

PFL PROJECT

Project Description

For this project, we were asked to:

- develop a set of functions to calculate the n th value of the fibonacci sequence using different techniques
- develop a module containing a new type called `BigNumber` and functions to associate it with. This new type's goal is to represent numbers as list, in order to make possible the representation of numbers without limitations of size

Function Description

Note: The time complexities described below are only rough estimates, as it is difficult to deduct given the number of prelude functions and operations with unknown or hard to picture costs and the unknown character of haskell's internal optimizations

BigNumber

`BigNumbers` are essentially numbers represented as lists, so to detach numbers from size limitations imposed by memory. We defined this type via the expression `data`, which enabled us to separate them into Positive (`Pos`) Negative (`Neg`) and Zero.

Main Functions

- **somaBN**
 - **Description:**
 - This function implements the sum of two `BigNumbers`
 - This is accomplished by implementing the same technique learned in primary school to execute the sum of two numbers digit by digit
 - First, the function divides the cases analyzing the signal of the numbers involved, utilizing properties of the sum and arithmetic expressions such as the fact that $Pos - Neg = Pos + Pos$ and $Pos + Neg = Pos - Pos$
 - Afterwards, calculations are performed, taking only the lists as arguments: the sum of the rightmost digits is executed and the carry (value that exceeds in the decimal case) is passed on to the next call
 - This process is repeated recursively until there is no more carry and one of the lists of digits has ended
 - **Time Complexity: $O(N)$**
 - N being the length of the list representing the biggest number, the sum of two digits (supposedly $O(1)$) is performed at most N times
 - Hence, $O(N) * O(1) = O(N)$
 - **Space Complexity: $O(1)$**
 - Only some variables are required to maintain the carry of the sums

- **subBN**

- **Description:**

- This function implements the subtraction between two BigNumbers
 - This is accomplished by implementing the same technique learned in primary school to execute the subtraction of two numbers digit by digit
 - First, the function divides the cases analyzing the signal of the numbers involved, utilizing properties of the subtraction and arithmetic expressions such as the fact that $\text{Pos} - \text{Neg} = \text{Pos} + \text{Pos}$ and $\text{Pos} + \text{Neg} = \text{Pos} - \text{Pos}$
 - For the classical subtraction case (same signal), the output can be predicted to be Negative, Positive or Zero by comparing the lists which represent the number
 - Afterwards, calculations are performed, taking only the lists as arguments: the procedure is similar to the sum, basing itself on digit by digit subtractions and the use of a carry
 - To clean up the list of digits of possible zeros in the beginning, a `dropWhile` is used
 - The end conditions are also the same as for the sum

- **Time Complexity: $O(N)$**

- N being the length of the list representing the biggest number, the subtraction of two digits (supposedly $O(1)$) is performed at most N times
 - The operation of comparing the two lists to determine the nature of the output costs, in the worst scenario, $O(N)$
 - $O(N) * O(1) + O(N) = 2 * O(N) = O(N)$

- **Space Complexity: $O(1)$**

- Only some variables are required to maintain the carries in the digit operations

- **mulBN**

- **Description:**

- This function implements the multiplication of two BigNumbers
 - This is accomplished by implementing the same technique learned in primary school to execute the multiplication of two numbers digit by digit
 - First, the function divides the cases analyzing the signal of the numbers involved, utilizing properties of the multiplication and arithmetic expressions such as the fact that $\text{Pos} * \text{Neg} = \text{Neg}$ and $\text{Anythin} * \text{Zero} = \text{Zero}$
 - For the calculations themselves, only the lists with digits are involved
 - The technique bases itself on multiplying the whole first number by each digit of the second number and summing the result
 - Each list arising from a multiplication between a digit and its elements, a left shift is executed, so to implement the different orders of magnitude of the digits in the second list
 - This technique is accomplished through the use of two auxiliary functions that apply recursion to achieve repetition:
 - one to execute the first step of multiplying
 - another to perform a shift to the first number and apply the sum to the results
 - Previously, another strategy was used, similar to the one applied to the division, but after some testing it was substituted due to the difference in performance

- **Time Complexity: $O(N * M)$**

- N and M being the sizes of the lists of the first and second BigNumbers respectively, the multiplication between two digits is going to be performed $N * M$ times
- **Time Complexity: $O(1)$**
 - Time complexity remains constant for the same reasons
- **divBN**
 - **Description:**
 - This functions implements the division between two BigNumbers, returning a tuple with the quocient and the remainder
 - This time, the conventional way won't help us, as it is too unreliable for big divisors
 - Instead, we used a slow division method aproach
 - Firstly, the signal of the result is asserted, as it was for the other functions, even if this time it is less relevant as divisions involving negative numbers result in unexpected results
 - Once again, for the calculations themselves, only the lists of digits are necessary. This time, the result is calculated by recursively calling a function which subtracts the the divisor to the dividend until it becomes smaller than it
 - **Time Complexity: $O(N^2/M)$**
 - This function performs subBN N/M times, being N the value of the dividend and M the value of the divisor
 - Ence, $O(N) * O(N / M) = O(N^2/M)$
 - **Space Complexity: $O(1)$**
 - Once again, no auxiliary structures needed

Auxiliary Functions

- **intToBN intToList bnToInt listToInt**
 - Conversion functions necessary to implement property testing
- **symmetricBN**
 - Function used in subBN to return the symmetric of a BigNumber, **time complexity** is $O(1)$
- **compareBN**
 - Function used in subBN to compare two lists representing bigNumbers, **time complexity** is $O(N)$ in worst scenario
- **finiteListGenBN** and **infiniListGenBN**
 - Used for the BigNumber version of the fibonacci functions, generate finite and infinit lists of BigNumbers respectively, **time complexity** is $O(N)$, N being the length of the lists (for the infinit, the length needed)
- **indexListBN**
 - Used for the BigNumber version of the fibonacci functions, accesses a certain index of a list of BigNumbers using a BigNumber index, **time complexity** is $O(I)$, I being the value of the index

Note: All these functions make use of subBN and somaBN which are $O(N)$ themselves. They are used in a best case scenario situation (sum and subtraction with and by 1), which makes them $O(1)$

Fibonacci

This module contains functions capable of calculating the nth element of the fibonacci sequence in different ways

- **fibRec**

- **Description:**

- Basic recursive formula for the fibonacci sequence. The function calculates the value of the fibonacci sequence's nth element by recursively calculating the values of the previous two elements and summing them.

- **Time Complexity: $O(2^N * N)$ for Integer, $O(2^N)$ for Int**

- $T(N) = T(N - 1) + T(N - 2) + N$, as deductible by the formula of the function (N as the cost of the sum of integers, for Int it would be 1 supposedly)
- $T(N - 1) \approx T(N - 2)$
- $T(N) = 2 * T(N - 1) + N$
- $T(N) = 2 * [2 * T(N-2) + N] + N = 4 * T(N-2) + 3N$
- $T(N) = 2^k * T(N-k) + 2^k * N - 1$
- As $T(0) = 1$, $N - k = 0 \Rightarrow N = k$
- Finally, $T(N) = 2^N + T(0) + 2^N * N - 1 = 2 * 2^N * N = 2^N * N$
- For Ints, the '+ N' in the beginning would be '+ 1', making it $O(2^N)$

- **Space Complexity: $O(1)$**

- The algorithm does not require any extra data structures

- **fibLista**

- **Description:**

- This function uses dynamic programming by saving the values of already calculated elements in a list, so to avoid the repetition of an operation. This technique of saving values that are expected to be needed later is called *Memoization*.
- The function will calculate the values of the numbers through a recursive list comprehension, where each element is equal to the sum of the previous two elements in the list.
- This technique incurs in a drastic reduction of the calculations needed and therefore a major increase in speed

- **Time Complexity: $O(N^2)$ for Integer, $O(N)$ for Int**

- The time complexity of the algorithm is expected to be quadratic as the cost is basically the cost of the sum operation times the length of the list (number of times this operation is performed), hence $O(N * N) = O(N^2)$ for Integers (sum is predicted to be $O(N)$) and $O(N)$ for Ints

- **Space Complexity: $O(N)$**

- List of size N is required for this algorithm

- **fibListaInfinita**

- **Description:**

- This function uses infinite lists to take advantage of haskell's laziness
- The list is incrementally calculated by summing itself to its tail using zipWith function (zipWith applies a certain function with two arguments to the elements of two lists)
- This leads to a similar strategy as the last function but implementing laziness

- This means haskell will only calculate the list until the element it is trying to access is defined
- **Time Complexity: $O(N^2)$ for Integer, $O(N)$ for Int**
 - Although the time of execution is much faster than the alternative before (again, hard to predict this behaviour because of haskell's internal optimizations), the time complexity of the algorithms should be the same
- **Space Complexity: $O(N)$**
 - List of size N is required for this algorithm
- **fibBigNumber variants**
 - The formulas of these functions are essentially the same as the ones described before but implementing the BigNumber type and its operations
 - Overall they are expected to be a bit slower but the principles behind the functions used for the BigNumbers should not stray too far from the ones used in the implementation of Integers

Data Comparison

While testing the Fibonacci sequence functions, we found that the best data type to use may differ with the circumstances, so we will compare these functions with 3 different types (**Int**, **Integer**, **BigNumber**).

- **Small to Medium Numbers**
 - For numbers within the range allowed for **Int**, they prove to be the best choice, since the efficiency of the algorithms is greater
 - This can be explained by the fact that both Integer and BigNumber require more complex operations to sum and subtract numbers due to their boundless nature
- **Big Numbers**
 - From the moment the result of the fibonacci functions surpasses the bounds of **Int** type (-9223372036854775808 to 9223372036854775807), the best choice would be the **Integer**
 - The BigNumber implemented in this project works fine for this case, only presenting lower efficiency

In general, while **Int** is more efficient than the others due to it's implementation, it is naturally limited, and so, not suited to use in bigger ranges.

The main difference between **Integer** and **BigNumber** is their implementation. **Integer** is a built-in data type so it's expected to be more efficiently built than our **BigNumber**.

Note: Int and Integer were tested by defining a variable of that type to save the result of the function and then comparing the results.

Fib Functions Comparison

For this project, we implemented three different functions to calculate the nth element of the fibonacci's sequence:

- Pure recursion (fibRec)
- Dynamic programming using list (fibLista)
- Using haskells infinit lists and lazy evaluation (fibListaInfinita)

As expected, their efficiency increases from first to last.

```
GHCI, version 8.8.4: https://www.haskell.org/ghc/  :? for help
Prelude> :load Fib.hs
[1 of 2] Compiling BigNumber          ( BigNumber.hs, interpreted )
[2 of 2] Compiling Fib                ( Fib.hs, interpreted )
Ok, two modules loaded.
*Fib> :set +s
*Fib> fibRec 30
832040
(3.60 secs, 1,341,618,976 bytes)
*Fib> fibLista 30
832040
(0.01 secs, 75,560 bytes)
*Fib> fibListaInfinita 30
832040
(0.01 secs, 65,752 bytes)
```

```
*Fib> fibLista 20000
2531162323732361242240155003520607291766356485802485278951929841991312781760541315230153423463758831637443488219211037689033673531462742885329724071555187618
0269316304491931589227713316423020303319710986892357808434782585027792002936356518974833096860428609963644435145587721560436914041558195729849717542785131124
798589271822959332948357853141914880538028162426090036299355691663861393997707468501618825858431232913952639355809684081297042295241855899185577230688244257
4855589237165219912238201311184749075137322987656049866305366913734924425822681338966507463855180236283582409861199212323835947891143765414913345008456022009
4557042108916377919112654751677697044773348591098225900537749329784656510238514479206013101062889578943015925020615605281312030727786774914434209218225907099
1044861732915613535546462089178845956608157282488951429635067095082420824517066760172641709112799999994114991301042453204688195828540946846321189758221507543
6515584016297874572183907949257286261608612401379639484713101138120404671732190451327881433201025184027541696124114463488665359385870910331476156665889459832
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077534028207257742577062782013083026426346781125910918430826657216971178387264317667411587435542988645609932555476084966868501858046597902171224265351332533
7142225068448611345734182791162551712881544732595854791211324236720199067223068130881919594101615600196195470024157655375073768155225684542115938685839943345
9045903975167084252876848848085910156941603293424067793097271128806817514906531652407763118308162377033463203514657531210413149191213595455280387631030665594
58918360157534002717299722428908163114472887362180552864876851136894863952297559046953957076889389788470846215864735295466789582262550423899987181413030556
3606077200388777303842236691382039774855079317816722019334601743002413449614114599189622774184251571899789862726991823692045349394665827387047326452311913376
5447653295022886429174942653014656521909469613184983671431465934965489425515981067546087342348350724207583544436107294087637975025147846254526938442435644928
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7215468122880442252343684547233668502313239328352671318130604247460452134121833305284398726438573787798499612760939462427922917659263046333084007208056631996
8563155396982340229534522115056756291536378672526950569253452200840200716112205757008412683026389952728421609942196326845753641801609918848850918582599962996
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460158324658852904105792472468108996135476637212057508192176910900422826969524389853320675970934540219240771017842159365396388086244201214597182860594018236
1421231432600427047175280272562581095378771389884614425699083511637123501952701318020403016760156706426857382069794886898263090416468516178308807650696431730
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07972198581671480395242930155809696820226169883709609037786301779702048804482662881746286685432123567873056356535776198779879981136679289548409720228335057085
87561902023411398915823487627297968947621416912816367516125096563705174220460639857683971213093125
(2.56 secs, 31,146,520 bytes)
```

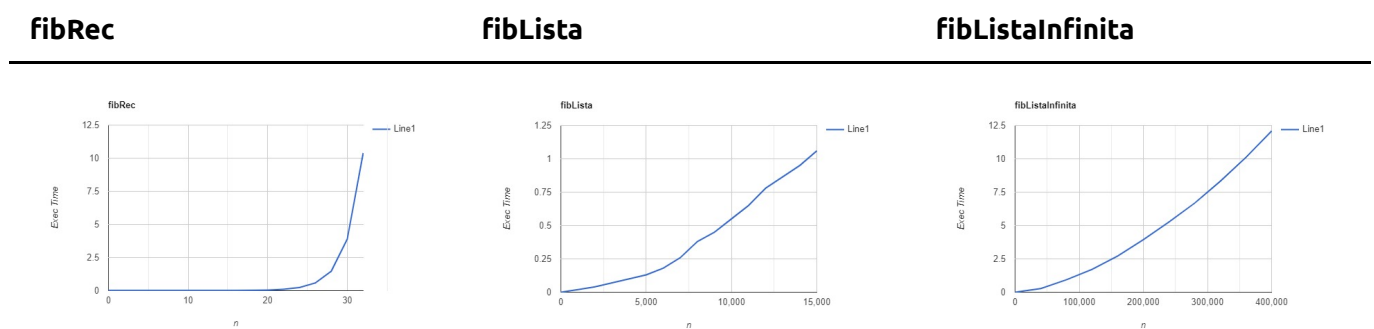
```
*Fib> fibListaInfinita 20000
2531162323732361242240155003520607291766356485802485278951929841991312781760541315230153423463758831637443488219211037689033673531462742885329724071555187618
0269316304491931589227713316423020303319710986892357808434782585027792002936356518974833096860428609963644435145587721560436914041558195729849717542785131124
798589271822959332948357853141914880538028162426090036299355691663861393997707468501618825858431232913952639355809684081297042295241855899185577230688244257
4855589237165219912238201311184749075137322987656049866305366913734924425822681338966507463855180236283582409861199212323835947891143765414913345008456022009
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1044861732915613535546462089178845956608157282488951429635067095082420824517066760172641709112799999994114991301042453204688195828540946846321189758221507543
6515584016297874572183907949257286261608612401379639484713101138120404671732190451327881433201025184027541696124114463488665359385870910331476156665889459832
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159651346656832893561887275494637328300758118515749615586692788473632798705953200998446768794571964325359733571283053902904713494802587518128903147792350810
4229525161740643984423978659638233074463100366500571977234508464710078102581304823235436518145074482824812996511614161933313389889630935320139507075992100561
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07972198581671480395242930155809696820226169883709609037786301779702048804482662881746286685432123567873056356535776198779879981136679289548409720228335057085
87561902023411398915823487627297968947621416912816367516125096563705174220460639857683971213093125
(0.02 secs, 24,688,936 bytes)
```

Table comparison

N	FibRec	FibLista	FibListaInfinita
10	0.01s	0.00s	0.00s
20	0.03s	0.00s	0.00s

N	fibRec	fibLista	fibListaInfinita
30	3.52s	0.01s	0.01s
1000	-	0.01s	0.01s
5000	-	0.14s	0.02s
10000	-	0.45s	0.02s
20000	-	3.11s	0.04s

Graphical comparison



Note: Graphics were made in a different compute, hence slightly different values; Graphs do not reflect $O(N)$ precisely due to small sample sizes

Testing

Testing of our functions was made in two ways:

- Property testing using QuickCheck
- Manual testing to ensure some edgier cases were treated of

Tests are defined in Test.hs and can be run by loading the file and running the main function

Prints of some functions, their calls, and the results of the tests


```

-- Specific tests
if not (testSomaBN 100 1) then putStrLn "Specific sum test 1 failed" else putStr ""
if not (testSomaBN 100 (-1)) then putStrLn "Specific sum test 2 failed" else putStr ""
if not (testSomaBN 1 100) then putStrLn "Specific sum test 3 failed" else putStr ""
if not (testSomaBN (-100) 1) then putStrLn "Specific sum test 4 failed" else putStr ""
if not (testSomaBN 97 25) then putStrLn "Specific sum test 5 failed" else putStr ""
if not (testSomaBN 103241 (-32)) then putStrLn "Specific sum test 6 failed" else putStr ""
if not (testSomaBN (-103241) (-32)) then putStrLn "Specific sum test 7 failed" else putStr ""
if not (testSomaBN 150 0) then putStrLn "Specific sum test 8 failed" else putStr ""
if not (testSomaBN 0 (-231)) then putStrLn "Specific sum test 9 failed" else putStr ""
if not (testSomaBN 0 0) then putStrLn "Specific sum test 10 failed" else putStr ""
if not (testSomaBN 1 (-1)) then putStrLn "Specific sum test 11 failed" else putStr ""
if not (testSomaBN 10 (-100)) then putStrLn "Specific sum test 12 failed" else putStr ""

if not (testSubBN 10 (-100)) then putStrLn "Specific subtraction test 1 failed" else putStr ""
if not (testSubBN 20 20) then putStrLn "Specific subtraction test 2 failed" else putStr ""
if not (testSubBN 100 20) then putStrLn "Specific subtraction test 3 failed" else putStr ""
if not (testSubBN (-10) (-100)) then putStrLn "Specific subtraction test 4 failed" else putStr ""
if not (testSubBN 50 20) then putStrLn "Specific subtraction test 5 failed" else putStr ""
if not (testSubBN 50 100) then putStrLn "Specific subtraction test 6 failed" else putStr ""
if not (testSubBN 100 (-100)) then putStrLn "Specific subtraction test 7 failed" else putStr ""
if not (testSubBN 24 3) then putStrLn "Specific subtraction test 8 failed" else putStr ""
if not (testSubBN 7890 3450) then putStrLn "Specific subtraction test 9 failed" else putStr ""
if not (testSubBN 9001 983) then putStrLn "Specific subtraction test 10 failed" else putStr ""

if not (testMulBN 24 3) then putStrLn "Specific multiplication test 1 failed" else putStr ""
if not (testMulBN 203 24) then putStrLn "Specific multiplication test 2 failed" else putStr ""
if not (testMulBN 2487 (-3)) then putStrLn "Specific multiplication test 3 failed" else putStr ""
if not (testMulBN 2 598) then putStrLn "Specific multiplication test 4 failed" else putStr ""
if not (testMulBN 19 (-54)) then putStrLn "Specific multiplication test 5 failed" else putStr ""
if not (testMulBN 0 20) then putStrLn "Specific multiplication test 6 failed" else putStr ""
if not (testMulBN 57 987) then putStrLn "Specific multiplication test 7 failed" else putStr ""
if not (testMulBN 124 0) then putStrLn "Specific multiplication test 8 failed" else putStr ""
if not (testMulBN 0 0) then putStrLn "Specific multiplication test 9 failed" else putStr ""
if not (testMulBN (-251) (-3)) then putStrLn "Specific multiplication test 10 failed" else putStr ""
if not (testMulBN (-24) 345) then putStrLn "Specific multiplication test 11 failed" else putStr ""

testBNToInt :: Int → Bool
testBNToInt n = show n == BigNumber.output (intToBN n)

testIntToBN :: Int → Bool
testIntToBN n = n == bnToInt (intToBN n)

testSomaBN :: Int → Int → Bool
testSomaBN a b = a + b == bnToInt (somaBN (intToBN a) (intToBN b))

testSubBN :: Int → Int → Bool
testSubBN a b = a - b == bnToInt (subBN (intToBN a) (intToBN b))

testMulBN :: Int → Int → Bool
testMulBN a b = a * b == bnToInt (mulBN (intToBN a) (intToBN b))

testDivBN :: Int → Int → Property
testDivBN a b = (a ≥ 0) && (b > 0) ⇒ div a b == bnToInt (fst res) && mod a b == bnToInt (snd res)
  where res = divBN (intToBN a) (intToBN b)

testDivBN2 :: Int → Int → Bool
testDivBN2 a b = div a b == bnToInt (fst res) && mod a b == bnToInt (snd res)
  where res = divBN (intToBN a) (intToBN b)

```



```
-- Property testing
quickCheck (withMaxSuccess 100 testBNTToInt)
quickCheck (withMaxSuccess 100 testIntToBN)
quickCheck (withMaxSuccess 10000 testSomaBN)
quickCheck (withMaxSuccess 10000 testSubBN)
quickCheck (withMaxSuccess 10000 testMulBN)
quickCheck (withMaxSuccess 10000 testDivBN)

*Fib> :load Test.hs
[1 of 3] Compiling BigNumber      ( BigNumber.hs, interpreted )
[2 of 3] Compiling Fib              ( Fib.hs, interpreted )
[3 of 3] Compiling Main            ( Test.hs, interpreted )
Ok, three modules loaded.
(0.18 secs,)
*Main> main
Output test passed
+++ OK, passed 100 tests.
+++ OK, passed 100 tests.
+++ OK, passed 10000 tests.
+++ OK, passed 10000 tests.
+++ OK, passed 10000 tests.
+++ OK, passed 10000 tests; 30700 discarded.
+++ OK, passed 20 tests; 117 discarded.
+++ OK, passed 20 tests; 133 discarded.
+++ OK, passed 20 tests; 108 discarded.
+++ OK, passed 20 tests; 122 discarded.
+++ OK, passed 20 tests; 92 discarded.
+++ OK, passed 20 tests; 119 discarded.
(4.06 secs, 1,605,710,168 bytes)
```

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