GEMINI test descriptions

Matthew D. Zettergren, PhD
Associate Professor of Engineering Physics
Center for Space and Atmospheric Physics
Physical Sciences Department
Embry-Riddle Aeronautical University
mattzett@gmail.com
zettergm@erau.edu

September 6, 2019

Contents

1 Purpose of this document

The **G**eospace **E**nvironment **M**odel of **I**on-**N**eutral **I**nteractions (GEMINI) is a general-purpose, three-dimensional (3D) terrestrial ionospheric model capable of describing most processes relevant to the ionosphere at medium to small spatial scales (200 m to 10000 km). The main source code repository for GEMINI can be found at `https://github.com/gemini3d/GEMINI`. This document describes the formulation of tests used to verify the GEMINI build and functioning.

2 Diffusion solver test problem

As discussed in the formulation document `https://github.com/gemini3d/GEMINI-docs/blob/master/formulation/GEMINI.pdf`, the parabolic portions of the energy equations are solved using implicit finite difference methods (including TRBDF2 and backward Euler). These are tested via solution of a simple heat equation describing the evolution of temperature $T(z,t)$ in space and time subject to uniform thermal conduction:

$$\frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial z^2} = 0. \tag{1}$$

For purposes of testing we solve this equation on the *bounded* domain $0 \leq x \leq 1$. Invoking separation of variables we presume $T(z,t) = Z(z)\mathcal{T}(t)$ and substitute back into the original equation:

$$\frac{1}{\lambda \mathcal{T}} \frac{\partial \mathcal{T}}{\partial t} - \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0. \tag{2}$$

Each term depends solely on one of the independent variables x, t, which implies that for this relation to be valid for all x, t then each term must be equal to a constant.

$$\frac{1}{\lambda \mathcal{T}} \frac{d\mathcal{T}}{dt} \quad = \quad -k^2 \tag{3}$$

$$-\frac{1}{Z} \frac{d^2 Z}{dz^2} \quad = \quad k^2. \tag{4}$$

Note also that since we have dependence on only one variable that we have converted the derivatives into *ordinary* derivatives. The solutions to these ODEs read:

$$\mathcal{T}(t) \quad = \quad A e^{-k^2 \lambda t} \tag{5}$$

$$Z(z) \quad = \quad A' \sin kz + B' \cos kz \tag{6}$$

The elemental solution (k arbitrary) is given by:

$$\tilde{T}(z,t) = Z(z)\mathcal{T}(t) = e^{-k^2 \lambda t} \left(A'' \sin kz + B'' \cos kz \right) \tag{7}$$

Further progress toward a general solution requires specific initial and boundary conditions. For our test problem we assume that the temperature goes to zero on the boundaries ($z \in \{0, 1\}$). Let us also assume that the initial temperature of the system is given by: $T(z,0) = f(z)$. First employing the condition $T(0,t) = 0$, we find that $B'' = 0$. The other boundary condition $T(1,t) = 0$ sets restrictions on the argument/eigenvalues of the sine function, namely that $k = n\pi, n \in \mathbb{Z}^+$ is a set of roots for the sine function. The elemental solution is then:

$$\tilde{T}_n(z,t) = A_n e^{-n^2\pi^2\lambda t} \sin(n\pi z) \tag{8}$$

Any integer value chosen for n results in a legitimate solution for the original partial differential equation; therefore, the general solution is a linear superposition of all such solutions.

$$T(z,t) = \sum_{n=1}^{\infty} A_n \; e^{-n^2\pi^2\lambda t} \sin(n\pi z). \tag{9}$$

The initial condition can now be applied to solve for the coefficients A_n by leveraging orthogonality of the sine functions. That is by making use of the fact that:

$$\left\langle \sin(n\pi z) | \sin(n'\pi z) \right\rangle = \int_0^1 \sin(n\pi z)\sin(n'\pi z) dz = \frac{1}{2}\delta_{nn'}, \tag{10}$$

we may produce a solution for A_n from the series solution. The initial condition is represented in summation form (a Fourer sine series) as:

$$T(z,0) = f(z) = \sum_{n=1}^{\infty} A_n \; \sin(n\pi z). \tag{11}$$

Taking the scalar product of both sides with $\sin(n'\pi z)$ gives:

$$\left\langle f(z) | \sin(n'\pi z) \right\rangle = \sum_{n=1}^{\infty} A_n \; \left\langle \sin(n\pi z) | \sin(n'\pi z) \right\rangle = \sum_n = \frac{A_n}{2}\delta_{nn'} = \frac{A_{n'}}{2} \tag{12}$$

Thus the coefficients are:

$$A_{n'} = 2\left\langle f(z) | \sin(n'\pi z) \right\rangle = 2\int_0^1 f(z)\sin(n'\pi z) dz \tag{13}$$

For purposes of testing it is easiest to pick a test problem with boundary conditions that are represented by a finite sum - one way this can be accomplished is by choosing a boundary condition that is an eigenfunction for this particular problem, say:

$$f(z) = \sin(2\pi z). \tag{14}$$

From this, and orthogonality of the sine function it follows that:

$$A_{n'} = \left\{ \begin{array}{ll} 0 & n \neq 2 \\ 1 & n = 2 \end{array} \right., \tag{15}$$

and that the general solution for this specific set of boundary and initial conditions is:

$$T(z,t) = e^{-4\pi^2\lambda t} \sin(2\pi z) \tag{16}$$

Energy diffusion in GEMINI is only performed in the field-aligned direction. Thus, for a 3D system we are effectively just executing a sequence of 1D diffusion equations.

3 Potential solver test problem

The elliptic potential solver is tested using a simplified 2D test problem, Laplace's equation:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \tag{17}$$

on the domain $0 \leq x \leq 1, 0 \leq y \leq 1$ with the boundary conditions $\Phi(x, 0) - = \Phi(0, y) = \Phi(1, y) = 0, \Phi(x, 1) = f(x)$. Exploying separation of variables $\Phi(x, y) = X(x)Y(y)$ we find the ODE solutions for $X(x)$ and $Y(y)$ to be:

$$X(x) = A \sin kx + B \cos kx \tag{18}$$

$$Y(y) = A' \sinh ky + B' \cosh ky \tag{19}$$

The boundary conditions dictate the following constraints: $\Phi(0, y) = 0 \implies B = 0, \Phi(x, 0) \implies B' = 0, \Phi(1, y) = 0 \implies k = n\pi$. Thus we have the general solution:

$$\Phi(x, y) = \sum_n A_n \sinh(n\pi y) \sin(n\pi x) \tag{20}$$

Again choosing our boundary conditions for this test problem so that only one term in the series survives we may choose:

$$f(x) = \sin(2\pi x). \tag{21}$$

Which gives:

$$f(x) = \sum_n A_n \sinh(n\pi) \sin(n\pi x), \tag{22}$$

for the potential evaluated at the non-grounded boundary. By orthogonality the coefficients are:

$$A_n = \frac{2}{\sinh(n\pi)} \langle f(x) | \sin(n\pi x) \rangle ; \tag{23}$$

however the only nonzero coefficient occurs for $n = 2$:

$$A_2 = \frac{2}{\sinh(2\pi)} \langle \sin(2\pi x) | \sin(2\pi x) \rangle = \frac{1}{\sinh(2\pi)}; \tag{24}$$

The solution for this set of boundary conditions is, thus:

$$\Phi(x, y) = \frac{\sinh(2\pi y)}{\sinh(2\pi)} \sin(2\pi x) \tag{25}$$

All potential solutions currently in GEMINI are two-dimensional so this particular test problem is representative of each.

4 Advection solver test problem

The advection solver in GEMINI deals with problems of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}\left(\rho v\right) = 0, \tag{26}$$

and higher dimensional equivalents (implemented through directional splitting). For constant velocity (assumed to be given) a simpler equation, which can be solved analytically, results.

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial z} = 0, \tag{27}$$

This equation can be seen to be equivalent to the wave equation by differentiating with respect to time and space respectively giving:

$$\frac{\partial^2 \rho}{\partial t^2} + v\frac{\partial^2 \rho}{\partial t \partial z} = 0 \tag{28}$$

$$\frac{\partial^2 \rho}{\partial z \partial t} + v\frac{\partial^2 \rho}{\partial z^2} = 0 \tag{29}$$

Eliminating the cross partial derivatives from these equations gives the familiar wave equation.

$$\frac{\partial^2 \rho}{\partial t^2} - v^2\frac{\partial^2 \rho}{\partial z^2} = 0, \tag{30}$$

The solution to this particular equation is a wave of the form:

$$\rho(z,t) = f(z - vt), \tag{31}$$

where the function f is arbitrary, generally speaking, but dictated by the specific initial conditions of the problem of interest. This solution can be verified by direct substitution, or derived by separation of variables - analogous to the test problems above. For an initial condition given by:

$$\rho(z,0) = e^{-\frac{z^2}{2\sigma_z^2}} \tag{32}$$

The solution at later times is:

$$\rho(z,t) = e^{-\frac{(z-vt)^2}{2\sigma_z^2}} \tag{33}$$

For testing purposes it is useful to numerical solve this on a periodic domain, e.g. $0 \leq x \leq 1$.

GEMINI advects mass, momentum, and energy in all three dimensions, representative of the equation:

$$\frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z} = 0 \tag{34}$$

For initial conditions given by:

$$\rho(z,0) = e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{y^2}{2\sigma_y^2}} e^{-\frac{z^2}{2\sigma_z^2}} \tag{35}$$

the solution at later time is:

$$\rho(z,t) = e^{-\frac{(x-v_x t)^2}{2\sigma_x^2}} e^{-\frac{(y-v_y t)^2}{2\sigma_y^2}} e^{-\frac{(z-v_z t)^2}{2\sigma_z^2}} \tag{36}$$

5 Error reporting

Please create an issue on our GitHub website `https://github.com/gemini3d/` if you find an error in our documentation or codes.

6 Contributors

Major contributors to GEMINI source code and testing include: M. Hirsch, G. Grubbs, and M. Burleigh.