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Polynomial approximation for the number of all possible endpoints of a random walk on a metric graph.

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Abstract

The asymptotics of the number of possible endpoints of a random walk on a metric graph with incommensurable edge lengths is found.

Keywords: metric graphs, graph polynomials, Barnes' multiple Bernoulli polynomials, random walks on graphs

1 Introduction

Let us consider a random walk (see, for example, [1]) on a finite compact metric graph (see, for example, [2]). The main difference with the often considered

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case (see, for example, [6]) is that the end point of a walk can be any point on an edge of a metric graph, and not just a vertex. Let one point move along the graph at the initial moment of time. The passage time for each individual edge is fixed. In each inner vertex, the point with some probability selects one of the edges for further movement. The reflection occurs in the vertices of valence one. Backward turns on the edges are prohibited. The aim is to analyze the asymptotic behavior of the number $N(T)$ of possible endpoints of such random walk on the metric graph as time T increases. We suppose that the probability of choosing an edge is non-zero for all edges. It is a situation of a general position. Such random walk is typical for evolution of wave packets, localized in a small neighborhood of one point at the initial moment of time (see [3] and references therein).

2 Main results

In the case of linearly independent over the rationals lengths the problem is related to the problem of counting the number of lattice points in expanding simplexes with real vertices. An asymptotic expansion for $N(T)$ using Barnes' multiple Bernoulli polynomials [9] (also known as Todd polynomials, see [8] for details) was found. Explicit formulas for the first two terms of the expansion for the counting function of the number of moving points are presented (see [5] for details). The leading term was found earlier (see [3]) and depends only on the number of vertices V, the number of edges E and the sum and the product of lengths of the edges t_j.

$$N(T) = \frac{1}{2^{V-2}(E-1)!}\frac{\sum\limits_{j=1}^{E} t_j}{\prod\limits_{j=1}^{E} t_j} T^{E-1}(1+o(1)). \tag{1}$$

The second term of the asymptotics is more complicated. It is determined by the quadratic form (i.e. E by E symmetric matrix) of the lengths of the edges of the metric graph. And it, generally speaking, depends on the starting vertex.

Let G denote a finite connected subgraph of the graph Γ, containing the starting vertex s. For the subgraph $G \subset \Gamma$ the vertex set is denoted by $V(G)$ and the set of edges by $E(G)$. For $v \in V(G)$, we denote by $\rho(G, v)$ the valency of the vertex v in the subgraph G. The vertex v is said to be the *end-vertex* in the subgraph G if $\rho(G, v) = 1$. Then the unique edge $e \in E(G)$ is called the *end-edge*. The edge $e \in E(G)$ is called the *isthmus*, if after deleting this edge the graph G splits into two connected components.

Theorem 2.1 *Suppose that the finite graph Γ has edge lengths $t_1, \dots, t_E$, linearly independent over $\mathbb{Q}$, and the starting vertex s is not an end-vertex, then the counting function has the decomposition $N(T) = N_1 T^{E-1} + N_2 T^{E-2} + o(T^{E-2})$, where*

$$N_2 = \frac{1}{2^{E-2}(E-2)! \prod_{i=1}^{E} t_i} \left[-\frac{1}{2} \sum_{j=1}^{E} \sum_{i=1, i\neq j}^{E} t_j t_i \gamma_{i,j} 2^{\beta_1(\Gamma \setminus e_i)} - 2^{\beta_1(\Gamma)-1} \sum_{e_j}^{(1)} t_j^2 \right.$$

$$\left. + \sum_{\{e_i,e_j\}}^{(2)} (4-m) 2^{\beta_1(G)} t_i t_j + \sum_{\{e_i,e_j\}}^{(3)} 2^{\beta_1(G)+\delta_{i,j}} t_i t_j - \sum_{\{e_i,e_j\}}^{(4)} 2^{\beta_1(G)} t_i t_j \right].$$

Here $\gamma_{i,j} = 1$ if, after removing the edge e_i, the edge e_j and the vertex s lie in one connected component, and $\gamma_{i,j} = 0$ otherwise. Next, $\delta_{i,j} = 1$ if e_i and e_j are cyclic edges, and $\delta_{i,j} = 0$ otherwise. Summation $\sum^{(1)}$ is taken over the non-end isthmuses e_j. The sum $\sum^{(2)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$, such that after removing these two edges, the graph $G = \Gamma \setminus \{e_i, e_j\}$ consists of m isolated vertices and another connected component. The sum $\sum^{(3)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$, such that after removing isolated vertices from the graph $G = \Gamma \setminus \{e_i, e_j\}$, we obtain two connected components. The sum $\sum^{(4)}$ is taken over all unordered pairs of edges $\{e_i, e_j\}$, such that they are incident to a vertex of valence 2 (where again $G = \Gamma \setminus \{e_i, e_j\}$).

Here $\beta_1(G)$ is the first Betti number of the graph G.

The idea of the proof consists in the following. The function $N(T)$ is non-decreasing and piecewise-constant. First, we find the points at which the jumps of $N(T)$ occur. These points are linear combinations of edge lengths with certain integer coefficients, defined first by a connected subgraph G containing the starting vertex, secondly by the vertex v of the subgraph G, and, thirdly, by a subset of the base cycles of the subgraph G . Then we find jumps of $N(T)$ at the points of discontinuity: the jump differs by at most one from the difference of the degrees of the vertex v in the original graph and the subgraph G. Thus, $N(T)$ is expressed as a linear combination of the number of solutions of certain inequalities. Then, using the result of D. Spencer (see [7]), we can replace the number of solutions of the inequality by a polynomial approximation and obtain a polynomial approximation for $N(T)$. Expanding each of the polynomials up to the first two terms of the expansion we obtain the Theorem.

Example. For the complete graph K_n (for which $E = V(V-1)/2$) the first two terms of the expansion of $N(T)$ do not depend on the position of the starting vertex and are equal to

$$N(T) = \frac{T^{E-1}}{2^{V-2}(E-1)!}\frac{\sum_{i=1}^{E} t_i}{\prod_{i=1}^{E} t_i} + \frac{T^{E-2}}{2^{V-2}(E-2)!}\frac{\sum_{1\le i<j\le E} t_i t_j}{\prod_{i=1}^{E} t_i} + o(T^{E-2})$$

The second term of the asymptotic expansion is connected with the graph structure. The graph can be recovered uniquely if the second term is known as a function of lengths in the case of a tree (see [4]).

In the case of rational lengths $N(T)$ reaches a plateau at a certain moment of time (stabilization time). The problem of finding the stabilization time is related to the problem of finding the Frobenius number (see [10] and references therein) and the Skolem-Malher-Lech theorem.

3 Further research

The problem could be similarly considered for the case of an infinite but locally finite graph. Besides, questions related to the first passage time (in the spirit of the article [11], but without backward turns on the edges) may be interesting.

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