Numerical Results

Given the specifications of the system described in section 1.1, remembering that parameter b is defined as $b \in [0.35, 0.65]$ and in accordance to the design previously described, considering the nominal system Σ_n we have the following matrices:

$$A_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.1196 & 25.5913 & -0.0609 \end{bmatrix}$$

$$A_{n,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.0793 & 25.5913 & -0.1130 \end{bmatrix}$$

Instead, the matrices of the adapted system Σ^ϵ are:

$$A_1^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0,0001 & 1,1196 & 25,5913 & -0,0609 \end{bmatrix}$$

$$A_2^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0,0002 & 2,0792 & 25,5913 & -0,1130 \end{bmatrix}$$

Thus:

$$\tilde{A}^\epsilon = \frac{A_1^\epsilon + A_2^\epsilon}{2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0002 & 1.5995 & 25.5913 & -0.0870 \end{bmatrix}$$

Moreover:

$$P = \begin{bmatrix} 6.2037 & 1.8805 & 0.0292 & 0.0003 \\ 1.8805 & 0.7736 & 0.0174 & 0.0002 \\ 0.0292 & 0.0174 & 0.0036 & 4,1650 \cdot 10^{-5} \\ 0.0003 & 0.0002 & 4,1650 \cdot 10^{-5} & 9.1882 \cdot 10^{-7} \end{bmatrix}$$

$$K^{opt} = 1 \cdot 10^4 \cdot \begin{bmatrix} -3.1989 & -1.9395 & -0.4165 & -0.0091 \end{bmatrix}$$

where the eigenvalues of P are $eig(P) = (4,2766 \cdot 10^{-7},\ 0,0031,\ 0,1863,\ 6,7915)$, thus $P > 0$. Plugging K^{opt} to real system Σ_n, the poles show that now it is asymptotically stable:

$$p_{1,2} = -43.1115 \pm i42.8612 \qquad p_{3,4} = -2.5228 \pm i1.5137$$

From equations 2.8 we obtain:

$$||T_{zw}(s)||_2^2 = trace(B_n' P B_n) = 9.1182 \cdot 10^{-7}$$

Now, in order to test if the system is quadratically stable we have to solve the following inequalities:

$$\begin{cases} \bar{P} > 0 \\ A'_{new,i}\bar{P} + \bar{P}A_{new,i} < 0 \;\; i = 1, 2 \end{cases}$$

where $A_{new,i} = A_{n,i} + B_n K^{opt}$. Hence, we find:

$$\bar{P} = \begin{bmatrix} 37.3256 & 15.6107 & 1.4606 & 0.0002 \\ 15.6107 & 13.0835 & 0.8964 & 0.0011 \\ 1.4606 & 0.8964 & 0.2433 & 0.0005 \\ 0.0002 & 0.0011 & 0.0005 & 4.5139 \cdot 10^{-5} \end{bmatrix}$$

where $eig(\bar{P}) = (4.3757 \cdot 10^{-5},\; 0.1733,\; 5.4456,\; 45.0335)$, making it positive definite. Therefore, we now have to test if:

$$\begin{aligned} i = 1 \;:\; A'_{new,1}\bar{P} + \bar{P}A_{new,1} < 0 \\ i = 2 \;:\; A'_{new,2}\bar{P} + \bar{P}A_{new,2} < 0 \end{aligned}$$

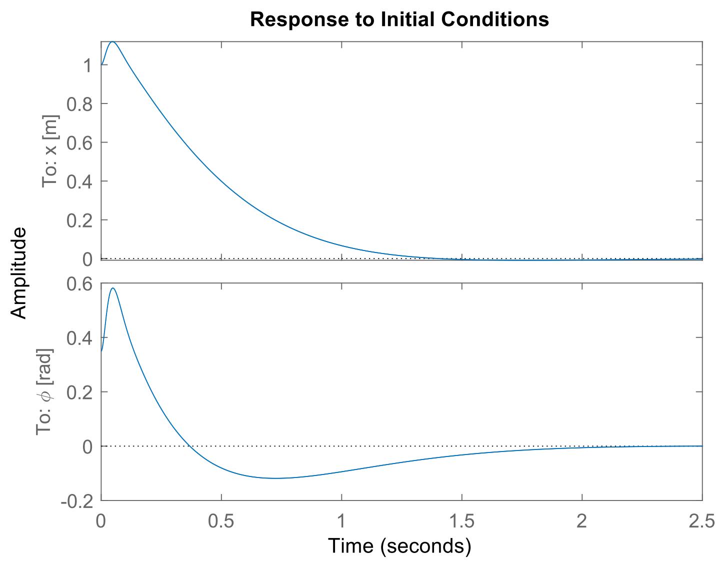
Hence:

$$eig(A'_{new,1}\bar{P} + \bar{P}A_{new,1}) = (-0.0066,\; -2.1941,\; -10.3640,\; -18.1419)$$
$$eig(A'_{new,2}\bar{P} + \bar{P}A_{new,2}) = (-0.0066,\; -2.1940,\; -10.3627,\; -18.1411)$$

Therefore, in accordance to theorem 7, it is possible to state that the system is also quadratically stable (stable $\forall A = A_n(b),\; b \in [b_{min}, b_{max}]$).Finally, having previously defined in section 1.9 vector $\underline{x}(0)$ as:

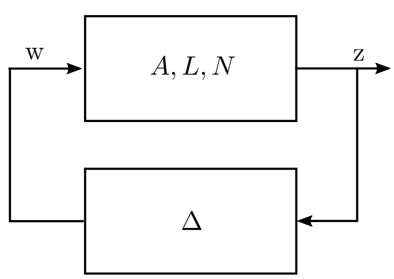
$$\underline{x}(0) = \begin{bmatrix} -0,2544 \\ 0 \\ 1,0705 \\ 0 \end{bmatrix}$$

The system response given the initial state is the following one:



2.4 Project 2 - H_2 and H_∞ Norm Bounded Control

Here will be compared the H_2 control and the H_∞ control taking into account norm bounded uncertainties. Given matrix $A_{un} = A + L\Delta N$, the system can be represented as follows:



$$\begin{cases} \dot{x} = Ax + Lw \\ z = Nz \\ w = \Delta x \end{cases}$$

Theorem 8 (Quadratic stability for norm bounded uncertain systems)**.** *Given a system $\Gamma : (A_{un}, B, C, D)$, with $A_{un} = A + L\Delta N$, A stable (with eigenvalues with strictly negative real part) and $||\Delta|| \leq \alpha$, it is quadratically stable if and only if:*

$$||N(sI - A)^{-1}L||_\infty \leq \alpha^{-1}$$

Proof. First, we observe that the following inequality holds:

$$(A + L\Delta N)'P + P(A + L\Delta N) =$$
$$= A'P + PA + N'\Delta'\Delta N + PLL'P - (N'\Delta' - PL)(\Delta N - L'P)$$
$$\leq A'P + PA + PLL'P + N'N\alpha^2 = \alpha^2(A'X + XA + \frac{XLL'X}{\alpha^{-2}} + N'N)$$

where $P\alpha^2 = X$. Hence, if $\exists X > 0$ satisfying previous equation, then $A+L\Delta N$ is asymptotically stable $\forall\Delta$, $||\Delta|| \leq \alpha$, with the same Lyapunov function (quadratic stability). The inequality

$$A'X + XA + \frac{XLL'X}{\alpha^{-2}} + N'N < 0$$

is true if $||N(sI - A)^{-1}L||_\infty < \alpha^{-1}$. In conclusion, we have proven that the condition $||N(sI - A)^{-1}L||_\infty < \alpha^{-1}$ implies that the system is quadratically stable and, in particular, that the system is robustly stable i.e. stable $\forall\Delta$ in the set $||\Delta|| \leq \alpha$. Hence, $\forall||\Delta|| \leq \alpha$ it results:

$$det\Big[I - \Delta' G(-s)'G(s)\Delta\Big] \neq 0, \ \ Re(s) \geq 0$$

Now, assume by contradiction that $||G(s)||_\infty \geq \alpha_{-1}$, i.e. $\exists b$ such that

$$\lambda_{max}\Big(I - \alpha G(-jb)'G(jb)\alpha\Big) \leq 0 \qquad (\star)$$

since $\lambda_{max}\Big(I - \alpha G(-\infty)'G(\infty)\alpha\Big) = 1 > 0$, we have that $\exists s = j\omega$ that violates $(\star)$, thus a contradiction. □

2.4.1 Design

Given system Σ^ϵ proposed in equation 2.6 we can notice that it is unstable, thus stabilization is required. The control law will be $u = K\underline{x}$. Now, in order to represent the uncertainty, dividing state matrix of $\Sigma_n : A_n = \hat{A_n} + L_n\Delta_n N_n$ we have that from matrix A:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bgmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b(J+mL^2)}{q} \end{bmatrix}$$

defining $\nu \to 0^+$, we obtain:

$$\hat{A}_n = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \nu & \frac{mgL(M+m)}{q} & \nu \end{bmatrix}$$

$$L_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Delta_n = b \qquad N_n = \begin{bmatrix} 0 & \frac{gmL}{q} & 0 & -\frac{(J+mL^2)}{q} \end{bmatrix}$$

Now, dividing also stability matrix of Σ^ϵ, we have that starting from:

$$A^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{bgmL\epsilon}{q} & \frac{bgmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b(J+mL^2)}{q} \end{bmatrix}$$

we obtain:

$$\Sigma^\epsilon : \ \ A^\epsilon = \hat{A}^\epsilon + L^\epsilon \Delta^\epsilon N^\epsilon$$

where $\hat{A}^\epsilon$ is unstable. Matrices $\hat{A}^\epsilon$, L^ϵ, Δ^ϵ and N^ϵ take the following form:

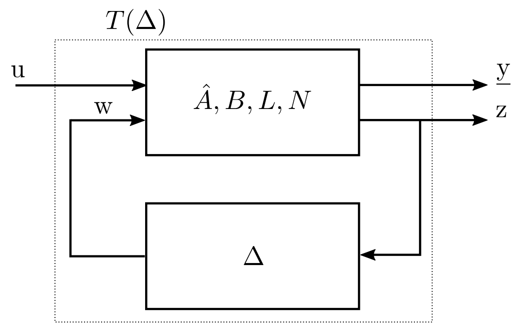
$$\hat{A}^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \nu & \nu & \frac{mgL(M+m)}{q} & \nu \end{bmatrix}$$

$$L^\epsilon = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Delta^\epsilon = b_{max} \qquad N^\epsilon = \begin{bmatrix} \frac{gmL\epsilon}{q} & \frac{gmL}{q} & 0 & -\frac{(J+mL^2)}{q} \end{bmatrix}$$

So, system $\Sigma^\epsilon : (A^\epsilon, B^\epsilon, C^\epsilon, D^\epsilon)$, $A^\epsilon = \hat{A}^\epsilon + L^\epsilon \Delta^\epsilon N^\epsilon$, can now be formalized as:

$$T^\epsilon(s, \Delta) = \begin{cases} \underline{\dot{x}} = \hat{A}^\epsilon \underline{x} + B^\epsilon u + L^\epsilon w \\ \underline{y} = C^\epsilon \underline{x} \\ z = N^\epsilon \underline{x} \\ w = \Delta^\epsilon z \end{cases} \tag{2.9}$$

Where system scheme is the following:



Where, with respect to the general formulation of the augmented plant:

$$\begin{cases} \dot{\underline{x}} = A\underline{x} + B_1 w + B_2 u \\ \underline{z} = C_1\underline{x} + D_{12}u \\ \underline{y} = C_2\underline{x} + D_{21}w \end{cases} \tag{2.10}$$

we have the following relation:

$$\begin{array}{lll} A = \hat{A}^\epsilon & B_1 = L^\epsilon & B_2 = B^\epsilon \\ C_1 = N^\epsilon & D_{12} = 0 & \\ C_2 = -C^\epsilon & D_{21} = -D^\epsilon & \end{array} \tag{2.11}$$

Once the controller will be designed, will be tested plugging it to the real system $\Sigma_n : (A_n, B_n, C_n, D_n)$

H_2 Controller

Given previous analysis, in accordance to proposition 2 and to Riccati equation:

$$A'P + PA + C'C = 0$$

By remembering that the control action is $u = K\underline{x}$, we get:

$$\left(\hat{A}^\epsilon + B_2K\right)'P + P\left(\hat{A}^\epsilon + B_2K\right) + \left(C_1 + D_{12}K\right)'\left(C_1 + D_{12}K\right) = 0$$

As in previous project, after some simple algebraic computation we obtain the following equation:

$$\begin{aligned} &\hat{A}^{\epsilon\prime}P + P\hat{A}^\epsilon + C_1'C_1 + \\ &\left[K' + \left(C_1'D_{12} + PB_2\right)\left(D_{12}'D_{12}\right)^{-1}\right]\left(D_{12}'D_{12}\right)\left[K + \left(D_{12}'D_{12}\right)^{-1}\left(D_{12}'C_1 + B_2'P\right)\right] \\ &\qquad - \left(C_1'D_{12} + PB_2\right)\left(D_{12}'D_{12}\right)^{-1}\left(C_1'D_{12} + PB_2\right)' = 0 \end{aligned}$$

Where the minimizing K is:

$$K^o = -\left(D_{12}'D_{12}\right)^{-1}\left(D_{12}'C_1 + B_2'P\right) \tag{2.12}$$

obtaining following Riccati equation:

$$\hat{A}^{\epsilon\prime}P + P\hat{A}^{\epsilon} + C_1'C_1 - \Big(C_1'D_{12} + PB_2\Big)\Big(D_{12}'D_{12}\Big)^{-1}\Big(C_1'D_{12} + PB_2\Big)' = 0$$

Now, substituting previous equation with matrices defined in 2.11, we get:

$$\begin{cases} \hat{A}^{\epsilon\prime}P + P\hat{A}^{\epsilon} + N^{\epsilon\prime}N^{\epsilon} - \Big(N^{\epsilon\prime}D_{12} + PB^{\epsilon}\Big)\Big(D_{12}'D_{12}\Big)^{-1}\Big(N^{\epsilon\prime}D_{12} + PB\Big)' = 0 \\ K^o = -\Big(D_{12}'D_{12}\Big)^{-1}\Big(D_{12}'N^{\epsilon} + B^{\epsilon\prime}P\Big) \end{cases} \tag{2.13}$$

We can now notice that in previous equation 2.13, a necessary condition for the existence of a solution is $\det(D_{12}'D_{12}) \neq 0$. Since in our model $D_{12} = 0$, I modify this matrix to:

$$D_{12} = \epsilon \tag{2.14}$$

With the small adjustment of previous equation 2.14 now exists a stabilizing controller so the following equations hold:

$$T^{\epsilon}(s) = T^{\epsilon}(s, \Delta^{\epsilon}, K^o) = \begin{cases} \dot{\underline{x}} = (\hat{A}^{\epsilon}\underline{x} + B^{\epsilon}K^o)\underline{x} + L^{\epsilon}w \\ \underline{y} = C^{\epsilon}\underline{x} \\ z = N^{\epsilon}\underline{x} \\ w = \Delta^{\epsilon}z \end{cases} \tag{2.15}$$

where:

$$\begin{cases} ||T_{zw}(s)||_2^2 = trace(L^{\epsilon\prime}PL^{\epsilon}) \\ u = K^o\underline{x} \\ \hat{A}^{\epsilon\prime}P + P\hat{A}^{\epsilon} + N^{\epsilon\prime}N^{\epsilon} - \Big(N^{\epsilon\prime}D_{12} + PB^{\epsilon}\Big)\Big(D_{12}'D_{12}\Big)^{-1}\Big(N^{\epsilon\prime}D_{12} + PB\Big)' = 0 \\ K^o = -\Big(D_{12}'D_{12}\Big)^{-1}\Big(D_{12}'N^{\epsilon} + B^{\epsilon\prime}P\Big) \end{cases} \tag{2.16}$$

So, attaching the controller to real system Σ_n, we have that it is now stable and the new state space is:

$$\Sigma_n : (\hat{A}_n + BK^o + L\Delta N, B, C, D)$$

where stability matrix of Σ_n can now be renamed as:

$$\begin{aligned} A_n &= \hat{A}_n + BK^o + L_n\Delta_n N_n \\ &= \tilde{A}_n + L_n\Delta_n N_n \end{aligned} \tag{2.17}$$

where $\tilde{A}_n = \hat{A}_n + B_nK^o$. In addition, the H_2 norm of the system is:

$$||\Sigma_n||_2^2 = trace(B'PB)$$

Now, as stated in theorem 8, the system is quadratically stable if and only if:

$$||N_n(sI - \tilde{A}_n)^{-1}L_n||_{\infty} \leq \alpha^{-1}$$

where α is chosen as $\alpha = \Delta^{\epsilon}$ (upper bound for the uncertainty). This property will be verified in following section entitled "Numerical Results".

H_∞ Controller

As introduced in section 2.4.1, the general formulation of the augmented plant is:

$$\begin{cases} \underline{\dot{x}} = A\underline{x} + B_1 w + B_2 u \\ \underline{z} = C_1\underline{x} + D_{12} u \\ \underline{y} = C_2\underline{x} + D_{21} w \end{cases}$$

(as written in equation 2.10) whereas the relation of matrices of our case of study is (as written in equation 2.11):

$$\begin{aligned} A &= \hat{A}^\epsilon \qquad B_1 = L^\epsilon \qquad B_2 = B^\epsilon \\ C_1 &= N^\epsilon \qquad D_{12} = 0 \\ C_2 &= -C'^\epsilon \qquad D_{21} = -D^\epsilon \end{aligned}$$

In accordance to what is stated by *small gain theorem* 6 and by *quadratic stability for norm bounded uncertain system theorem* 8, in order to prove the quadratic stability property it is necessary to verify that:

$$||N(sI - A)^{-1}L||_\infty \leq \alpha^{-1}$$

where $||\Delta||_\infty \leq \alpha$ by definition. Hence, since $||\Delta||_\infty = b_{max}$, we can set

$$\alpha = b_{max} \implies \gamma = \alpha^{-1}$$

so to find a controller such that:

$$||T_{zw}(s)||_\infty < \gamma$$

Similarly to what has been done for previous controller, considering now the non formal proof of bounded real lemma 5 it is possible to write:

$$A'P + PA + \Big(PB + C'D\Big)\Big(\gamma^2 I - D'D\Big)^{-1}\Big(B'P + DC'\Big) + C'C = 0$$

Where the desired control law is $u = K\underline{x}$. Thus, substituting the matrices of the general formulation of the augmented plant, we have:

$$\begin{aligned} \Big(\hat{A}^\epsilon + B_2K\Big)'P + P\Big(\hat{A}^\epsilon + B_2K\Big) + \Big(C_1 + D_{12}K\Big)'\Big(C_1 + D_{12}K\Big) \\ + \Big(PB_1 + C_1'D_{12}\Big)\Big(\gamma^2 I - D_{12}'D_{12}\Big)^{-1}\Big(B_1'P + D_{12}C_1'\Big) = 0 \end{aligned}$$

And, as it has been done for H_2 controller, after some algebraic computation we obtain:

$$\begin{aligned} &\hat{A}^{\epsilon\prime}P + P\hat{A}^\epsilon + C_1'C_1 + \\ &\Big(PB_1 + C_1'D_{12}\Big)\Big(\gamma^2 I - D_{12}'D_{12}\Big)^{-1}\Big(B_1'P + D_{12}C_1'\Big) + \\ &\Big[K' + \Big(C_1'D_{12} + PB_2\Big)\Big(D_{12}'D_{12}\Big)^{-1}\Big]\Big(D_{12}'D_{12}\Big)\Big[K + \Big(D_{12}'D_{12}\Big)^{-1}\Big(D_{12}'C_1 + B_2'P\Big)\Big] \\ &- \Big(C_1'D_{12} + PB_2\Big)\Big(D_{12}'D_{12}\Big)^{-1}\Big(B_2'P + D_{12}C_1'\Big) = 0 \end{aligned}$$

Where, as in previous case, the minimizing gain K is:

$$K^o = -(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1)$$

By attaching it to previous equation we obtain:

$$\begin{aligned}
&\hat{A}^{\epsilon\prime}P + P\hat{A}^{\epsilon} + C_1'C_1 \\
&\qquad + \left(C_1'D_{12} + PB_1\right)\left(\gamma^2 I - D_{12}'D_{12}\right)^{-1}\left(B_1'P + D_{12}C_1'\right) \\
&\qquad\qquad - \left(C_1'D_{12} + PB_2\right)\left(D_{12}'D_{12}\right)^{-1}\left(D_{12}'C_1 + B_2'P\right) = 0
\end{aligned}$$

As for previous controller, in order to find a solution it is necessary to modify D_{12} to $D_{12} = \epsilon$. Now, substituting the matrices of the real system defined in 2.11 we obtain the following optimal gain:

$$K^o = -(D_{12}'D_{12})^{-1}(B^{\epsilon'}P + D_{12}'N^{\epsilon}) \tag{2.18}$$

and the following Riccati equation:

$$\begin{aligned}
&\hat{A}^{\epsilon\prime}P + P\hat{A}^{\epsilon} + N^{\epsilon'}N^{\epsilon} \\
&\qquad + \left(N^{\epsilon'}D_{12} + PL^{\epsilon}\right)\left(\gamma^2 I - D_{12}'D_{12}\right)^{-1}\left(L^{\epsilon'}P + D_{12}N^{\epsilon'}\right) \\
&\qquad\qquad - \left(N^{\epsilon'}D_{12} + PB^{\epsilon}\right)\left(D_{12}'D_{12}\right)^{-1}\left(D_{12}'N^{\epsilon} + B^{\epsilon'}P\right) = 0
\end{aligned} \tag{2.19}$$

With this small modification, the controlled augmented plant can now be written as:

$$T(s) = T(s, \Delta, K^o) = \begin{cases} \dot{\underline{x}} = (\hat{A}\underline{x} + BK^o)\underline{x} + Lw \\ \underline{y} = C\underline{x} \\ z = N\underline{x} \\ w = \Delta z \end{cases} \tag{2.20}$$

where:

$$\begin{cases} ||T_{zw}(s)||_{\infty} < \gamma \\ P \ : \ \textit{equation } 2.19 \\ u = K^o\underline{x} \\ K^o = -(D_{12}'D_{12})^{-1}(B^{\epsilon'}P + D_{12}'N^{\epsilon}) \end{cases} \tag{2.21}$$

The system is now stable and the original state space can be rewritten as:

$$\Sigma_n : (\hat{A}_n + B_nK^o + L_n\Delta_nN_n,\ B_n,\ C_n,\ D_n)$$

As before, defining $\tilde{A} = \hat{A}_n + B_nK^o$, in order to verify if the system is quadratically stable we need to test:

$$||N(sI - \tilde{A})^{-1}L||_{\infty} \leq \alpha^{-1}$$

This property will be verified in following section entitled "Numerical Results".

2.4.2 Numerical Results

Given previous analysis, setting $\nu = 1 \cdot 10^{-10}$, the matrices of Σ_n are:

$$\hat{A}_n = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 \cdot 10^{-10} & 25.5913 & 1 \cdot 10^{-10} \end{bmatrix}$$

$$L_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Delta_n = 0,5 \qquad N_n = \begin{bmatrix} 0 & 3.1989 & 0 & -0.1739 \end{bmatrix}$$

And, for completeness, the other system matrices are:

$$B_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad C_n = \begin{bmatrix} -3,1989 & 0 & 0,1739 & 0 \\ 0 & 0 & 0,3261 & 0 \end{bmatrix} \qquad D_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Considering adapted system Σ^ϵ we have:

$$\hat{A}^\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 \cdot 10^{-10} & 1 \cdot 10^{-10} & 25.5913 & 1 \cdot 10^{-10} \end{bmatrix}$$

$$L^\epsilon = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Delta^\epsilon = 0,65 \qquad N^\epsilon = \begin{bmatrix} 0.0003 & 3.1989 & 0 & -0.1739 \end{bmatrix}$$

where the other system matrices are:

$$B^\epsilon = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad C^\epsilon = \begin{bmatrix} -3.1989 & 0 & 0.1739 & 0 \\ 0 & 3,2609 \cdot 10^{-5} & 0.3261 & 0 \end{bmatrix} \qquad D^\epsilon = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In addition, for proving quadratic stability, it has been chosen:

$$\alpha = \Delta^\epsilon = 0.65$$

hence, the upper bound for the uncertain parameter b.

Numerical Results of H_2 Control

From equations 2.13 we obtain:

$$P = \begin{bmatrix} 4,786 \cdot 10^{-8} & 0,0005 & 0,0001 & -3,6406 \cdot 10^{-19} \\ 0,0005 & 4,7862 & 1,1154 & 1,4961 \cdot 10^{-8} \\ 0,0001 & 1,1154 & 0,2606 & 0,0001 \\ -3,6406 \cdot 10^{-19} & 1,4961 \cdot 10^{-8} & 0,0001 & 3,4868 \cdot 10^{-5} \end{bmatrix}$$

$$K^o = \begin{bmatrix} -3,1989 & -31990,6266 & -14961,7225 & -1747,7123 \end{bmatrix}$$

With P and K^o the poles of the controlled system Σ_n, obtained from $A + BK^o$, are:

$$p_1 = -1739,2220 \qquad p_2 = -4,2906$$
$$p_3 = -4,2865 \qquad p_4 = -0,0001$$

hence, the system is stable. Matrix $\tilde{A} = \hat{A_n} + B_n K^o$ takes the following form:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0003 & -3.1991 & -1.4936 & -0.1748 \end{bmatrix}$$

and since $L^\epsilon = B_n \implies ||T_{zw}(s)||_2^2 = trace(L^{\epsilon'} P L^\epsilon) = ||\Sigma_n||_2^2 = trace(B_n' P B_n) = 3.4868 \cdot 10^{-5}$. To verify if the system satisfies also property of *quadratic stability*, in accordance with theorem 8, the following condition must hold:

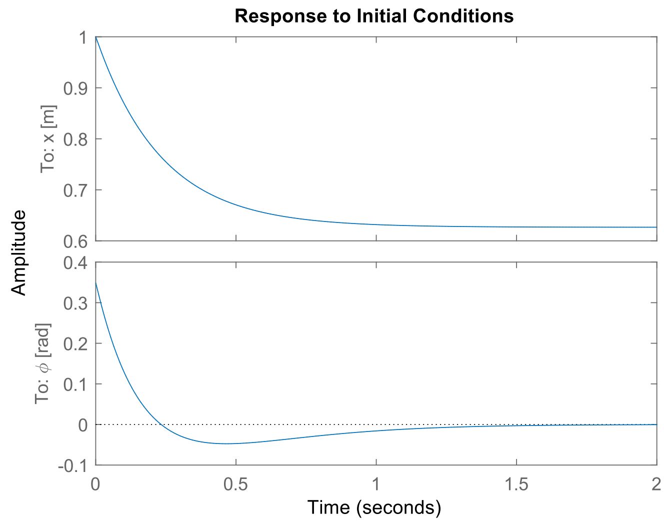
$$||N_n(sI - \tilde{A})^{-1} L_n||_\infty \leq \alpha^{-1}$$

Thus, substituting the numerical data we have:

$$||N_n(sI - \tilde{A})^{-1} L_n||_\infty = 1 \cdot 10^{-4} < \frac{1}{\alpha} = 1.5385$$

Therefore, in accordance with theorem 8, the system is also quadratically stable.

Now, defined initial state x_0 as in equation 1.21, system response is:



As it may be noticed, the system is stable but output x doesn't converge to zero. To solve this we can change Riccati equation

$$\hat{A}^{\epsilon\prime} P + P\hat{A}^{\epsilon} + N^{\epsilon\prime} N^{\epsilon} - \left(N^{\epsilon\prime} D_{12} + PB^{\epsilon}\right)\left(D_{12}' D_{12}\right)^{-1}\left(N^{\epsilon\prime} D_{12} + PB\right)' = 0$$

We give more weight to the output (thus, the error) by substituting $N^{\epsilon\prime} N^{\epsilon}$ with $C^{\epsilon\prime} C^{\epsilon}$, obtaining following relation:

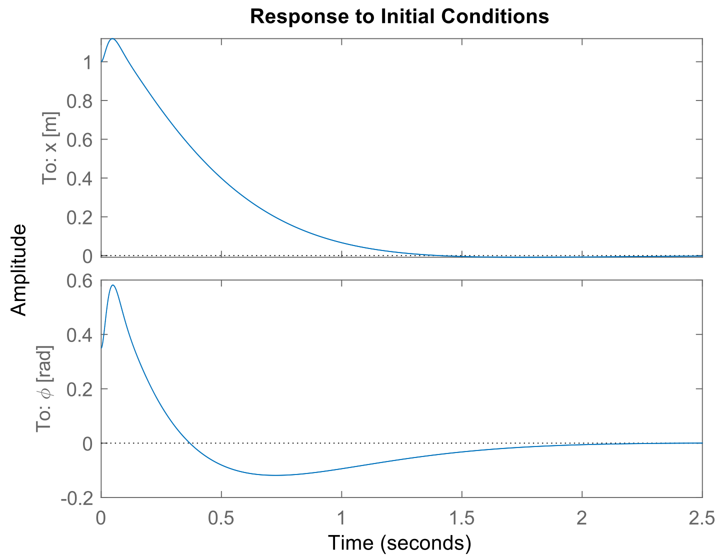
$$\begin{cases} \hat{A}^{\epsilon\prime} P + P\hat{A}^{\epsilon} + C^{\epsilon\prime} C^{\epsilon} - \left(N^{\epsilon\prime} D_{12} + PB^{\epsilon}\right)\left(D_{12}' D_{12}\right)^{-1}\left(N^{\epsilon\prime} D_{12} + PB^{\epsilon}\right)' = 0 \\ K^{o} = -\left(D_{12}' D_{12}\right)^{-1}\left(D_{12}' N^{\epsilon} + B^{\epsilon\prime} P\right) \end{cases} \tag{2.22}$$

With this new formalization, we have following results:

$$P = \begin{bmatrix} 6,2037 & 1,8805 & 0,0292 & 0,0003 \\ 1,8805 & 0,7817 & 0,0175 & -0,0001 \\ 0,0292 & 0,0174 & 0,0035 & 4,1650 \cdot 10^{-5} \\ 0,0003 & -0,0001 & 4,1650 \cdot 10^{-5} & 1,8304 \cdot 10^{-5} \end{bmatrix}$$

$$K^{o} = \begin{bmatrix} -31989,1304 & -19393,2307 & -4164,9769 & -91,2686 \end{bmatrix}$$
$$||T_{zw}(s)||_2^2 = trace(L^{\epsilon\prime} P L^{\epsilon}) = ||\Sigma_n||_2^2 = trace(B_n' P B_n) = 1.8304 \cdot 10^{-5}$$

We can notice that with this new design $||\Sigma_n||_2^2$ is lower than the previous case. System response to initial conditions are:



We can appreciate that with this design that both system outputs converge to zero. With this design system poles are:

$$p_{1,2} = -43.1550 \pm i42.8123 \qquad p_{3,4} = -2.5228 \pm i1.5141$$

where matrix $\tilde{A} = \hat{A_n} + B_n K^o$ takes the following form:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3.1989 & -1.9393 & -0.4139 & -0.0091 \end{bmatrix}$$

As before, to test the quadratic stability property, we need to test the following equation:

$$||N_n(sI - \tilde{A})^{-1} L_n||_\infty \leq \alpha^{-1}$$

And, substituting our data we have:

$$||N_n(sI - \tilde{A})^{-1} L_n||_\infty = 0.0020 < \frac{1}{\alpha} = 1.5385$$

Hence the system is asymptotically and quadratically stable. As additional confirmation of the quadratic stability property of system Σ_n we will exploit the first equation of the proof; in particular:

$$\begin{cases} \bar{P} > 0 \\ (\tilde{A} + L\Delta N)'\bar{P} + \bar{P}(\tilde{A} + L\Delta N) < 0 \end{cases}$$

And it will be verified for second controller only (to avoid redundancy). Hence, we obtain:

$$\bar{P} = \begin{bmatrix} 77.1150 & 32.1293 & 3.8810 & 0.0005 \\ 32.1293 & 27.4142 & 2.3460 & 0.0023 \\ 3.8810 & 2.3460 & 0.6695 & 0.0011 \\ 0.0005 & 0.0023 & 0.0011 & 0.0001 \end{bmatrix}$$

with $eig(\bar{P}) = (0.0001,\ 0.4347,\ 11.6604,\ 93.1035)$, thus $\bar{P} > 0$. As last, we need to check second inequality:

$$eig\big((\tilde{A} + L\Delta N)'\bar{P} + \bar{P}(A + L\Delta N)\big) = (-0.0171,\ -3.3858,\ -21.8731,\ -40.0200)$$

hence, as expected, this inequality is negative definite and thus the system is quadratically stable.

Numerical Results of H_∞ Control

In accordance with previous design, setting $\gamma = \alpha^{-1} = 1.5385$, from equations 2.18, 2.19 we obtain:

$$P = \begin{bmatrix} 4,7860 \cdot 10^{-8} & 0,0005 & 0,0001 & -3,7897 \cdot 10^{-16} \\ 0,0005 & 4,7862 & 1,1154 & 1,4958 \cdot 10^{-8} \\ 0,0001 & 1,1154 & 0,2606 & 0,0001 \\ -3,7897 \cdot 10^{-16} & 1,4958 \cdot 10^{-8} & 0,0001 & 3,4868 \cdot 10^{-5} \end{bmatrix}$$

$$K^o = \begin{bmatrix} -3,1989 & -31990,6262 & -14961,7223 & -1747,7122 \end{bmatrix}$$

Given P and K^o, the poles of the controlled system Σ_n now are:

$$p_1 = -1739,2219 \qquad p_2 = -4,2906$$
$$p_3 = -4,2865 \qquad p_4 = -0,0001$$

Matrix $\tilde{A} = \hat{A_n} + B_n K^o$ takes the following form:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0003 & -3.1991 & -1.4936 & -0.1748 \end{bmatrix}$$

As before, to test the quadratic stability property we have to check the following inequality:

$$||N_n(sI - \tilde{A})^{-1}L_n||_\infty \leq \gamma$$

Hence, substituting the values:

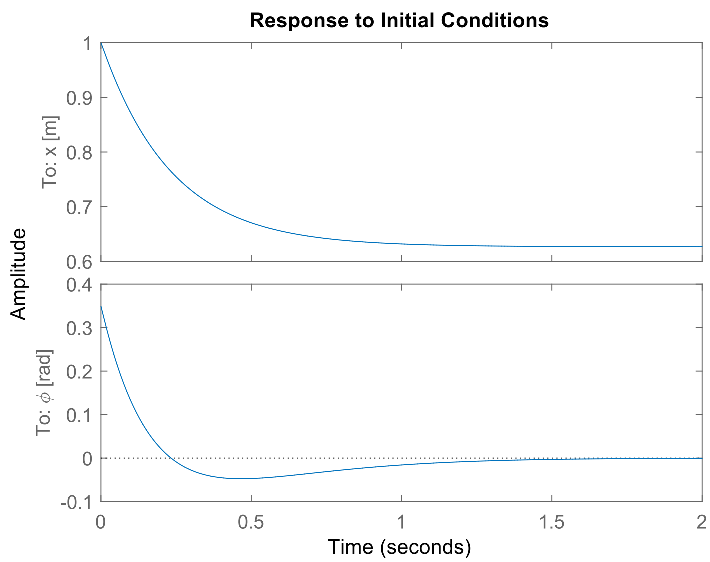
$$||N_n(sI - \tilde{A})^{-1}L_n||_\infty = 1 \cdot 10^{-4} < \gamma = 1.5385$$

Therefore, the system is quadratically stable. We could guess the system had this property by noticing that, even though the controller has been designed with H_∞ control strategy, system poles coincide with the equivalent controller for the H_2 case.

Considering now the H_2 norm of the system, we have:

$$||\Sigma_n||_2^2 = trace(B_n' P B_n) = 3.4868 \cdot 10^{-5}$$

System response is:

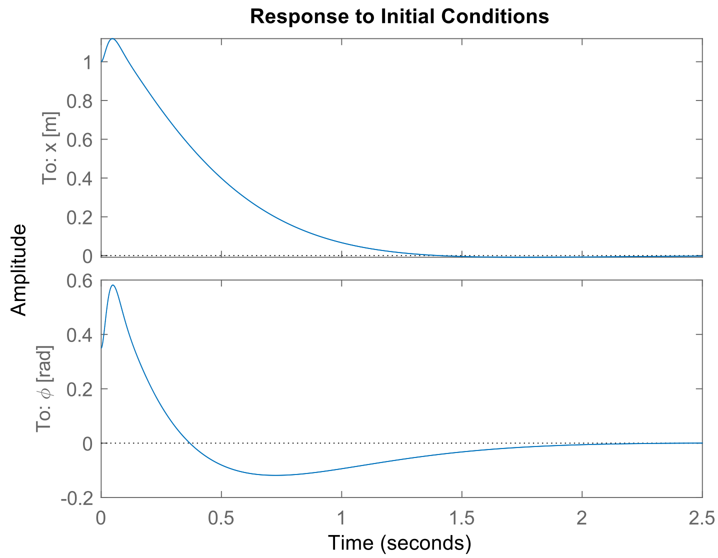


As in previous case, in order to have also output x to converge to zero it is necessary to substitute in equation 2.19 the term $N^{\epsilon'} N^{\epsilon}$ with $C^{\epsilon'} C^{\epsilon}$. By doing so, we obtain:

$$P = \begin{bmatrix} 6,2037 & 1,8805 & 0,0292 & 0,0003 \\ 1,8805 & 0,7817 & 0,0175 & -0,0001 \\ 0,0292 & 0,0174 & 0,0035 & 4,1650 \cdot 10^{-5} \\ 0,0003 & -0,0001 & 4,1650 \cdot 10^{-5} & 1,8304 \cdot 10^{-5} \end{bmatrix}$$

$$K^o = \begin{bmatrix} -31989,1304 & -19393,2307 & -4164,9769 & -91,2686 \end{bmatrix}$$

System response to initial condition x_0 now is:



We can notice that now both system outputs converges to zero. With this design, system poles are:

$$p_{1,2} = -43.1550 \pm i42.8123 \qquad p_{3,4} = -2.5228 \pm i1.5141$$

Again, to prove quadratic stability we have to check the following inequality:

$$||N_n(sI - \tilde{A})^{-1}L_n||_\infty \leq \gamma$$

Hence, substituting the values:

$$||N_n(sI - \tilde{A})^{-1}L_n||_\infty = 0.0020 < \gamma = 1.5385$$

The same considerations of before can be done also here. Since the poles of the system coincide with the second controller for H_2 case, system properties are the same (and also $||N_n(sI - \tilde{A})^{-1}L_n||_\infty$). In addition, matrix $\tilde{A} = \hat{A}_n + B_n K^o$ takes the following form:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3.1989 & -1.9393 & -0.4139 & -0.0091 \end{bmatrix}$$

The H_2 norm of the system is:

$$||\Sigma_n||_2^2 = trace(B_n' P B_n) = 1.8304 \cdot 10^{-5}$$

2.5 Project 3 - H_2 Control with Polytopic Uncertainties via LMI

In this section will be solved the problem of controlling the system considering polytopic uncertainties by making use of the LMI formulation. In order to do so, it is necessary to remember Shur Theorem:

Theorem 9 (Shur Theorem)**.** *Given matrix M such that:*

$$M = \begin{bmatrix} x & z \\ z' & y \end{bmatrix} \quad x = x\text{'},\ y = y\text{'}$$

then:

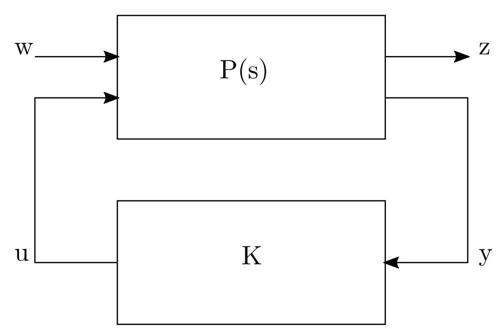
$$det(M) > 0 \iff \begin{matrix} \begin{cases} x > 0 \\ y - z'x^{-1}z > 0 \end{cases} \\ or \\ \begin{cases} y > 0 \\ x - z'y^{-1}z > 0 \end{cases} \end{matrix}$$

2.5.1 Design

Given state space $\Sigma^\epsilon : (A^\epsilon, B^\epsilon, C^\epsilon, D^\epsilon)$ and initial condition x_0 defined in section 1.9 such that:

$$\Sigma^\epsilon : \begin{cases} \dot{\underline{x}} = A^\epsilon \underline{x} + B^\epsilon u \\ \underline{y} = C^\epsilon \underline{x} + D^\epsilon u \\ \underline{x}(0) = x_0 \end{cases}$$

As done for previous cases, we can represent the system as an augmented plant with scheme:



and state space:

$$\begin{cases} \dot{\underline{x}} = A\underline{x} + B_1 w + B_2 u \\ \underline{z} = C_1 \underline{x} + D_{12} u \\ \underline{y} = C_2 \underline{x} + D_{21} w \end{cases}$$

with:

$$\begin{aligned} A = A^{\epsilon} \qquad & B_1 = x_0 \qquad B_2 = B^{\epsilon} \\ C_1 = C^{\epsilon} \qquad & D_{12} = D^{\epsilon} \\ C_2 = -C^{\epsilon} \qquad & D_{21} = -D^{\epsilon} \end{aligned}$$

where $w(t)$ can be seen as an impulse signal. The goal is to find $u = K\underline{x}$ such that:

$$u : J^o = \min \int_0^{\infty} (\underline{x}' Q \underline{x} + u' R u) dt \qquad Q \geq 0, R > 0 \tag{2.23}$$

Thus, given theorem 7 and given $b \in [b_{min}, b_{max}]$, with respect to state space Σ^{ϵ} I define:

$$A_1^{\epsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{b_{min} gmL\epsilon}{q} & \frac{b_{min} gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_{min}(J+mL^2)}{q} \end{bmatrix}$$

$$A_2^{\epsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{b_{max} gmL\epsilon}{q} & \frac{b_{max} gmL}{q} & \frac{mgL(M+m)}{q} & -\frac{b_{max}(J+mL^2)}{q} \end{bmatrix}$$

And, similarly to project 1 of section 2.3, in order to test quadratic stability we need to verify theorem 7. Given the control law and the state space, I can reformulate the latter as:

$$\begin{cases} \dot{\underline{x}} = (A + B_2 K)\underline{x} + B_1 w \\ \underline{z} = (C_1 + D_{12} K)\underline{x} \end{cases}$$

From proposition 3 it is possible write:

$$\begin{cases} ||T_{zw}||_2^2 = trace(CSC') \\ AS + SA' + BB' = 0 \end{cases}$$

Substituting the matrices we have:

$$\begin{cases} ||T_{zw}||_2^2 = trace\Big((C_1 + D_{12}K)S(C_1 + D_{12}K)'\Big) \\ (A + B_2K)S + S(A + B_2K)' + B_1 B_1' = 0 \end{cases}$$

Defining $W = KS$, modifying D_{12} to $D_{12} = \epsilon$ and distributing the terms we have:

$$\begin{cases} ||T_{zw}||_2^2 = trace\Big((C_1 S + D_{12}W)S^{-1}(C_1 S + D_{12}W)'\Big) \\ AS + SA' + B_2 W + W' B_2' + B_1 B_1' = 0 \end{cases}$$

Hence, I can write:

$$\begin{cases} A_iS + SA_i' + B_2W + W'B_2' + B_1B_1' < 0 \qquad i = 1,2 \\ (C_1S + D_{12}W)S^{-1}(C_1S + D_{12}W)' < \alpha I \\ W = KS \\ S > 0 \\ \alpha > 0 \\ \min \alpha \end{cases}$$

Therefore, taking into account polytopic uncertainties and making use of theorem 9 I can write:

$$\begin{cases} \begin{bmatrix} A_1S + SA_1' + B_2W + W'B_2' & B_2 \\ B_2' & -I \end{bmatrix} < 0 \\ \\ \begin{bmatrix} A_2S + SA_2' + B_2W + W'B_2' & B_2 \\ B_2' & -I \end{bmatrix} < 0 \\ \\ \begin{bmatrix} \alpha I & (C_1S + D_{12}W) \\ (C_1S + D_{12}W)' & S \end{bmatrix} > 0 \\ \\ W = KS \\ S > 0 \\ \alpha > 0 \\ \min \alpha \end{cases}$$

And thus, substituting real system matrices of Σ^ϵ:

$$\begin{cases} \begin{bmatrix} A_1^\epsilon S + SA_1^{\epsilon'} + B^\epsilon W + W'B^{\epsilon'} & B^\epsilon \\ B^{\epsilon'} & -I \end{bmatrix} < 0 \\ \\ \begin{bmatrix} A_2^\epsilon S + SA_2^{\epsilon'} + B^\epsilon W + W'B^{\epsilon'} & B^\epsilon \\ B^{\epsilon'} & -I \end{bmatrix} < 0 \\ \\ \begin{bmatrix} \alpha I & (C^\epsilon S + D_{12}W) \\ (C^\epsilon S + D_{12}W)' & S \end{bmatrix} > 0 \\ \\ W = KS \\ S > 0 \\ \alpha > 0 \\ \min \alpha \end{cases}$$

Hence, the optimal control is $u^o = W^o(S^o)^{-1}\underline{x}$, thus the optimal gain is:

$$K^o = W^o(S^o)^{-1} \tag{2.24}$$

With respect to the cost function 2.23 we have $Q = C^{\epsilon'}C^{\epsilon}$ and $R = D_{12}'D_{12}$. In addition, in order to test the controller performances we need to plug K^o to the real system (so to Σ_n), obtaining the following state space:

$$\Sigma_n : (A_n + B_n K^o, B_n, C_n, D_n)$$

And, with the solution previously found, as stated from theorem 2.3, the system is quadratically stable if and only if $\exists \bar{P} > 0$ such that:

$$A_{n,i}'\bar{P} + \bar{P}A_{n,i} < 0 \;\; i = 1, 2$$

where $A_{n,1} = A_n(b_{min})$ and $A_{n,2} = A_n(b_{max})$, matrix A of state space Σ_n.

2.5.2 Numerical Results

Stability matrices used for finding the controller are:

$$A_{n,1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.1196 & 25.5913 & -0.0609 \end{bmatrix}$$

$$A_{n,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.0792 & 25.5913 & -0.1130 \end{bmatrix}$$

Instead, the matrices of the adapted system Σ^{ϵ} are:

$$A_1^{\epsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0,0001 & 1,1196 & 25,5913 & -0,0609 \end{bmatrix}$$

$$A_2^{\epsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0,0002 & 2,0792 & 25,5913 & -0,1130 \end{bmatrix}$$

Given previous design and solving the LMIs we obtain:

$$W^o = \begin{bmatrix} 1,6256 \cdot 10^{-6} & -9,4068 \cdot 10^{-6} & -0,0058 & -0,4997 \end{bmatrix}$$

$$S^o = \begin{bmatrix} 8,3930 \cdot 10^{-10} & -2,0578 \cdot 10^{-12} & -6,8261 \cdot 10^{-9} & 5,0706 \cdot 10^{-12} \\ -2,0578 \cdot 10^{-12} & 6,8365 \cdot 10^{-9} & -2,1290 \cdot 10^{-11} & -1,451 \cdot 10^{-6} \\ -6,8261 \cdot 10^{-9} & -2,1290 \cdot 10^{-11} & 1,4509 \cdot 10^{-6} & -2,3581 \cdot 10^{-9} \\ 5,0706 \cdot 10^{-12} & -1,4508 \cdot 10^{-6} & -2,3581 \cdot 10^{-9} & 0,0058 \end{bmatrix}$$

And thus:

$$K^o = 1 \cdot 10^4 \begin{bmatrix} -3.1983 & -1.9392 & -0.4165 & -0.0091 \end{bmatrix}$$

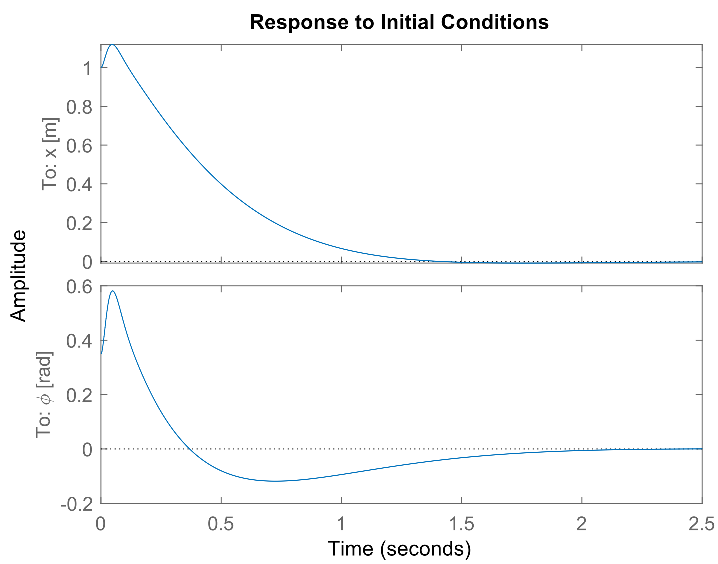
So, plugging K^o to state space Σ_n the new stability matrix takes the form $A_n + B_n K^o$ and system poles are:

$$p_{1,2} = -43.1243 \pm i42.8454 \qquad p_{3,4} = -2.5225 \pm i1.5138$$

And, the norm of the system is:

$$||T_{zw}||_2^2 = trace\Big((C_1 S + D_{12} W) S^{-1} (C_1 S + D_{12} W)'\Big) = 4.9933 \cdot 10^{-9}$$

So, given initial state x_0 previously defined, system response is:



And, to test quadratic stability we first check if $\exists \bar{P} > 0$ such that:

$$A'_{new,i} \bar{P} + \bar{P} A_{new,i} < 0 \;\; i = 1, 2$$

where $A_{new,i} = A_{n,i} + B_n K^{opt}$. Therefore, we obtain:

$$\bar{P} = \begin{bmatrix} 37.3321 & 15.6147 & 1.4615 & 0.0002 \\ 15.6147 & 13.0866 & 0.8970 & 0.0011 \\ 1.4615 & 0.8970 & 0.2435 & 0.0005 \\ 0.0002 & 0.0011 & 0.0005 & 4,5175 \cdot 10^{-5} \end{bmatrix}$$

where $eig(\bar{P}) = (4.3793 \cdot 10^{-5},\ 0.1734,\ 5.4463,\ 45.0425)$, hence $\bar{P} > 0$. Now we can check if:

$$A'_{new,i} \bar{P} + \bar{P} A_{new,i} < 0 \;\; i = 1, 2$$

And, by substituting we obtain:

$$eig(A'_{new,1}\bar{P} + \bar{P}A_{new,1}) = (-0.0066, \ -2.1945, \ -10.3645, \ -18.1427)$$
$$eig(A'_{new,2}\bar{P} + \bar{P}A_{new,2}) = (-0.0066, \ -2.1944, \ -10.3632, \ -18.1418)$$

Thus, the system is also quadratically stable.

Chapter 3

Conclusions

In this document has been proposed the formalization of the single inverted pendulum on a cart. In order to apply robust control theory it had been necessary to move the pole on the imaginary axis. Then three different state feedback controllers have been designed and implemented on the adapted system, in order to test the true state space have been used.

The controllers proposed in this document guarantee asymptotic stability and also quadratic stability with respect to the uncertain friction parameter. It's important to notice that the three different controllers solve the same problem (state feedback stabilization) and also the poles of the system are the same. In addition, with respect to the H_2/H_∞ controller for the norm bounded stabilization case, the two different approaches share the same solution. This happens because the γ defined in the problem is already big enough to transform the H_∞ control problem in an H_2 problem.

Chapter 4

Matlab Code

4.1 Transformation Matrix Code

Hereunder there is the code for obtaining the transformation matrix:

```
clear all;
close all;
clc;

M = 5;
m = 3;
L = 0.4;
l = L;
J = m*L^2/3;
I=J;
g = 9.81;
b = 0.5;
bmin = 0.7*b;
bmax = 1.3*b;
q = (M+m)*(J+m*L^2)-(m*L)^2;
p = q;

eps = 1*10^(-4);

%% State Space from differential equations

Aph = [0 1 0 0;
       0 -(I+m*l^2)*b/p m^2*g*l^2/p 0;
       0 0 0 1;
       0 -m*l*b/p m*g*l*(M+m)/p 0];
Bph = [0; (I+m*l^2)/p; 0; m*l/p];
Cph = [1 0 0 0; 0 0 1 0];
Dph = [0;0];

states = {'x' 'x_dot' 'phi' 'phi_dot'};
inputs = {'u'};
outputs = {'x'; 'phi'};

```

```
ssph = ss(Aph,Bph,Cph,Dph,'statename',states,'inputname',...
    inputs,'outputname',outputs);
Mph = [Bph Aph*Bph Aph^2*Bph Aph^3*Bph];

%% Canonical State Space
An =  [0 1 0 0;
       0 0 1 0;
       0 0 0 1;
       0 (b*g*m*L)/q m*g*L*(M+m)/q -b*(J+m*L^2)/q];
Bn = [0; 0; 0; 1];
Cn = [-m*g*L/q, 0, (J+m*L^2)/q, 0;
0, 0, m*L/q, 0];
Dn = [0;0];

states = {'x1' 'x2' 'x3' 'x4'};
inputs = {'u'};
outputs = {'x'; '\phi'};

ssCan = ss(An,Bn,Cn,Dn,'statename',states,'inputname',...
    inputs,'outputname',outputs);
Mn = [Bn An*Bn An^2*Bn An^3*Bn];

%% Transformation Matrix
T = Mn*inv(Mph);

%% Initial states
xph0 = [1;0;0.349066;0];
x0 = T*xph0

%% Open loop system response
figure(1)
initial(ssCan,x0,3)
```

4.2 Project 1 Code

Matlab code for project of section 2.3 is shown in the following:

```
clear all;
close all;
clc;

%% Data
M = 5;
m = 3;
L = 0.4;
J = m*L^2/3;
g = 9.81;
b = 0.5;
bmin = 0.7*b;
bmax = 1.3*b;
q = (M+m)*(J+m*L^2)-(m*L)^2;
p = q;
eps = 1*10^(-4);

```

```
%% Canonical State Space - Adapted (eps inserted)
Aeps1 = [0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        bmin*g*m*L*eps/q, bmin*g*m*L/q, ...
                m*g*L*(M+m)/q,-bmin*(J+m*L^2)/q];
Aeps2 = [0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        bmax*g*m*L*eps/q, bmax*g*m*L/q, ...
                m*g*L*(M+m)/q,-bmax*(J+m*L^2)/q];

Beps = [0;0;0;1];

Ceps = [-m*g*L/q, 0, (J+m*L^2)/q, 0;
        0, m*L*eps/q, m*L/q,0];

Deps = [0; 0];

Aeps = 0.5*(Aeps1+Aeps2);

%% H2 Optimal Control
% Select yhe matrices Q and R as follows
Q = Ceps'*Ceps;
R = eps^2;

P = care(Aeps,Beps,Q,R);
Ko = -inv(R)*(Beps'*P);
eigP = eig(P);

%% Plugging the controller to real system
An = [0 1 0 0;
      0 0 1 0;
      0 0 0 1;
      0 (b*g*m*L)/q m*g*L*(M+m)/q -b*(J+m*L^2)/q];
An1 = [0 1 0 0;
       0 0 1 0;
       0 0 0 1;
       0 (bmin*g*m*L)/q m*g*L*(M+m)/q -bmin*(J+m*L^2)/q];
An2 = [0 1 0 0;
       0 0 1 0;
       0 0 0 1;
       0 (bmax*g*m*L)/q m*g*L*(M+m)/q -bmax*(J+m*L^2)/q];
Bn = [0; 0; 0; 1];

Cn = [-m*g*L/q 0 (J+m*L^2)/q 0;
       0 0 m*L/q  0];
Dn = [0;0];

states = {'x1' 'x2' 'x3' 'x4'};
inputs = {'u'};
outputs = {'x [m]'; '\phi [rad]'};

ssControlled = ss(An+Bn*Ko,Bn, Cn, ...
    Dn,'statename',states,'inputname',...
inputs,'outputname',outputs);
eig(An+Bn*Ko)
```

```

%% Plotting system response
x0 = [-0.2544;0; 1.0705;0];
initial(ssControlled,x0)

%% H2 norm
H2Norm = trace(Bn'*P*Bn)

%% Quadratic Stability
% Stability Matrices
Aun1 = An1+Bn*Ko;
Aun2 = An2+Bn*Ko;

%Inequality
nstate = size(Aun1,1);
setlmis([]);    % Initialization of the LMI
P=lmivar(1, [nstate,1]);

% Subject function, LMI #1
% Aun1'P + PAun1 < 0
lmiterm([1 1 1 P], 1,Aun1 ,'s'); % LMI #1: Aun1'P + PAun1

% Subject function, LMI #2
% Aun2'P + P Aun2 < 0
lmiterm([2 1 1 P],1, Aun2,  's'); % LMI #1: Aun2'P + PAun2

% Subject function, LMI #3
% P>0
lmiterm([-3 1 1 P], 1, 1, 's'); % LMI #1: P>0

% Solving the LMI feasability problem
lmis = getlmis;
[tmin,xfeas] = feasp(lmis);
Popt = dec2mat(lmis,xfeas,P);

% Test for quadratic stability
eigPopt = eig(Popt)
quad1 = eig(Aun1'*Popt+Popt*Aun1)
quad2 = eig(Aun2'*Popt+Popt*Aun2)
```

4.3 Project 2 Code

Matlab code for project of section 2.4 is shown here.

4.3.1 H_2 Controller

There follows Matlab code for H_2 controller.

```
close all;
clear all;
clc;

M = 5;
```

```
m = 3;
L = 0.4;
J = m*L^2/3;
g = 9.81;
b = 0.5;
q = (M+m)*(J+m*L^2)-(m*L)^2;

eps = 1*10^(-4);
nu = 1*10^(-10);

Ahateps = [0 1 0 0;
           0 0 1 0;
           0 0 0 1;
           nu, nu, m*g*L*(M+m)/q, nu];

Delta_eps = b*1.3;

L_eps = [0;0;0;1];
N_eps = [g*m*L*eps/q, g*m*L/q, 0, -1*(J+m*L^2)/q];

B_eps = [0;0;0;1];

C_eps = [-m*g*L/q, 0, (J+m*L^2)/q, 0;
0, m*L*eps/q, m*L/q, 0];

D = [0; 0];

%% Real System
An = [  0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        0, (b*g*m*L)/q, m*g*L*(M+m)/q, -b*(J+m*L^2)/q];


AHatn = [0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        0, nu, m*g*L*(M+m)/q, nu];

L_n = [0;0;0;1];
Delta_n = b;
N_n = [0, (g*m*L)/q, 0, -1*(J+m*L^2)/q];

Bn = [0; 0; 0; 1];

Cn = [-m*g*L/q 0 (J+m*L^2)/q 0;
        0 0 m*L/q 0];
Dn = [0;0];
states = {'x1' 'x2' 'x3' 'x4'};
inputs = {'u'};
outputs = {'x [m]'; '\phi [rad]'};

%% ------------------- H2 Controller ----------------------- %%
AH2 = Ahateps;
B2 = B_eps;
B1 = L_eps;
C1 = N_eps;
```

```
C2 = C_eps;
D12 = eps;

P_H2 = care(AH2, B2, C1'*C1, D12'*D12, C1'*D12);
K_o = -inv(D12'*D12)*(B2'*P_H2+D12'*C1);

P_H2_v2 = care(AH2, B2, C2'*C2, D12'*D12, C1'*D12);
K_o_v2 = -inv(D12'*D12)*(B2'*P_H2_v2+D12'*C1);
%% H2 Controller - Plugging the controller to real system

ssH2v1 = ss(An+Bn*K_o,Bn, Cn, Dn,'statename',states,'inputname',...
inputs,'outputname',outputs);
eig(An+Bn*K_o)

ssH2_v2 = ss(An+Bn*K_o_v2,Bn, Cn, ...
    Dn,'statename',states,'inputname',...
inputs,'outputname',outputs);
eig(An+Bn*K_o_v2)
%% H2 Controller - Plotting H2 system response
x0 = [-0.2544;0; 1.0705;0];
figure(1);
initial(ssH2v1,x0)

figure(2);
initial(ssH2_v2,x0);

%% H2 Controller - H2 Norm
H2Tzw = trace(L_eps'*P_H2*L_eps)
H2Norm = trace(Bn'*P_H2*Bn)
H2Normv2 = trace(Bn'*P_H2_v2*Bn)

%% Quadratic Stability
s = tf('s');

quad = norm((N_n*inv(s*eye(4)-(An+Bn*K_o))*L_n),Inf)<inv(b)
quadv2 = norm((N_n*inv(s*eye(4)-(An+Bn*K_o_v2))*L_n),Inf)<inv(b)

% LMI double check for H2v2 controller -
% must be (A+LDN)'P+P(A+LDN)<0 with P>0

% Stability Matrix
Aun = AHatn+Bn*K_o_v2+L_n*Delta_n*N_n;

%Inequality
nstate = size(Aun,1);
setlmis([]);    % Initialization of the LMI
P=lmivar(1, [nstate,1]);

% Subject function, LMI #1
% Aun1'P + PAun1 < 0
lmiterm([1 1 1 P], 1,Aun ,'s'); % LMI #1: Aun1'P + PAun1


% Subject function, LMI #3
% P>0
lmiterm([-2 1 1 P], 1, 1, 's'); % LMI #1: P>0

```

```
% Solving the LMI feasability problem
lmis = getlmis;
[tmin,xfeas] = feasp(lmis);
Popt = dec2mat(lmis,xfeas,P)

% Test for quadratic stability
eigPopt = eig(Popt)
quadv2INEQ = eig(Aun'*Popt+Popt*Aun)
```

4.3.2 H_∞ Controller

There follows Matlab code for H_∞ controller.

```
close all;
clear all;
clc;

M = 5;
m = 3;
l = 0.4;
J = m*l^2/3;
g = 9.81;
b = 0.5;
q = (M+m)*(J+m*l^2)-(m*l)^2;

eps = 1*10^(-4);
nu = 1*10^(-10);

AhatEPS = [ 0 1 0 0;
            0 0 1 0;
            0 0 0 1;
            nu, nu, m*g*l*(M+m)/q, nu];

Delta_EPS = b*1.3;

L_EPS = [0;0;0;1];
N_EPS = [g*m*l*eps/q, g*m*l/q, 0, -1*(J+m*l^2)/q];

B_EPS = [0;0;0;1];

C_EPS = [-m*g*l/q, 0, (J+m*l^2)/q, 0;
          0, m*l*eps/q, m*l/q, 0];

D = [0; 0];

%% Real System
An = [  0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        0 (b*g*m*l)/q m*g*l*(M+m)/q -b*(J+m*l^2)/q];


AHatn = [   0 1 0 0;
            0 0 1 0;
            0 0 0 1;
```

```
             0 nu m*g*l*(M+m)/q nu];
L_n = [0;0;0;1];
Delta_n = b;
N_n = [0 (g*m*l)/q 0 -1*(J+m*l^2)/q];

Bn = [0; 0; 0; 1];

Cn = [-m*g*l/q 0 (J+m*l^2)/q 0;
0 0 m*l/q 0];
Dn = [0;0];
states = {'x1' 'x2' 'x3' 'x4'};
inputs = {'u'};
outputs = {'x [m]'; '\phi [rad]'};

%% ------------------- HInf Controller ----------------------- %%
AH2 = AhatEPS;
B2 = B_EPS;
B1 = L_EPS;
C1 = N_EPS;
C2 = C_EPS;
D12 = eps;

alpha = Delta_EPS;
gamma = inv(alpha);


Matr_B = [B2, B1];
Matr_R = [D12'*D12, 0; 0, inv(gamma^2-D12'*D12)];
Matr_S = [C1'*D12, C1'*D12];

P_HInf = care(AH2, Matr_B, C1'*C1, Matr_R, Matr_S);
K_o = -inv(D12'*D12)*(B2'*P_HInf+D12'*C1);

P_HInfv2 = care(AH2, Matr_B, C2'*C2, Matr_R, Matr_S);
K_ov2 = -inv(D12'*D12)*(B2'*P_HInfv2+D12'*C1);

%% Hinf Controller - Plugging the controller to real system

ssHInfv1 = ss(An+Bn*K_o,Bn, Cn, ...
    Dn,'statename',states,'inputname',...
inputs,'outputname',outputs);
eig(An+Bn*K_o)

ssHInfv2 = ss(An+Bn*K_ov2,Bn, Cn, ...
    Dn,'statename',states,'inputname',...
inputs,'outputname',outputs);
eig(An+Bn*K_ov2)

%% Hinf Controller - Plotting H2 system response
x0 = [-0.2544;0; 1.0705;0];
figure(1);
initial(ssHInfv1,x0);
figure(2);
initial(ssHInfv2,x0);

%% H2 norm
H2Tzw = trace(L_EPS'*P_HInf*L_EPS)
```

```
H2Norm = trace(Bn'*P_HInf*Bn)
H2Normv2 = trace(Bn'*P_HInfv2*Bn)

%% Quadratic Stability
s = tf('s');

quad = norm((N_n*inv(s*eye(4)-(An+Bn*K_o))*L_n),Inf)<inv(b)
quadv2 = norm((N_n*inv(s*eye(4)-(An+Bn*K_ov2))*L_n),Inf)<inv(b)
```

4.4 Project 3

The code for project of section 2.5 is shown in the following:

```
close all;
clear all;
clc;

M = 5;
m = 3;
L = 0.4;
J = m*L^2/3;
g = 9.81;
b = 0.5;
bmin = b*0.7;
bmax = b*1.3;
q = (M+m)*(J+m*L^2)-(m*L)^2;

eps = 1*10^(-4);

A1_Eps = [  0 1 0 0;
            0 0 1 0;
            0 0 0 1;
            bmin*g*m*L*eps/q, bmin*g*m*L/q, ...
                m*g*L*(M+m)/q,-bmin*(J+m*L^2)/q];
A2_Eps = [  0 1 0 0;
            0 0 1 0;
            0 0 0 1;
            bmax*g*m*L*eps/q, bmax*g*m*L/q, ...
                m*g*L*(M+m)/q,-bmax*(J+m*L^2)/q];

B_Eps = [0;0;0;1];

C_Eps = [   -m*g*L/q, 0, (J+m*L^2)/q, 0;
            0, m*L*eps/q, m*L/q,0];

D_Eps = [0; 0];

%% Real System
An = [  0 1 0 0;
        0 0 1 0;
        0 0 0 1;
        0 (b*g*m*L)/q m*g*L*(M+m)/q -b*(J+m*L^2)/q];

A1n = [ 0 1 0 0;
```

```
         0 0 1 0;
         0 0 0 1;
         0 (bmin*g*m*L)/q m*g*L*(M+m)/q -bmin*(J+m*L^2)/q];
A2n = [ 0 1 0 0;
         0 0 1 0;
         0 0 0 1;
         0 (bmax*g*m*L)/q m*g*L*(M+m)/q -bmax*(J+m*L^2)/q];

Bn = [0; 0; 0; 1];

Cn = [  -m*g*L/q 0 (J+m*L^2)/q 0;
         0 0 m*L/q 0];
Dn = [0;0];

states = {'x1' 'x2' 'x3' 'x4'};
inputs = {'u'};
outputs = {'x [m]'; '\phi [rad]'};

%% Control Matrices
D12 = eps;
Q = C_Eps'*C_Eps;
R = D12'*D12;

%% Enforcing simmetry
Q = (Q+Q.')/2;
Q_half = Q^(1/2);
Q_half=(Q_half+Q_half.')/2;

R = (R+R.')/2;
R_half = R^(1/2);
R_half=(R_half+R_half.')/2;

%% LMI configuration
nstate = size(A1_Eps,1);

% Initialization of the LMI
setlmis([]);
S=lmivar(1, [nstate,1]);
Z=lmivar(2, [1,1]);
W=lmivar(2, [1,nstate]);

% Subject function, LMI #1
% A1S + BW + SA1' + W'B' + BB' < 0
lmiterm([1 1 1 S], A1_Eps, 1, 's'); % LMI #1: A1S + SA1'
lmiterm([1 1 1 W], B_Eps, 1, 's'); % LMI #1: BW + W'B'
lmiterm([1 1 1 0],B_Eps*B_Eps'); % LMI #1: BB'

% Subject function, LMI #2
% A1S + BW + SA1' + W'B' + BB' < 0
lmiterm([2 1 1 S], A2_Eps, 1, 's'); % LMI #1: A2S + SA2'
lmiterm([2 1 1 W], B_Eps, 1, 's'); % LMI #1: BW + W'B'
lmiterm([2 1 1 0],B_Eps*B_Eps'); % LMI #1: BB'


% Subject function, LMI #3:
% [ Z Sqrt(R)W ]
% [ ] > 0
```

```
% [W'Sqrt(R) S ]
lmiterm([-3 1 1 Z], 1, 1); % LMI #2: Z
lmiterm([-3 2 1 -W], 1, R_half); % LMI #2: W'*sqrt(R)
lmiterm([-3 2 2 S], 1, 1); % LMI #2: S

% Subject function, LMI #4:
% S>0
lmiterm([-4 1 1 S], 1, 1, 's'); % LMI #4: S>0

% Create the LMI system
lmisys = getlmis;
n = decnbr(lmisys);
c = zeros(n,1);
for i=1:n
[Sj, Zj, Wj] = defcx(lmisys, i, S, Z, W);
c(i) =  trace(Zj)+trace(Q*Sj);
end

% Solving LMIs
options = [1e-20, 100, -1, 5, 1];
[copt, xopt] = mincx (lmisys, c, options);
% Results
Zopt = dec2mat(lmisys, xopt, Z);
Wopt = dec2mat(lmisys, xopt, W);
Sopt = dec2mat(lmisys, xopt, S);

% K stabilizing obtained via LMI
Klmi = Wopt*inv(Sopt);
eigSopt = eig(Sopt)

%% New System
Anew = An + Bn*Klmi;
eig(Anew)
ssNew = ss(Anew, Bn, Cn, Dn,'statename',states,'inputname',...
inputs,'outputname',outputs);
figure(1)
x0 = [-0.2544;0; 1.0705;0];
initial(ssNew,x0)

%% H2 norm
H2Norm =trace((Cn*Sopt+D12*Wopt)*inv(S)*(Cn*Sopt+D12*Wopt)')

%% Quadratic Stability
Aun1 = A1n+Bn*Klmi;
Aun2 = A2n+Bn*Klmi;

%Inequality
nstate = size(Aun1,1);
setlmis([]);    % Initialization of the LMI
P=lmivar(1, [nstate,1]);

% Subject function, LMI #1
% Aun1'P + PAun1 < 0
lmiterm([1 1 1 P], 1,Aun1 ,'s'); % LMI #1: Aun1'P + PAun1

% Subject function, LMI #2
% Aun2'P + P Aun2 < 0
```

```
lmiterm([2 1 1 P],1, Aun2,  's'); % LMI #1: Aun2'P + PAun2

% Subject function, LMI #3
% P>0
lmiterm([-3 1 1 P], 1, 1, 's'); % LMI #1: P>0

% Solving the LMI feasability problem
lmis = getlmis;
[tmin,xfeas] = feasp(lmis);
Popt = dec2mat(lmis,xfeas,P)

% Test for quadratic stability
eigP = eig(Popt)
quad1 = eig(Aun1'*Popt+Popt*Aun1)
quad2 = eig(Aun2'*Popt+Popt*Aun2)
```