# THE LATTICE POINTS OF AN $n$-DIMENSIONAL TETRAHEDRON

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In this paper we consider the problem of determining the number of lattice points inside or on the boundary of the $n$-dimensional simplex or "tetrahedron" bounded by the $n$ coördinate hyperplanes

$$x_1 = 0, x_2 = 0, \cdots, x_n = 0$$

and the hyperplane

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n = \lambda,$$

where $\omega_i$ are positive and $\lambda$ is a non-negative parameter. Points on the boundary are given the same weight as interior points. The total number of such points we denote by

$$N_n = N_n(\lambda) = N_n(\lambda \mid \omega_1, \omega_2, \cdots, \omega_n).$$

In other words, $N_n$ is the number of sets $(x_1, x_2, \cdots, x_n)$ of non-negative integers for which the inequality

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n \leqq \lambda \tag{1}$$

holds. Although the right triangle case ($n = 2$) has been considered by many writers,[1] there is as yet no published account of the general problem for $n > 2$. There are a number of isolated problems, however, which have been treated from time to time and which may be considered as special cases of the higher dimensional tetrahedron. It is the purpose of this paper to present a workable method for obtaining inequalities for the function $N_n(\lambda)$.

Three special tetrahedra may be mentioned as outstanding examples: (1) the equilateral tetrahedron, (2) the "additive" tetrahedron, in which the $\omega$'s are distinct integers, and (3) the "multiplicative" tetrahedron in which the $\omega$'s are logarithms of primes.

The first of these cases is the only one in which a really simple formula for $N_n(\lambda)$ can be given, and is useful for comparing approximate formulas. This case is interesting also as being that in which $N_n(\lambda)$ has the greatest discontinuity. The other two cases, which are interesting on account of their applications, will be considered briefly in what follows.

Before considering any special tetrahedra, however, we set down a fundamental recursion formula for the general tetrahedron obtained by dissecting the tetrahedron by the parallel hyperplanes

$$x_n = k \qquad (k = 0, 1, 2, \cdots).$$

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[1] See Koksma, *Diophantische Approximationen*, Berlin, 1936, pp. 102–110.

We obtain in this way a set of $(n-1)$-dimensional tetrahedra, so that we may write

$$(2) \qquad N_n(\lambda \mid \omega_1, \cdots, \omega_n) = \sum_{k=0} N_{n-1}(\lambda - k\omega_n \mid \omega_1, \cdots, \omega_{n-1}),$$

or, what is the same thing,

$$(3) \quad N_n(\lambda \mid \omega_1, \cdots, \omega_n) - N_n(\lambda - \omega_n \mid \omega_1, \cdots, \omega_n) = N_{n-1}(\lambda \mid \omega_1, \cdots, \omega_{n-1}).$$

This exhibits $N_n(\lambda)$ as the solution of a linear difference equation in which $N_{n-1}(\lambda)$ is thought of as known.

In considering the equilateral tetrahedron there is no loss in generality in assuming that all the $\omega$'s are equal to unity.[2] Then by a simple induction on $n$ it follows easily from (2) or (3) that

$$(4) \qquad N_n(\lambda \mid 1, 1, \cdots, 1) = \binom{[\lambda] + n}{n},$$

where $[\lambda]$ is the greatest integer not exceeding $\lambda$. In this case $N_n$ is the number of "compositions" (rather than partitions) of the integers $\leqq \lambda$ as sums of not more than $n$ positive integers. The step function $N_n(\lambda)$ has, for every $n$, discontinuities whenever $\lambda$ is an integer $m$, and by (4) the jump is precisely

$$\binom{m + n - 1}{n - 1} = O(m^{n-1})$$

as $m \to \infty$. The ratio of this jump to the volume of the tetrahedron is asymptotic to $n/m$ and hence tends to zero only rather slowly.

In the case of the additive tetrahedron, in which the $\omega$'s are distinct positive integers, $N_n(\lambda \mid \omega_1, \omega_2, \cdots, \omega_n)$ is clearly the total number of partitions of the numbers $\leqq \lambda$ into the parts $\omega_1, \omega_2, \cdots, \omega_n$. For many choices of the $\omega$'s it is possible to give exact formulas for $N_n(\lambda)$ as well as certain asymptotic results when $\lambda$ is a function of $n$. This rather special topic will be dealt with in a separate note.

The most useful tetrahedron is the multiplicative one in which $\omega_k = \log p_k$, where $p_1, p_2, \cdots, p_n$ are distinct primes (especially the first $n$ primes). Its importance is due to the following observation: Let P be a property of integers preserved under multiplication. Then if $p_1, p_2, \cdots, p_n$ have this property, at least

$$N_n(\log x \mid \log p_1, \log p_2, \cdots, \log p_n)$$

integers $\leqq x$ have P. The theory of numbers abounds with instances of properties P, and it is in many of these instances that it becomes desirable to find usable upper or lower bounds for $N_n(\lambda)$.

Quite recently in a Cambridge dissertation D. C. Spencer has considered the

[2] In the general tetrahedron one could, by a trivial change in the variable $\lambda$, assume that $\omega_1 = 1$, say.

problem of the $n$-dimensional tetrahedron. As approximating function he takes the polynomial

$$R_n(\lambda \mid \omega_1, \cdots, \omega_n) = \frac{B_n^{(n)}(\lambda + \Omega \mid \omega_1, \cdots, \omega_n)}{n!\, \omega_1\omega_2 \cdots \omega_n},$$

where $\Omega$ denotes $\omega_1 + \omega_2 + \cdots + \omega_n$, and where $B_n^{(n)}(x \mid \omega_1, \cdots, \omega_n)$ is Nörlund's generalized Bernoulli polynomial,[3] and proves among other things that the difference

$$N_n(\lambda) - R_n(\lambda) \tag{5}$$

is $o(\lambda^{n-1})$ for arbitrary irrational $\omega$'s; that it is $o(\lambda^{n-1}/\log \lambda)$ for the multiplicative tetrahedron, $\omega_i = \log p_i$; and is $O((\log \lambda)^{n+\epsilon})$ for all $\epsilon > 0$ and for almost all $(\omega_1, \omega_2, \cdots, \omega_n)$.

The problem of getting actual numerical bounds for the difference (5) appears beset with grave practical difficulties, however. Even determining the sign of this difference in a particular case would seem to be nearly impossible.

The polynomial $R_n(\lambda \mid \omega_1, \cdots, \omega_n)$ is of the $n$-th degree in $\lambda$ with coefficients which are complicated symmetric functions of the $\omega$'s. For $n = 5$, for example, we have

$$\begin{aligned} 5!\,\sigma_5 R_5(\lambda \mid \omega_1, \cdots, \omega_5) &= \lambda^5 + \tfrac{5}{2}\sigma_1\lambda^4 + \tfrac{5}{3}(\sigma_1^2 - \sigma_2)\lambda^3 + \tfrac{5}{2}\sigma_1\sigma_2\lambda^2 \\ &\quad + \{\tfrac{5}{6}(\sigma_2^2 + \sigma_1\sigma_3 - \sigma_4) + \tfrac{1}{6}s_4\}\lambda \\ &\quad - \tfrac{1}{12}\{\sigma_1^3\sigma_2 - 3\sigma_2^2\sigma_1 + 5\sigma_3\sigma_2 - \sigma_1^2\sigma_3 + \sigma_1\sigma_4 - 5\sigma_5\}, \end{aligned} \tag{6}$$

where $\sigma_k$ denotes the sum of the products of the five $\omega$'s taken $k$ at a time, and $s_4$ is the sum of their 4-th powers. This polynomial will be compared with others at the end of this paper.

Upper bounds for the general tetrahedron may be found in the form $Ae^{\alpha\lambda}$ by quite another method suggested by an inequality device used by Rankin.[4]

Let $\theta_i = e^{\omega_i}$ so that $\theta_i > 1$. Further let $\epsilon > 0$. Then the inequality (1) is equivalent to

$$\theta_1^{x_1}\theta_2^{x_2} \cdots \theta_n^{x_n} \leqq e^{\lambda}.$$

Hence we can write

$$\begin{aligned} N_n(\lambda \mid \omega_1, \cdots, \omega_n) &= \sum_{x_1\omega_1 + \cdots + x_n\omega_n \leqq \lambda} 1 = \sum_{\theta_1^{x_1}\cdots\theta_n^{x_n} \leqq e^{\lambda}} 1 \\ &\leqq e^{\epsilon\lambda} \sum (\theta_1^{x_1} \cdots \theta_n^{x_n})^{-\epsilon} < e^{\epsilon\lambda} \prod_{\nu=1}^{n} (1 - \theta_\nu^{-\epsilon})^{-1} = A_\epsilon e^{\epsilon\lambda}. \end{aligned} \tag{7}$$

[3] See N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924, pp. 129–137.

[4] R. A. Rankin, *The difference between consecutive prime numbers*, London Mathematical Society Journal, vol. 13(1938), pp. 242–247.

As $\epsilon$ is made to approach zero, the product $A_\epsilon$ tends to infinity. The best results are obtained by making $\epsilon$ depend on $\lambda$ in such a way that

$$\lambda = \sum_{\nu=1}^{n} \omega_\nu(\theta_\nu^\epsilon - 1)^{-1}.$$

For example, for the equilateral tetrahedron ($\omega_i = 1$) this condition reduces to $\epsilon = \log(1 + n/\lambda)$, and we find in this case that

$$\text{(8)}\qquad N_n(\lambda \mid 1, 1, \cdots, 1) < (\lambda + n)^{\lambda+n} \lambda^{-\lambda} n^{-n} \sim (2\pi n)^{\frac{1}{2}} \binom{\lambda + n}{n} \left(1 + \frac{n}{\lambda}\right)^{-\frac{1}{2}},$$

a result too large by a factor of nearly $(2\pi n)^{\frac{1}{2}}$.

The method presented herewith consists in constructing two approximating polynomials $P_n(\lambda \mid \omega_1, \omega_2, \cdots, \omega_n)$ and $Q_n(\lambda \mid \omega_1, \omega_2, \cdots, \omega_n)$ each of degree $n$ in $\lambda$ with coefficients depending on $\omega_1, \omega_2, \cdots, \omega_n$ and such that the inequalities

$$P_n(\lambda \mid \omega_1, \cdots, \omega_n) < N_n(\lambda \mid \omega_1, \cdots, \omega_n) < Q_n(\lambda \mid \omega_1, \cdots, \omega_n)$$

hold for all $\lambda \geqq 0$. There are, of course, infinitely many ways of doing this. The method adopted here is one in which $P_n(\lambda)$ and $Q_n(\lambda)$ are obtained recursively in such a way as to minimize, for all $\lambda$, the discrepancies between $N_k(\lambda)$ and its lower and upper bounds $P_k(\lambda)$ and $Q_k(\lambda)$. In other words, $P_1(\lambda)$, $P_2(\lambda)$, $\cdots$, $P_n(\lambda)$ are obtained as successive solutions of a sequence of linear difference equations, the $n$ additive constants of these solutions being determined as the best possible. This is best accomplished by introducing Bernoulli polynomials.

We begin by stating a few facts, proved elsewhere,[5] about the maximum $M_\nu$ and the minimum $m_\nu$ of the $\nu$-th Bernoulli polynomials $B_\nu(x)$ in the unit interval $0 \leqq x \leqq 1$. The notation employed is that in which

$$\begin{aligned} B_\nu(x) &= (B + x)^\nu = \sum_{k=0}^{\nu} \binom{\nu}{k} x^{\nu-k} B_k \\ &= x^\nu - \frac{\nu}{2} x^{\nu-1} + \frac{\nu(\nu-1)}{12} x^{\nu-2} - \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{720} x^{\nu-4} + \cdots, \end{aligned}$$

where the notation for Bernoulli numbers $B_k$ is such that symbolically

$$(B + 1)^\nu = B^\nu \qquad (B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_{2k+1} = 0,\ k > 0).$$

When $\nu$ is even, exact formulas for $M_\nu$ and $m_\nu$ may be given as follows:

$$\begin{aligned} M_0 &= m_0 = 1, \\ M_{4h} &= B_{4h}(\tfrac{1}{2}) = (1 - 2^{1-4h}) \mid B_{4h} \mid \qquad (h > 0), \\ m_{4h} &= B_{4h}(0) = - \mid B_{4h} \mid \qquad (h > 0), \\ M_{4h+2} &= B_{4h+2}(0) = B_{4h+2}, \\ m_{4h+2} &= B_{4h+2}(\tfrac{1}{2}) = -(1 - 2^{-1-4h}) B_{4h+2}. \end{aligned}$$

[5] American Mathematical Monthly, vol. 47(1940), pp. 533–538.

For $\nu$ odd, no exact formula for $M_{2k+1}$ or $m_{2k+1}$ exists. However, we have

$$M_{2k+1} = -m_{2k+1} < (2k+1)2^{-4k-2}\left\{|E_{2k}| + \frac{2}{\pi}(1 - 2^{-2k+1})|B_{2k}|\right\},$$

where $E_{2k}$ are Euler numbers. Here the difference between the right side and $M_{2k+1}$ tends rapidly to zero as $k \to \infty$. For $2k + 1 = 13$, for instance, this difference is less than $10^{-9}$. A considerably simpler, though a little less precise, result is

$$M_\nu = -m_\nu < 2\nu!(2\pi)^{-\nu} \qquad (\nu \text{ odd}).$$

Here the right member when $\nu = 13$ exceeds the left by about $3 \cdot 10^{-7}$. When $\nu < 13$, these results are less satisfactory and values of $M_{2k+1} = -m_{2k+1}$ as actually computed may be used. What we shall need, however, is not so much $M_n$ and $m_n$ as the functions

$$W_n = (B_n - M_n)/n \quad \text{and} \quad w_n = (B_n - m_n)/n.$$

These may be tabulated for $n \leqq 16$ as follows:

| $k$ odd | | $k$ even | |
|---|---|---|---|
| $k$ | $w_k = -W_k$ | $k$ | $w_k$ |
| | | 2 | $.1250000000 = 1/2^3$ |
| 3 | .01603750748 | 6 | $.0078125000 = 1/2^7$ |
| 5 | .004891638174 | 10 | $.01513671875 = 31/2^{11}$ |
| 7 | .003723587751 | 14 | $.1666564941 = 5461/2^{15}$ |
| 9 | .005283395737 | | |
| 11 | .012045150767 | | $-W_k$ |
| 13 | .1611223890 | 4 | $.0156250000 = 1/2^6$ |
| 15 | .1856693827 | 8 | $.00830078125 = 17/2^{11}$ |
| | | 12 | $.04217529297 = 691/2^{14}$ |
| | | 16 | $.8865060806 = 929569/2^{20}$ |

$$W_{4k+2} = w_{4k} = 0, \qquad W_1 = -1, \qquad w_1 = 0.$$

In what follows, we shall need estimates for sums of the type

$$S_k(x, \omega) = \sum_{1 \leqq \mu \leqq x/\omega} (x - \mu\omega)^k.$$

Hence we give the following lemma.

LEMMA. *If $x$ and $\omega$ are positive, and if $k$ is a non-negative integer, then*

$$\omega^k(B_{k+1}(x/\omega) - M_{k+1}) \leqq (k+1)S_k(x, \omega) \leqq \omega^k(B_{k+1}(x/\omega) - m_{k+1}). \tag{9}$$

*Proof.* If in the well-known difference equation

$$B_{k+1}(z+1) - B_{k+1}(z) = (k+1)z^k \tag{10}$$

we set $z = x/\omega - 1, x/\omega - 2, \cdots, x/\omega - [x/\omega]$ and add the resulting equations, we obtain

$$(k+1) \sum_{1 \leqq \mu \leqq x/\omega} \left(\frac{x}{\omega} - \mu\right)^k = (k+1)S_k(x, \omega)\omega^{-k} = B_{k+1}\left(\frac{x}{\omega}\right) - B_{k+1}\left(\frac{x}{\omega} - \left[\frac{x}{\omega}\right]\right).$$

Since by definition

$$m_{k+1} \leqq B_{k+1}\left(\frac{x}{\omega} - \left[\frac{x}{\omega}\right]\right) \leqq M_{k+1},$$

the lemma now follows at once.

It is worth noting that for $k > 0$, either of the two equal signs in (9) holds for an infinity of $x$ in arithmetic progression, in fact for

$$x = \omega(h + r_{k+1}) \qquad (h = 0, 1, 2, \cdots),$$

where $r_{k+1}$ is that point on the interval $0 \leqq t < 1$ at which $B_{k+1}$ attains the maximum or minimum value. For $k = 0$, however, the lemma states simply that

$$\frac{x}{\omega} - 1 \leqq \left[\frac{x}{\omega}\right] \leqq \frac{x}{\omega}. \tag{11}$$

Here the second equality sign holds infinitely often. The first never holds, but fails to do so by an arbitrarily small margin infinitely often.

We are now in a position to consider the problem of constructing $P_n(\lambda \mid \omega_1, \omega_2, \cdots, \omega_n)$. To begin with, we take the trivial case of $n = 1$ in which obviously

$$N_1(\lambda \mid \omega_1) = 1 + \left[\frac{\lambda}{\omega_1}\right].$$

By (11) the best possible choice of the polynomial $P_1(\lambda \mid \omega_1)$ is

$$P_1(\lambda \mid \omega_1) = \frac{\lambda}{\omega_1}. \tag{12}$$

Let us suppose that we have already constructed a polynomial of degree $k - 1$, say,

$$P_{k-1}(\lambda) = P_{k-1}(\lambda \mid \omega_1, \omega_2, \cdots, \omega_{k-1}) = \sum_{\nu=0}^{k-1} p_\nu^{(k-1)}\lambda^\nu$$

such that for all $\lambda > 0$

$$P_{k-1}(\lambda \mid \omega_1, \cdots, \omega_{k-1}) < N_{k-1}(\lambda \mid \omega_1, \cdots, \omega_{k-1}),$$

and where the coefficients $p_\nu^{(k-1)}$ depend only on $\omega_1, \omega_2, \cdots, \omega_{k-1}$. By (12),

$$p_0^{(1)} = 0, \qquad p_1^{(1)} = \frac{1}{\omega_1}. \tag{13}$$

To construct recursively the next polynomial $P_k(\lambda)$, we write the fundamental inequality (1) in the form

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_{k-1} x_{k-1} \leqq \lambda - \omega_k x_k \qquad (k > 1).$$

Allowing $x_k$ to range over the integers $0, 1, \cdots, [\lambda/\omega_k]$, we obtain by definition

$$N_k(\lambda \mid \omega_1, \omega_2, \cdots, \omega_k) = \sum_{0 \leqq \mu \leqq \lambda/\omega_k} N_{k-1}(\lambda - \mu\omega_k \mid \omega_1, \cdots, \omega_{k-1}).$$

Hence

$$\begin{aligned} N_k(\lambda) > \sum_{0 \leqq \mu \leqq \lambda/\omega_k} P_{k-1}(\lambda - \mu\omega_k) &= P_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} p_\nu^{(k-1)} \sum_{1 \leqq \mu \leqq \lambda/\omega_k} (\lambda - \mu\omega_k)^\nu \\ &= P_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} p_\nu^{(k-1)} S_\nu(\lambda, \omega_k). \end{aligned} \tag{14}$$

If we define for convenience the function $K_n(t)$ by

$$K_n(t) = \begin{cases} tm_n/n & \text{if } t \leqq 0, \\ tM_n/n & \text{if } t \geqq 0, \end{cases}$$

then the result of applying the lemma to (14) may be written

$$N_k(\lambda) > P_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} p_\nu^{(k-1)} \frac{\omega_k^\nu}{\nu + 1} B_{\nu+1}(\lambda/\omega_k) - \sum_{\nu=0}^{k-1} \omega_k^\nu K_{\nu+1}(p_\nu^{(k-1)}). \tag{15}$$

The right member, which is a polynomial in $\lambda$ of degree $k$, whose coefficients depend only on $\omega_1, \omega_2, \cdots, \omega_k$, we take for

$$P_k(\lambda) = P_k(\lambda \mid \omega_1, \cdots, \omega_k) = \sum_{\nu=0}^{k} p_\nu^{(k)} \lambda^\nu.$$

Expanding the Bernoulli polynomials and collecting the coefficients of the various powers of $\lambda$, we have the following recursion formula for $p_\nu^{(r)}$ or rather for

$$c_\nu^{(r)} = \omega_1 \omega_2 \cdots \omega_r p_\nu^{(r)}. \tag{16}$$

When $\nu > 0$

$$\nu c_\nu^{(k)} = c_{\nu-1}^{(k-1)} + \omega_k \frac{\nu}{2} c_\nu^{(k-1)} + \sum_{j=1}^{[\frac{1}{2}(k-\nu)]} \omega_k^{2j} A_j(\nu)\, c_{\nu+2j-1}^{(k-1)}, \tag{17}$$

where the coefficients

$$A_j(\nu) = \binom{\nu + 2j - 1}{2j} B_{2j} \tag{18}$$

do not depend upon $k$, and when once computed may be used in each successive determination of $P_k(\lambda)$ $(k = 1, 2, 3, \cdots, n)$. When $\nu = 0$, we have

$$c_0^{(k)} = \sum_{\nu=2}^{k-1} \omega_k^{\nu+1} V_{\nu+1}(c_\nu^{(k-1)}), \tag{19}$$

where

$$V_h(t) = \begin{cases} tW_h & \text{if } t \geqq 0, \\ tw_h & \text{if } t \leqq 0. \end{cases} \tag{20}$$

From (17) and (19) the $c_\nu^{(k)}$ may be found recursively starting from the initial values

$$c_1^{(1)} = 1, \qquad c_0^{(1)} = 0.$$

The first few polynomials $P_n(\lambda)$ are thus found to be

$$\begin{aligned}
\omega_1 P_1(\lambda \mid \omega_1) &= \lambda, \\
2!\omega_1\omega_2 P_2(\lambda \mid \omega_1, \omega_2) &= \lambda^2 + \omega_2\lambda, \\
3!\omega_1\omega_2\omega_3 P_3(\lambda \mid \omega_1, \omega_2, \omega_3) &= \lambda^3 + \tfrac{3}{2}(\omega_2 + \omega_3)\lambda^2 + \tfrac{1}{2}(\omega_3^2 + 3\omega_2\omega_3)\lambda + 3\omega_3^3 W_3, \\
4!\omega_1\omega_2\omega_3\omega_4 P_4(\lambda \mid \omega_1, \omega_2, \omega_3, \omega_4) &= \lambda^4 + 2(\omega_2 + \omega_3 + \omega_4)\lambda^3 \\
&\quad + (\omega_3^2 + \omega_4^2 + 3(\omega_2\omega_3 + \omega_3\omega_4 + \omega_2\omega_4))\lambda^2 \\
&\quad + (\omega_3^2\omega_4 + \omega_4^2\omega_2 + \omega_4^2\omega_3 + 3\omega_2\omega_3\omega_4 + 12\omega_3^3 W_3)\lambda \\
&\quad + 2\omega_4^3(3W_3(\omega_2 + \omega_3) + 2W_4\omega_4),
\end{aligned}$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$\begin{aligned}
n!\omega_1\omega_2 \cdots \omega_n P_n(\lambda \mid \omega_1, \cdots, \omega_n) &= \lambda^n + \frac{n}{2}(\omega_2 + \omega_3 + \cdots + \omega_n)\lambda^{n-1} \\
&\quad + \frac{n(n-1)}{12}(\omega_3^2 + \cdots + \omega_n^2 + 3\sum \omega_2\omega_3)\lambda^{n-2} + \cdots.
\end{aligned} \tag{21}$$

It is seen that $P_n(\lambda \mid \omega_1, \cdots, \omega_n)$ is not a symmetric function of the $\omega$'s although $N_n(\lambda \mid \omega_1, \cdots, \omega_n)$ is. This raises the question as to the order in which the $\omega$'s should be introduced to maximize $P_n(\lambda \mid \omega_1, \cdots, \omega_n)$. That this choice of order will in general depend on $\lambda$ is seen in the formula for $P_3(\lambda)$. In fact, if $\lambda$ is large, the order should be obviously $\omega_1 \leqq \omega_2 \leqq \omega_3$ in spite of the fact that this will minimize the negative constant term $3\omega_3^3 W_3$. In general it is seen that for all large $\lambda$ the value of $P_n(\lambda \mid \omega_1, \cdots, \omega_n)$ will be largest when

$$\omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_n$$

since this has the effect of maximizing the coefficients of $\lambda^{n-1}$ and $\lambda^{n-2}$.

Another question that arises is: In what sense does $P_n(\lambda)$ approximate $N_n(\lambda)$? We may state the following answer.

THEOREM 1. *There exist for each $n > 0$ infinitely many $n$-dimensional tetrahedra for which*

$$N_n(\lambda) - P_n(\lambda) < C\lambda^{n-3},$$

*where $C$ is a positive constant depending on $n$.*

*Proof.* Consider those $n$-dimensional equilateral tetrahedra in which $\omega_1 = \omega_2 = \cdots = \omega_n = 1$. By (4) we have

$$n!N_n(\lambda) = n!\binom{[\lambda] + n}{n} = \lambda^n + \left\{\frac{n(n+1)}{2} - n\delta\right\}\lambda^{n-1}$$
$$+ \left\{\frac{n(n-1)}{2}\delta^2 - \frac{n(n^2-1)}{2}\delta + \frac{n(n^2-1)(3n+2)}{24}\right\}\lambda^{n-2} + O(\lambda^{n-3}),$$

where we have written

$$[\lambda] = \lambda - \delta.$$

By (21), however, we have in this case

$$n!P_n(\lambda) = \lambda^n + \frac{n(n-1)}{2}\lambda^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{24}\lambda^{n-2} + O(\lambda^{n-3}).$$

Hence

$$n!\{N_n(\lambda) - P_n(\lambda)\} = n(1-\delta)\lambda^{n-1} + \frac{n(n-1)}{2}(1-\delta)(n-\delta)\lambda^{n-2} + O(\lambda^{n-3}).$$

If we now choose $\lambda$ of such a form that

$$1 - \delta = O(\lambda^{-2}),$$

as, for instance,

$$\lambda = k - k^{-2}, \qquad k \text{ an integer,}$$

then

$$0 < N_n(\lambda) - P_n(\lambda) = O(\lambda^{n-3}).$$

This proves the theorem.

The polynomial $Q_n(\lambda)$ is constructed in a similar way. In fact since

$$N_1(\lambda \mid \omega_1) = 1 + \left[\frac{\lambda}{\omega_1}\right],$$

we may set

$$Q_1(\lambda \mid \omega_1) = 1 + \frac{\lambda}{\omega_1},$$

so that

$$Q_1(\lambda \mid \omega_1) \geqq N_1(\lambda \mid \omega_1),$$

the equality holding only when $\lambda$ is an integer multiple of $\omega_1$. If

$$Q_{k-1}(\lambda \mid \omega_1, \cdots, \omega_{k-1}) = \sum_{\nu=0}^{k-1} q_\nu^{(k-1)}\lambda^\nu$$

has already been determined in such a way that

$$Q_{k-1}(\lambda) \geqq N_{k-1}(\lambda \mid \omega_1, \cdots, \omega_{k-1}) \qquad (\lambda \geqq 0),$$

then

$$\begin{aligned} N_k(\lambda \mid \omega_1, \cdots, \omega_k) &= \sum_{\mu=0}^{[\lambda/\omega_k]} N_{k-1}(\lambda - \mu\omega_k \mid \omega_1, \cdots, \omega_{k-1}) \leqq \sum_{\mu=0}^{[\lambda/\omega_k]} Q_{k-1}(\lambda - \mu\omega_k) \\ &= Q_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} \sum_{\mu=1}^{[\lambda/\omega_k]} q_\nu^{(k-1)} (\lambda - \mu\omega_k)^\nu \\ &= Q_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} q_\nu^{(k-1)} S_\nu(\lambda, \omega_k). \end{aligned}$$

Applying our lemma we have

$$N_k(\lambda \mid \omega_1, \cdots, \omega_k) \leqq Q_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} q_\nu^{(k-1)} \frac{\omega_k^\nu}{\nu+1} B_{\nu+1}(\lambda/\omega_k) + \sum_{\nu=0}^{k-1} \omega_k^\nu K_{\nu+1}(-q_\nu^{(k-1)}).$$

The polynomial on the right is taken as

$$Q_k(\lambda) = \sum_{\nu=0}^{k} q_\nu^{(k)} \lambda^\nu.$$

Expanding the Bernoulli polynomials as before and setting

$$d_\nu^{(k)} = \omega_1\omega_2 \cdots \omega_k q_\nu^{(k)},$$

we obtain recursion formulas for $d_\nu^{(k)}$ as follows:

For $\nu > 0$ we have corresponding to (17)

$$\nu d_\nu^{(k)} = d_{\nu-1}^{(k-1)} + \frac{\nu\omega_k}{2} d_\nu^{(k-1)} + \sum_{j=1}^{[\frac{1}{2}(k-\nu)]} \omega_k^{2j} A_j(\nu)\, d_{\nu+2j-1}^{(k-1)}, \tag{22}$$

where $A_j(\nu)$ is given in (18).

When $\nu = 0$ we have

$$d_0^{(k)} = \omega_k d_0^{(k-1)} - \sum_{\nu=0}^{k-1} \omega_k^{\nu+1} V_{\nu+1}(-d_\nu^{(k-1)}), \tag{23}$$

where the function $V_h(t)$ is given by (20).

Starting with $Q_1(\lambda \mid \omega_1)$ we find for the first few values of $n$

$$\begin{aligned} \omega_1 Q_1(\lambda \mid \omega_1) &= \lambda + \omega_1, \\ 2!\,\omega_1\omega_2 Q_2(\lambda \mid \omega_1, \omega_2) &= \lambda^2 + (2\omega_1 + \omega_2)\lambda + \tfrac{1}{4}\omega_2^2 + 2\omega_1\omega_2, \\ 3!\,\omega_1\omega_2\omega_3 Q_3(\lambda \mid \omega_1, \omega_2, \omega_3) &= \lambda^3 + \tfrac{3}{2}(2\omega_1 + \omega_2 + \omega_3)\lambda^2 \\ &\quad + (6\omega_1\omega_2 + 3\omega_1\omega_3 + \tfrac{3}{2}\omega_2\omega_3 + \tfrac{3}{4}\omega_2^2 + \tfrac{1}{2}\omega_3^2)\lambda \\ &\quad + 6\omega_1\omega_2\omega_3 + \tfrac{3}{4}\omega_2^2\omega_3 + \tfrac{3}{4}\omega_3^2\omega_1 + \tfrac{3}{8}\omega_3^2\omega_2 + \frac{3^{\frac{1}{2}}}{36}\omega_3, \end{aligned}$$

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$$n!\omega_1 \cdots \omega_n Q_n(\lambda \mid \omega_1, \cdots, \omega_n) = \lambda^n + \frac{n}{2}(2\omega_1 + \omega_2 + \cdots + \omega_n)\lambda^{n-1}$$

$$(24) \qquad + \frac{n(n-1)}{2}\{2\omega_1\omega_2 + \omega_1(\omega_3 + \cdots + \omega_n) + \tfrac{1}{2}\sum \omega_2\omega_3 + \tfrac{1}{4}\omega_2^2$$
$$+ \tfrac{1}{6}(\omega_3^2 + \cdots + \omega_n^2)\}\lambda^{n-2} + \cdots.$$

As in the case of $P_n(\lambda)$, it is seen that if $\omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_n$, the best results are obtained for large $\lambda$ since this minimizes the coefficients of $\lambda^{n-1}$ and $\lambda^{n-2}$.

As an analogue of Theorem 1 we have the less satisfactory result.

THEOREM 2. *There exist for each $n > 0$ infinitely many $n$-dimensional tetrahedra for which*

$$Q_n(\lambda) - N_n(\lambda) = \frac{\lambda^{n-2}}{8(n-2)!} + O(\lambda^{n-3}).$$

*Proof.* As before take the equilateral case

$$\omega_1 = \omega_2 = \cdots = \omega_n = 1$$

and let $\lambda$ be an integer so that

$$n!N_n(\lambda) = \lambda^n + \frac{n(n+1)}{2}\lambda^{n-1} + \frac{n(n^2-1)(3n+2)}{24}\lambda^{n-2} + O(\lambda^{n-3}),$$

whereas by (24)

$$n!Q_n(\lambda) = \lambda^n + \frac{n(n+1)}{2}\lambda^{n-1} + \frac{n(n-1)}{24}(3n^2+5n+5)\lambda^{n-2} + O(\lambda^{n-3}).$$

The theorem now follows at once from subtracting the right sides.

The fact that the coefficients of $P_n(\lambda)$ and $Q_n(\lambda)$ are complicated functions of $\omega_1, \omega_2, \cdots, \omega_n$ does not mean that the actual values of these coefficients cannot be found readily when numerical values of $\omega_1, \omega_2, \cdots, \omega_n$ are given. In fact the recurrence formulas (17), (19), (22) and (23) enable one to compute readily the successive numerical values of the coefficients $c_\nu^{(k)}$ and $d_\nu^{(k)}$, and hence $p_\nu^{(k)}$ and $q_\nu^{(k)}$. It has been quite feasible to compute these coefficients up to as high as $n = 13$ and 14, in connection with an investigation into the first case of Fermat's last theorem,[6] which involved the multiplicative tetrahedron.

A quite valuable check at each stage of the work is afforded by

THEOREM 3.

$$(25) \qquad P_k(-\omega_k) = P_k(0) - P_{k-1}(0),$$

$$(26) \qquad Q_k(-\omega_k) = Q_k(0) - Q_{k-1}(0).$$

[6] To appear shortly in the Bulletin of the American Mathematical Society.

*Proof.* To prove the first relation substitute first $\lambda = -\omega_k$ and then $\lambda = 0$ into the right member of (15), and subtract the results obtained so as to get

$$P_k(-\omega_k) - P_k(0) = P_{k-1}(-\omega_k) - P_{k-1}(0) \tag{27}$$
$$+ \sum_{\nu=0}^{k-1} p_\nu^{(k-1)} \frac{\omega_k^\nu}{\nu+1} \{B_{\nu+1}(-1) - B_{\nu+1}(0)\}.$$

But from (10) with $k = \nu$ it follows that

$$B_{\nu+1}(-1) - B_{\nu+1}(0) = -(-1)^\nu(\nu + 1).$$

Hence the sum in the right member of (27) becomes

$$-\sum_{\nu=0}^{k-1} p_\nu^{(k-1)}(-\omega_k)^\nu = -P_{k-1}(-\omega_k),$$

so that (25) follows at once. (26) follows in precisely the same way.

In considering the multiplicative case J. B. Rosser[7] obtained a lower bound for $N_n(\lambda)$ as a polynomial $f_n(\lambda)$ which, when extended to the general tetrahedron, may be written

$$f_n(\lambda) = \frac{1}{n!\omega_1\omega_2\cdots\omega_n}\left(\lambda^n + \frac{n}{2}\sigma_1'\lambda^{n-1} + \frac{n(n-1)}{2^2}\sigma_2'\lambda^{n-2} + \cdots + \frac{n!}{2^{n-1}}\sigma_{n-1}'\lambda\right),$$

where $\sigma_k'$ is the sum of the products $k$ at a time of $\omega_2, \omega_3, \cdots, \omega_n$. The first two coefficients of $f_n(\lambda)$ will be seen to agree with those of $P_n(\lambda)$. In fact

$$P_n(\lambda) - f_n(\lambda) = \frac{\lambda^{n-2}}{12(n-2)!\omega_1\omega_2\cdots\omega_n}(\omega_3^2 + \omega_4^2 + \cdots + \omega_n^2) + O(\lambda^{n-3}).$$

Hence for $n > 2$ and $\lambda > \lambda_0$

$$f_n(\lambda) < P_n(\lambda) < N_n(\lambda).$$

By way of comparison of the various approximations to $N_n(\lambda)$ discussed above, we give the actual polynomials in the typical example of $N_5(\lambda \mid \log_{10} 2, \log_{10} 3, \log_{10} 5, \log_{10} 7, \log_{10} 11)$, i.e., the number of positive integers $\leqq 10^\lambda$ divisible by no prime exceeding 11, and compare their values at several points with the exact values of $N_5(\lambda)$, as kindly furnished by Dr. A. E. Western, who has prepared extensive tables of $N_n(\lambda)$ in the multiplicative case. From these tables he has constructed an approximating polynomial $\phi_n(\lambda)$ by applying the method of least squares. The polynomial $\phi_5$ is also given below, and compared with the others.

[7] *On the first case of Fermat's last theorem*, Bulletin of the American Mathematical Society, vol. 45(1939), pp. 636–640.

$$R_5(\lambda) = .094319\lambda^5 + .79313\lambda^4 + 2.46300\lambda^3 + 3.59621\lambda^2 - 6.36020\lambda - .037937,$$

$$P_5(\lambda) = .094319\lambda^5 + .72215\lambda^4 + 1.97819\lambda^3 + 2.26936\lambda^2 + .87536\lambda - .082148,$$

$$Q_5(\lambda) = .094319\lambda^5 + .86411\lambda^4 + 3.03689\lambda^3 + 5.27786\lambda^2 + 5.01395\lambda + 2.69600,$$

$$f_5(\lambda) = .094319\lambda^5 + .72215\lambda^4 + 1.61864\lambda^3 + 1.17723\lambda^2 + .20762\lambda,$$

$$\phi_5(\lambda) = .0033629\lambda^5 + 5.14087\lambda^4 - 71.79074\lambda^3 + 596.2170\lambda^2 - 2245.997\lambda + 3327.38.$$

The following table gives the exact values of $N_n(\lambda)$ together with the discrepancies of the approximating polynomials.

| $\lambda$ | $N_5(\lambda)$ | $N - R$ | $N - P$ | $Q - N$ | $N - f$ | $\phi - N$ |
|---|---|---|---|---|---|---|
| 1 | 10 | 9.45 | 4.13 | 6.98 | 6.18 | 589.95 |
| 2 | 55 | 17.96 | 13.86 | 19.97 | 22.36 | 651.29 |
| 3 | 192 | 25.09 | 34.21 | 48.15 | 55.67 | 209.22 |
| 5 | 1197 | 40.60 | 142.61 | 177.14 | 218.11 | .53 |
| 8 | 7838 | 58.38 | 624.42 | 727.55 | 884.53 | 1.65 |
| 10 | 20193 | 70.77 | 1325.78 | 1497.59 | 1801.20 | 140.41 |
| 10.5 | 24932 | 72.75 | 1567.04 | 1761.97 | 2110.69 | 239.94 |

To illustrate the use of equation (7) in this case suppose we attempt to represent $N_n(\lambda)$ by an exponential function near the value 20193. Then the $\lambda$ of equation (7) is $10 \log_e 10 = 23.026 \cdots$. The best value of $\epsilon$ is found to be about .18889, and the value of $A$ is about 14797. Hence

$$N_5(\lambda) < 14797 \cdot 10^{.18889\lambda}.$$

Substituting $\lambda = 10$ and 10.5 we find 114540 and 142360, values which are too large by the factors 5.67 and 5.71 respectively, which do not differ greatly from $(2\pi n)^{\frac{1}{2}} = (10\pi)^{\frac{1}{2}} = 5.61$.

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