Completely Random Measure

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1 Poisson Process

Let $(S, \mathcal{S})$ be a measurable space and Π be a random countable collection of points on S. Let counting process $N(A) = |\Pi \cap A|$ for any measurable set A. Π is a Poisson process if $N(A)$ and $N(B)$ are independent for every measurable disjoint sets A and B and $N(A)$ is Poisson distributed with mean $\mu(A)$ for a σ-finite measure μ (also called mean measure).

Let f be a measurable function from S to $\mathbb{R}$, then by the Campbell's theorem (Kingman, 1993) $\sum_{x\in\Pi} f(x)$ is absolutely convergent with probability one if and only if

$$\int_S \min(|f(x), 1|)\mu(dx) < \infty. \tag{1}$$

The Laplace functional of Poisson process for any $f \geq 0$ is then

$$\mathbb{E}_\Pi[e^{-\sum_{x\in\Pi} f(x)}] = \exp\left\{ -\int_S (1 - e^{-f(x)})\mu(dx)\right\}. \tag{2}$$

2 Completely Random Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space, $(M(S), \mathcal{B})$ be the space of all σ-finite measures on $(S, \mathcal{S})$. A completely random measure (CRM) Λ on $(S, \mathcal{S})$ is a measurable function from Ω to $M(S)$ [1] such that

1. $\mathbb{P}(\Lambda(\emptyset) = 0)$
2. For any disjoint countable collection of sets A_i, the random variable $\Lambda(A_i)$ are independent, and $\Lambda(\cup A_i) = \sum_i \Lambda(A_i)$ a.s. (also known as independent increments)

CRMs with random masses at random locations can be represented as $\Lambda = \sum_{i=1}^\infty w_i \delta_{x_i}$ where x_i is a location and w_i is a mass on that location.

[1] This corresponds to a measure-theoretic definition of a random variable. So one can define a probability over a set random measures $\mathbb{P}(\Lambda^{-1}(A))$ where $A \in \mathcal{B}$. However, in the rest of the paper, we use $\Lambda(A)$ as a measure on S where $A \in \mathcal{S}$.

2.1 Completely Random Measure and Poisson Process

The most important characteristic of CRM is its relation to the Poisson process. For any CRM Λ on $(S, \mathcal{S})$ without any deterministic component, there is a corresponding Poisson Process Π on $(\mathbb{R}_+ \times S, \mathcal{B}_{\mathbb{R}_+} \times \mathcal{S})$[2] such that

$$\Lambda(A) = \sum_{(w,x)\in\Pi} w\mathbf{1}_{[x\in A]} = \int_{\mathbf{R}\times A} w\Pi(dw, dx). \tag{3}$$

Let $\nu(dw, dx)$ be a mean measure of Poisson Process Π, From the Campbell's theorem, one can easily derive the Laplace transform of $\Lambda(A)$ in $t \geq 0$ for a measurable set A:

$$\mathbb{E}_\Lambda[e^{-t\Lambda(A)}] = \mathbb{E}_\Pi[e^{-t\int_{\mathbf{R}\times A} w\Pi(dw,dx)}] \tag{4}$$

$$= \exp\left(- \int_{\mathbb{R}_+\times A} (1 - e^{-tw})\nu(dw, dx)\right), \tag{5}$$

which is derived from Laplace functional of Poison process where $f(w, x) = tw$. If the mean measure $\nu(dw, dx) = \rho(dw)H_0(dx)$ where ρ and H_0 is both σ-finite measures, then Λ is known as homogeneous CRM, or if $\nu(dw, dx) = \rho(dw|dx)H_0(dx)$ then this is non-homogeneous CRM which implies that the masses (w_i) of atoms in Λ are dependent on the locations. In homogeneous case, the masses are independent of the locations and are distributed according to a Poisson process over $\mathbb{R}_+$ with mean intensity ρ, while the locations are i.i.d. from H_0. In practice, H_0 is usually referred as a base distribution (or base measure) which has some parametric probability density on S (e.g. Gaussian distribution on $\mathbb{R}$).

By using some known properties about Poisson process, we can also deduce some known properties of CRM. For example, the expected number of points on $\mathbb{R}_+ \times S$ is computed as

$$\mathbb{E}_\Pi[\Pi(\mathbb{R}_+ \times S)] = \int_{\mathbb{R}_+\times S} \nu(dw, dx). \tag{6}$$

Sometimes (in most of the useful cases), the expected number of points might be diverge (i.e. $\mathbb{E}[\Pi(\mathbb{R}_+ \times S)] = \infty$, a.s.), however, even in this case the total mass of CRM $\Lambda(S)$ could be positive and finite with probability one if the following condition is satisfied[3]:

$$\int_{\mathbb{R}_+\times A} (1 - e^{-w})\rho(dw)H_0(S) < \infty. \tag{7}$$

If the above two conditions are satisfied, then Λ has an infinite number of atoms (again, which corresponds to the expected number of points). This property is also important to construct a normalised random measure (NRM); since the total mass of CRM is positive and finite almost surely, one can construct a NRM through the normalisation of CRM.

3 Special Case

CRM shows different characteristics based on the choice of intensity on weights $\rho(dw)$.

[2]Unlike Section 1, now the Poisson process is on the product space where each point corresponds to a pair (w, x)

[3]This condition is from the Laplace transform of $\lambda(S)$ where $t = 1$ so that the exponent of Laplace transform does not diverge.

3.1 Generalised Gamma Process

The Lévy intensity measure ρ of the generalised Gamma process (GGP) is

$$\rho_{\alpha,\sigma,\tau}(dw) = \frac{\alpha}{\Gamma(1-\sigma)} w^{-\sigma-1} e^{-\tau w} dw \tag{8}$$

GGP encompasses several well-known processes based on the different configuration on parameter σ and τ:

- Finite activity case: $\int_w \rho_{\alpha,\sigma,\tau}(dw) < \infty$
  - $(\sigma \leq 0, \tau > 0)$: weights w_i are i.i.d. from Gamma($-\sigma$,τ).
- Infinite activity case : $\int_w \rho_{\alpha,\sigma,\tau}(dw) = \infty$
  - $(\sigma = 0, \tau > 0)$: the Gamma process. Normalised Gamma process = Dirichlet process.
  - $(\sigma = \frac{1}{2}, \tau > 0)$: the inverse-Gaussian process.
  - $(\sigma \in (0,1), \tau = 0)$: the stable process

Sum of weights from GGP: As we saw in the previous section, for some intensity measure, the total mass $\Lambda(S)$ is finite a.s. If we consider $\Lambda(S)$ as a random variable, then the Laplace transform of the variable is

$$\mathbb{E}[e^{-t\Lambda(S)}] = \exp\left\{ -\int_{\mathbb{R}_+} (1 - e^{-tw})\rho_{\alpha,\sigma,\tau}(dw) \right\} = \exp\left\{ -\frac{\alpha}{\sigma}\big((t+\tau)^{\sigma} - \tau^{\sigma}\big) \right\}, \tag{9}$$

which corresponds to the Laplace transform of the exponentially tilted stable distribution where the exact sampler exists (Devroye, 2009; Hofert, 2011). Taking the derivative of t and set $t = 0$ shows the expected sum of weights is $\alpha\tau^{\sigma-1}$.

3.1.1 Gamma Process

If $\sigma = 0$, then the GGP will be the Gamma process of which normalisation is well known Dirichlet process.

The expected number of partitions $\mathbb{E}[N_k] = \sum_{i=1}^{n} \frac{\alpha}{\alpha+i-1} = \alpha(\Psi(\alpha+n) - \Psi(\alpha)) = O(\alpha \log n)$, where Ψ is a digamma function.

3.2 Beta Process

$$\nu(dw, dx) = \alpha w^{-1}(1-w)^{\alpha-1} dw H_0(dx) \tag{10}$$

4 Auxiliary

- Stirling's approximation $\Gamma(n+1) \approx \sqrt{2\pi n}(n/e)^n$

References

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