A Macaulay 2 Package for Computing Rational Sum of Squares Decompositions

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Outline

1. Introduction: Sum of squares (SOS)
2. Computing rational SOS
3. Macaulay 2 SOS package
4. Outlook

Polynomial Nonnegativity

Given a polynomial $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d, consider the question

$$p(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n \quad ?$$

If true, $p(x)$ is called positive semidefinite (PSD).

- Problem NP-hard, but decidable
- Many applications

A sufficient condition: there exist polynomials $g_i(x)$ s.t.

$$p(x) = \sum_i g_i(x)^2 ,$$

i. e., $f(x)$ can be written as a sum of squares (SOS).

SOS - Background Information

Hilbert 1888: SOS=PSD only in 3 special cases

- $d = 1$, i.e., univariate polynomials
- $n = 2$, i.e., quadratic polynomials
- $d = 4, n = 2$, i.e., quartic polynomials in two variables

Famous counter examples for SOS$\neq$PSD found by

- Motzkin, Robinson, Choi and Lam

The good news: SOS condition is “easily checkable”.
$\Rightarrow$ Semidefinite Programming (SDP)

goes back to Shor (1987) and Choi, Lam, Reznick (1995)

SDP Formulation

Suppose $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ with degree $2d$.

Express $p(x)$ as a quadratic form in the monomials:

$$p(x) = z^T Q z$$

z: vector of monomials of degree less than or equal to d

- Q not unique
- Matching coefficients yields linear constraints on Q (denote this affine space by $\mathscr{L}$)

$p(x)$ is SOS if and only if there exists a Q s.t.:

$$\begin{array}{rcl} Q & \succeq & 0 \\ p(x) & = & z^T Q z \end{array}$$

SDP Formulation

$$\begin{aligned} Q &\succeq 0 \\ p(x) &= z^T Q z \end{aligned}$$

This is an SDP

- solvable in polynomial time by interior point methods (for fixed d or n)
- efficient free solvers available (e.g. SeDuMi, SDPT3, CSDP, ...)
- problem is linear in coefficients of $p(x)$
  $\Rightarrow$ optimization over coefficients possible

Numerical SOS Packages

Available numerical SOS packages:

- YALMIP (Löfberg)
  `http://control.ee.ethz.ch/~joloef/yalmip.php`
- SOSTOOLS (Prajna, Papachristodoulou, Seiler, Parrilo)
  `http://www.cds.caltech.edu/sostools/`

Both packages

- are free
- depend on MATLAB
- give numerical answers

Rational SOS

- Interior point solvers attain only finite accuracy
- We want an exact solution

Rational SOS

Given $p(x) \in \mathbb{Q}[x]$, if there exist $g_i(x) \in \mathbb{Q}[x]$ s.t.

$$p(x) = \sum g_i(x)^2,$$

then we call $p(x)$ a *rational sum of squares*.

Rational SOS - Some Background

Is every SOS $p(x) \in \mathbb{Q}[x]$ a rational SOS?

Some results:

- Landau 1906: true for $n = 1$
- Schweighofer 1999: algorithmic proof, $n = 1$
- Hillar 2007: If $p(x)$ is SOS in $K[x]$, K being a totally real number field, then $p(x)$ is a rational SOS.

In general: open problem (to our knowledge)

Rational SOS - SDP connection

Rational SOS $\Longleftrightarrow$ rational Gram matrix Q

Assume $Q \in \mathbb{Q}^{k \times k}$. Consider $L^T DL$ decomposition:

$$p(x) = z^T Q z = z^T L^T D L z = \sum_{i=1}^{k} d_i (L_i z)^2 = \sum_{i=1}^{k} d_i g_i(x)^2,$$

where $0 < d_i \in \mathbb{Q}$ and $g_i(x) \in \mathbb{Q}[x]$.

Lagrange's Theorem: d_i is a sum of at most 4 squares.

$\Longrightarrow p(x)$ is a sum of at most $4k$ squares in $\mathbb{Q}[x]$.

Question: How do we obtain a rational Q?

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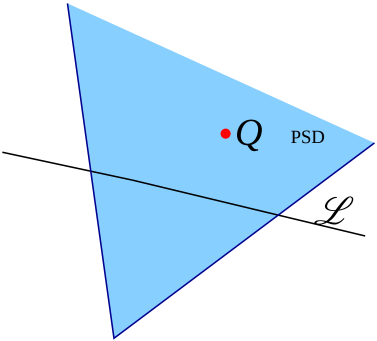
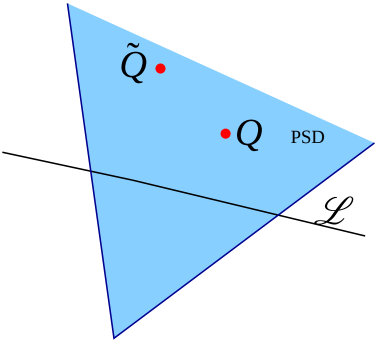
Computing Rational SOS

1. Compute numerical approximate solution Q



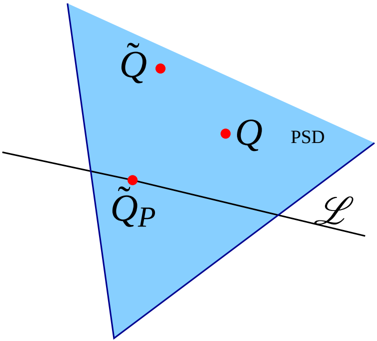
Computing Rational SOS

1. Compute numerical approximate solution Q
2. Round to rational matrix $\tilde{Q}$



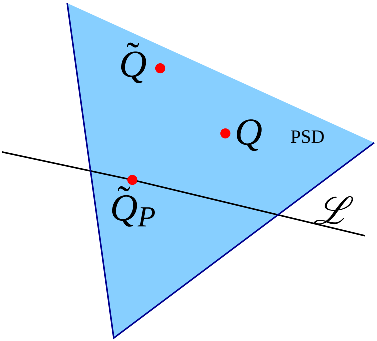
Computing Rational SOS

1. Compute numerical approximate solution Q
2. Round to rational matrix $\tilde{Q}$
3. Project back onto $\mathscr{L}$



Computing Rational SOS

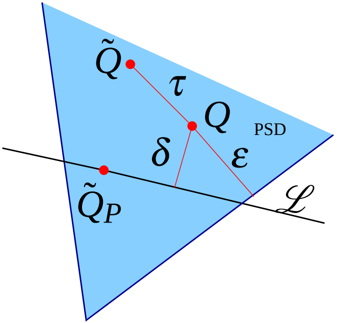
1. Compute numerical approximate solution Q
2. Round to rational matrix $\tilde{Q}$
3. Project back onto $\mathscr{L}$
4. Check whether $Q \succeq 0$
   - yes: done
   - no: increase precision back to step 2.



Computing Rational SOS - Remarks

Proposition

Assume $\delta < \varepsilon$, $\tau \leq \sqrt{\varepsilon^2 - \delta^2}$. Then the projected matrix $\tilde{Q}_P$ describes a valid a rational SOS decomposition of $p(x)$.



Hence: if SDP is strictly feasible (and $\delta < \varepsilon$), rational SOS (in principle) always possible by using sufficiently many digits.

Computing Rational SOS - Remarks

Catch: strict feasibility crucial

Partial remedy: eliminate unnecessary monomials in z

- Newton polytope
- Symmetry reduction

Caveat: not strictly feasible problems usually fail

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SOS.m2 - A M2 package

- Algorithm implemented in MACAULAY 2

  `http://www.math.uiuc.edu/Macaulay2/`
- Code may be downloaded for free from

  `http://www.control.ee.ethz.ch/~hpeyrl`

Main package: `SOS.m2`

exports the functions:

`getSOS`	computes weighted SOS decomposition
`findSOS`	same as `getSOS`, returns Gram matrix
`sumSOS`	just for checks

Usage – getSOS, sumSOS

Usage: `(g,d) = getSOS(f)`

Example

Is $p(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ a SOS?

Usage – getSOS, sumSOS

Usage: `(g,d) = getSOS(f)`

Example

Is $p(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ a SOS?

```
i1 : loadPackage"SOS";
i2 : R = QQ[x,y];
i3 : p = 4*x^4+4*x^3*y-7*x^2*y^2
        -2*x*y^3+10*y^4;
i4 : (g,d) = getSOS (p)

     ... omitted output ...
```

Usage – getSOS, sumSOS

Usage: `(g,d) = getSOS(f)`

Example

Is $p(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ a SOS?

```
             11   2     1            2    2    65
o4 = ((- --*x  - --*x*y + y , x  + --*x*y,
             25         10                     86
                  258  112
     x*y), (10, ---, ---))
                  125  215
```

Usage – getSOS, sumSOS

Usage: `(g,d) = getSOS(f)`

Example

Is $p(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ a SOS?

Hence:

$$p = 10\left(-\frac{11}{25}x^2 - \frac{1}{10}xy + y^2\right)^2 + \frac{258}{125}\left(x^2 + \frac{65}{86}xy\right)^2 + \frac{112}{215}(xy)^2$$

Let's check …

Usage – getSOS, sumSOS

Usage: `p = sumSOS(g,d)`

Example

Is $p(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ a SOS?

```
i5 : sumSOS (g,d)

        4      3      2 2       3      4
o5 = 4x  + 4x y - 7x y  - 2x*y  + 10y

o5 : QQ [x, y]
```

Usage – More Possibilities

- Linearly parameterized SOS

  Usage: `(g,d,tQ) = getSOS(p,t)`

`t`	a list of parameters
`tQ`	a list in $\mathbb{Q}$ s.t. p is SOS

- Minimizing a linear functional

  Usage: `(g,d,tQ) = getSOS(p,t,ob,rndTol=>x)`

`t`	a list of parameters
`ob`	a linear function in `t`
`rndTol`	optional minimal rounding precision in x binary digits
`tQ`	a list in $\mathbb{Q}$ s.t. p is SOS

Usage – More Possibilities

Example

Compute a lower bound on $p(x) = x^4 - 2x$

```
i5 : R = QQ[x,t];
i6 : f = x^4 - 2*x - t;
i7 : (g,d,tval) = getSOS (f,{t},-t,rndTol=>10)

     ... omitted output ...
```

Usage – More Possibilities

Example

Compute a lower bound on $p(x) = x^4 - 2x$

```
                 20480     665794809  2        2
o4 = ({x - -----, - ---------*x  + 1, x },
                 25803     419422495
  25803   83884499      41938573          9753
 {-----, ---------, --------------}, {- ----})
  20480  211378176  17179545395200        8192
```

Future

Planned features:

- Support multiple SOS constraints
  (for e.g., Lyapunov stability problems)
- Interface 3rd party SDP solvers (e.g., CSDP)
- Exploit symmetries to reduce SDP size
- Your feature request ???

The last slide

Thanks for your attention!