RoboJackets RoboNav

URC Manipulation - From an Electrical Viewpoint

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Contents

1 Introduction

As RoboNav moves away from IGVC and towards URC, understanding of the underlying principles for manipulation would be important for the electrical team. The study of manipulation has a long history in ECE and is arguably one of the most well-studied types of robot due to the ability to model the robot and the environment it operates in. Whether the task of planning (and / or) control of the manipulators eventually falls on the electrical team or not, a good grasp of the underlying principles of manipulation would always be a handy tool at hand for any unexpected happenings.

This guide borrowed much from the book "A Mathematical Introduction to Robotic Manipulation" by Dr. Richard Murray, Dr. Zexiang Li, and Dr. S. Shankar Sastry. It is meant to provide a simplified introduction to manipulation that freshmen wouldn't need to spent more than a semester understanding the basis of manipulation. We will start from rigid body motion, the right representation for rigid body motion and how it applies to the representation and planning of a manipulator.

2 Rigid Body Motions

The study of robot kinematics and controls has its core in the study of rigid body motions [1] , and we will attempt to approach rigid body motion using linear algebra and screw theory.

Michel Chases proved that a rigid body can be moved from any position to any other by a movement consisting of:

- A movement consisting of rotation about a straight line
- followed by translation parallel to that line.

One such motion is called a **screw motion**. The time derivative version of screw motion is called a **Twist**. Screw motion and twist play the central roles in the formulation of robot kinematics.

2.1 Rotational Motion in $\mathbb{R}^3$

2.1.1 Representation

Rotation Matrices

We begin the study by considering only the rotation aspect of rigid body motion. One of the most common method to describe the orientation of coordinate frame $\mathcal{B}$ relative to inertial frame $\mathcal{A}$ is to sequentially rotate about the z-axis of $\mathcal{B}$ by α, then y-axis of $\mathcal{B}$ by β, and finally along z-axis by γ. This yields a net rotation of $R(\alpha, \beta, \gamma)$ and α, γ, β are called the ZYZ Euler angles.

We use rotation matrix to represent a, well, rotation of coordinates. There are a number of reasons we want to use matrices to represent such transformation:

[1] think that the position of each joint on the manipulator is a movement away from the last joint

- Modern computers are highly optimized for linear algebras.
- Rotation matrix is the cononical form for special orthogonal groups, which has proven properties that are useful to the representation, which we will briefly go over in section 2.1.2.

$$R_{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin \\ 0 & \sin & \cos \end{bmatrix}, R_{\mathbf{y}} = \begin{bmatrix} \cos & 0 & \sin \\ 0 & 1 & 0 \\ -\sin & 0 & \cos \end{bmatrix}, R_{\mathbf{z}} = \begin{bmatrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

A ZYZ Euler angle rotation would result in

$$R_{ab} = R_{\mathbf{z}}(\alpha) R_{\mathbf{y}}(\beta) R_{\mathbf{z}}(\gamma)$$

We use R_{ab} in the sense that it can transform coorindate points such that for point q currently attached on coordinate frame $\mathcal{A}$, and represented as $\mathbf{q}_a = [x, y, z]$. When coordinate frame $\mathcal{B}$ rotate to frame $\mathcal{B}$, point q's relative position in the frame $\mathcal{B}$ is still $\mathbf{q}_b = [x, y, z]$, since the point moved with the reference frame. However the current location of $\mathbf{q}_a = R_{ab}\mathbf{q}_b$

If point q rotates with frame $\mathcal{B}$ to a new frame $\mathcal{C}$, then

$$\mathbf{q}_a = R_{ab}\mathbf{q}_b = R_{ab}R_{bc}\mathbf{q}_c$$

There exists other types of euler angle parameterizations by using different ordered sets of rotation axes, including ZYX[2] and YZX. They avoided singularity at identity orientation, however do contain singularity at other orientations. We do not cover the details of singularity at this point, but more info at gimbal lock[3].

Quaternion

Quaternion works in a similar way that complex number works on the unit circle to represent planar rotations. They give a global parameterization of $SO(3)$ at the cost of using 4 numbers.

$$Q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

w is the scalar component and $\mathbf{q} = (x, y, z)$ being vector component, a convenient shorthand notation being $Q = (w, \mathbf{q})$. Vector component satisfies the following relationship

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = -1$$

$$\mathbf{i} \cdot \mathbf{j} = -\mathbf{j} \cdot \mathbf{i} = \mathbf{k}, \mathbf{j} \cdot \mathbf{k} = -\mathbf{k} \cdot \mathbf{j} = \mathbf{i}, \mathbf{k} \cdot \mathbf{i} = -\mathbf{i} \cdot \mathbf{k} = \mathbf{j}$$

The *conjugate* of a quaternion (recall imaginery numbers) $Q = (w, \mathbf{q})$ is $Q^* = (w, -\mathbf{q})$, and that the magnitude of a quaternion satisfies $||Q|| = Q \cdot Q^*$, and the inverse of a quaternion is

$$Q^{-1} = Q^*/||Q||^2$$

[2] roll pitch yaw

[3] `https://en.wikipedia.org/wiki/Gimbal_lock`

Unit quaternions are the subset of $Q \in \mathbb{Q}$ that $||Q|| = 1$, and typically are the only type of quaternion we operate on.

Similar to rotation matrices, if Q_{ab}, Q_{bc}, and Q_{ac} represents rotation between frame $\mathcal{AB}$, $\mathcal{BC}$, and $\mathcal{AC}$ respectively, the following equation holds.

$$Q_{ac} = Q_{ab} \cdot Q_{bc}$$

2.1.2 Understanding

Figure here

Consider that we have a inertial frame $\mathcal{A}$, and a body frame $\mathcal{B}$ that has undergone a rotation about the origin point in inertial frame and no translation. $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}$ are the coordinates of the principle axes of $\mathcal{B}$ in the inertial frame. Three vectors be vertical vector and the concatenation coordinate vectors obtain

$$R_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$$

Since the column vectors of R_{ab} are from principle axes, dot product of the column vector with itself is 1, otherwise 0, yielding

$$R^T R = RR^T = 1 \Rightarrow \det R = \pm 1$$

Since $\det R = \mathbf{x}^T(\mathbf{y} \times \mathbf{z})$ and that $\mathbf{x} = \mathbf{y} \times \mathbf{z}$ under right-handed coordinate system, $\det R = 1$.

Therefore **right-handed coordinate frame are represented by orthogonal matrices with determinant 1**. The set of all matrices in $n \times n$ dimension are denoted by $SO(n)$, as special orthogonal. $SO(3) \in \mathbb{R}^{3\times 3}$ is a **group** under matrix multiplication. A set G, with a binary operation $\circ$ defined on elements of G is considered a group if it satisfies the following axioms:

- Closure: $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$
- Identity: exists e for every $g \in G$ that $g \circ e = e \circ g = g$
- Inverse: for each $g \in G$ there exists a unique inverse, $g^{-1} \in G$ that $g \circ g^{-1} = g^{-1} \circ g = e$
- Associativity.

In the case of $SO(3)$, it satiefies the above criteria in that

- $AB = C$ where $A, B \in \mathbb{R}^3$ gives $C \in \mathbb{R}^3$ by definition.
- $IR = RI = R$.
- $R^T R = RR^T = I$.
- Associativity from matrix multiplication.

Figure 2.2 here

Consider the above scenario that $\omega \in \mathbb{R}^3$ be a unit vector specifying direction of rotation (rotation axis) and $\theta \in \mathbb{R}$ be the angle of rotation (Rad). The velocity of the point can be written as

$$\dot{q}(t) = \omega \times q(t) = \widehat{w}q(t)$$
$$w = [w_1, w_2, w_3] \Rightarrow \widehat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

$\widehat{\omega}$ is a conversion for cross product to be represented as a typical matrix multiplication, and the equation is a time-invariant linear differential that can be integrated to:

$$q(t) = e^{\widehat{\omega}t}q(0)$$
$$\therefore R(\omega, \theta) = e^{\widehat{\omega}\theta} \tag{2}$$

$R\omega, \theta$ denotes rotation about axis ω at unit velocity for ω units of time. Expand $e^{\widehat{\omega}\theta}$ using taylor expansion:

$$e^{\widehat{\omega}\theta} = I + \widehat{\omega}\theta + \frac{(\widehat{\omega}\theta)^2}{2!} + \frac{(\widehat{\omega}\theta)^3}{3!} + ... \tag{3}$$

Since $\widehat{\omega}$ are skew-symmetric matrices, denote such matrices for $\mathbb{R}^3$ as $so(3), \forall S \in so(3) : S^T = -S$. Through this property, $\forall \widehat{a} \in so(3)$:

$$\widehat{a}^2 = aa^T - ||a||^2 I$$
$$\widehat{a}^3 = -||a||^2 \widehat{a}$$

Apply the above to equation 3:

$$e^{\widehat{\omega}\theta} = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\right)\widehat{\omega} + \left(\frac{\theta^2}{2!} - \frac{\omega^4}{4!} + \frac{\omega^6}{6!} - ...\right)\widehat{\omega}^2$$

and hence

$$\boxed{e^{\widehat{\omega}\theta} = I + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1 - \cos\theta)} \tag{4}$$

Commonly known as *Rodrigue's formula*, we may relate this back to the rotation matrix in equation 1 of section 2.1.1 that

$$R_{\mathbf{x}}(\theta) = e^{\widehat{\mathbf{x}}\theta} = I + \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sin\theta + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} (1 - \cos\theta)\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin \\ 0 & \sin & \cos \end{bmatrix} \tag{5}$$

This similarly holds for rotation along other axis.

2.1.3 Application

Have people write MATLAB code that transforms coordinates between frames, rotation only.

2.2 Rigid Motion in $\mathbb{R}^3$

2.2.1 Representation

As a screw motion requires a rotation and a translation component, we denote the pair of (p_{ab}, R_{ab}) as $SE(3)$ (special euclidean group) that $p \in \mathbb{R}^3, R \in SO(3)$. Similar to the case of rotation, for q_a, q_b be coordinate of q in frame A and B,

$$q_a = p_{ab} + R_{ab} q_b \tag{6}$$

We represent transformation in equation 6 in its homogenous form that

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{7}$$

and if we append 1 to the coordinate of a point to yield the *homogeneous coordinates*

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \tag{8}$$

$$\bar{q}_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} p_{ab} + R_{ab} q_b \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = g_{ab} \bar{q}_b \tag{9}$$

Similar to rotation components, given g_{ab}, g_{ac}, g_{ac} can be given by the following equation

$$g_{ac} = g_{ab} g_{bc} = \begin{bmatrix} R_{ab} R_{bc} & R_{ab} p_{bc} + p_{ab} \\ 0 & 1 \end{bmatrix} \tag{10}$$

2.2.2 Understanding

We will first show that the set of rigid transformation is a group, and then link rigid motions and twists through exponential coordinates.

2.2.3 Application

Similar to rotation subsection.

2.3 Velocity of a Rigid Body

Adjoint Transformation, but questionable applicability.

3 Manipulator Kinematics

3.1 Forward Kinematics

Products of exponential formula 1. Uniform treatment of revolute and prismatic joints 2. Only two reference frame (base and tool) 3. Easy geometric intepretation

3.2 Inverse Kinematics

3.3 Manipulator Jacobian

4 Building A Manipulator in MATLAB