

1 三角函数相关公式

$$\sin 2x = 2 \sin x \cos x$$

$$\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x)$$

$$\sin 3x = -4 \sin^3 x + 3 \sin x$$

$$\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos x)$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

$$\sin^2 z = \frac{1}{2}(1 - \cos 2z)$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$\cot \frac{x}{2} = \frac{\sin x}{1 - \cos x} = \frac{\sin x}{1 + \cos x} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

2 泰勒展开式

$$e^x = 1 + x + \frac{1}{2!}x^2$$

$$\arcsin x = x + \frac{1}{3!}x^3$$

$$\ln(1 + x) = x - \frac{1}{2!}x^2$$

$$\tan x = x + \frac{1}{3}x^3$$

$$\sin x = x - \frac{1}{3!}x^3$$

$$\arctan x = x - \frac{1}{3}x^3$$

$$\cos x = 1 - \frac{1}{2!}x^2$$

$$\frac{1}{1+x} = 1 - x + x^2$$

$$\frac{1}{1-x} = 1 + x + x^2$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2$$

3 基本求导公式

$$(x^k)' = kx^{k-1}$$

$$(\ln |\cos x|)' = -\tan x$$

$$(\ln x)' = \frac{1}{x} \quad (x > 0)$$

$$(\ln |\sec x + \tan x|)' = \sec x$$

$$(e^x)' = e^x$$

$$(\ln |\csc x - \cot x|)' = -\csc x$$

$$(a^x)' = a^x \ln a \quad (a > 0, a \neq 1)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\sin x)' = \cos x$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$$

$$(\cos x)' = -\sin x$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\tan x)' = \sec^2 x$$

$$(\operatorname{arccot} x)' = \frac{-1}{1+x^2}$$

$$(\cot x)' = -\csc^2 x$$

$$[\ln(x + \sqrt{x^2 + a^2})]' = \frac{1}{\sqrt{x^2 + a^2}}$$

$$(\sec x)' = \sec x \tan x$$

$$[\ln(x + \sqrt{x^2 - a^2})]' = \frac{1}{\sqrt{x^2 - a^2}}$$

$$(\csc x)' = -\csc x \cot x$$

4 微分的几何应用

水平渐近线: $\lim_{x \rightarrow +\infty} f(x) = y_1$ 和 $\lim_{x \rightarrow -\infty} f(x) = y_2 \Rightarrow y = y_1$ 和 $y = y_2$

斜渐近线: $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = k_1, \lim_{x \rightarrow +\infty} [f(x) - k_1x] = b_1 \Rightarrow y = k_1x + b_1$

斜渐近线: $\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = k_2, \lim_{x \rightarrow -\infty} [f(x) - k_2x] = b_2 \Rightarrow y = k_2x + b_2$

旋转曲面的侧面积, 绕 x 轴: $S = 2\pi \int_a^b |y(x)| \sqrt{1 + [y'(x)]^2} dx$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad \alpha \leq t \leq \beta \Rightarrow S = 2\pi \int_{\alpha}^{\beta} |y(t)| \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

曲率及曲率半径: $k = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}}, R = \frac{1}{k}$

5 基本积分公式

$$\int x^k dx = \frac{1}{k+1} x^{k+1} + C \quad k \neq -1$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \frac{1}{x^2} = -\frac{1}{x} + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{x}} = 2\sqrt{x} + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{x} = \ln |x| + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \operatorname{arccot} \frac{x}{a} + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\int \tan x dx = -\ln |\cos x| + C$$

$$\int \tan^2 x dx = \tan x - x + C$$

$$\int \cot x dx = \ln |\sin x| + C$$

$$\int \cot^2 x dx = -\cot x + x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int u dv = uv - \int v du + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\int \frac{dx}{\cos x} = \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \frac{dx}{\sin x} = \int \csc x dx = \ln |\csc x - \cot x| + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln (x + \sqrt{x^2 + a^2}) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$

$$\int_a^b f(x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx$$

$$\int_a^b f(x) dx = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx$$

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

$$\int_0^\pi x f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\int_0^{\frac{\pi}{2}} f(\sin x, \cos x) dx = \int_0^{\frac{\pi}{2}} f(\sin x, \cos x) dx$$

$$\int_a^b f(x) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\left(\frac{a+b}{2} + \frac{b-a}{2} \sin t\right) \cdot \frac{b-a}{2} \cos x dx$$

$$\int_a^b f(x) dx = \int_0^1 (b-a) f[a + (b-a)t] dt$$

$$\int_{-a}^a f(x) dx = \int_0^a [f(x) + f(-x)] dx (a > 0)$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{2}{3} \cdot 1 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{cases}$$

$$\int_0^\pi \sin^n x dx = \begin{cases} 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{2}{3} \cdot 1 \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{2}{3} \cdot \frac{\pi}{2} \end{cases}$$

$$\int_0^\pi \cos^n x dx = \begin{cases} 0 \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \frac{\pi}{2} \end{cases}$$

$$\int_0^{2\pi} \sin^n x dx = \int_0^{2\pi} \cos^n x dx = \begin{cases} 0 \\ 4 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \frac{\pi}{2} \end{cases}$$

6 积分的简单应用

面积:

直角坐标系下: $S = \int_a^b |f(x) - g(x)| dx$

极坐标下: $S = \int_a^b \frac{1}{2} |r_2^2(\theta) - r_1^2(\theta)| d\theta$

参数方程下: $S = \int_a^b f(x) dx = \int_\alpha^\beta y(t) dx(t)$

旋转体体积:

绕 x 轴: $V_x = \int_a^b \pi y^2(x) dx$

绕 y 轴: $V_y = \int_a^b 2\pi x |y(x)| dx$ (柱壳法)

平面曲线的弧长:

直角坐标系下: $s = \int_a^b \sqrt{1 + [y'(x)]^2} dx$

参数方程下: $s = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$

极坐标方程下: $s = \int_\alpha^\beta \sqrt{[r'(\theta)]^2 + [r(\theta)]^2} d\theta$

旋转曲面的面积 (侧面积):

直角坐标系下，绕 x 轴： $S = 2\pi \int_a^b |y(x)| \sqrt{1 + [y'(x)]^2} dx$

参数方程下，绕 x 轴： $S = 2\pi \int_\alpha^\beta |y(t)| \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$

曲边梯形的形心公式：

$$\bar{x} = \frac{\iint_D x d\sigma}{\iint_D d\sigma} = \frac{\int_a^b dx \int_0^{f(x)} x dy}{\int_a^b dx \int_0^{f(x)} dy} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}$$
$$\bar{y} = \frac{\iint_D y d\sigma}{\iint_D d\sigma} = \frac{\int_a^b dx \int_0^{f(x)} y dy}{\int_a^b dx \int_0^{f(x)} dy} = \frac{\frac{1}{2} \int_a^b f^2(x) dx}{\int_a^b f(x) dx}$$

平行截面面积已知的立体体积： $V = \int_a^b S(x) dx$

总路程： $S = \int_{t_1}^{t_2} v(t) dx$

变力沿直线做功： $W = \int_a^b F(x) dx$

提取物体做功： $W = \rho g \int_a^b s A(x) dx$

静水压力： $P = \int_a^b \rho g x \cdot [f(x) - h(x)] dx$

细杆质心： $\bar{x} = \frac{\int_a^b x \rho(x) dx}{\int_a^b \rho(x) dx}$

7 二重积分

普通对称性：

若 D 关于 y 轴对称

$$\iint_D f(x, y) d\sigma = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma & f(x, y) = f(-x, y) \\ 0 & f(x, y) = -f(-x, y) \end{cases}$$

若 D 关于 x 轴对称

$$\iint_D f(x, y) d\sigma = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma, & f(x, y) = f(x, -y) \\ 0 & f(x, y) = -f(x, -y) \end{cases}$$

若 D 关于原点对称

$$\iint_D f(x, y) d\sigma = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma, & f(x, y) = f(-x, -y) \\ 0 & f(x, y) = -f(-x, -y) \end{cases}$$

若 D 关于 $y = x$ 对称

$$\iint_D f(x, y) d\sigma = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma, & f(x, y) = f(y, x) \\ 0 & f(x, y) = -f(y, x) \end{cases}$$

若 D 关于 $y = a$ 对称

$$\iint_D f(x, y) d\sigma = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma, & f(x, y) = f(x, 2a - y) \\ 0 & f(x, y) = -f(x, 2a - y) \end{cases}$$

若 D 关于 $x = a$ 轴对称

$$\iint_D f(x, y) d\sigma = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma, & f(x, y) = f(2a - x, y) \\ 0 & f(x, y) = -f(2a - x, y) \end{cases}$$

轮换对称性

若将 D 中的 x, y 对调后, D 不变, 则有

$$I = \iint_D f(x, y) dx dy = \iint_D f(y, x) dx dy$$

若 $f(x, y) + f(y, x) \underset{>}{=} a$ 则

$$I = \frac{1}{2} \iint_D [f(x, y) + f(y, x)] dx dy \underset{>}{=} \frac{1}{2} \iint_D a dx dy = \frac{a}{2} S_D$$

8 微分方程

$$y' = f(x) \cdot g(y) \text{ 型 : } \Rightarrow \frac{dx}{g(y)} = f(x)dx \Rightarrow \int \frac{dx}{g(y)} = \int f(x)dx$$

$$y' = f(ax + by + c) \text{ 型 : } \text{令 } u = ax + by + c \Rightarrow u' = a + bf'(u) \Rightarrow$$

$$\frac{dx}{a+bf(u)} = dx \Rightarrow \int \frac{dx}{a+bf(u)} = \int dx$$

$$y' = f\left(\frac{y}{x}\right) \text{ 型 : } \text{令 } \frac{y}{x} = u \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx} \text{ 原方程}$$

$$x\frac{du}{dx} + u = f(u) \Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x} \Rightarrow \int \frac{du}{f(u)-u} = \int \frac{dx}{x}$$

$$\frac{1}{y'} = f\left(\frac{x}{y}\right) \text{ 型 : } \text{令 } \frac{x}{y} = u \Rightarrow x = uy \Rightarrow \frac{dx}{dy} = u + y\frac{du}{dy} \text{ 原方程}$$

$$y\frac{du}{dy} + u = f(u) \Rightarrow \frac{du}{f(u)-u} = \frac{dy}{y} \Rightarrow \int \frac{du}{f(u)-u} = \int \frac{dy}{y}$$

$$y' + p(x)y = q(x) \text{ 型 : } \text{方程两边同时乘上 } e^{\int p(x)dx} \Rightarrow e^{\int p(x)dx} \cdot y' + e^{\int p(x)dx} p(x) \cdot y = e^{\int p(x)dx} \cdot q(x) \Rightarrow \left[e^{\int p(x)dx} \cdot y \right]' = e^{\int p(x)dx} \cdot q(x) \Rightarrow e^{\int p(x)dx} \cdot y = \int e^{\int p(x)dx} \cdot q(x)dx + C$$

$$\text{得 } y = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} \cdot q(x)dx + C \right]$$

$$y' + p(x)y = q(x)y^n \text{ 型 : } \text{先变形到 } y^{-n} \cdot y' + p(x)y^{1-n} = q(x) \xrightarrow{z=y^{1-n}} \text{得}$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}, \text{ 则 } \frac{1}{1-n}\frac{dz}{dx} + p(x)z = q(x)$$

$$y'' = f(x, y') \text{ 型 : } \text{令 } y' = p \Rightarrow y'' = p' \Rightarrow \frac{dp}{dx} = f(x, p) \text{ 若解得}$$

$$p = \varphi(x, C_1) \text{ 即 } y' = \varphi(x, C_1) \text{ 则通解为 } y = \int \varphi(x, C_1)dx + C_2$$

$$y'' = f(y', y'') \text{ 型 : } \text{令 } y' = p \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy}p \text{ 得 } p\frac{dp}{dy} =$$

$$f(y, p) \text{ 若解得 } p = \varphi(y, C_1) \text{ 则由 } p = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \varphi(y, C_1) \text{ 分离变量得}$$

$$\frac{dy}{\varphi(y, C_1)} = dx \Rightarrow \int \frac{dy}{\varphi(y, C_1)} = x + C_2$$

$y'' + py' + qy = f(x)$ 型 :

$$\begin{cases} \lambda^2 + p\lambda + q = 0 \Rightarrow \lambda_1, \lambda_2 \Rightarrow \text{写出齐次方程的通解} \\ \text{设特解 } y'' \Rightarrow \text{回代, 求待定系数} \Rightarrow \text{特解} \end{cases} \Rightarrow \text{写出通解}$$

$y'' + py' + qy = f_1(x) + f_2(x)$ 型 :

$$\begin{cases} \text{写 } \lambda^2 + p\lambda + q = 0 \Rightarrow \text{齐次方程的通解} \\ y'' + py' + q = f_1(x) \text{ 写特解 } y_1^* \Rightarrow \text{通解} \\ y'' + py' + qy = f_2(x) \text{ 写特解 } y_2^* \Rightarrow \text{故 } y_1^* + y_2^* \text{ 为特解} \end{cases}$$

(1) 齐次方程的通解

$$p^2 - 4q > 0, \text{ 即 } \lambda_1 \neq \lambda_2 \quad \text{通解为 } y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$p^2 - 4q = 0, \text{ 即 } \lambda_1 = \lambda_2 = \lambda \quad \text{通解为 } y = (C_1 + C_2 x) e^{\lambda x}$$

$$p^2 - 4q < 0, \text{ 共轭复根为 } \alpha \pm \beta i \quad \text{通解为 } y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

(2) 非齐次方程的特解

自由项 $f(x) = P_n(x)e^{ax}$ 时, 特解 $y^* = e^{ax}Q_n(x)x^k$

$$\begin{cases} e^{ax} \text{ 照抄} \\ Q_n(x) \text{ 为 } x \text{ 的 } n \text{ 次一般多项式} \\ k = \begin{cases} 0 & \alpha \neq \lambda_1, \alpha \neq \lambda_2 \\ 1 & \alpha \neq \lambda_1 \text{ 或 } \alpha \neq \lambda_2 \\ 2 & \alpha = \lambda_1 = \lambda_2 \end{cases} \end{cases}$$