1　三角函数相关公式

$\sin 2x = 2\sin x\cos x$

$\sin 3x = -4\sin^3 x + 3\sin x$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\cot 2x = \frac{\cot^2 x-1}{2\cot x}$

$\sin^2 \frac{x}{2} = \frac{1}{2}(1-\cos x)$

$\cos^2 \frac{x}{2} = \frac{1}{2}(1+\cos x)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

$\cot(\alpha\pm\beta) = \frac{\cot\alpha\cot\beta\mp1}{\cot\beta\pm\cot\alpha}$

$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

$\cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{\sin x}{1+\cos x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

万能公式，若$u = \tan\frac{x}{2}(-\pi < x < \pi)$，则$\sin x = \frac{2u}{1+u^2}, \cos x = \frac{1-u^2}{1+u^2}$

2 泰勒展开式

$e^x = 1 + x + \frac{1}{2!}x^2$

$\ln(1+x) = x - \frac{1}{2!}x^2$

$\sin x = x - \frac{1}{3!}x^3$

$\arcsin x = x + \frac{1}{3!}x^3$

$\tan x = x + \frac{1}{3}x^3$

$\arctan x = x - \frac{1}{3}x^3$

$\cos x = 1 - \frac{1}{2!}x^2$

$\frac{1}{1-x} = 1 + x + x^2$

$\frac{1}{1+x} = 1 - x + x^2$

$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2$

3 基本求导公式

$(x^k)' = kx^{k-1}$

$(\ln x)' = \frac{1}{x} \quad (x > 0)$

$(e^x)' = e^x$

$(a^x)' = a^x \ln a \quad (a > 0, a \neq 1)$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$

$(\csc x)' = -\csc x \cot x$

$(\ln|\cos x|)' = -\tan x$

$(\ln|\sec x + \tan x|)' = \sec x$

$(\ln|\csc x - \cot x|)' = -\csc x$

$(\arcsin x)' = \frac{1}{\sqrt{1+x^2}}$

$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\operatorname{arccot} x)' = \frac{-1}{1+x^2}$

$[\ln(x + \sqrt{x^2 + a^2})]' = \frac{1}{\sqrt{x^2+a^2}}$

$[\ln(x + \sqrt{x^2 - a^2})]' = \frac{1}{\sqrt{x^2-a^2}}$

4 微分的几何应用

水平渐近线: $\lim\limits_{x\to+\infty} f(x)=y_1$和$\lim\limits_{x\to-\infty} f(x)=y_2 \Rightarrow y=y_1$和$y=y_2$

斜渐近线: $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=k_1, \lim\limits_{x\to+\infty}[f(x)-k_1x]=b_1 \Rightarrow y=k_1x+b_1$

斜渐近线: $\lim\limits_{x\to-\infty}\frac{f(x)}{x}=k_2, \lim\limits_{x\to-\infty}[f(x)-k_2x]=b_2 \Rightarrow y=k_2x+b_2$

旋转曲面的侧面积，绕x轴: $S=2\pi\int_a^b|y(x)|\sqrt{1+[y'(x)]^2}\mathrm{d}x$

$$\begin{cases} x=x(t) \\ y=y(t) \end{cases} \alpha\le t\le\beta \Rightarrow S=2\pi\int_\alpha^\beta|y(t)|\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t$$

曲率及曲率半径: $k=\frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}}, R=\frac{1}{k}$

5 基本积分公式

$\int x^k\mathrm{d}x=\frac{1}{k+1}x^{k+1}+C \quad k\ne 1$

$\int\frac{1}{x^2}=-\frac{1}{x}+C$

$\int\frac{1}{\sqrt{x}}=2\sqrt{x}+C$

$\int\frac{1}{x}=\ln|x|+C$

$\int e^x\mathrm{d}x=e^x+C$

$\int a^x\mathrm{d}x=\frac{a^x}{\ln a}+C$

$\int\sin x\mathrm{d}x=-\cos x+C$

$\int\cos x\mathrm{d}x=\sin x+C$

$\int\tan x\mathrm{d}x=-\ln|\cos x|+C$

$\int\cot x\mathrm{d}x=\ln|\sin x|+C$

$\int\sec^2 x\mathrm{d}x=\tan x+C$

$\int\csc^2 x\mathrm{d}x=-\cot x+C$

$\int\sec x\tan x\mathrm{d}x=\sec x+C$

$\int\csc x\cot x\mathrm{d}x=-\csc x+C$

$\int\frac{1}{1+x^2}\mathrm{d}x=\arctan x+C$

$\int\frac{1}{a^2+x^2}\mathrm{d}x=\frac{1}{a}\operatorname{arccot}\frac{x}{a}+C$

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \arcsin x + C$$

$$\int \tan^2 x\mathrm{d}x = \tan x - x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin \frac{x}{a} + C$$

$$\int \cot^2 x\mathrm{d}x = -\cot x + x + C$$

$$\int \sin^2 x\mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u + C$$

$$\int \cos^2 x\mathrm{d}x = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x$$

$$\int \frac{\mathrm{d}x}{\cos x} = \int \sec x\mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\int \frac{\mathrm{d}x}{\sin x} = \int \csc x\mathrm{d}x = \ln|\csc x - \cot x| + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x = \ln\left|x + \sqrt{x^2-a^2}\right| + C$$

$$\int \frac{1}{x^2-a^2}\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2-x^2}\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$$

$$\int \sqrt{a^2-x^2}\mathrm{d}x = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

$$\int_a^b f(x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)]\mathrm{d}x$$

$$\int_0^\pi xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\mathrm{d}x$$

$$\int_0^\pi xf(\sin x)\mathrm{d}x = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$$

$$\int_0^{\frac{\pi}{2}} f(\sin x, \cos x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\sin x, \cos x)\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\frac{a+b}{2} + \frac{b-a}{2}\sin t)\cdot\frac{b-a}{2}\cos x\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x = \int_0^1 (b-a)f[a+(b-a)t]\mathrm{d}t$$

$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x)+f(-x)]\mathrm{d}x (a>0)$

$$\int_{0}^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \int_{0}^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{2}{3} \cdot 1 \\ \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{cases}$$

$$\int_{0}^{\pi} \sin^n x\mathrm{d}x = \begin{cases} 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1 \\ \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \frac{\pi}{2} \end{cases}$$

$$\int_{0}^{\pi} \cos^n x\mathrm{d}x = \begin{cases} 0 \\ \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \frac{\pi}{2} \end{cases}$$

$$\int_{0}^{2\pi} \sin^n x\mathrm{d}x = \int_{0}^{2\pi} \cos^n x\mathrm{d}x = \begin{cases} 0 \\ \\ 4 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \frac{\pi}{2} \end{cases}$$

6　积分的简单应用

面积:

直角坐标系下: $S = \int_{a}^{b} |f(x) - g(x))|\mathrm{d}x$

极坐标下: $S = \int_{a}^{b} \frac{1}{2}|r_2^2(\theta) - r_1^2(\theta))|\mathrm{d}\theta$

参数方程下: $S = \int_{a}^{b} f(x)\mathrm{d}x = \int_{\alpha}^{\beta} y(t)\mathrm{d}x(t)$

旋转体体积:

绕x轴: $V_x = \int_{a}^{b} \pi y^2(x)\mathrm{d}x$

绕y轴: $V_y = \int_{a}^{b} 2\pi x|y(x)|\mathrm{d}x$(柱壳法)

平面曲线的弧长:

直角坐标系下: $s=\int_a^b\sqrt{1+[y'(x)]^2}\mathrm{d}x$

参数方程下: $s=\int_\alpha^\beta\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t$

极坐标方程下: $s=\int_\alpha^\beta\sqrt{[r'(\theta)]^2+[r'(\theta)]^2}\mathrm{d}\theta$

旋转曲面的面积 (侧面积):

直角坐标系下，绕x轴: $S=2\pi\int_a^b|y(x)|\sqrt{1+[y'(x)]^2}\mathrm{d}x$

参数方程下，绕x轴: $S=2\pi\int_\alpha^\beta|y(t)|\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t$

曲边梯形的形心公式:

$$\bar{x}=\frac{\iint\limits_D x\mathrm{d}\sigma}{\iint\limits_D \mathrm{d}\sigma}=\frac{\int_a^b\mathrm{d}x\int_0^{f(x)}x\mathrm{d}y}{\int_a^b\mathrm{d}x\int_0^{f(x)}\mathrm{d}y}=\frac{\int_a^b xf(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}$$

$$\bar{y}=\frac{\iint\limits_D y\mathrm{d}\sigma}{\iint\limits_D \mathrm{d}\sigma}=\frac{\int_a^b\mathrm{d}x\int_0^{f(x)}y\mathrm{d}y}{\int_a^b\mathrm{d}x\int_0^{f(x)}\mathrm{d}y}=\frac{\frac{1}{2}\int_a^b f^2(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}$$

平行截面面积已知的立体体积: $V=\int_a^b S(x)\mathrm{d}x$

总路程: $S=\int_{t_1}^{t_2}v(t)\mathrm{d}x$

变力沿直线做功: $W=\int_a^b F(x)\mathrm{d}x$

提取物体做功: $W=\rho g\int_a^b sA(x)\mathrm{d}x$

静水压力: $P=\int_a^b\rho gx\cdot[f(x)-h(x)]\mathrm{d}x$

细杆质心: $\bar{x}=\frac{\int_a^b x\rho(x)\mathrm{d}x}{\int_a^b\rho(x)\mathrm{d}x}$

普通对称性:

若D关于y轴对称

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \begin{cases} 2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma & f(x,y)=f(-x,y) \\ 0 & f(x,y)=-f(-x,y) \end{cases}$$

若D关于x轴对称

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \begin{cases} 2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma, & f(x,y)=f(x,-y) \\ 0 & f(x,y)=-f(x,-y) \end{cases}$$

若D关于原点对称

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \begin{cases} 2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma, & f(x,y)=f(-x,-y) \\ 0 & f(x,y)=-f(-x,-y) \end{cases}$$

若D关于$y=x$对称

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \begin{cases} 2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma, & f(x,y)=f(y,x) \\ 0 & f(x,y)=-f(y,x) \end{cases}$$

若D关于$y=a$对称

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \begin{cases} 2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma, & f(x,y)=f(x,2a-y) \\ 0 & f(x,y)=-f(x,2a-y) \end{cases}$$

若D关于$x=a$轴对称

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \begin{cases} 2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma, & f(x,y)=f(2a-x,y) \\ 0 & f(x,y)=-f(2a-x,y) \end{cases}$$

轮换对称性

若将D中的x,y对调后,D不变，则有

$$I=\iint\limits_{D} f(x,y)\mathrm{d}x\mathrm{d}y=\iint\limits_{D} f(y,x)\mathrm{d}x\mathrm{d}y$$

若$f(x,y)+f(y,x)\underset{>}{=}a$则

$$I=\frac{1}{2}\iint\limits_{D}[f(x,y)+f(y,x)]\mathrm{d}x\mathrm{d}y\underset{>}{=}\frac{1}{2}\iint\limits_{D} a\mathrm{d}x\mathrm{d}y=\frac{a}{2}S_D$$

8 微分方程

$y'=f(x)\cdot g(y)$ 型： $\Rightarrow \frac{\mathrm{d}x}{g(y)}=f(x)\mathrm{d}x\Rightarrow\int\frac{\mathrm{d}x}{g(y)}=\int f(x)\mathrm{d}x$

$y'=f(ax+by+c)$ 型： 令 $u=ax+by+c\Rightarrow u'=a+bf'(u)\Rightarrow \frac{\mathrm{d}x}{a+bf(u)}=\mathrm{d}x\Rightarrow\int\frac{\mathrm{d}x}{a+bf(u)}=\int\mathrm{d}x$

$y'=f(\frac{y}{x})$ 型： 令 $\frac{y}{x}=u\Rightarrow y=ux\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ 原方程 $x\frac{\mathrm{d}u}{\mathrm{d}x}+u=f(u)\Rightarrow\frac{\mathrm{d}u}{f(u)-u}=\frac{\mathrm{d}x}{x}\Rightarrow\int\frac{\mathrm{d}u}{f(u)-u}=\int\frac{\mathrm{d}x}{x}$

$\frac{1}{y'}=f(\frac{x}{y})$ 型： 令 $\frac{x}{y}=u\Rightarrow x=uy\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}y}=u+y\frac{\mathrm{d}u}{\mathrm{d}y}$ 原方程 $y\frac{\mathrm{d}u}{\mathrm{d}y}+u=f(u)\Rightarrow\frac{\mathrm{d}u}{f(u)-u}=\frac{\mathrm{d}y}{y}\Rightarrow\int\frac{\mathrm{d}u}{f(u)-u}=\int\frac{\mathrm{d}y}{y}$

$y'+p(x)y=q(x)$ 型：方程两边同时乘上 $e^{\int p(x)\mathrm{d}x}\Rightarrow e^{\int p(x)\mathrm{d}x}\cdot y'+e^{\int p(x)\mathrm{d}x}p(x)\cdot y=e^{\int p(x)\mathrm{d}x}\cdot q(x)\Rightarrow\left[e^{\int p(x)\mathrm{d}x}\cdot y\right]'=e^{\int p(x)\mathrm{d}x}\cdot q(x)\Rightarrow e^{\int p(x)\mathrm{d}x}\cdot y=\int e^{\int p(x)\mathrm{d}x}\cdot q(x)\mathrm{d}x+C$

得 $y=e^{-\int p(x)\mathrm{d}x}\left[\int e^{\int p(x)\mathrm{d}x}\cdot q(x)\mathrm{d}x+C\right]$

$y'+p(x)y=q(x)y^n$ 型：先变形到 $y^{-n}\cdot y'+p(x)y^{1-n}=q(x)\overset{z=y^{1-n}}{\Longrightarrow}$ 得 $\frac{\mathrm{d}z}{\mathrm{d}x}=(1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$，则 $\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x}+p(x)z=q(x)$

$y''=f(x,y')$ 型：令 $y'=p\Rightarrow y''=p'\Rightarrow\frac{\mathrm{d}p}{\mathrm{d}x}=f(x,p)$ 若解得 $p=\varphi(x,C_1)$ 即 $y'=\varphi(x,C_1)$ 则通解为 $y=\int\varphi(x,C_1)\mathrm{d}x+C_2$

$y''=f(y',y'')$ 型：令 $y'=p\Rightarrow y''=\frac{dp}{\mathrm{d}x}=\frac{\mathrm{d}p}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}p}{\mathrm{d}y}p$ 得 $p\frac{\mathrm{d}p}{\mathrm{d}y}=f(y,p)$ 若解得 $p=\varphi(y,C_1)$ 则由 $p=\frac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\varphi(y,C_1)$ 分离变量得 $\frac{\mathrm{d}y}{\varphi(y,C_1)}=\mathrm{d}x\Rightarrow\int\frac{\mathrm{d}y}{\varphi(y,C_1)}=x+C_2$

$y''+py'+qy=f(x)$ 型：

$$\left\{\begin{array}{l}\lambda^2+p\lambda+q=0\Rightarrow\lambda_1,\lambda_2\Rightarrow\text{写出齐次方程的通解}\\ \text{设特解}y''\Rightarrow\text{回代，求待定系数}\Rightarrow\text{特解}\end{array}\right.\Rightarrow\text{写出通解}$$

$y''+py'+qy=f_1(x)+f_2(x)$ 型：

$$\left\{\begin{array}{l}\text{写}\lambda^2+px+q=0\Rightarrow\text{齐次方程的通解}\\ \begin{array}{l}y''+py'+q=f_1(x)\text{写特解}y_1^*\\ y''+py'+qy=f_2(x)\text{写特解}y_2^*\end{array}\Rightarrow\text{故}y_1^*+y_2^*\text{为特解}\end{array}\right.\Rightarrow\text{通解}$$

(1) 齐次方程的通解

$p^2-4q>0,\text{即}\lambda_1\neq\lambda_2$	通解为$y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$
$p^2-4q=0,\text{即}\lambda_1=\lambda_2=\lambda$	通解为$y=(C_1+C_2x)e^{\lambda x}$
$p^2-4q<0,\text{共轭复根为}\alpha\pm\beta\mathrm{i}$	通解为$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

(2) 非齐次方程的特解

自由项$f(x)=P_n(x)e^{ax}$时, 特解$y^*=e^{ax}Q_n(x)x^k$

$$
\begin{cases}
e^{ax}\text{照抄} \\
Q_n(x)\text{为}x\text{的}n\text{次一般多项式} \\
k = \begin{cases} 0 \quad \alpha \neq \lambda_1, \alpha \neq \lambda_2 \\ 1 \quad \alpha \neq \lambda_1\text{或}\alpha \neq \lambda_2 \\ 2 \quad \alpha = \lambda_1 = \lambda_2 \end{cases}
\end{cases}
$$