ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 24, No. 3, 2017, pp. 290–298. © Pleiades Publishing, Ltd., 2017.

Correction to the Leading Term of Asymptotics in the Problem of Counting the Number of Points Moving on a Metric Tree

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Received April 4, 2017

Abstract. In the problem of determining the asymptotics for the number of points moving along a metric tree, a polynomial approximation that uses Barnes' multiple Bernoulli polynomials is found. The connection between the second term of the asymptotic expansion and the graph structure is discussed.

DOI 10.1134/S1061920817030025

1. INTRODUCTION. STATEMENT OF THE PROBLEM

Let a metric tree $\Gamma = (V, E)$ be given, where, in the standard way, V is the set of vertices and E of edges. Assume that the edges $e_1, \dots, e_{|E|}$ have lengths $t_1, \dots, t_{|E|}$, respectively, and these numbers are linearly independent over $\mathbb{Q}$, which corresponds to the situation of general position. Let one of the vertices of the tree be marked; in this paper it is called the "root".

Let us consider the following dynamical system (see [1, 4]) whose investigation is motivated by the problem of studying the behavior of wave packets that are localized in a small neighborhood of a single point at the initial time moment and evolve on metric graphs or hybrid spaces (see [5, 6] and the references therein). At the initial time moment points issue from the root along all incident edges and move with unit speed. When k points, where k can take values from 1 to the valence v of the vertex v_j, occur at the vertex of the graph, v new points occur which go along all edges incident to the vertex v_j. Let $N(T)$ be the number of points that move along the graph at the time point T. The function $N(T)$ is piecewise constant. Suppose that the value of $N(T)$ at the points of discontinuity is equal to the half-sum of the limits from the left and from the right.

The leading part of the asymptotics of $N(T)$ as $T \to \infty$ for an arbitrary metric graph with incommensurable edge lengths was found earlier [2, 5].

In this paper a description of the polynomial approximation to $N(t)$ for an arbitrary tree is given. It should be noted that the corrections to the the leading coefficient of the expansion have been previously made and discussed, for some examples, in [4].

The general structure of the paper is as follows. Firstly, the results related to the approximation of the number of integer points in a simplex with real vertices are discussed. Then it is described how to reduce, in the case of a tree, the problem of calculating $N(t)$ to the calculation of the number of integer points in some family of simplices and to construct by a graph a polynomial approximating the number of moving points. The next section is devoted to the discussion of the location of the second term of the expansion and its relation to the structure of the graph. Finally, two examples, namely, a star graph and an H-graph (see [4]) are considered.

2. THE NUMBER OF POSITIVE INTEGER POINTS OF A SIMPLEX WITH REAL VERTICES

In what follows, we need the Barnes' multiple Bernoulli polynomials and the results describing the connection of these polynomials with the problem of approximating the number of points of an integer lattice that lies within an expanding simplex with real vertices. Therefore, in this section, we recall the basic definitions and give some references.

Let $w_1, \dots, w_k$ be chosen positive reals. Denote by $N_k(\lambda \,|\, w_1, \dots, w_k)$ the number of positive integer solutions $(n_1, \dots, n_k)$ of the inequality $\sum_{i=1}^k n_i w_i \leqslant \lambda$. Here the number of solutions of the equation $\sum_{i=1}^k n_i w_i = \lambda$ is taken with the weight $1/2$, i.e., at the points of discontinuity, the function $N_k(\lambda| \, w_1, \dots, w_k)$ is equal to the half-sum of the limits from the left and from the right. Sometimes later, if this does not lead to confusion, we would prefer not to write all the arguments of to ensure that the formulas are not too bulky.

As it was proved by Spencer in his thesis (see [8, 9]), for almost all $w_1, \dots, w_k$, the function $N_k(\lambda)$ is approximated by some polynomial $R_k(\lambda)$ to within a power of the logarithm,

$$N_k(\lambda) - R_k(\lambda) = O((\log \lambda)^{k+\varepsilon}) \ (\lambda \to \infty) \ \forall \varepsilon > 0. \tag{1}$$

It is important to highlight that these polynomials, apparently, first appeared in the works by Barnes (see [11] and the references therein). Spencer [8] called these polynomials "Nørlund generalizations of Bernoulli polynomials"; however, now another object is often called Nørlund polynomials (see [17]), and therefore, we use the more common version (see, e.g., [19] and the references therein): Barnes' multiple Bernoulli polynomials.

The role of these polynomials (up to relations of the form (6)) in the approximation of the number of integer points was independently discovered in the case of a polyhedron with rational vertices (see Chapter 14 of [13]) and the papers [15, 16]). In these works, the term "Todd polynomials" is used, since these polynomials were introduced (see [18]) to describe a particular case of characteristic classes, which became known later on as the Todd characteristic classes (see [20]).

In the recent paper [3], the polynomials under consideration arose again; however, in that case, estimates of the number of integer points in an expanding simplex were carried out for all algebraic w_i rather than for almost all real w_i, using the results by W. M. Schmidt [21].

It should be noted that the study of the questions related to the calculation of the number of points in expanding polyhedra is being actively continued now (see [22] and references therein).

Let us now discuss the idea of decomposition (1). Let $\{\lambda_k\}_{k=1}^\infty$ be all numbers representable in the form $n_1 w_1 + \dots + n_k w_k$ $(n_i \in \mathbb{N})$ and let a_k be the number of decompose λ_k into this sum. Then

$$\sum_{k=1}^\infty a_k e^{-\lambda_k z} = \frac{e^{-z(w_1+\dots+w_k)}}{\prod_{i=1}^k (1 - e^{-w_i z})}.$$

By the Perron formula [12], for λ such that $\lambda_n < \lambda < \lambda_{n+1}$, we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{z(\lambda - w_1 - \dots - w_k)}}{z \prod_{i=1}^k (1 - e^{-w_i z})} \, dz = \sum_{i=1}^n a_i = N_k(\lambda \,|\, w_1, \dots, w_k),$$

where $c > 0$, and the integral is taken in the sense of the principal value. Thus formula uses the integral $\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs} \frac{ds}{s}$, which is equal to $1, 1/2, 0$ for $x > 0, x = 0, x < 0$, respectively.

Further, the integration contour is deformed in such a way that it covers the negative real semiaxis and contains no other singularities of the integrand except for the origin. Under the deformation, the sum of the residues at the poles lying on the imaginary axis appears. Spencer [8] proved that, for almost all $w_1, \dots, w_k$, the sum of these residues on the imaginary axis grows not faster than $(\log \lambda)^{k+\varepsilon}$. The residue at the origin (which is a pole of order $k+1$) is a polynomial $R_k(\lambda)$ of degree k whose coefficients are symmetric functions of $w_1, \dots, w_k$. The polynomial $R_k(\lambda)$ can explicitly be expressed using the Bernoulli numbers:

$$R_k(\lambda) = \operatorname{res}_0 \frac{e^{z(\lambda - w_1 - \dots - w_k)}}{z \prod_{i=1}^k (1 - e^{-w_i z})} = [z^k] e^{z\lambda} \prod_{i=1}^k \frac{z e^{-w_i z}}{1 - e^{-w_i z}} = \frac{1}{\prod_{i=1}^k w_i} [z^k] e^{z\lambda} \prod_{i=1}^k \frac{w_i z}{e^{w_i z} - 1},$$

where $[z^k]$ stands for the coefficient at z^k. Now let us expand every factor of the form $\frac{w_i z}{e^{w_i z} - 1}$ in the Taylor series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty \frac{B_n}{n!} x^n.$$

Multiply the series:

$$\prod_{i=1}^{k} \frac{w_i z}{e^{w_i z}-1} = \sum_{n=0}^{\infty} \widetilde{B}_n(w_1,\dots,w_k) z^n, \tag{2}$$

where

$$\widetilde{B}_n(w_1,\dots,w_k) = \sum_{s_i \geqslant 0,\, s_1+\dots+s_k=n} \frac{B_{s_1}\cdots B_{s_k}}{s_1!\cdots s_k!} w_1^{s_1}\cdots w_k^{s_k}.$$

Then

$$R_k(\lambda) = \frac{1}{\prod_{i=1}^k w_i}[z^k]e^{z\lambda}\sum_{n=0}^{\infty}\widetilde{B}_n(w_1,\dots,w_k)z^n = \frac{1}{\prod_{i=1}^k w_i}\sum_{s=0}^{k}\frac{\lambda^s}{s!}\widetilde{B}_{k-s}(w_1,\dots,w_k)$$

Up to factor, this polynomial coincides with the polynomial $B_k^{(k)}(\lambda \,|\, w_1,\dots,w_k)$, which was considered by Nørlund in [10] (this polynomial arose from other considerations, as a solution of some difference equations):

$$R_k(\lambda \,|\, w_1,\dots,w_k) = \frac{B_k^{(k)}(\lambda \,|\, w_1,\dots,w_k)}{k!\,\prod_{i=1}^k w_i}. \tag{3}$$

Note that

$$R_k(\lambda \,|\, w_1,\dots,w_k) = \frac{1}{\prod_{i=1}^k w_i}\left(\frac{\lambda^k}{k!} - \frac{1}{2}(w_1+\dots+w_k)\,\frac{\lambda^{k-1}}{(k-1)!}\right) + O(\lambda^{k-2}). \tag{4}$$

One can now similarly define $\bar{N}_k(\lambda \,|\, w_1,\dots,w_k)$ as the number of nonnegative integral solutions $(n_1,\dots,n_k)$ of the inequality $\sum_{i=1}^k n_i w_i \leqslant \lambda$; here λ is the number of solutions of the equation $\sum_{i=1}^k n_i w_i = \lambda$ is taken here with the weight 1/2. Then $\bar{N}_k(\lambda)$, up to a power of logarithm, is approximated by the polynomial

$$\bar{R}_k(\lambda) = \frac{1}{\prod_{i=1}^k w_i}\sum_{s=0}^{k}\frac{\lambda^s}{s!} td_{k-s}(w_1,\dots,w_k), \tag{5}$$

where td_i are the Todd polynomials [13] defined by the equation

$$\prod_{i=1}^{k}\frac{w_i z}{1-e^{-w_i z}} = \sum_{s=0}^{\infty} z^s td_s(w_1,\dots,w_k).$$

Comparing this formula with (2), we obtain

$$\widetilde{B}_n(w_1,\dots,w_k) = (-1)^n td_n(w_1,\dots,w_k). \tag{6}$$

3. POLYNOMIAL APPROXIMATION FOR $N(T)$.

Let us choose a vertex $v \in V$ and find all the time moments at which the points issued at the initial time moment from the root r arrive at the vertex v. To this end, we consider the edge progression passed by the point from $v_{i_1} = r$ to $v_{i_k} = v$: $\mu = (v_{i_1}, e_{i_1}, v_{i_2}, \dots, e_{i_k}, v_{i_{k+1}})$. Introduce the *time of passage* of the edge progression μ:

$$t(\mu) = \sum_{i=1}^{k} t_{i_k}$$

Proposition 1. *Let $\Gamma = (V, E)$ be a tree whose lengths of edges $t_1,\dots,t_{|E|}$ are linearly independent over $\mathbb{Q}$. Then the set of times of passage of all edge progressions from $r \in V$ to $v \in V$ is $\bigsqcup_{E'} D_{E',l}$, where*

$$D_{E',l} = \left\{\sum_{e_i\in E'} 2n_i t_i - \sum_{e_i \in l} t_i : n_i > 0, e_i \in E'\right\},$$

and the union is taken over all subsets $E' \subset E$ of the edges that form subtrees $\Gamma' = (V', E')$ such that $v, r \in V'$ and $l \subset E'$ is the set of edges of a unique path from r to v (it is possible that $l = \emptyset$).

Proof. Let us consider an arbitrary edge progression μ from r to v. The set of edges of the edge progression μ forms a subtree $\Gamma' = (V', E')$ of the tree $\Gamma = (V, E)$. In this case $r, v \in V'$, and hence E' contains all edges l of the path that connect r with v (but possibly $r = v$, and then $l = \emptyset$). Note that the edge progression μ passes $2n_i - 1\,(n_i \in \mathbb{N})$ times along the edges $e_i \in l$, and passes $2n_i\,(n_i \in \mathbb{N})$ times along the edges $e_i \in E' \setminus l$. Thus, the time of passage over the edge progression is

$$t(\mu) = \sum_{e_i \in E' \setminus l} 2n_i t_i + \sum_{e_i \in l} (2n_i - 1) t_i = \sum_{e_i \in E'} 2n_i t_i - \sum_{e_i \in l} t_i, \quad \text{i.e.,} \quad t(\mu) \in D_{E',l}. \tag{7}$$

Suppose now that, conversely, a number $t' \in D_{E',l}$ is given. Since $\{t_i\}_{e_i \in E}$ are linearly independent over $\mathbb{Q}$, it follows that the number t' determines uniquely the set of edges E', l and the coefficients $n_i \in E'$. This implies that the union $\bigsqcup_{E'} D_{E',l}$ is disjoint. It is easy to show that, to a linear combination t' of this kind, there corresponds at least one edge progression μ with time of passage $t(\mu) = t'$ which begins at the root and ends at v, which passes $2n_i$ times along the edges of $E' \setminus l$ and $2n_i - 1$ times along the the edges of l. To construct such an edge progression, we first construct an edge progression μ_0 from r to v which passes exactly once along the edges of l and exactly twice along the edges of $E' \setminus l$. Adding to the edge progression μ_0 several passes forward and back along the same edge if necessary, we obtain an edge progression μ with the given time of passage $t' = t(\mu)$.

For edges of some path l, let us write $\mathrm{end}(l)$ for the end vertex of the path l, $\mathrm{last}(l)$ for the edge of l incident to $\mathrm{end}(l)$. For $E' \subset E$ and $v \in V$, we also write $\rho(E', v)$ for the number of edges in E' incident to the vertex v.

Let us write a formula for the polynomial approximation $R(T)$ of the function $N(T)$.

Theorem 1. *Let $\Gamma = (V, E)$ be a tree with lengths of edges $t_1, \dots, t_{|E|}$. Then, for almost all $t_1, \dots, t_{|E|}$ and for all $\varepsilon > 0$, $N(T) = R(T) + O((\log T)^{|E|-1+\varepsilon})$, where $R(T) = R'(T) + R''(T)$ and the polynomials R' and R'' are defined as follows:*

$$R'(T) = \sum_{E'} \sum_{l} \left(\rho(E, \mathit{end}(l)) - \rho(E', \mathit{end}(l))\right) R_{|E'|}\left(T + \sum_{e_i \in l} t_i \,\middle|\, \{2t_i\}_{e_i \in E'}\right),$$

where the first summation is carried out over all subsets of edges $E' \subset E$ that form a subtree Γ' of Γ containing the root of Γ, and the other summation is carried out over all subsets of edges $l \subset E'$ that form a path (possibly of zero length) from the root in the subtree Γ';

$$R''(T) = \sum_{E'} \sum_{l} R_{|E'|-1}\left(T + \sum_{e_i \in l \setminus \mathrm{last}(l)} t_i - t_{\mathrm{last}(l)} \,\middle|\, \{2t_i\}_{e_i \in E' \setminus \mathrm{last}(l)}\right), \tag{8}$$

where the first summation is carried out over all subsets of edges $E' \subset E$ that form a subtree Γ' of Γ containing the root of Γ, and the other summation is carried out over all subsets of edges $l \subset E'$ that form a path beginning at the root in the subtree Γ' and are such that $\rho(E', \mathit{end}(l)) > 1$. The polynomials R_k are defined by formula (3).

Proof. Let us find the number of all times in $D_{E',l}$ not exceeding T (taking into account that the number of times equal to T is taken with the weight $1/2$). To this end, one should find the number of solutions $n_i \in \mathbb{N}$ $(e_i \in E')$ of the inequality

$$\sum_{e_i \in E'} 2n_i t_i - \sum_{e_i \in l} t_i \leqslant T. \tag{9}$$

By definition, the number of positive integer solutions of this inequality is

$$N_{|E'|}\left(T + \sum_{e_i \in l} t_i \,\middle|\, \{2t_i\}_{e_i \in E'}\right).$$

Now we must multiply these numbers by the coefficients equal to the number of new points formed at these moments at the point $v = \mathrm{end}(l)$.

At times from $D_{E',l}$, new points are always formed on the edges in $E \setminus E'$ that are incident to the vertex $v = \mathrm{end}(l)$, since the edge progressions with the passage times in $D_{E',l}$ do not pass along the edges in $E \setminus E'$. That is, from the vertex v, $\rho(E, v) - \rho(E', v)$ new points start to move (where $\rho(E, v)$ is the number of edges in the set E incident to the vertex v) in the direction "from the root".

In the direction "to the root" (i.e., along the last edge $\mathrm{last}(l)$) one new point can be issued only if there is no edge progression with the time of passage $T \in D_{E',l}$ which terminates along the edge $\mathrm{last}(l)$. There is no edge progression of this kind only if the coefficient n_i in the expansion (7), where $e_i = \mathrm{last}(l)$, is equal to one (i.e., the point must pass along the edge $\mathrm{last}(l)$ only once) and $\rho(E', v) > 1$ (i.e., there are edges below the vertex v along which the point must still pass).

In accordance with these two cases, we partition $N(T)$ into the sum $N(T) = N'(T) + N''(T)$, where $N'(T)$ is the total number of new points formed at the vertices of V at moments of time less than T in the direction "from the root", and $N''(T)$ is the total number of new points formed at the vertices of V at the time moments less than T in the direction "to the root".

To find $N'(T)$ and $N''(T)$, we sum up all vertices v or, which is the same, sum up all paths l. We obtain

$$N'(T) = \sum_{E'} \sum_{l} \left(\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l))\right) N_{|E'|} \left(T + \sum_{e_i \in l} t_i \,\middle|\, \{2t_i\}_{e_i \in E'} \right)$$

where the first summation ranges over all subsets of edges $E' \subset E$ that form a subtree Γ' of Γ containing the root of Γ and the other summation ranges over all subsets of the edges $l \subset E'$ that form a path (possibly of zero length) from the root in the subtree Γ'. We also obtain

$$N''(T) = \sum_{E'} \sum_{l} N_{|E'|-1} \left(T + \sum_{e_i \in l \setminus \mathrm{last}(l)} t_i - t_{\mathrm{last}(l)} \,\middle|\, \{2t_i\}_{e_i \in E' \setminus \mathrm{last}(l)} \right), \tag{10}$$

where the first summation ranges over all subsets of edges $E' \subset E$ that form a subtree Γ' of Γ containing the root of Γ, and the other summation ranges over all nonempty subsets of edges $l \subset E'$ that form a path beginning at the root in the subtree Γ' and are such that $\rho(E', \mathrm{end}(l)) > 1$.

Replacing $N_k(T)$ by their polynomial approximation (1), we obtain the desired formula.

In the next section, we find two leading coefficients of this polynomial.

4. SECOND TERM OF THE EXPANSION

Let $|E| > 2$. Let us write the first two terms in the expansion $N(T) = N_1 T^{|E|-1} + N_2 T^{|E|-2} + o(T^{|E|-2})$. It is important to highlight that for the number of integers in the expanding simplex be representable in the same form, it is sufficient (see [8, 14]) that there are incommensurable real numbers among the lengths of the edges. In our case, this is always true, and, by the results of the previous section, this ensures the presence of such a representation for $N(t)$. The leading coefficient of the asymptotic expansion for an arbitrary metric graph with incommensurable edge lengths was found earlier (see [2, 5] and [7]). Let us write it down in the case of a tree graph:

$$N_1 = \frac{1}{(|E| - 1)!\, 2^{|E|-1}} \frac{\sum_{e_i \in E} t_i}{\prod_{e_i \in E} t_i}.$$

Let us introduce the following notation:

H is the set of end-edges of the graph Γ,

$\mathrm{up}(e)$ is the set of edges of the path leading from the root to the end of the edge e that is closer to the root ($e \notin \mathrm{up}(e)$).

Theorem 2. *Let $\Gamma = (V, E)$ be a tree with the lengths of the edges $t_1, \ldots, t_{|E|}$ linearly independent over $\mathbb{Q}$. Then*

$$N_2 = \frac{1}{(|E| - 2)!\, 2^{|E|-2} \prod_{e_i \in E} t_i} P_2(t_1, \ldots, t_{|E|}),$$

where

$$P_2(t_1,\dots,t_{|E|}) = -\frac{1}{2}\sum_{e\in E}\sum_{e_i\in E\setminus(\mathrm{up}(e)\cup e)} t_e t_i - \frac{1}{2}\sum_{e\in E\setminus H} t_e^2 + \sum_{e\in H}\sum_{f\in E\setminus e} t_e t_f.$$

Proof. Let us turn to Theorem 1 and begin by finding the first two terms in the expansion of $R'(T)$ and $R''(T)$.

I. Expansion of $R'(T)$.

In the expression for $R'(T)$, there is a summation over the subtrees $\Gamma' = (V', E')$. Note that if $\Gamma' = \Gamma$, then $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) = 0$.

Let us consider case 1: $|E'| = |E| - 1$. That is, Γ' is Γ without an end-edge e. In order to have $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) \neq 0$, it is necessary that the path from the edges of l be ended at the beginning of the end-edge $e = E \setminus E'$. Then $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) = 1$. To this case 1, there correspond the following summands in the formula for $R'(T)$:

$$\sum_{e\in H} R_{|E|-1}\left(T + \sum_{e_i\in\mathrm{up}(e)} t_i \,\middle|\, \{2t_i\}_{e_i\in E\setminus e}\right). \tag{11}$$

Using (4), we can expand (11):

$$\sum_{e\in H}\frac{1}{(|E|-1)!\prod_{e_i\in E\setminus e}2t_i}\left(\left(T+\sum_{e_i\in\mathrm{up}(e)}t_i\right)^{|E|-1} - (|E|-1)\sum_{e_i\in E\setminus e}t_i\,T^{|E|-2}\right) + O(T^{|E|-3})$$

$$= \sum_{e\in H}\frac{1}{(|E|-1)!\prod_{e_i\in E\setminus e}2t_i}\left(T^{|E|-1} - (|E|-1)\sum_{e_i\in E\setminus(\mathrm{up}(e)\cup e)}t_i\;T^{|E|-2}\right) + O(T^{|E|-3}).$$

Let us consider case 2: $|E'| = |E| - 2$. Here, to define Γ', we must delete two edges, e and f. Here we have more cases; however, in the expansion of the corresponding polynomials, we can restrict ourselves to the leading part only. Consider the following cases:

Case 2a: e and f are end-edges beginning at a common vertex. In order to have the inequality $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) \neq 0$, it is necessary that the path from the edges of l end at the common beginning of the edges e and f. Then $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) = 2$. This case gives the following contribution to $R'(T)$:

$$2\sum_{e,f} R_{|E|-2}\left(T + \sum_{e_i\in\mathrm{up}(e)} t_i \,\middle|\, \{2t_i\}_{e_i\in E\setminus\{e,f\}}\right),$$

where the summation ranges over all unordered pairs of end-edges (e, f) having a common beginning.

Case 2b: e and f are end-edges having no common beginning. In order to have the inequality $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) \neq 0$, it is necessary that the path from the edges of l be ended either at the beginning of the edge e or at the beginning of the edge f. In both cases $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) = 1$. This case gives the following contribution to $R'(T)$:

$$\sum_{e,f}\left(R_{|E|-2}\left(T + \sum_{e_i\in\mathrm{up}(e)} t_i \,\middle|\, \{2t_i\}_{e_i\in E\setminus\{e,f\}}\right) + R_{|E|-2}\left(T + \sum_{e_i\in\mathrm{up}(f)} t_i \,\middle|\, \{2t_i\}_{e_i\in E\setminus\{e,f\}}\right)\right),$$

where the sum is taken over all unordered pairs of end-edges having no common beginning.

The contributions of cases 2a and 2b can be combined by writing out the single formula

$$\sum_{e\in H}\sum_{f\in H, f\neq e}\frac{1}{(|E|-2)!\prod_{e_i\in E\setminus\{e,f\}}2t_i}\;T^{|E|-2} + O(T^{|E|-3}) \tag{12}$$

Case 2c: e and f is a *pair of end-edges*, i.e., e is an end-edge, the beginning of e coincides with the end of f and, except for f, there are no other edges incident to the beginning of the edge e. In order to have $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) \neq 0$, it is necessary that the path from the edges of l ends at the beginning of the edge f. Then $\rho(E, \mathrm{end}(l)) - \rho(E', \mathrm{end}(l)) = 1$. This case gives the following contribution to $R'(T)$:

$$\sum_{(e,f)\in W} R_{|E|-2}\left(T+\sum_{e_i\in \mathrm{up}(f)} t_i \,\Big|\, \{2t_i\}_{e_i\in E\setminus\{e,f\}}\right), \tag{13}$$

where W stands for the set of (ordered) pairs of end-edges. Expression (13) can be expanded:

$$\sum_{(e,f)\in W} \frac{1}{(|E|-2)!\prod_{e_i\in E\setminus\{e,f\}} 2t_i}\; T^{|E|-2}+O(T^{|E|-3})$$

II. Expansion of $R''(T)$.

Case 1: $\Gamma'=\Gamma$. In the formula (8), it is necessary to sum up l such that $\rho(E,\mathrm{end}(l))>1$, i.e., the last edge of l should be not an end-edge. This case gives the following contribution to $R''(T)$:

$$\sum_{e\notin H} R_{|E|-1}\left(T+\sum_{e_i\in \mathrm{up}(e)} t_i - t_e \,\Big|\, \{2t_i\}_{e_i\in E\setminus e}\right)$$

$$=\sum_{e\notin H} \frac{1}{(|E|-1)!\prod_{e_i\in E\setminus e} 2t_i}\left(\left(T+\sum_{e_i\in \mathrm{up}(e)} t_i - t_e\right)^{|E|-1} - (|E|-1)\sum_{e_i\in E\setminus e} t_i\; T^{|E|-2}\right)+O(T^{|E|-3})$$

$$=\sum_{e\notin H} \frac{1}{(|E|-1)!\prod_{e_i\in E\setminus e} 2t_i}\left(T^{|E|-1} - (|E|-1)\sum_{e_i\in E\setminus \mathrm{up}(e)} t_i\; T^{|E|-2}\right)+O(T^{|E|-3}).$$

Case 2: $|E'|=|E|-1$. The subtree Γ' is Γ without any end-edge e. The last edge $f=\mathrm{last}(l)$ in Γ' cannot be an end-edge, i.e., the edge f cannot be an end-edge in the graph Γ, and the pair of edges (e,f) must be not a pair of end-edges. Case 2 gives the following contribution to $R''(T)$:

$$\sum_{e\in H}\sum_{f\notin H,(e,f)\notin W} R_{|E|-2}\left(T+\sum_{e_i\in \mathrm{up}(f)} t_i - t_f \,\Big|\, \{2t_i\}_{e_i\in E\setminus\{e,f\}}\right)$$

$$=\sum_{e\in H}\sum_{f\notin H,(e,f)\notin W} \frac{1}{(|E|-2)!\prod_{e_i\in E\setminus\{e,f\}} 2t_i} T^{|E|-2}+O(T^{|E|-3})$$

III. Expansion of $N(T)$. Let us now sum the expansions obtained in parts I and II. The coefficient at the leading power $T^{|E|-1}$ is of the form

$$N_1=\sum_{e\in E}\frac{1}{(|E|-1)!\prod_{e_i\in E\setminus e} 2t_i}=\frac{1}{(|E|-1)!\,2^{|E|-1}}\frac{\sum_{e_i\in E} t_i}{\prod_{e_i\in E} t_i},$$

The next coefficient in the expansion, after multiplying by $(|E|-2)!\,2^{|E|-2}\prod_{e_i\in E} t_i$, is of the form

$$\begin{aligned}N_2\,(|E|-2)!\,2^{|E|-2}\prod_{e_i\in E} t_i &= -\frac{1}{2}\sum_{e\in H} t_e \sum_{e_i\in E\setminus(\mathrm{up}(e)\cup e)} t_i \;-\frac{1}{2}\sum_{e\notin H} t_e \sum_{e_i\in E\setminus \mathrm{up}(e)} t_i\\ &\quad+\sum_{e\in H}\sum_{f\in H, f\neq e} t_e t_f+\sum_{(e,f)\in W} t_e t_f+\sum_{e\in H}\sum_{f\notin H,(e,f)\notin W} t_e t_f\\ &= -\frac{1}{2}\sum_{e\in E}\sum_{e_i\in E\setminus(\mathrm{up}(e)\cup e)} t_e t_i-\frac{1}{2}\sum_{e\notin H} t_e^2+\sum_{e\in H}\sum_{f\neq e} t_e t_f.\end{aligned}$$

Corollary 1. *Let the homogeneous polynomial P_2 in the lengths of the edges of the graph be known* (*see the statement of Theorem* 2); *then the tree Γ and its root can be recovered uniquely.*

Proof. Indeed, first, by looking at the polynomial, we can find out what edges are end-edges: only the variables corresponding to the end-edges have no squares in the polynomial P_2. Then we subtract from P_2 the polynomial

$$\sum_{e\in H}\sum_{f\in E\setminus e} t_e t_f-\frac{1}{2}\sum_{e\in E\setminus H} t_e^2$$

(we know this polynomial because we know what edges are end-edges and what are not) and obtain a polynomial $\sum_{e\in E}\sum_{e_i\in E\setminus(\mathrm{up}(e)\cup e)} t_e t_i$. This means that for any vertex $v\in V$, we know the path from v to the root. Hence, the tree and its root are uniquely determined.

5. EXAMPLES

5.1. Star Graph.

Let us consider a star graph with n edges (in the graph, there is a vertex of degree n, and the remaining vertices are of valency 1). Let the root be a vertex of valency n.

A formula that expresses $N(T)$ in terms of the number of points in some family of expanding simplices was obtained in [5] in the case of a star graph. Now it can be obtained as a special case of the general theorem, Theorem 1. Also we can write the polynomial approximating $N(T)$.

The summand N'' of Theorem 1 vanishes (since there are no nonzero paths l ending at a non-end vertex), and, in the formula for N', the path l can be only zero. Let us sum up all subtrees and obtain

$$N(T) = \sum_{k=1}^{n-1}(n-k) \sum_{I\subset\{1,\dots,n\},|I|=k} \#\Big\{\sum_{i\in I} 2t_i n_i \leqslant T, n_i > 0\Big\},$$

where #{system of inequalities} stands for the number of integer solutions of the system of inequalities. It is more convenient to rewrite this formula in terms of the numbers of nonnegative integer solutions

$$N(T) = \sum_{j=1}^{n} \#\Big\{\sum_{i\in\{1,\dots,n\}\setminus\{j\}} 2t_i n_i \leqslant T, n_i \geqslant 0\Big\} + const$$

and then apply formula (5):

$$N(T) = \frac{1}{\prod_{i=1}^n t_i} \sum_{s=0}^{n-1} \frac{T^s}{2^s s!} \sum_{i=1}^{n} t_i \,\mathrm{td}_{n-s-1}(t_1,\dots,\widehat{t_i},\dots,t_n) + O((\log T)^{n-1+\varepsilon}).$$

In particular, since $\mathrm{td}_1(\xi_1,\dots,\xi_k) = \frac{1}{2}\sum_{i=1}^k \xi_i$, then the coefficient at T^{n-2} is equal to

$$N_2 = \frac{1}{(n-2)!\,2^{n-2}\prod_{i=1}^n t_i} \sum_{1\leqslant i<j\leqslant n} t_i t_j.$$

5.2. H-Graph.

Let us consider a tree with the edges $e_1,\dots,e_5$ such that the edges e_1, e_2, e_3 join at the vertex v_1 of valency 3 and the edges e_3, e_4, e_5 join at another vertex v_2 of valency 3. The remaining 4 vertices have the valency one. This is so-called “H-junction” or “H-graph”. This graph, in particular, was considered in [4].

Let us assume that at the initial time moment the point issues from the vertex v_1. Then

$$N_1 = \frac{1}{2^4 4!\prod_{i=1}^5 t_i} \sum_{i=1}^{5} t_i,$$

$$N_2 = \frac{1}{2^3 3!\prod_{i=1}^5 t_i}\left(-\frac{1}{2}t_3^2 + t_2t_4 + t_2t_5 + t_4t_5 + \frac{1}{2}t_3t_4 + \frac{1}{2}t_3t_5 + t_1t_2 + t_1t_4 + t_1t_5\right).$$

We can see from this formula how the counting function is changed when transposing the edges of the H-graph. For example, if we transpose the edges e_1 and e_5, then the difference between the counting functions of the original graph and the new graph is equal to

$$\frac{1}{96t_2t_4}\left(\frac{1}{t_1} - \frac{1}{t_5}\right)T^3 + o(T^3).$$

This formula was obtained in [4] without using Theorem 2.

6. ACKNOWLEDGMENTS

The authors express their gratitude to A. Barvinok, S. Robins, B. Borda, A. I. Esterov, A. I. Shafarevich, M. M. Skriganov, and G. Berkolaiko for very productive discussions.

The research was financially supported by the grant 16-11-10069 of the Russian Science Foundation.

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