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Over a thousand new periodic orbits of a planar three-body system with unequal masses

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Abstract

The three-body problem is common in astronomy, examples of which are the solar system, exoplanets, and stellar systems. Due to its chaotic characteristic, discovered by Poincaré, only three families of periodic three-body orbits were found in 300 years, until 2013 when Šuvakov and Dmitrašinović (2013, Phys. Rev. Lett., 110, 114301) found 13 new periodic orbits of a Newtonian planar three-body problem with *equal* mass. Recently, more than 600 new families of periodic orbits of triple systems with equal mass were found by Li and Liao (2017, Sci. China-Phys. Mech. Astron., 60, 129511). Here, we report 1349 new families of planar periodic orbits of the triple system where two bodies have the same mass and the other has a different mass. None of the families have ever been reported, except the famous "figure-eight" family. In particular, 1223 among these 1349 families are entirely new, i.e., with newly found "free group elements" that have been never reported, even for three-body systems with equal mass. It has been traditionally believed that triple systems are often unstable if they are non-hierarchical. However, all of our new periodic orbits are in non-hierarchical configurations, but many of them are either linearly or marginally stable. This might inspire the long-term astronomical observation of stable non-hierarchical triple systems in practice. In addition, using these new periodic orbits as initial guesses, new periodic orbits of triple systems with three unequal masses can be found by means of the continuation method, which is more general and thus should have practical meaning from an astronomical viewpoint.

Key words: celestial mechanics — chaos — methods: numerical

1 Introduction

The three-body problem was first formulated by Newton (1687). Due to its sensitive dependence on initial conditions (SDIC), first discovered by Poincaré (1890) and which was called the "butterfly-effect" later by Lorenz (1963), it is very difficult to obtain the trajectories of a three-body system accurately: in the 300 years, only three families of periodic three-body orbits were reported—until 2013, when Šuvakov and Dmitrašinović (2013) discovered 13 new periodic orbits of a Newtonian planar three-body problem with *equal* mass. Dmitrašinović, Šuvakov, and Hudomal (2014) further investigated gravitational waves of these periodic three-body systems. Iasko and Orlov (2014) found nine new periodic orbits of three-body systems with *equal* mass, and

Šuvakov (2014) gained 11 solutions in the vicinity of the figure-eight orbits with *equal* mass. In addition, Hudomal (2015) reported 25 families of periodic orbits of Newtonian planar triple systems with *equal* mass. Rose (2016) obtained 90 periodic, collisionless orbits with *equal* mass in the case of isosceles collinear configurations. Recently, Li and Liao (2017) found more than 600 new families of planar triple systems with *equal* mass. Note that all of these researchers focused on periodic three-body problems with *equal* mass. However, the triple system with *unequal* mass is more general. Galán et al. (2002) studied the stability of a figure-eight orbit (Moore 1993; Chenciner & Montgomery 2000) with unequal masses. Doedel et al. (2003) obtained periodic orbits from the figure-eight as the mass of one body is varied. Yan (2015) investigated the spatial isosceles three-body problem with unequal masses in the case of one body moving up and down on a vertical line. It is a pity that little attention has been paid searching for the collisionless periodic orbits of a planar triple system with *unequal* mass. So, in this paper, we present some new periodic orbits of Newtonian planar triple systems with unequal masses.

2 Numerical searching for periodic orbits

The motions of the planar triple system are described by the Newtonian second law and gravitational law:

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1,\, j\neq i}^{3} \frac{G m_j (\boldsymbol{r}_j - \boldsymbol{r}_i)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}, \tag{1}$$

where $\boldsymbol{r}_i$ and m_i denote vector position and mass of the ith body ($i = 1, 2, 3$), G is the Newtonian gravity coefficient, and the dot denotes the derivative with respect to the time t. We consider a planar three-body system with zero angular momentum and two *unequal* masses ($m_1 = m_2 \neq m_3$) in the case of $G = 1$ and the initial conditions in the case of the isosceles collinear configurations:

$$\begin{cases} \boldsymbol{r}_1(0) = (x_1, x_2) = -\boldsymbol{r}_2(0),\ \boldsymbol{r}_3(0) = (0, 0), \\ \dot{\boldsymbol{r}}_1(0) = \dot{\boldsymbol{r}}_2(0) = (v_1, v_2),\ \dot{\boldsymbol{r}}_3(0) = -\frac{m_1+m_2}{m_3}\dot{\boldsymbol{r}}_1(0). \end{cases} \tag{2}$$

With these configurations, if $\boldsymbol{r}_i(t)$ ($i = 1, 2, 3$) denotes a periodic orbit with the period T of a triple system, then

$$\boldsymbol{r}'_i(t') = \boldsymbol{r}_i(t),\ \boldsymbol{v}'_i(t') = \alpha \boldsymbol{v}_i(t),\ t' = \frac{t}{\alpha},\ m'_i = \alpha^2 m_i, \tag{3}$$

has the same periodic orbit with a different period $T' = T/\alpha$ for arbitrary $\alpha > 0$. Therefore, we need consider only the case of $m_1 = m_2 = 1$ with various values of m_3 and the Newtonian constant of gravitation is equal to unity. Without loss of generality, we investigated only the seven cases of $m_3 = 0.5, 0.75, 2, 4, 5, 8$, and 10. We search for periodic three-body orbits within the period $T \in (0, 200]$ by means of the grid-search method and the Newton–Raphson method. Obviously, more periodic orbits can be found in a similar way if more values of m_3 are considered. Thus, in theory, an unlimited number of families of periodic orbits of three-body systems should exist in the case of $m_1 = m_2 \neq m_3$.

For any given m_3, the orbits are determined by the four parameters (x_1, x_2, v_1, v_2) of the initial condition equation (2). We write $y(t) = [\boldsymbol{r}_1(t), \dot{\boldsymbol{r}}_1(t)]$. We search for approximated initial conditions to satisfy the equation $y(T_0) - y(0) = 0$. We set $x_1 = -1$, $x_2 = 0$ and search for initial velocities in a square plane: $v_1 \in [0, 1]$ and $v_2 \in [0, 1]$. We set 4000×4000 grid points in the search plane. With these 16 million initial conditions, the differential equations of a triple system are integrated to time $T_0 = 200$ by means of the ODE solver dop853 (Hairer et al. 1993). We choose the initial conditions and the period T_0 as the candidates when the return proximity function

$$|y(t) - y(0)| = \sqrt{\sum_{i=1}^{4} [y_i(t) - y_i(0)]^2} \tag{4}$$

is less than 10^{-1}. Then we use the Newton–Raphson method (Farantos 1995; Lara & Pelaez 2002; Abad et al. 2011) to improve these candidates. The candidates are corrected until equation (4) is less than 10^{-3}. As mentioned by Li and Liao (2017), many periodic orbits might be lost by means of traditional algorithms in double precision. Thus, we further integrate the equations of motion by means of a "clean numerical simulation" (CNS) (Liao 2009, 2014, 2017; Liao & Wang 2014; Liao & Li 2015; Lin et al. 2017) with negligible numerical noises in a finite interval of time, which is based on the arbitrary order of the Taylor expansion method (Barton et al. 1971; Corliss & Chang 1982; Chang & Corliss 1994; Barrio et al. 2005) in a multiple precision (MP) (Oyanarte 1990; Viswanath 2004), plus a convergence verification using an additional computation with smaller numerical noises. We consider a periodic orbit to be located if equation (4) is lower than 10^{-6}.

At this time, the initial positions $\boldsymbol{r}_1 = (x_1, x_2)$ are in the neighborhood of $(-1, 0)$. There are scaling laws for the three-body problem: $\boldsymbol{r} \to \beta \boldsymbol{r}$, $t \to \beta^{3/2} t$, and then $\boldsymbol{v} \to \boldsymbol{v}/\sqrt{\beta}$. Therefore, with coordinate transformation and the above scaling laws, the initial positions of the body-1, 2, and 3 can be enforced to $(-1, 0), (1, 0)$, and (0,0), respectively. In this way, the periodic orbits are only dependent upon two physical parameters (v_1, v_2), i.e., the initial velocity of body-1.

Table 1. Initial conditions and periods of six new periodic three-body orbits with unequal mass.*

Class and number	v_1	v_2	T	T^*	L_f	Stability	ν_1	ν_2	μ
$\mathrm{I.A}_{34}^{\mathrm{i.c.}}(0.5)$	0.2668455153	0.0138391891	63.2419174415	77.282	68	S	0.347100	0.000669	—
$\mathrm{I.A}_{68}^{\mathrm{i.c.}}(0.5)$	0.213841083	0.054293840	83.847290765	118.112	104	M	0.407680	—	0.001088
$\mathrm{I.B}_{59}^{\mathrm{i.c.}}(0.75)$	0.410137872	0.134189417	121.097636144	183.067	102	U	—	—	2.719062
$\mathrm{I.A}_{1}^{\mathrm{i.c.}}(2)$	0.664910758	0.832416786	12.648906151	42.121	8	M	0.459173	—	0.000668
$\mathrm{II.D}_{2}^{\mathrm{i.c.}}(2)$	0.305722433	0.521512426	8.823706765	64.567	12	U	—	—	7.849260
$\mathrm{I.A}_{2}^{\mathrm{i.c.}}(4)$	0.991198122	0.711947212	17.650780784	276.852	24	M	0.439114	—	0.001243

*In the case of $\boldsymbol{r}_1(0) = (-1, 0) = -\boldsymbol{r}_2(0)$, $\dot{\boldsymbol{r}}_1(0) = (v_1, v_2) = \dot{\boldsymbol{r}}_2(0)$, and $\boldsymbol{r}_3(0) = (0, 0)$, $\dot{\boldsymbol{r}}_3(0) = (-2v_1m_1/m_3, -2v_2m_2/m_3)$ when $G = 1$ and $m_1 = m_2 = 1$, where $T^* = T|E|^{3/2}$ is its scale-invariant period, L_f is the length of the free group word (element). Here, the superscript "i.c." means the initial conditions in the case of a "isosceles collinear" configuration. The two largest winding numbers ν_1, ν_2 are defined by the linear stability coefficients $\lambda_j = \exp(2\pi i\nu_j)$, and the largest Lyaponuv exponent μ is defined by $\lambda = \exp(\pm\mu)$. The stability of periodic orbits can be classified as linear stable (S), marginal (M), and linear unstable (U).

3 Results

The periodic orbits are identified by Montgomery's topological method (Montgomery 1998; Šuvakov 2014). The shape sphere coordinates of the positions of the triple system are defined by the unit vector:

$$(n_x, n_y, n_z) = \left[\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{R^2}, \frac{\lambda^2-\rho^2}{R^2}, \frac{2(\boldsymbol{\rho}\times\boldsymbol{\lambda})\cdot\boldsymbol{e}_z}{R^2}\right] \tag{5}$$

where $\boldsymbol{\rho} = (1/\sqrt{2})(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, $\boldsymbol{\lambda} = (1/\sqrt{6})(\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3)$, and the hyper-radius $R = \sqrt{\rho^2 + \lambda^2}$. With these definitions, a periodic trajectory of the three-body system corresponds to a closed curve on the sphere surface with three punctures (two-body collision points). A map projection is obtained by projecting points on the sphere from a "north pole" (one of three punctures) to a plane. A closed curve on the sphere surface is mapped to a closed curve on the plane with the other two punctures. Then, the topology of a closed curve on the plane is defined by the letters a (clockwise around the right-hand side puncture), b (counter-clockwise around the left-hand side puncture), and their inverses $a^{-1} = A$, $b^{-1} = B$.

The periodic orbits can be divided into four classes based on their geometric and algebraic symmetries:

- (A) symmetry with $a\leftrightarrow A$ and $b\leftrightarrow B$,
- (B) symmetry with $a\leftrightarrow b$ and $A\leftrightarrow B$,
- (C) free group elements are not symmetric with either (A) or (B) or (D),
- (D) symmetry with $a\leftrightarrow B$ and $b\leftrightarrow A$.

Note that the symmetries (A), (B), and (C) were reported by Šuvakov and Dmitrašinović (2013), but (D) has been never reported and thus is completely new.

Within the period $T \in (0, 200]$, we find 565, 401, 237, 85, 35, 17, and nine families of periodic orbits in the cases of $m_3 = 0.5, 0.75, 2, 4, 5, 8$, and 10, respectively. To the best of our knowledge, nearly all of them have never been reported, except the figure-eight family (Doedel et al. 2003). In particular, 1223 of these 1349 families of periodic orbits are entirely new, i.e., with completely new "free group elements" that have been never reported even for a three-body system with equal mass (Li & Liao 2017). These 1349 families of the collisionless periodic orbits can be divided into seven classes (I.A, I.B, I.C, II.A, II.B, II.C, and II.D), as listed in table S I–LXXXI in supplementary data.[1] The initial velocities of the periodic three-body orbits are listed in tables S I–XXIX in the supplementary data.[1] The free group elements of the 1349 families of periodic three-body obits are listed in tables S XXIX–LXXXI in the supplementary data.[1] We show six new periodic three-body solutions in table 1, and their trajectories are shown in figure 1. The movies of these 1349 periodic three-body orbits are shown online.[2] It should be emphasized that all of these periodic orbits are in non-hierarchical configuration. Note that Doedel et al. (2003) found periodic orbits which have the same topology as the figure-eight as the mass of one body is varied. Here we also find that some periodic orbits have the same free group element for different m_3 values. For instance, as shown in figure 2, the periodic orbits have the same free group element ($BaBabAbA$) for $m_3 = 0.5, 0.75$, and 1.

As mentioned by Li and Liao (2017), the scale-invariant average period $\bar{T}^* = \bar{T}|E|^{3/2}$ is close to a *universal* constant in the case of $m_1 = m_2 = m_3 = 1$, i.e., $\bar{T}^* \approx 2.433 \pm 0.075$, with the average period $\bar{T} = T/L_f$, where L_f is the length of free group element of periodic three-body orbits. Here, the scale-invariant average periods are approximately equal to 1.14, 1.77, 2.43, 5.39, 11.53, 14.64, 23.99, and 30.25 in the case of $m_1 = m_2 = 1$ and $m_3 = 0.5, 0.75, 2, 4, 5, 8$, and 10, respectively. They agree well with the formula $\bar{T}^* = 3.074m_3 - 0.617$, as shown in figure 3, with the standard deviation $\sigma = 0.135$. It suggests that the scale-invariant average period should increase linearly with respect to m_3 (when $m_1 = m_2 = 1$), which can be regarded as a

[1] Supplementary data is available only on the online edition.

[2] ⟨http://numericaltank.sjtu.edu.cn/three-body/three-body-unequal-mass.htm⟩.

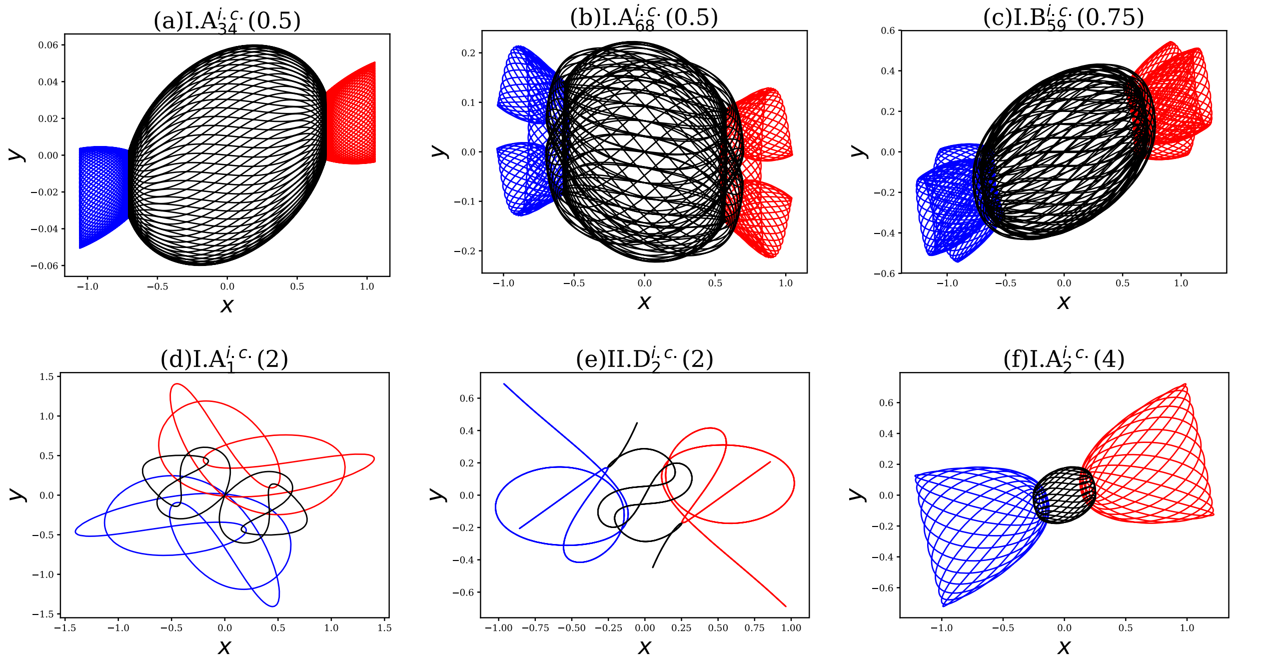

Fig. 1. Trajectory of six new families of periodic orbits in the case of $m_1 = m_2 \neq m_3$. Blue line: body-1, red line: body-2, black line: body-3. (Color online)

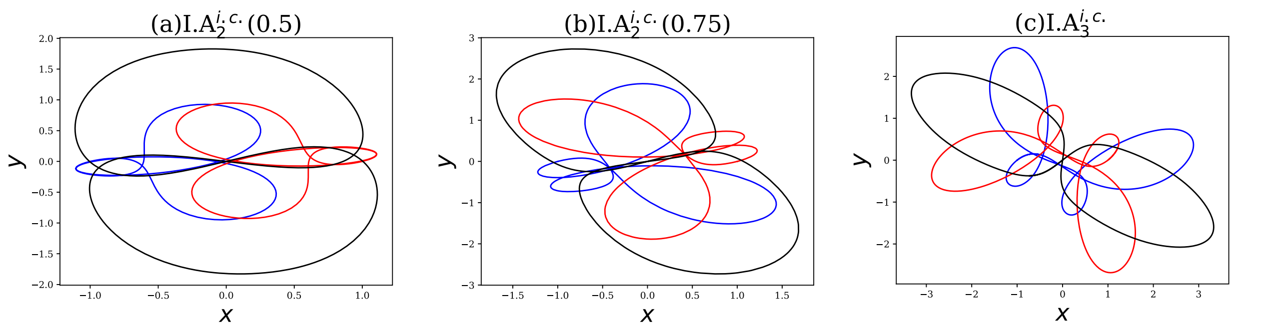

Fig. 2. Periodic orbits with the same free group element in the case of $m_1 = m_2 = 1$ and $m_3 = 0.5$, 0.75, and 1. Blue line: body-1, red line: body-2, black line: body-3. (Color online)

kind of generalized Kepler's third law (Li & Liao 2017; Dmitrašinović & Šuvakov 2015).

4 Stability

The linear stability of a periodic orbit is determined by the eigenvalues λ of the monodromy matrix (Richter et al. 1993). The eigenvalues λ are in pairs or quadruplets (λ, λ^{-1}, λ^*, λ^{*-1}). They fall into the following categories.

- (1) Elliptical, if $\lambda = \exp(\pm 2\pi i\nu)$ with the winding number ν being real. All the eigenvalues are on the unit circle.
- (2) Marginally stable if $\lambda = +1$ ($\lambda = -1$).
- (3) Hyperbolic if $\lambda = \exp(\pm\mu)$ [$\lambda = -\exp(\pm\mu)$] where $\mu > 0$ (real) is the Lyapunov exponent of the periodic orbit.
- (4) Loxodromic if $\lambda = \exp(\pm\mu \pm i\nu)$ with μ, ν being real numbers.

Simó (2002) announced that the figure-eight has elliptical stability with $\lambda_j = \exp(2\pi i\nu_j)$, with $\nu_1 = 0.00842272$, $\nu_2 = 0.29809253$. Dmitrašinović et al. (2017) found 18 linearly stable periodic orbits of the three-body problem with equal mass. We list the two largest winding numbers ν_1, ν_2 and the largest Lyapunov exponent μ of these 1349 families of periodic orbits in the supplementary data.[1] Here, the periodic orbits are regarded to be marginally stable if the largest Lyapunov exponent $\mu \leq 0.05$ and unstable if $\mu > 0.05$. Among these 1349 families of periodic orbits, there are 23 linearly stable orbits, 545 marginally stable orbits, and 781 linearly unstable orbits, as shown in the supplementary data.[1] In particular, 23 families are linearly stable and 545 families have marginal stability, and all of them are in non-hierarchical configuration. Note that these 23 linearly stable periodic orbits have different topological classes to those linearly stable periodic orbits with equal mass reported by Dmitrašinović et al. (2017).

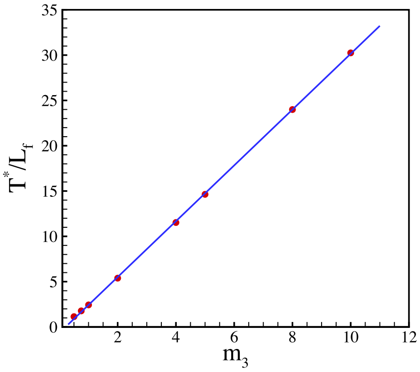

Fig. 3. Scale-invariant average period $\bar{T}^* = T^*/L_f$ versus m_3 in the case of $m_1 = m_2 = 1$, where L_f is the length of free group element and $T^* = T|E|^{3/2}$. Symbols: computed results; line: $T^*/L_f = 3.074\, m_3 - 0.617$. (Color online)

It is traditionally believed that triple systems are often unstable if they are non-hierarchical (Reipurth & Mikkola 2012). However, all of our newly found periodic orbits are non-hierarchical; in addition, 23 families among them are linearly stable and 545 families have marginal stability. Note that many of these 1349 new periodic orbits are rather complicated: they are seemingly in disorder if the observation time is not long enough. Therefore, our new findings of the 1349 periodic orbits might inspire the long-term astronomical observation of stable non-hierarchical triple systems.

5 Finding periodic orbits of a three-body system with different masses

Note that our newly found 1349 periodic orbits of the triple system have two bodies with equal mass, say, $m_1 = m_2 = 1 \neq m_3$. Using each of these orbits as an initial approximation, we can search for periodic orbits of a three-body system with completely different masses (say, $m_1 \neq m_2 \neq m_3$) by means of the numerical continuation method (Allgower & Georg 1990).

The continuation method (Allgower & Georg 1990) of a natural parameter is a way of computing periodic solutions of a nonlinear equation, $\boldsymbol{x}' = F(\boldsymbol{x}, \lambda)$, where λ is a physical parameter. All that is required is that an initial solution $\boldsymbol{x}_0$ at $\lambda = \lambda_0$ can be given, say, $\boldsymbol{x}'_0 = F(\boldsymbol{x}_0, \lambda_0)$. Fortunately, we now have the 1349 newly found periodic orbits of triple systems, which can be used as the initial guess of the continuation method (Allgower & Georg 1990). By means of the numerical continuation method (Allgower & Georg 1990), the known periodic orbits of triple system in the case of $m_1 = m_2 \neq m_3$ can be used as an initial guess for the periodic solution at $m_1 + \Delta m$. When Δm is sufficiently small, a new periodic orbit at $m'_1 = m_1 + \Delta m$ with the corresponding masses $m'_2 \neq m_2$ and $m'_3 \neq m_3$ can be obtained by means of the Newton–Raphson method (Farantos 1995; Lara & Pelaez 2002; Abad et al. 2011). In this way, m_1 can be changed to a given value, and as a result the masses of the triple system are totally different from each other.

Nowadays, it is impractical to use the grid search method to search for periodic orbits of a three-body system with unequal mass. Thus we suggest here a new strategy: we first gain periodic orbits of triple system with two bodies having the same mass by means of the grid search method, and then use these periodic orbits as initial approximations for the continuation method to gain periodic orbits of three-body problem with unequal mass. It should be emphasized that the 1349 known periodic orbits of triple system reported in this paper are the key for the application of the continuation method to gain periodic orbits of three-body systems with unequal masses. For example, starting from the periodic orbits $\mathrm{I.A}_1^{\mathrm{i.c.}}(0.5)$ with masses $m_1 = m_2 = 1$ and $m_3 = 0.5$, we gain periodic orbits with mass $m_2 = 1$ and $m_1 = 0.975$, 0.95, 0.925, and 0.9, respectively, as shown in figure 4, by means of the numerical continuation method (Allgower & Georg 1990).

For example, we gain 65 periodic orbits in the case of $m_1 = 0.9$, $m_2 = 1$, and $m_3 \neq m_1 \neq m_2$ in this way. The initial condition and periods of these periodic orbits are shown in tables S LXXXII–LXXXVI in the supplementary data.[1] Note that each of these 65 periodic orbits correspond to a continuous path of m_1 varying from 1 to 0.9, and each point $m_1 \in (0.9, 1)$ of this path corresponds to a periodic orbit of a three-body system with unequal mass! Limited by the length, all of them are neglected in this paper. This illustrates the validity of this approach well. In theory, we can similarly search for periodic orbits of a three-body system with arbitrary ratio of m_1/m_2 in the case of $m_1 \neq m_2 \neq m_3$ by means of the numerical continuation method (Allgower & Georg 1990). It implies that there should exist an infinite number of families of periodic orbits of triple systems with unequal mass. Thus, our 1349 newly found periodic orbits lay a foundation for searching periodic orbits of three-body systems with unequal masses ($m_1 \neq m_2 \neq m_3$).

6 Conclusions

In this paper we report 1349 new families of planar periodic orbits of the triple system, with two of the bodies having the same mass and the other having a different mass. In particular, 1223 among these 1349 families are entirely new, i.e., with newly found "free group elements" that have never

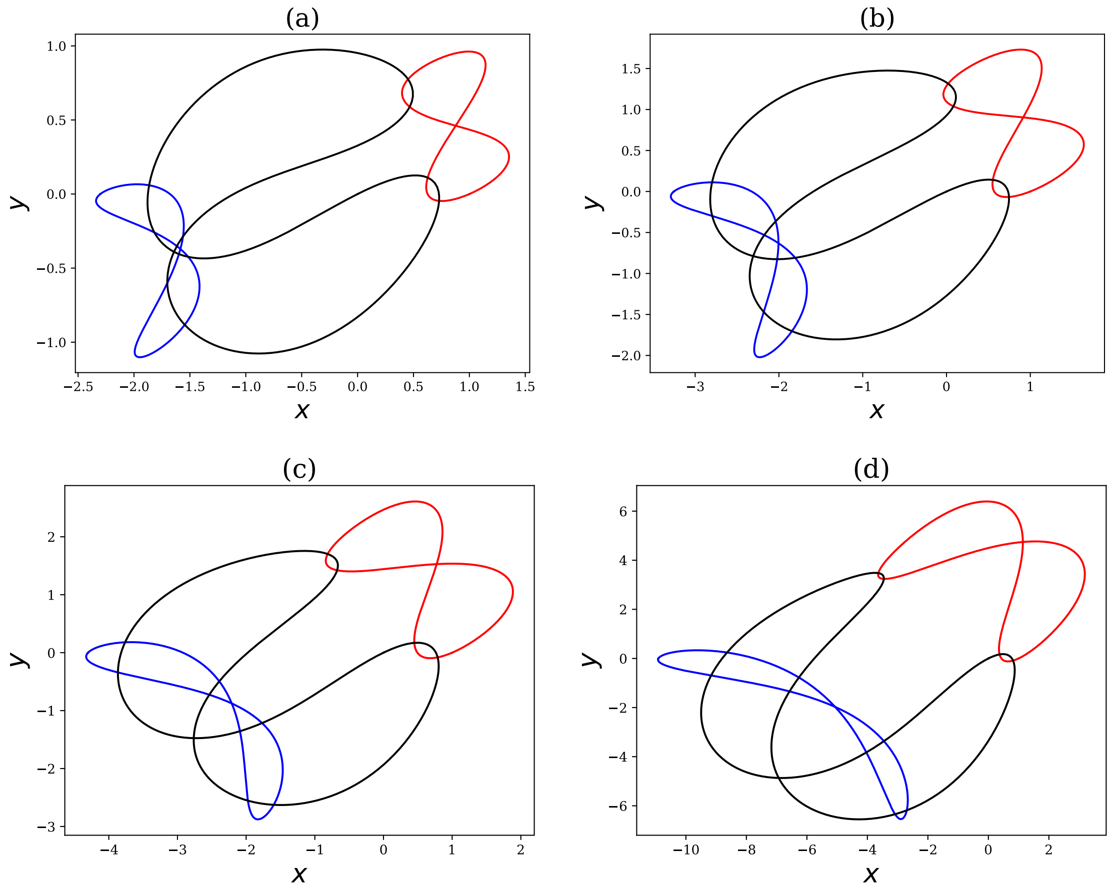

Fig. 4. Trajectory of periodic orbits with three unequal masses: (a) $m_1 = 0.975$, $m_2 = 1$, and $m_3 = 0.5764$; (b) $m_1 = 0.95$, $m_2 = 1$, and $m_3 = 0.6454$; (c) $m_1 = 0.925$, $m_2 = 1$, and $m_3 = 0.7169$; (d) $m_1 = 0.9$, $m_2 = 1$, and $m_3 = 0.7569$. The initial condition and periods of these orbits are shown in table S LXXXII in the supplementary data.[1] Blue line: body-1, red line: body-2, black line: body-3. (Color online)

been reported even for three-body system with equal mass. It has been traditionally believed that triple systems are often unstable if they are in a non-hierarchical configuration. However, all of our new periodic orbits are in non-hierarchical configurations, but many of them are either linearly or marginally stable. This might inspire the long-term astronomical observation of stable non-hierarchical triple systems in practice. In addition, using these new periodic orbits as initial guesses, new periodic orbits of triple systems with three unequal masses can be found by means of the continuation method.

What is more, many researchers (Imai et al. 2007; Torigoe et al. 2009; Asada 2009; Dmitrašinović et al. 2014) studied general relativistic effects and gravitational waves of previously found Newtonian periodic orbits with equal mass. Therefore, all of these newly found Newtonian periodic orbits with unequal mass may encourage and facilitate further areas of research, such as the general relativistic corrections of these orbits, the gravitational waves they generate, and the possibility of detection by gravitational wave observations (Meiron et al. 2017).

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Supplementary data

Supplementary data are available at *PASJ* online.

Table S I–LXXXVI.

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