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The Second Term in the Asymptotics for the Number of Points Moving Along a Metric Graph

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Abstract—We consider the problem of determining the asymptotics for the number of points moving along a metric graph. This problem is motivated by the problem of the evolution of wave packets, which at the initial moment of time are localized in a small neighborhood of one point. It turns out that the number of points, as a function of time, allows a polynomial approximation. This polynomial is expressed via Barnes' multiple Bernoulli polynomials, which are related to the problem of counting the number of lattice points in expanding simplexes.
In this paper we give explicit formulas for the first two terms of the expansion for the counting function of the number of moving points. The leading term was found earlier and depends only on the number of vertices, the number of edges and the lengths of the edges. The second term in the expansion shows what happens to the graph when one or two edges are removed. In particular, whether it breaks up into several connected components or not. In this paper, examples of the calculation of the leading and second terms are given.

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1. INTRODUCTION

Let Γ be an undirected, connected, locally finite graph (possibly with loops and multiple edges). We assume that edges e_i have lengths t_i, respectively, and these numbers are linearly independent over $\mathbb{Q}$, which corresponds to the situation of the general position.

Let s be a fixed vertex $s \in V(\Gamma)$, which we will call *the source*.

Let us consider the following dynamical system (see [9, 13]), the study of which is motivated by the problem of studying the behavior of wave packets that are localized in a small neighborhood of a single point at the initial moment and evolve on metric graphs or hybrid spaces (see the articles [4, 11] and references therein). At the initial moment of time points emerge from the source s along all edges incident to s and move with the unit speed. At that time when k points, where k can take values from 1 to the valence v of the vertex v_j, occur at the vertex of the graph, v new points emerge which go along all edges incident to the vertex v_j. Let $N(T)$ be the number of points that move along the graph at the time point T. The function $N(T)$ is piecewise constant. The propagation of the points corresponds to the dynamics of the centers of narrow Gaussian wave

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packets if we consider the evolution of the semiclassical solution of the Cauchy problem for the time-dependent Schrödinger equation on a metric graph.

The leading part of the asymptotics of $N(T)$ as $T \to \infty$ for an arbitrary finite metric graph with incommensurable edge lengths was found earlier [4, 11].

In [5] a polynomial approximation was given for finite trees, a correction to the leading term was written and it was shown that if we know the correction as a function of the lengths of the edges, then the tree and the source can be recovered uniquely.

In this paper we find the general formula for $N(T)$ and, in the case of an arbitrary finite graph, we find a polynomial approximation to $N(T)$ and explicitly write out the first two terms of this approximation. Also we show (see the Examples section) that in the case of a general graph the source cannot be recovered uniquely only from the second term of the asymptotic expansion.

We use the results on approximating the number of lattice points that lie within an expanding simplex with real vertices. Namely, the results of D. C. Spencer about Barnes' multiple Bernoulli polynomials (see [6, 7, 10]) also known as Todd polynomoials (see [1]). For more detailed references see [5].

2. NECESSARY NOTATIONS AND DEFINITIONS

Let G denote a finite connected subgraph of the graph Γ, containing the vertex s. For the subgraph $G \subset \Gamma$ the vertex set is denoted by $V(G)$ and the set of edges by $E(G)$.

For $v \in V(G)$, we denote by $\rho(G, v)$ the valency of the vertex v in the subgraph G.

Definition 1. *The vertex v is said to be the end-vertex in the subgraph G if $\rho(G, v) = 1$. Then the unique edge $e \in E(G)$ is called the end-edge.*

Definition 2. *The edge $e \in E(G)$ is called the isthmus if after deleting this edge the graph G splits into two connected components.*

Definition 3. *The edge progression from s to v is a finite sequence of the form $\mu = (s, e_{i_1}, v_{i_1}, \ldots, v_{i_{m-1}}, e_{i_m}, v)$, where to the left and right of each edge there are its two ends. The edges and vertices can be repeated in this sequence.*

Definition 4. *The multiplicity of the edge progression μ along the edge e_i is the number $k_i(\mu)$, which is equal to the number of times this edge occurs in the sequence μ.*

Definition 5. *The set of multiplicities of the edge progression μ is the set $k(\mu) = \big(k_i(\mu)\big)_{e_i \in E(G)}$, where G is the connected subgraph of the graph Γ, consisting of all edges, which are in the edge progression μ.*

Definition 6. *For a given set of numbers $k = (k_i)_{e_i \in E(G)}$, indexed by edges G, let us define the graph G_k, which is obtained from G by adding to each edge $e_i \in E(G)$ another $k_i - 1$ edges (that is, each edge e_i in G corresponds to the edges $e_i^1, \ldots, e_i^{k_i}$ in the graph G_k).*

In particular, k can be a set of multiplicities of some edge progression μ, then we write $G_{k(\mu)}$. It is obvious that the graph $G_{k(\mu)}$ is unicursal.

Definition 7. *The time of passage of the edge progression μ is the number $\sum_{e_i \in E(G)} k_i(\mu) t_i$, where G is the connected subgraph of the graph Γ, consisting of all edges, which are in the edge progression μ.*

Definition 8. *A simple chain is the edge progression, in which all vertices and edges are distinct.*

Definition 9. *A cycle is the edge progression in which all edges are different, and the initial and final vertices coincide.*

Definition 10. *A simple cycle is a cycle that does not pass through one vertex twice.*

We denote by $\beta_1(G)$ the first Betti number of the graph G, that is, a sum of numbers of edges that are not in the maximal tree over all connected components of the graph G.

The edges of any edge progression form a finite connected subgraph Γ. Let us choose some finite connected subgraph $G \subset \Gamma$ containing the vertex s. We also choose a vertex $v \in V(G)$ (possibly, $v = s$) and find times of passage of all edge progressions from s to v that pass only along the edges in G and pass at least once along each edge in G. Let us denote the set of such edge progressions by $U_{G,v}$.

3. THE SET OF PASSAGE TIMES OF EDGE PROGRESSIONS FROM $U_{G,v}$

We assign to each $\mu \in U_{G,v}$ a set of labels on the edges of a subgraph G: $c = (c_i)_{e_i \in E(G)}$, where $c_i = 2 - \big(k_i(\mu) \bmod 2\big)$.

Let us define the set of all different labels for all edge progressions from $U_{G,v}$:

$$C_{G,v} = \cup_{\mu \in U_{G,v}} \{c(\mu)\}.$$

The set of all possible multiplicities of edge progressions from $U_{G,v}$ is defined as

$$K_{G,v} = \cup_{\mu \in U_{G,v}} \{k(\mu)\}.$$

Statement 1.

$$K_{G,v} = \cup_{c \in C_{G,v}} \{(c_i + 2n_i)_{e_i \in G} | n_i \geqslant 0\}.$$

Proof (*of Statement* 1). The inclusion of the left set into the right set obviously follows from the definition of the set of labels $C_{G,v}$. Let us prove the inclusion in the opposite direction.

1) Let us prove that for any $c \in C_{G,v}$ there exists an edge progression $\mu \in U_{G,v}$ such that $k(\mu) = c$. By definition, there is an edge progression $\mu_1 \in U_{G,v}$ such that $k(\mu_1) = (c_i + 2n_i)_{e_i \in E(G)}$ for some $n_i \geqslant 0$. Consider the graph $G_{k(\mu_1)}$ (see Definition 6). All vertices of the graph $G_{k(\mu_1)}$, with the exception of s and v, have an even valency. If $s = v$, then s also has an even valency. If $s \neq v$, then s and v have an odd valency. For each $e_i \in E(G)$ we remove $2n_i$ edges from multiple edges $e_i^1, \ldots, e_i^{c_i+2n_i}$ of the graph $G_{k(\mu_1)}$. There remain two multiple edges e_i^1, e_i^2 for $c_i = 2$ or one edge e_i^1 for $c_i = 1$. We obtain a new graph that is unicursal, and the edge progression from s to v in this graph determines the edge progression μ in G with $k(\mu) = c$.

2) Now for any $e_i \in E(G)$ and any $n_i \geqslant 0$ we can add to the edge progression μ constructed in step 1 $2n_i$–multiple passes forward and back along the edge e_i, and we obtain an edge progression with a set of multiplicities $(c_i + 2n_i)_{e_i \in E(G)}$. □

Since all passage times of the edges are linearly independent over $\mathbb{Q}$, we obtain

Statement 2.

$$T_{G,v} = \{t(\mu) | \mu \in U_{G,v}\} = \sqcup_{c \in C_{G,v}} \left\{ \sum_{e_i \in E(G)} (c_i + 2n_i) t_i \big| n_i \geqslant 0 \right\}.$$

If we assume that the passage time of an edge is the weight of an edge, then using the formula found in the statement, we can obtain the counting function $f(T)$ for the number of weighted edge progressions whose weight is less than T.

Note that the closest classical problem of discrete mathematics is the problem of counting the sum of the weights of edge progressions leading from one vertex to another, but only if the number of these edge progressions is finite and the graph is directed (see [2]). In our case, the graph is undirected and the number of edge progressions is infinite, so the sum of weights of edge progressions is infinite, but we can find the counting function of weights of edge progressions.

Now let us find how many different labels exist in the subgraph G.

Statement 3. $\# C_{G,v} = 2^{\beta_1(G)}$.

Proof (*of Statement* 3). Two edge progressions μ_1 and μ_2, beginning at the vertex s and ending at the vertex v, are *discrete homotopic* if one is obtained from the other by applying a finite number of operations of inserting or deleting a sequence of the form $(v_i, e_j, v_l, e_j, v_i)$ (that is, double passes forward and back along the edge). It is obvious that edge progressions are discrete homotopic if and only if the corresponding continuous paths are homotopic as paths with common ends. For two discrete homotopic paths μ_1 and μ_2: $c(\mu_1) = c(\mu_2)$.

Let us choose in the graph G the maximal tree with the root at the vertex s. For any edge progression $\mu \in U_{G,v}$ we can make a discrete homotopy into an edge progression μ_1 from s to v, which does not necessarily pass through all edges of G, but which consists of the union of simple cycles, each of which contains only one cross connection, and possibly a simple chain that passes from s to v along the edges of the maximal tree. Then $c(\mu_1)$ is uniquely determined by the parity of the passage of the edge progression μ_1 along the cross connections. The parity of the multiplicities of the passage along the cross connections can be set by $2^{\beta_1(G)}$ ways. □

3.1. The Formula for $N(T)$

The function $N(T)$ is piecewise constant and the jump of this function can occur only in times of the form $T_{G,v}$ for some subgraph G and $v \in V(G)$.

To know the jump of the function $N(T)$ at the time moment $t_0 \in T_{G,v}$, we need to know at which edges the edge progressions with time of passage t_0 end. Let edge progressions end at l edges from $E(G)$, incident to v. This means that at the time moment t_0, l points enter the vertex v and at the time moment $t_0 + \varepsilon$, $\rho(\Gamma, v)$ points emerge. Hence, the jump of the function $N(T)$ at the point t_0 is equal to $\delta = \rho(\Gamma, v) - l$.

For convenience, let us decompose the $\delta = \delta' + \delta'$, where $\delta' = \rho(\Gamma, v) - \rho(G, v)$, and $\delta'' = \rho(G, v) - l$. In the following statement we show that δ'' can be 0 or 1.

Now we will find at which edges the edge progressions with a fixed time of passage $t_0 \in T_{G,v}$ can end. Actually, we will find end edges for the edge progressions with a fixed set of multiplicities of the passage $k \in K_{G,v}$.

Statement 4. *Let $k \in K_{G,v}$ and e_j be an edge from G, incident to v. The following statements are equivalent:*

A) *There is no edge progression from $U_{G,v}$ with a set of multiplicities of passage k, such that e_j is the last edge of the edge progression.*

B) *$k_j = 1$ and e_j is such an isthmus of the graph G that after its removal the vertices v and s are in different connected components and the vertex v does not become isolated.*

Proof (*of Statement* 4). $B \Rightarrow A$. Suppose that after removing e_j there are two connected components: G', which contains s, and G'', which contains v and at least one edge. Then, having passed once along the edge e_j from G' to G'', we cannot end the edge progression at the edge e_j, since there is at least one edge in G'' along which it is still necessary to pass.

$A \Rightarrow B$. Firstly, let us prove that $v \neq s$. Suppose that $v = s$, then in the graph G_k all vertices have an even valency. Then there is an Eulerian cycle in it. In the Eulerian cycle, there is a sequence of the form $e_j^1, v, \ldots$. We cut this cycle in the place where there is a vertex v and get an edge progression in G, which ends at the edge e_j. A contradiction with condition A occurs.

Secondly, let us prove that $k_j = 1$. Suppose that $k_j > 1$ and consider two cases: a) $e_j = (v, s)$, b) $e_j \neq (v, s)$. In case a) we remove from the graph G_k an edge e_j^1. The resulting graph will be connected as before and all its vertices will have an even valency. In it, we construct an Eulerian cycle from s to s and add an edge e_j to it. We obtain an edge progression in G that ends at the edge e_j. That is a contradiction. In case b) we add an edge $g = (v, s)$ to the graph G_k, and then remove the edges g and e_j^1 from the vertex v and glue them in the new vertex v'. The resulting graph will be connected, all vertices will have an even valency. Let us build an Eulerian cycle. In the Eulerian cycle there is a sequence of the form e_j^1, v', g, s. We cut the cycle at the vertex v', remove the edge g and obtain an edge progresson in G of the form $(s, \dots, e_j, v)$, which contradicts condition A.

Thirdly, let us prove that e_j is an isthmus. Assume that after deleting the edge e_j the vertices v and s are in the same connected component. We consider two cases: a) $e_j = (v, s)$, b) $e_j \neq (v, s)$. In case a) we remove the edge e_j^1 from the graph G_k and obtain a connected graph, all vertices of which have an even valency. We take an Eulerian cycle in this graph, cut it at the vertex s and add the edge e_j to the end. We obtain an edge progression from s to v, which ends at the edge e_j. In case b) we remove e_j^1 from the graph G_k and obtain a connected graph whose only two vertices s and v have an odd valency. Hence, there is an edge progression from s to v, and if we add an edge e_j to its end, we get an edge progression in the graph G from s to v, which ends at the edge e_j. That is a contradiction.

Fourthly, let us show that after removing the edge e_j the vertices v and s must appear in different connected components. Suppose that after removing the edge e_j they are in one connected component. Then we have to go through the edge e_j at least twice, which contradicts the fact that $k_j = 1$.

Fifthly, let us show that after removing the edge e_j the vertex v cannot turn out to be an isolated vertex. Indeed, this would mean that v is an end-vertex in the graph G. Then any edge progression with the multiplicity of passing $k_j = 1$ must end at the vertex v, which contradicts A. □

Corollary 1. *The jump δ'' can only be 0 or 1, since for given v, G there is at most one edge defined by condition B.*

In accordance with the decomposition of the jump into two terms $\delta = \delta' + \delta''$ we decompose $N(T) = N'(T) + N''(T)$ into two terms in which we calculate the corresponding jumps.

If we now sum over all connected subgraphs G containing s, then sum over all vertices $v \in V(G)$ and sum over all times of passage of the edge progressions $T_{G,v}$, then we obtain

Theorem 1. *For a locally finite graph with edge lengths that are linearly independent over $\mathbb{Q}$, the counting function has the form $N(T) = N'(T) + N''(T)$, where*

$$N'(T) = \sum_{G \subset \Gamma}^{(1)} \sum_{v \in V(G)} \big(\rho(\Gamma, v) - \rho(G, v)\big) \sum_{c \in C_{G,v}} \# \left\{ \sum_{e_i \in E(G)} (c_i + 2n_i) t_i \leqslant T \,|\, n_i \geqslant 0 \right\}, \tag{3.1}$$

where the sum $\sum^{(1)}$ is taken over all connected finite subgraphs $G \subset \Gamma$ containing the vertex s.

$$N''(T) = \sum_{G \subset \Gamma}^{(1)} \sum_{v \in V(G)}^{(2)} \sum_{c \in C_{G,v}} \# \left\{ \sum_{e_i \in E(G), i \neq j} (c_i + 2n_i) t_i + t_j \leqslant T \,|\, n_i \geqslant 0 \right\}, \tag{3.2}$$

where the sum $\sum^{(1)}$ is taken over all connected finite subgraphs G containing s, and the sum $\sum^{(2)}$ is taken over all non-end vertices $v \in V(G)$ such that v is the end of some isthmus e_j, and when e_j is deleted, the vertices v and s are located in different connected components.

If we have an infinite graph but such that the set of edge lengths is separated from zero $(t_i > C > 0\ \forall i)$, then for each fixed T in the sum over the subgraphs $G \subset \Gamma$ there will be only a finite number of nonzero terms.

This formula for $N(T)$ differs from the formula that was presented for trees in [5]. The summation over all possible arrangements of labels on the edges is added.

To convert the formula for $N(T)$ into a more explicit form, we need to have a formula for the number of integer points in the expanding simplex. The next section will be devoted to this problem.

3.2. The Number of Natural Points of a Simplex with Real Vertices

We define the function $N_k(\lambda \,|\, w_1, \dots, w_k)$ equal to the number of nonnegative integer solutions $(n_1, \dots, n_k)$ of inequality $\sum_{i=1}^{k} n_i w_i \leqslant \lambda$ (the number of solutions of the equation $\sum_{i=1}^{k} n_i w_i = \lambda$ is taken with the weight $\frac{1}{2}$). Then for almost all $w_1, \dots, w_k$ the function $N_k(\lambda)$ is approximated to the power of the logarithm by a polynomial (see [6, 7]). Namely: $N_k(\lambda) - R_k(\lambda) = O((\log \lambda)^{k+\varepsilon})(\lambda \to \infty)$ $\forall \varepsilon > 0$, where

$$R_k(\lambda) = \frac{1}{\prod_{i=1}^{k} w_i} \sum_{s=0}^{k} \frac{\lambda^s}{s!} td_{k-s}(w_1, \dots, w_k),$$

where td_i are the Todd polynomials [1], defined by

$$\prod_{i=1}^{k} \frac{w_i z}{1 - e^{-w_i z}} = \sum_{s=0}^{\infty} z^s td_s(w_1, \dots, w_k).$$

$$R_k(\lambda \,|\, w_1, \dots, w_k) = \frac{1}{\prod_{i=1}^{k} w_i} \left(\frac{\lambda^k}{k!} + \frac{1}{2}(w_1 + \ldots + w_k) \frac{\lambda^{k-1}}{(k-1)!} \right) + o(\lambda^{k-1}).$$

This polynomial (see [7, 8]) is called Barnes' multiple Bernoulli polynomials.

Note that in order that $N(\lambda)$ can be approximated by the first two powers, it is sufficient that there are at least two incommensurate numbers w_i, w_j (see [6, 10]).

3.3. The Polynomial Approximation for $N(T)$

In what follows we assume that the graph Γ is finite. Then we can apply the result of D. C. Spencer (see [7]) and obtain the following theorem.

Theorem 2. *For a finite graph with linearly independent over $\mathbb{Q}$ edge lengths $t_1, \dots, t_E$ the counting function has a polynomial approximation $R(T)$ such that $N(T) - R(T) = O\big((\log T)^{E-1}\big)$. The polynomial $R(T)$ can be decomposed as $R(T) = R'(T) + R''(T)$, where*

$$R'(T) = \sum_{G \subset \Gamma}^{(1)} \sum_{v \in V(G)} \big(\rho(\Gamma, v) - \rho(G, v)\big) \sum_{c \in C_{G,v}} R_{|E(G)|} \left(T + \sum_{c_i = 1} t_i \,\middle|\, \{2t_i\}_{e_i \in E(G)} \right),$$

where the sum $\sum^{(1)}$ is taken over all connected finite subgraphs $G \subset \Gamma$ containing the vertex s.

$$R''(T) = \sum_{G \subset \Gamma}^{(1)} \sum_{v \in V(G)}^{(2)} \sum_{c \in C_{G,v}} R_{|E(G)|-1} \left(T + \sum_{c_i = 1, i \neq j} t_i - t_j \,\middle|\, \{2t_i\}_{e_i \in E(G), i \neq j} \right),$$

where the sum $\sum^{(1)}$ is taken over all connected finite subgraphs G containing s, and the sum $\sum^{(2)}$ is taken over all the non-end vertices $v \in V(G)$ such that v is the end of some isthmus e_j and when e_j is deleted, the vertices v and s are located in different connected components.

3.4. The Leading Coefficient $N(T)$ for a Finite Graph

Theorem 3. *Suppose that the finite graph Γ has V vertices, E edges and edge lengths $t_1, \dots, t_E$, linearly independent over $\mathbb{Q}$, then the counting function has the decomposition $N(T) = N_1 T^{E-1} + o(T^{E-1})$, where*

$$N_1 = \frac{1}{2^{V-2}(E-1)!} \frac{\sum_{i=1}^{E} t_i}{\prod_{i=1}^{E} t_i}.$$

Proof (*of Theorem* 3). If we let $G = \Gamma$ in the expression (3.1) for $N'(T)$, then $\rho(\Gamma, v) - \rho(G, v) = 0$, therefore the leading part of $N'(T)$ is determined by those connected subgraphs G for which $|E(G)| = E - 1$, that is, $N'(T) = N_1' T^{E-1} + o(T^{E-1})$. But the leading part of $N''(T)$ (see expression (3.2)) is determined by the term with $G = \Gamma$ and $N''(T) = N_1'' T^{E-1} + o(T^{E-1})$. Let us find the coefficients N_1', N_2' and $N_1 = N_1' + N_1''$.

For simplicity, we assume that s is a non-end vertex, otherwise we can shift the argument $N(T)$ to the time of the passage of the edge incident to s and assume that the source is in the neighboring vertex — this shift does not change the leading coefficient N_1.

1) Let us find N_1'. To do this, we consider all connected subgraphs of G such that $|E(G)| = E - 1$. This means that G is obtained from Γ by either deleting the end-edge e_j and an isolated vertex, or by deleting the edge e_j, which is in the cycle. In the first case, for only one vertex v: $\rho(\Gamma, v) - \rho(G, v) \neq 0$, and $\beta_1(G) = \beta_1(\Gamma)$. In the second case, for two vertices $\rho(\Gamma, v) - \rho(G, v) \neq 0$, and $\beta_1(G) = \beta_1(\Gamma) - 1$. We obtain

$$N_1' = \sum_{e_j}^{(1)} 2^{\beta_1(\Gamma)} \frac{1}{\prod_{i \neq j} 2t_i (E-1)!} + \sum_{e_j}^{(2)} 2 \cdot 2^{\beta_1(\Gamma \setminus e_j)} \frac{1}{\prod_{i \neq j} 2t_i (E-1)!},$$

where $\sum^{(1)}$ is taken over the end-edges, $\sum^{(2)}$ is taken along the edges in the cycles.

2) Let us find N_1''. The leading term is obtained if we set $G = \Gamma$ and take the sum over all isthmuses that are non-end edges. Then $\beta_1(G) = \beta_1(\Gamma)$.

$$N_1'' = \sum_{e_j} 2^{\beta_1(\Gamma)} \frac{1}{\prod_{i \neq j} 2t_i (E-1)!}.$$

3) As a result,

$$N_1 = N_1' + N_1'' = 2^{\beta_1(\Gamma)} \frac{\sum_{i=1}^{E} t_i}{2^{E-1}(E-1)! \prod_{i=1}^{E} t_i} = \frac{1}{2^{V-2}(E-1)!} \frac{\sum_{i=1}^{E} t_i}{\prod_{i=1}^{E} t_i}.$$

$\square$

Note that the coefficient N_1 has been found earlier from different considerations (see [4] and references therein). Also cf. with the theorem on the number of degeneracy classes in [12] and Sylvester's denumerant (see [3] and references therein).

3.5. *Correction to the Leading Coefficient for the Finite Graph*

Let us calculate the second term of asymptotic expansion.

Theorem 4. *Suppose that the finite graph Γ has edge lengths $t_1, \ldots, t_E$, linearly independent over $\mathbb{Q}$, and the source s is not an end-vertex, then the counting function has the decomposition $N(T) = N_1 T^{E-1} + N_2 T^{E-2} + o\big(T^{E-2}\big)$, where*

$$N_2 = \frac{1}{2^{E-2}(E-2)! \prod_{i=1}^{E} t_i} \left[-\frac{1}{2} \sum_{j=1}^{E} \sum_{i=1, i \neq j}^{E} t_j t_i \gamma_{i,j} 2^{\beta_1(\Gamma \setminus e_i)} - 2^{\beta_1(\Gamma)-1} \sum_{e_j}^{(1)} t_j^2 \right.$$
$$\left. + \sum_{\{e_i, e_j\}}^{(2)} (4-m) 2^{\beta_1(G)} t_i t_j + \sum_{\{e_i, e_j\}}^{(3)} 2^{\beta_1(G) + \delta_{i,j}} t_i t_j - \sum_{\{e_i, e_j\}}^{(4)} 2^{\beta_1(G)} t_i t_j \right].$$

Here $\gamma_{i,j} = 1$ if, after removing the edge e_i, the edge e_j and the vertex s lie in one connected component, and $\gamma_{i,j} = 0$ otherwise. Next, $\delta_{i,j} = 1$ if the unordered pair $\{e_i, e_j\}$ consists of a cyclic edge and an isthmus, and $\delta_{i,j} = 0$ otherwise. Summation $\sum^{(1)}$ is taken over the non-end isthmuses e_j. The sum $\sum^{(2)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$ such that after *removing these two edges, the graph $G = \Gamma \setminus \{e_i, e_j\}$ consists of m isolated vertices and another connected component. The sum $\sum^{(3)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$ such that after removing isolated vertices from the graph $G = \Gamma \setminus \{e_i, e_j\}$ we obtain two connected components. The sum $\sum^{(4)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$ such that they are incident to a vertex of valency 2 (where again $G = \Gamma \setminus \{e_i, e_j\}$).*

Proof (*of Theorem* 4). The coefficient N_2' is obtained as the sum $N_{2,1}' + N_{2,2}'$, where $N_{2,1}'$ is obtained from the summands with $|E(G)| = E - 1$ and $N_{2,2}'$ is obtained from the summands with $|E(G)| = E - 2$.

1) $N_{2,1}'$ is the coefficient of T^{E-2} in the decomposition

$$\sum_{e_j}^{(1)} \sum_{v \in V(G)} \big(\rho(\Gamma, v) - \rho(G, v)\big) \sum_{c \in C_{G,v}} \# \left\{ \sum_{e_i \in E(G)} 2n_i t_i \leqslant T + \sum_{c_i = 1} t_i \,|\, n_i > 0 \right\},$$

where the summation $\sum^{(1)}$ is taken over all edges e_j that are either an end-edge or an edge in the cycles. By $G = \Gamma \setminus e_j$ we denote the graph obtained by removing an edge e_j and isolated vertices from Γ:

$$\# \left\{ \sum_{e_i \in E(G)} 2n_i t_i \leqslant T + \sum_{c_i = 1} t_i \,|\, n_i > 0 \right\} = \frac{1}{\prod_{e_i \in G} 2t_i} \left(\frac{(T + \sum_{c_i=1} t_i)^{E-1}}{(E-1)!} \right.$$
$$\left. - \frac{1}{2} \sum_{e_i \in G} 2t_i \frac{T^{E-2}}{(E-2)!} + o\big(T^{E-2}\big) \right).$$

Whence the coefficient of T^{E-2} is equal to

$$- \frac{1}{2^{E-1}(E-2)!} \frac{1}{\prod_{i=1}^{E} t_i} \sum_{c_i = 2, i \neq j} t_j t_i.$$

We find that $N_{2,1}'$ is equal to

$$N_{2,1}' = \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_i t_i} \left(-\frac{1}{2} \sum_{e_j = (u,v)}^{(1)} \sum_{c \in C_{G,u} \cup C_{G,v}, c_i = 2} t_j t_i - \frac{1}{2} \sum_{e_j = (u,v)}^{(2)} \sum_{c \in C_{G,v}, c_i = 2} t_j t_i \right),$$

where the summation $\sum^{(1)}$ is taken over all edges e_j of the graph Γ contained in the cycles, and the summation $\sum^{(2)}$ is taken over the end-edges $e_j = (u, v)$, where u denotes an end-vertex.

2) The coefficient $N_{2,2}'$ is obtained from those summands in the sum over G for which $\#E(G) = E - 2$:

$$N_{2,2}' = \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_{i=1}^{E} t_i} \sum_{\{e_j, e_l\}}^{(1)} \sum_{v \in V(G)} \big(\rho(\Gamma, v) - \rho(G, v)\big) 2^{\beta_1(G)} t_j t_l,$$

where the summation $\sum^{(1)}$ is taken over all unordered pairs of edges $\{e_j, e_l\}$ such that, after removing these two edges and isolated vertices from the graph Γ, we obtain the graph G, which contains the vertex s and is connected.

Now note that the value $\sum_{v \in V(G)} \big(\rho(\Gamma, v) - \rho(G, v)\big)$ can be expressed in terms of the number of isolated points that were formed after removing from the graph Γ two edges e_i, e_j. We obtain

$$N_{2,2}' = \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_{i=1}^{E} t_i} \left(\sum_{\{e_j, e_l\}}^{(1)} (4 - m) 2^{\beta_1(G)} t_j t_l - \sum_{\{e_j, e_l\}}^{(2)} 2^{\beta_1(G)} t_j t_l \right),$$

where the sum $\sum^{(1)}$ is taken over all unordered pairs of edges $\{e_j, e_l\}$ such that, after removing these two edges from the graph Γ, we obtain a graph consisting of m isolated vertices and a graph G that contains the vertex s and is connected. The sum $\sum^{(2)}$ is taken over all unordered pairs of edges $\{e_j, e_l\}$ that are incident to a vertex of valency 2, which does not coincide with s.

3) The coefficient $N''_{2,1}$. Let $G = \Gamma$. The coefficient $N''_{2,1}$ is the coefficient of T^{E-2} in the expansion for

$$\sum_{e_j}^{(1)} \sum_{c \in C_{\Gamma,v}} \#\left\{ \sum_{e_i \in E(\Gamma), i \neq j} 2n_i t_i \leqslant T + \sum_{c_i=1, i\neq j} t_i - t_j \,\Big|\, n_i > 0 \right\},$$

where the sum $\sum^{(1)}$ is taken over all the non-end isthmuses $e_j = (v, u)$ such that, after removing the edge e_j, the vertices v and s lie in different connected components. The coefficient of T^{E-2} in the expansion of

$$\#\left\{ \sum_{e_i \in E(\Gamma), i \neq j} 2n_i t_i \leqslant T + \sum_{c_i=1, i\neq j} t_i - t_j \,\Big|\, n_i > 0 \right\}$$

is equal to

$$\frac{1}{2^{E-1}(E-2)!} \frac{1}{\prod_{i\neq j} t_i} \left(\sum_{c_i=1, i\neq j} t_i - t_j - \sum_{i \neq j} t_i \right) = -\frac{1}{2} \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_{i=1}^{E} t_i} \left(\sum_{c_i=2} t_i t_j + t_j^2 \right)$$

$$N''_{2,1} = -\frac{1}{2} \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_i t_i} \sum_{v \in V(\Gamma)}^{(1)} \sum_{c \in C_{\Gamma,v}} \left(\sum_{c_i=2} t_i t_j + t_j^2 \right),$$

where the sum $\sum^{(1)}$ is taken over all the non-end isthmuses $e_j = (v, u)$ such that, after removing the edge e_j, the vertices v and s lie in different connected components.

4) The coefficient $N''_{2,2}$. This is the coefficient of T^{E-2} in the expansion of the term, which corresponds to $|E(G)| = G - 1$.

$$N''_{2,2} = \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_i t_i} \sum_{e_j}^{(1)} \sum_{e_l}^{(2)} 2^{\beta_1(\Gamma \setminus e_j)} t_j t_l,$$

where $\sum^{(1)}$ is taken along the edges e_j, which are end-edges or are in cycles (recall that we assume that s is not an end-vertex), and the sum $\sum^{(2)}$ is taken over all isthmuses e_l in the graph $\Gamma \setminus e_j$ such that, after removing them, a component that does not contain s, contains at least one edge.

Note that in $N'_{2,1}$ the sum is taken over the end-edges and cyclic edges of the graph Γ, and in the expression for $N''_{2,1}$ the sum is taken over the non-end isthmuses. It can be combined into a single sum over all edges.

$$N'_{2,1} + N''_{2,1} = -\frac{1}{2} \frac{1}{2^{E-2}(E-2)!} \frac{1}{\prod_i t_i} \left(\sum_{e_j \in E(\Gamma)} t_j \sum_{c \in \mathbf{C}} \sum_{c_i=2} t_i + 2^{\beta_1(\Gamma)} \sum_{e_j\text{–non-end isthmus}} t_j^2 \right),$$

where

$$\mathbf{C} = \begin{cases} C_{\Gamma\setminus e_j, u}, & \text{if } e_j \text{ is an end-edge with non-end end } u \\ C_{\Gamma\setminus e_j, u} \cup C_{\Gamma\setminus e_j, v}, & \text{if } e_j = (u, v) \text{ is a cyclic edge} \\ C_{\Gamma, v}, & \text{if } e_j \text{ is a non-end isthmus and } v \text{ is the farthest from } s \text{ end.} \end{cases}$$

This expression can be further simplified.

Statement 5.

$$\sum_{e_j\in E(\Gamma)} t_j \sum_{c\in \mathbf{C}} \sum_{c_i=2} t_i = \sum_{j=1}^{E} \sum_{i=1, i\neq j}^{E} t_j t_i \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)},$$

where $\gamma_{i,j}=1$ if, after removing the edge e_i, the edge e_j and the vertex s lie in one connected component, and $\gamma_{i,j}=0$ otherwise.

Proof (*of Statement* 5). For each ordered pair of edges (e_j, e_i) we find the number of labels from $\mathbf{C}$ such that the edge e_i has label 2. Consider 9 cases, depending on whether each edge of a pair is an end-edge, cyclic, or a non-end isthmus.

1) e_j, e_i are end-edges. Then e_i with the label 2 is contained in any marking of the edges. This case gives the following contribution $2^{\beta_1(\Gamma)} t_j t_i = t_j t_i \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

2) e_j is an end-edge, e_i is cyclic. The number of markings with the label 2 on the edge e_i is equal to $2^{\beta_1(\Gamma)-1} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

3) e_j is an end-edge, e_i is a non-end isthmus. On the edge e_i there is the label 2 if after removing e_i an edge e_j and a source s lie in one connected component. Otherwise there is a label 1 on the edge e_i. Therefore, the number of markings with the label 2 on the edge e_i is equal to $\gamma_{i,j} 2^{\beta_1(\Gamma)} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

4) e_j is a cyclic edge, e_i is an end-edge. The number of markings with the label 2 on the edge e_i is equal to $2^{\beta_1(\Gamma)} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

5) $e_j \neq e_i$ are cyclic edges. The number of markings with the label 2 on the edge e_i is equal to $2^{\beta_1(\Gamma)-1} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

6) e_j is a cyclic edge, e_i is a non-end isthmus. On the edge e_i there can be the label 2 only if after removing e_i the vertex s and the edge e_j lie in one connected component. The number of markings with the label 2 on the edge e_i is equal to $2^{\beta_1(\Gamma)} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

7) e_j is a non-end isthmus, e_i is an end-edge. The number of markings with the label 2 on the edge e_i is equal to $2^{\beta_1(\Gamma)} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

8) e_j is a non-end isthmus, e_i is a cyclic edge. The number of markings with the label 2 on the edge e_i is equal to $2^{\beta_1(\Gamma)-1} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$.

9) $e_j \neq e_i$ are non-end isthmuses. The number of markings with the label 2 on the edge e_i is equal to $\gamma_{i,j} 2^{\beta_1(\Gamma)} = \gamma_{i,j} 2^{\beta_1(\Gamma\setminus e_i)}$. □

Similarly, we can prove the following statement.

Statement 6.

$$N'_{2,2} + N''_{2,2} = \frac{1}{2^{E-2}(E-2)! \prod_{l=1}^{E} t_l} \left[\sum_{\{e_i,e_j\}}^{(1)} (4-m) 2^{\beta_1(G)} t_i t_j + \sum_{\{e_i,e_j\}}^{(2)} 2^{\beta_1(G)+\delta_{i,j}} t_i t_j - \sum_{\{e_i,e_j\}}^{(3)} 2^{\beta_1(G)} t_i t_j \right],$$

where $\sum^{(1)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$ such that, after removing these two edges, the graph $G=\Gamma\setminus\{e_i,e_j\}$ consists of m isolated vertices and another connected component. The sum $\sum^{(2)}$ is taken over all unordered pairs of edges $\{e_i,e_j\}$ such that, after removing isolated vertices from the graph $G=\Gamma\setminus\{e_i,e_j\}$, we obtain two connected components. The sum $\sum^{(3)}$ is taken over all unordered pairs of edges $\{e_i,e_j\}$ such that they are incident to a vertex of valency 2 (where again $G=\Gamma\setminus\{e_i,e_j\}$).

In particular, $N'_{2,2}+N''_{2,2}$ does not depend on s.

Putting together all 4 summands $N_2 = N'_{2,1} + N'_{2,2} + N''_{2,1} + N''_{2,2}$, we obtain a theorem. □

4. EXAMPLES

4.1. The Complete Graph K_n, $n \geqslant 3$ on n edges

For the graph K_n (for which $V = n, E = n(n-1)/2$) the first two terms of the expansion of $N(T)$ do not depend on the position of the source and are equal to

$$N(T) = \frac{T^{E-1}}{2^{V-2}(E-1)!} \frac{\sum_{i=1}^{E} t_i}{\prod_{i=1}^{E} t_i} + \frac{T^{E-2}}{2^{V-2}(E-2)!} \frac{\sum_{1 \leqslant i < j \leqslant E} t_i t_j}{\prod_{i=1}^{E} t_i} + o\big(T^{E-2}\big).$$

4.2. The Cycle C_n on n Vertices

For the cycle on n vertices ($V = n, E = n$) the first two terms of the expansion of $N(T)$ do not depend on the position of the source and are equal to

$$N(T) = \frac{T^{n-1}}{2^{n-2}(n-1)!} \frac{\sum_{i=1}^{n} t_i}{\prod_{i=1}^{n} t_i} + \frac{T^{n-2}}{2^{n-2}(n-2)!} \frac{\sum_{1 \leqslant i < j \leqslant n} t_i t_j}{\prod_{i=1}^{n} t_i} + o\big(T^{n-2}\big).$$

4.3. Multiple Edges Between Two Vertices

Consider a graph for which $V = 2$, $E = n \geqslant 3$. Then

$$N(T) = \frac{T^{n-1}}{(n-1)!} \frac{\sum_{i=1}^{n} t_i}{\prod_{i=1}^{n} t_i} + \frac{T^{n-2}}{(n-2)!} \frac{\sum_{1 \leqslant i < j \leqslant n} t_i t_j}{\prod_{i=1}^{n} t_i} + o\big(T^{n-2}\big).$$

4.4. Triangle with Tail of Two Edges

Consider a triangle consisting of edges e_1, e_2, e_3. We glue the edge e_4 to the vertex where the edges e_1, e_2 meet, and then we glue the end-edge e_5 to the end of end-edge e_4. The source s is located at the vertex of valency 3, where the edges e_1, e_2, e_4 meet. Then

$$\begin{aligned} N(T) = \frac{T^4}{192} \frac{\sum_{i=1}^{5} t_i}{\prod_{i=1}^{5} t_i} + \frac{T^3}{48 \prod_{i=1}^{5} t_i} \Bigg(t_1 \left(t_2 + t_3 - \frac{1}{2} t_4 + \frac{3}{2} t_5 \right) + t_2 \left(t_3 - \frac{1}{2} t_4 + \frac{3}{2} t_5 \right) \\ + t_3 \left(-\frac{1}{2} t_4 + \frac{3}{2} t_5 \right) + t_4(-t_4 + t_5) \Bigg) + o\big(T^3\big). \end{aligned}$$

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