THE LATTICE POINTS OF TETRAHEDRA

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1. This note is a sequel to my paper "On a Hardy-Littlewood problem of Diophantine approximation," *Proc. Cambridge Phil. Soc.*, XXXV (1939), 527–547, in which the two dimensional problem was considered. The consideration of higher dimensions leads to interesting new problems in an unexplored domain of Diophantine analysis. My purpose here is to point out a few of these problems.

2. Suppose that $\eta, \omega_1, \omega_2, \cdots, \omega_s$ are positive real numbers and that $r \geq 0$. Let

$$N_r^{(s)}(\eta) = \Sigma(\eta - m_1\omega_1 - \cdots - m_s\omega_s), \tag{2.1}$$

where the summation is over all $m_k \geq 1$ $(k = 1, 2, \cdots, s)$ for which

$$m_1\omega_1 + \cdots + m_s\omega_s \leq \eta. \tag{2.2}$$

Then $N_0^{(s)}(\eta)$ is the number of solutions of the inequality (2.2). In geometrical language $N_0^{(s)}(\eta)$ is the number of lattice points lying inside or on the boundary of the s-dimensional tetrahedron bounded by the s coördinate hyperplanes $x_1 = 1, x_2 = 1, \cdots, x_s = 1$ and the hyperplane

$$\omega_1x_1 + \omega_2x_2 + \cdots + \omega_sx_s = \eta. \tag{2.3}$$

$N_r^{(s)}(\eta)$ is the r − pl (perhaps fractional) integral of $N_0^{(s)}(\eta)$.

Suppose that $\alpha \geq 0$ and let $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ $(\lambda_n < \lambda_{n+1})$ be the set of numbers which are representable in the form

$$\lambda_n = \alpha + m_1\omega_1 + \cdots + m_s\omega_s \tag{2.4}$$

where $m_k \geq 0$ $(k = 1, 2, \cdots, s)$. Let a_n be the number of representations of λ_n. Then for $Re(z) > 0$

$$\frac{e^{-\alpha z}}{\prod_{k=1}^{s}(1 - e^{-\omega_k z})} = \sum_1^\infty a_n e^{-\lambda_n z} \tag{2.5}$$

where the Dirichlet's series is absolutely convergent. Taking $\alpha = \sum_1^s w_k$ we have by Perron's formula[1]

$$N_r^{(s)}(\eta) = \frac{\Gamma(1+r)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{(\eta - \sum_1^s \omega)z}}{z^{1+r}\prod_1^s(1 - e^{-\omega_k z})}\,dz \tag{2.6}$$

[1] See G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series," Cambridge Tract No. 18.

where $c > 0$. Here we suppose that $z^{1+r} = \exp\{(1+r)\log z\}$, that branch of $\log z$ being taken which is real on the real axis. When $r = 0$ the integral is defined as

$$\lim_{T\to\infty}\int_{c-iT}^{c+iT}, \tag{2.7}$$

in which case lattice points on the hyperplane (2.3) are counted with weight $\frac{1}{2}$.

Deforming the line of integration in (2.6) into a contour embracing the real axis and containing no singularity of the integrand other than the origin, we see that

$$N_r^{(s)}(\eta) = (-1)^s\zeta_s(-r, \eta \mid \omega_1, \omega_2, \cdots, \omega_s) + T_r^{(s)}(\eta) \tag{2.8}$$

where ζ_s is the s-pl ζ-function of Barnes[2] and $T_r^{(s)}(\eta)$ is a sum of terms resulting from the residues at the poles of the integrand which lie on the imaginary axis. When r is integral, $\zeta_s(-r, \eta)$ reduces to a polynomial of degree $s + r$ whose coefficients are symmetric functions of the ω's; in general $\zeta_s(-r, \eta)$ has the form

$$K_0\eta^{s+r} + K_1\eta^{s+r-1} + \cdots + K_{s+[r]}\eta^{(r)} + O(1)$$

where $(r) = r - [r]$.

The problem of approximating to $N_0^{(s)}(\eta)$ by polynomials has been considered by D. H. Lehmer[3]—in particular the polynomial

$$R_s(\eta \mid \omega_1, \omega_2, \cdots, \omega_s) = (-1)^s\zeta_s(0, \eta \mid \omega_1, \omega_2, \cdots, \omega_s) \tag{2.9}$$

is compared numerically in special cases with several others constructed by Lehmer, J. B. Rosser, and A. E. Western. We shall see below (Theorem II) that for almost all $(\omega_1, \omega_2, \cdots, \omega_s)$ R_s is asymptotically the best approximating polynomial.

We note in passing the two relations:

$$N_r^{(s)}(\eta + \omega_s) - N_r^{(s)}(\eta) = N_r^{(s-1)}(\eta), \tag{2.10}$$

$$N_r^{(s)}(\eta) = \sum_{m\geq 1} N_r^{(s-1)}(\eta - m\omega_s), \tag{2.11}$$

immediate consequences of (2.6) but obvious *a priori* from geometrical considerations.

3. The delicate features of the problem appear in connection with the term $T_r^{(s)}(\eta)$, the behavior of which is entirely dependent on the arithmetical nature of the ω's.

We divide the ω's into n classes C_k $(k = 1, 2, \cdots, n)$, ω and ω' belonging to the same class if and only if the ratio ω/ω' is rational, and denote the ω's of C_k by $\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_{m_k}^{(k)}$. The numbers $\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_{m_k}^{(k)}$ are either all rational

[2] E. W. Barnes, "On the theory of the multiple Gamma function," *Trans. Cambridge Phil. Soc.*, XIX (1904), 374–425.

[3] D. H. Lehmer, "The lattice points of an n-dimensional tetrahedron," *Duke Math. Journal*, 7(1940), 341–353.

or all irrational; if all irrational each is a rational multiple of $\omega_1^{(k)}$. Therefore in either case these numbers are of essentially the same arithmetical structure.

If $n > 1$ let $f_j(x)$ $(j = 1, 2, \cdots, n)$ be a continuous increasing function satisfying

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} \left| \sin \frac{\lambda \omega_1^{(k)} \pi}{\omega_1^{(j)}} \right|^{m_k} \geq \frac{1}{f_j(\lambda)} \tag{3.1}$$

for all integral $\lambda \geq 1$, and let $g_j(x)$ be the function inverse to f_j. If $n = 1$ we define $g(x) = g_1(x) = 1$. It is probable that the following inequality is generally valid:

$$T_r^{(s)}(\eta) = O\left[\sum_{j=1}^{n} \eta^{m_j - 1} \cdot \max\left\{\frac{\eta^{s-m_j}}{[g_j(\eta^{s-m_j})]^{1+r}}, [\nu_r(\eta)]^{s-m_j}\right\}\right] \tag{3.2}$$

where

$$\nu_0(\eta) = \log \eta, \tag{3.3}$$

and

$$\nu_r(\eta) = \begin{cases} 1, & \text{if, for some } \epsilon > 0, \quad \dfrac{\eta^{s-m_j}}{[g_j(\eta^{s-m_j})]^{1+r}} = O(\eta^{-\epsilon}) \\ \log \log \eta, & \text{otherwise} \end{cases} \tag{3.4}$$

if $r > 0$.

4. If for all $\epsilon > 0$

$$g_j(x) \prec x^{\epsilon}, \tag{4.1}$$

that is to say if $g_j(x)/x^{\epsilon} \to 0$ as $x \to \infty$, the inequality (3.2) is true. But if (4.1) is satisfied, the analysis is neither interesting nor difficult, and we therefore omit a proof of (3.2) in this case.

We have therefore, in particular, the following (best possible) result:

Theorem I. *If* $n = 1$,

$$T_r^{(s)}(\eta) = O(\eta^{s-1}); \tag{4.2}$$

if $n > 1$,

$$T_r^{(s)}(\eta) = o(\eta^{s-1}). \tag{4.3}$$

It is perhaps worthwhile to point out another result of this type for the case in which $\omega_k = \log p_k$, where $p_1, p_2, \cdots, p_s$ are distinct primes. Then $n = s$, and for any $\epsilon > 0$ we may take[4]

$$f_j(x) = Ae^{\epsilon x} \qquad (j = 1, 2, \cdots, s)$$

[4] See G. H. Hardy, *Ramanujan*, Cambridge University Press, p. 78.

where A depends on ϵ but not on x. Since (4.1) is satisfied, we have by (3.2):

$$T_r^{(s)}(\eta) = 0\left\{\frac{\eta^{s-1}}{(\log \eta)^{1+r}}\right\}. \tag{4.4}$$

This inequality, though certainly far from the real truth, is interesting for the reason that it is the best that can be deduced from the known arithmetical properties of the logarithms of primes.

5. The main interest attaches to the cases where $T_r^{(s)}(\eta)$ is relatively small, and in this direction the following result is not difficult to prove:

Theorem II. *For almost all points* $\Omega = (\omega_1, \omega_2, \cdots, \omega_s)$ *and any* $\epsilon > 0$:

$$T_0^{(s)}(\eta) = O\{(\log \eta)^{s+\epsilon}\}; \tag{5.1}$$

$$T_r^{(s)}(\eta) = O(1) \ (r > 0). \tag{5.2}$$

Since the Ω for which $n < s$ are obviously of measure zero, we may suppose in Theorem II that $n = s$. Now it is not difficult to prove that for any $\epsilon > 0$ and almost all points $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ in k-dimensional space

$$\prod_{j=1}^{k} \left| \sin m\pi\theta_j \right| \geq \frac{K}{m \log^{k+\epsilon} m} \qquad (m \geq 1; k \geq 1 \text{ and fixed}), \tag{5.3}$$

where K depends on Θ and k but not on m. Therefore if we assume (3.2) we have in place of (5.1) the stronger inequality

$$T_0^{(s)}(\eta) = O\{(\log \eta)^{s-1+\epsilon}\}. \tag{5.4}$$

But (5.4) has not been proved for $s > 2$.

6. The term $T_r^{(s)}(\eta)$ is the sum of the residues at the poles, other than the origin, of the function

$$\Gamma(1 + r) \frac{e^{(\eta - \Sigma\omega)z}}{z^{1+r} \prod_{1}^{s} (1 - e^{-\omega_k z})}. \tag{6.1}$$

The multiplicity p of the pole at $z = 2\lambda\pi i/\omega_{\nu_1}^{(j)}$ (λ integral) is equal to the number of the $\omega_\mu^{(j)}$ for which $\lambda\omega_\mu^{(j)}/\omega_{\nu_1}^{(j)}$ is an integer, and therefore $1 \leq p \leq m_j$. We denote this set of $\omega_\mu^{(j)}$ by $(\omega_{\nu_1}^{(j)}, \omega_{\nu_2}^{(j)}, \cdots, \omega_{\nu_p}^{(j)})$. Combining the residue at this pole with that at the conjugate pole $z = -2\lambda\pi i/\omega_{\nu_1}^{(j)}$, we find that the dominant term of the residue is

$$\left\{\frac{\Gamma(1 + r)}{2^{s+r-p}\pi^{1+r}\Gamma(p)} \cdot \frac{(\omega_{\nu_1}^{(j)})^{1+r}}{\prod_{l=1}^{p} \omega_{\nu_l}^{(j)}} \cdot \frac{\cos\left(\frac{2\lambda\pi}{\omega_{\nu_1}^{(j)}}\left(\eta - \frac{1}{2}\sum_{1}^{s}\omega\right) - (s - p + r + 1)\pi\right)}{\lambda^{1+r} \prod_{k=1}^{n} \prod_{\substack{\mu_k=1 \\ \mu_j \neq \nu_l}}^{m_k} \sin \frac{\lambda\omega_{\mu_k}^{(k)}\pi}{\omega_{\nu_1}^{(j)}}}\right\}\eta^{p-1}.$$

The other terms give contributions of orders η^{p-2}, η^{p-3}, $\cdots$, η, and $O(1)$. Summing over all p-pl poles it is easily shown that the resulting contribution to $T_r^{(s)}(\eta)$ is at least of order η^{p-1} for a sequence of η tending to infinity.

If $n = s$, all poles (other than the origin) are simple, and in this case writing

$$(6.2) \qquad (\varphi(\omega_1, \omega_2, \cdots, \omega_s))^* = \sum_{j=1}^{s} \varphi(\omega_j, \omega_{j+1}, \cdots \omega_s, \omega_1, \omega_2, \cdots, \omega_{j-1})$$

we have:

Theorem III. *If* $n = s$,

$$(6.3) \quad T_r^{(s)}(\eta) = \frac{\Gamma(1+r)}{2^{r+s-1}\pi^{1+r}} \left(\omega_j^r \sum_{\lambda_j=1}^{\infty} \frac{\cos\left(\dfrac{2\lambda_j \pi}{\omega_j}\left(\eta - \dfrac{1}{2}\sum_1^s \omega\right) - \dfrac{1}{2}(r+s)\pi\right)}{\lambda_j^{1+r} \prod\limits_{\substack{k=1 \\ k \neq j}}^{s} \sin \dfrac{\lambda_j \omega_k \pi}{\omega_j}} \right)^* .$$

The summation is to be effected as follows: the partial sums are formed of those terms of the s series for which $\lambda_j < \dfrac{\omega_j Y}{2\pi}$ $(j = 1, 2, \cdots, s)$, *and the limit is taken as* $Y \to \infty$ *through a sequence* (Y_j) *all of whose members differ by a number* $K(\omega_1, \omega_2, \cdots, \omega_s)$ *from any of the numbers* $\dfrac{2\lambda_j \pi}{\omega_j}$.

7. We digress in order to make one or two remarks of a formal nature, and we suppose always that $n = s$. First, if $0 < \eta < \sum_1^s \omega$, $N_r^{(s)}(\eta) = 0$, and so by (2.8) for this range of η

$$(7.1) \quad \begin{aligned} &\zeta_s(-r, \eta \mid \omega_1, \omega_2, \cdots, \omega_s) \\ &= (-1)^{s-1} \frac{\Gamma(1+r)}{2^{r+s-1}\pi^{1+r}} \left(\omega_j^r \sum_{\lambda_j=1}^{\infty} \frac{\cos\left(\dfrac{2\lambda_j\pi}{\omega_j}(\eta - \tfrac{1}{2}\sum\omega) - \tfrac{1}{2}(r+s)\pi\right)}{\lambda_j^{1+r} \prod\limits_{k \neq j} \sin \dfrac{\lambda_j \omega_k \pi}{\omega_j}} \right)^* \\ &= (S^{(j)})^* \end{aligned}$$

say. Secondly, writing $\eta = \eta_1 + \eta_2 + \cdots + \eta_s$, we suppose that $0 < \eta_k < \omega_k$. It can be shown that ζ_s is expressible as an s-pl Fourier series:

$$(7.2) \quad \begin{aligned} &\zeta_s(-r, \eta \mid \omega_1, \omega_2, \cdots, \omega_s) \\ &\qquad = \sum_{-\infty}^{+\infty} c_{\mu_1, \mu_2, \cdots, \mu_s} \exp\left\{-2\pi i \left(\frac{\mu_1 \eta_1}{\omega_1} + \frac{\mu_2 \eta_2}{\omega_2} + \cdots + \frac{\mu_s \eta_s}{\omega_s}\right)\right\} \end{aligned}$$

where

$$c_{\mu_1, \mu_2, \cdots, \mu_s} = c_{\mu_1, \mu_2, \cdots, \mu_s}(-r \mid \omega_1, \omega_2, \cdots, \omega_s)$$

$$= \frac{\Gamma(1+r)}{(2\pi)^{r+s}(i)^{s-1}} \left(\omega_j^r \frac{\exp\left\{\frac{(1+r)\pi s}{2}\right\}}{\mu_j^{1+r} \prod_{\substack{1 \\ k \neq j}}^{s} \left(\mu_j \frac{\omega_k}{\omega_j} - \mu_k \right)} \right)^* = (c^{(j)}_{\mu_1, \mu_2, \cdots, \mu_s})^*, \tag{7.3}$$

say. (7.1) and (7.2) are connected by the formal identity

$$\sum_{-\infty}^{+\infty} c^{(j)}_{\mu_1, \mu_2, \cdots, \mu_s} \exp\left\{-2\pi i \left(\frac{\mu_1 \eta_1}{\omega_1} + \frac{\mu_2 \eta_2}{\omega_2} + \cdots + \frac{\mu_s \eta_s}{\omega_s} \right)\right\} = S^{(j)}, \tag{7.4}$$

both sides of which may be infinite. These relations were pointed out by Hardy[5] in the case $s = 2$.

Simple transformations of (6.3) yield different forms. For example,

$$T_0^{(2)}(\eta) = -\frac{\omega_1 \omega_2}{\pi^2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \left(\frac{\cos 2\pi m_1 \eta/\omega_1 - \cos 2\pi m_2 \eta/\omega_2}{m_1^2 \omega_2^2 - m_2^2 \omega_1^2} \right) + O(1) \tag{7.5}$$

if the summation is effected in the right way. One is also led to (7.5) by a direct summation of lattice points using Euler's summation formula.

8. We now prove Theorem II, and base the proof on the following lemma:

Lemma I. *The series*

$$\sum_{m=2}^{\infty} \frac{1}{m \log^{k+1+\epsilon} m \prod_1^k |\sin m\pi\theta_j|} \tag{8.1}$$

is convergent for every $\epsilon > 0$ *and almost all* $(\theta_1, \theta_2, \cdots, \theta_k)$.

In fact

$$\int_0^1 \cdots \int_0^1 \sum_2^{\infty} \frac{d\theta_1 d\theta_2 \cdots d\theta_k}{m \log^{1+\epsilon} m \prod_1^k |\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon}}$$

$$< K(\epsilon) \sum_2^{\infty} \frac{1}{m \log^{1+\epsilon} m} < K(\epsilon)$$

since

$$\int_0^1 \frac{d\theta_j}{|\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon}} < K(\epsilon).$$

Hence for almost all $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ the series

$$\sum_2^{\infty} \frac{1}{m \log^{1+\epsilon} m \prod_1^k |\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon}}$$

[5] G. H. Hardy, "On double Fourier series, and especially those which represent the double Zeta-function with real and incommensurable parameters," *Quart. Journal*, XXXVII (1905), 53–79.

is convergent. From this it follows in particular that for almost all Θ there exists a $K(\Theta)$ such that

$$|\sin m\pi\theta_j| \geq Km^{-2} \qquad (j = 1, 2, \cdots, k),$$

and so

$$\left|\prod_1^k \log|\sin m\pi\theta_j|\right|^{1+\epsilon} \leq K \log^{k(1+\epsilon)} m.$$

Therefore for $\epsilon' = \epsilon/(k+1)$ and almost all Θ we have

$$\sum_2^\infty \frac{1}{m \log^{k+1+\epsilon} m \prod_1^k |\sin m\pi\theta_j|}$$

$$\leq K \sum_2^\infty \frac{1}{m \log^{1+\epsilon'} m \prod_1^k |\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon'}} < \infty,$$

and this proves the lemma.

As immediate consequences of Lemma I we have, for almost all Θ,

$$\sum_1^N \frac{1}{m \prod_1^k |\sin m\pi\theta_j|} = O(\log^{k+1+\epsilon} N); \tag{8.2}$$

$$\sum_1^\infty \frac{1}{m^{1+r} \prod_1^k |\sin m\pi\theta_j|} = O(1) \qquad (r > 0). \tag{8.3}$$

(5.2) of Theorem II follows from Theorem III and (8.3).

To prove (5.1) we write

$$T_1^{(s)}(\eta) = \int^\eta T_0^{(s)}(\eta)\, d\eta$$

$$= \frac{1}{2^s\pi^2}\left(\omega_j \sum_1^\infty \frac{\cos\left(\frac{2\lambda_j\pi}{\omega_j}\left(\eta - \frac{1}{2}\sum_1^s \omega\right) - \frac{1}{2}(s+1)\pi\right)}{\lambda_j^2 \prod_{k\neq j} \sin\frac{\lambda_j\omega_k\pi}{\omega_j}}\right)^* \tag{8.4}$$

$$= \frac{1}{2^s\pi^2}(\omega_j \sum_{\lambda_j\delta<1})^* + \frac{1}{2^s\pi^2}(\omega_j \sum_{\lambda_j\delta\geq 1})^* = I_1 + I_2,$$

and let $\Delta f = f(x+\delta) - f(x)$. Then

$$\Delta I_1 = O\left(\delta \sum_{m\delta<1} \frac{1}{m \prod_{k\neq j}\left|\sin\frac{m\pi\omega_k}{\omega_j}\right|}\right) = O\left(\delta \log^{s+\epsilon}\frac{1}{\delta}\right) \tag{8.5}$$

by (8.2). Next, summing by parts,

$$I_2 = O\left(\delta \log^{s+\epsilon} \frac{1}{\delta}\right), \tag{8.6}$$

and so, combining (8.5) and (8.6),

$$\Delta T_1^{(s)}(\eta) = O\left(\delta \log^{s+\epsilon} \frac{1}{\delta}\right). \tag{8.7}$$

Finally since $N_0^{(s)}(\eta)$ is non-decreasing, we have

$$\delta N_0^{(s)}(\eta) \leq \Delta N_1^{(s)}(\eta) \leq \delta N_0^{(s)}(\eta + \delta). \tag{8.8}$$

Also

$$\Delta \zeta_s(-1, \eta) = \delta \zeta_s(0, \eta) + O(\delta^2 \eta^{s-1}). \tag{8.9}$$

Hence by (8.7), (8.8), and (8.9):

$$\begin{aligned} N_0^{(s)}(\eta) \leq \frac{\Delta N_1^{(s)}}{\delta} &= (-1)^s \zeta_s(0, \eta) + O(\delta \eta^{s-1}) + O\left(\frac{\Delta T_1^{(s)}}{\delta}\right) \\ &= (-1)^s \zeta_s(0, \eta) + O(\delta \eta^{s-1}) + O\left(\log^{s+\epsilon} \frac{1}{\delta}\right), \end{aligned} \tag{8.10}$$

and the opposite inequality comes similarly. (5.1) follows by choosing $\delta = 1/\eta^{s-1}$.

Easy extensions of the latter half of the proof show in particular that the study of $T_r^{(s)}(\eta)$ is reduced to the consideration of the partial sums

$$\sum_1^N \frac{\cos\left(\frac{2\lambda_j \pi}{\omega_j}(\eta - \frac{1}{2}\sum \omega) - \frac{1}{2}(r + s)\pi\right)}{\lambda_j^{1+r} \prod_{k \neq j} \sin \frac{\lambda_j \omega_k \pi}{\omega_j}}. \tag{8.11}$$

9. The above analysis (though superficial) raises several interesting questions; for example:

(i) What is the smallest possible order of magnitude of $T_0^{(s)}(\eta)$ for all $\Omega = (\omega_1, \omega_2, \cdots, \omega_s)$? It is known that $| T_0^{(s)}(\eta) | > K \log \eta$ for a sequence of η tending to infinity and all Ω.

(ii) Do there exist points $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ such that

$$\prod_1^k | \sin m\pi\theta_j | \geq \frac{K}{m} \qquad (m \geq 1)$$

when $k > 1$? If not, what is the function $\varphi_k(m)$ of slowest growth for which the inequality

$$\prod_1^k | \sin m\pi\theta_j | \geq \frac{K}{m\varphi_k(m)}$$

is true for at least one Θ and all $m \geq 1$?

(iii) Are there Θ for which, uniformly in η,

$$\left| \sum_{1}^{N} \frac{e^{im\pi\eta}}{m \prod_{1}^{k} \sin m\pi\theta_j} \right| = O(\log^k N)$$

when $k > 1$? Are there Θ such that

$$\left| \sum_{1}^{N} \frac{1}{m \prod_{1}^{k} \sin m\pi\theta_j} \right| = O(1)$$

when $k > 1$?

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