EE 709 Assignment 1
Boolean Algebras

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1 Solutions to the provided problem set

1.1 Problem 1

The 0 and 1 elements in a Boolean Algebra are unique

Proof. Let $\exists$ two zero elements $0_1, 0_2 \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$$\begin{aligned}
&0_1 + 0_2 = 0_1 && (a + 0 = a) \text{ - } (1)\\
&0_2 + 0_1 = 0_2 && (a + 0 = a) \text{ - } (2)\\
&0_1 + 0_2 = 0_2 + 0_1 && (\text{commutativity of } +)\\
&0_1 = 0_2 && \text{from (1) and (2)}\\
&\iff 0_1 = 0_2 \equiv 0
\end{aligned}$$

□

Proof. Let $\exists$ two one elements $1_1, 1_2 \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$$\begin{aligned}
&1_1 \cdot 1_2 = 1_1 && (a \cdot 1 = a) \text{ - } (1)\\
&1_2 \cdot 1_1 = 1_2 && (a \cdot 1 = a) \text{ - } (2)\\
&1_1 \cdot 1_2 = 1_2 \cdot 1_1 && (\text{commutativity of . })\\
&1_1 = 1_2 && \text{from (1) and (2)}\\
&\iff 1_1 = 1_2 \equiv 1
\end{aligned}$$

□

1.2 Problem 2

a·0 = 0 for each element a

Proof. Let $a \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$$\begin{aligned}
a \cdot 0 &= a \cdot 0 + 0 && (a + 0 = a)\\
&= a \cdot 0 + a \cdot \bar{a} && (a \cdot \bar{a} = 0)\\
&= a \cdot (0 + \bar{a}) && (\text{distributivity of .})\\
&= a \cdot \bar{a} && (a + 0 = a)\\
&\equiv 0 && (a \cdot \bar{a} = 0)
\end{aligned}$$

□

1.3 Problem 3

a + 1 = 1 for each element a

Proof. Let $a \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$$\begin{aligned} a + 1 &= (a+1)\cdot 1 && (a \cdot 1 = a) \\ &= (a+1)\cdot(a+\bar{a}) && (a + \bar{a} = 1) \\ &= a + 1\cdot\bar{a} && (\text{distributivity of } +) \\ &= a + \bar{a} && (a \cdot 1 = a) \\ &\equiv 1 && (a + \bar{a} = 1) \end{aligned}$$

□

1.4 Problem 4

For each element a, a + a = a, and a·a = a

Proof. Let $a \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$$\begin{aligned} a + a &= a + a \cdot 1 && (a \cdot 1 = a) \\ &= a \cdot (1+1) && (\text{distributivity of .}) \\ &= a \cdot 1 \\ &\equiv a && (a \cdot 1 = a) \end{aligned}$$

□

Proof. Let $a \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$$\begin{aligned} a \cdot a &= a \cdot a + 0 && (a + 0 = a) \\ &= a \cdot a + a \cdot \bar{a} && (a \cdot \bar{a} = 0) \\ &= a \cdot (a + \bar{a}) && (\text{distributivity of .}) \\ &= a \cdot 1 && (a + \bar{a} = 1) \\ &\equiv a && (a \cdot 1 = a) \end{aligned}$$

□

1.5 Problem 5

The second De Morgan's Law $(\overline{a \cdot b} = \bar{a} + \bar{b})$

Proof. Let $a, b \in \mathbb{B} = \{\Omega, +, \cdot\}$, then

$\forall\ x \in \mathbb{B},$ we know that $\bar{x} \cdot x = 0,\ \bar{x} + x = 1$

we need $(a \cdot b) \cdot (\bar{a} + \bar{b}) = 0$ and $(a \cdot b) + (\bar{a} + \bar{b}) = 1$

$$(a \cdot b) \cdot (\bar{a} + \bar{b}) = a \cdot b \cdot \bar{a} + a \cdot b \cdot \bar{b} \qquad (\text{distributivity of } +)$$

$$= b \cdot 0 + a \cdot 0 \qquad (a \cdot \bar{a} = 0)$$

$$\equiv 0 \qquad (a \cdot 0 = 0 \text{ proved in } \textbf{Problem 3})$$

$$(a \cdot b) + (\bar{a} + \bar{b}) = (a \cdot b + \bar{a}) + \bar{b} \qquad (\text{commutativity of } +)$$

$$= ((a + \bar{a}) \cdot (\bar{a} + b)) + \bar{b} = \bar{a} + (\bar{b} + b) \qquad (\text{distributivity of } +)$$

$$= \bar{a} + 1 \qquad (a + \bar{a} = 1)$$

$$\equiv 1 \qquad (a + 1 = 1 \text{ proved in } \textbf{Problem 3})$$

□

1.6 Problem 6

In finite Boolean Algebra with at least two elements, $\mathbf{0 \neq 1}$

Proof. Suppose $\mathbb{B}$ is a finite boolean algebra with n elements $n \geq 2$ and suppose $0 \equiv 1 \equiv m$. Let $a \in \mathbb{B} \neq m$, the existence of a is justified as $n \geq 2$.

$$\begin{aligned} a + m &= a && (a + 0 = a) \\ a + m &= m && (a + 1 = 1 \text{ proved in } \textbf{Problem 3}) \\ \implies a &= m && (\text{contradiction to a} \neq m) \end{aligned}$$

Hence proved $0 \neq 1$ by contradiction □

1.7 Problem 7

In a finite Boolean algebra with at least two elements, the set of atoms is non-empty.

Proof. Let $\mathbb{B}$ is a finite boolean algebra with n elements $n \geq 2$. Let A denote the set of all atoms a. Suppose for contradiction, the set of atoms A is empty. Suppose $x \in \mathbb{B}$ is an element then $\exists\ a \leq x$, where a is an atom (From **Problem 11**)

$$\implies \mathcal{N}(A) \geq 1$$

Hence proved that the set A is non empty. □

1.8 Problem 8

If a is an atom, then for any element $x \neq 0$, either $a \leq x$ or $a \leq \bar{x}$. However, it is not possible that $a \leq x$ and $a \leq \bar{x}$.

Proof. Let $A = \{b_1, b_2, b_3 ...\}$ be the set of all atoms in $\mathbb{B}$. Let $A_x = \{p_i\}$ be the set of all atoms $m \leq x$ and $A_{\bar{x}} = \{q_i\}$ be the set of all atoms $m \leq \bar{x}$. The existence of these sets is proved in **Problem 7**

Since A is the set of all atoms in $\mathbb{B}$, we have $A_x \subseteq A$ and $A_{\bar{x}} \subseteq A$. We also have that all atoms $a \in \mathbb{B}$ satisfy $a \leq 1$ as $a \cdot 1 = a$. So we have $A_1 \equiv A$

$$\begin{aligned} x + \bar{x} &= 1 \\ &= (\sum_{p_i \in A_x} p_i) + (\sum_{q_i \in A_{\bar{x}}} q_i) = 1 \end{aligned}$$

$x + \bar{x}$ will contain atoms $m \in A_x \cup A_{\bar{x}}$ as if $\exists$ terms $p_i = q_j$ we have $p_i + q_j = p_i = q_j$. As we have $x + \bar{x} = 1$, we have

$$A_x \cup A_{\bar{x}} = A$$

Thus if we have $a \in A \implies a \in A_x$ **or** $a \in A_{\bar{x}}$. So we have $a \leq x$ or $a \leq \bar{x}$

Suppose $\exists\ m \neq 0 \in A_x$ and $m \in A_{\bar{x}}$

$$
\begin{aligned}
& m \cdot x = m && \text{(definition of } \leq\text{)} \\
& m \cdot \bar{x} = m && \\
& (m \cdot x) \cdot (m \cdot \bar{x}) = m \cdot m && \text{(using } \cdot \text{ in above two relations)} \\
& \implies (m \cdot m) \cdot (x \cdot \bar{x}) = (m \cdot m) && \\
& \implies m \cdot 0 = m \implies m = 0 && (a \cdot a = a \text{ proved in } \textbf{Problem 4}\ ) \\
& && (a \cdot 0 = 0 \text{ proved in } \textbf{Problem 2}) \\
& && (a \cdot \bar{a} = 0)
\end{aligned}
$$

Which is a contradiction hence it is not possible that $\exists\ a : a \leq x$ and $a \leq \bar{x}$.

Thus we can conclude that every atom a satisfies either $a \leq x$ or $a \leq \bar{x}$ but not both together.

□

1.9 Problem 9

If a is an atom, then $a \leq x + y \iff a \leq x$ or $a \leq y$

Proof. Let $a \neq 0$ be an atom $\in \mathbb{B}$ s.t $a \leq x + y$. Let A_x and A_y be the set of atoms s.t $x = \sum_{p_i \in A_x} p_i$ and $y = \sum_{q_i \in A_y} q_i$

$$
\begin{aligned}
& a \cdot (x + y) = a && \text{(definition of } \leq\text{)} \\
& x + y = (\sum_{p_i \in A_x} p_i) + (\sum_{q_i \in A_y} q_i)
\end{aligned}
$$

$x + y$ will only contain atoms $m \in A_x \cup A_y$ as for all the terms $p_i = q_j$ we have $p_i + q_j = p_i = q_j$. Also we know from set theory that $A_x \cup A_y \supseteq A_x$ and $A_x \cup A_y \supseteq A_y$. So the atoms satisfy the condition $m \in A_x$ **or** $m \in A_y$

Thus $a \cdot (x \cdot y) = a \implies a \in m \implies a \in A_x$ **or** $a \in A_y$

$$
\begin{aligned}
& a \cdot x = a \implies a \leq x && (a \in A_x) \\
& a \cdot y = a \implies a \leq y && (a \in A_y)
\end{aligned}
$$

Thus proved $a \leq x + y \implies a \leq x$ **or** $a \leq y$

Let $\underbrace{a \leq x}_{p_1}$ **or** $\underbrace{a \leq y}_{p_2}$, for $p_1 \vee p_2$, we check that $p_1 \wedge p_2, \neg p_1 \wedge p_2$ and $p_1 \wedge \neg p_2$ are true

1) For $p_1 \wedge p_2$

$$
\begin{aligned}
& a \leq x \implies a \cdot x = a && \text{(definition of } \leq\text{)} \\
& a \leq y \implies a \cdot y = a && \\
& a \cdot (x + y) = a \cdot x + a \cdot y = a + a = a && \text{(distributivity of } \cdot \text{ and } a + a = a \text{ proved in } \textbf{Problem 4}\text{)}
\end{aligned}
$$

2) For $p_1 \wedge \neg p_2$

$$
\begin{aligned}
& a \leq x \implies a \cdot x = a && \text{(definition of } \leq\text{)} \\
& a \cdot (x + y) = a \cdot x + a \cdot y = a + a \cdot y && \text{(distributivity of } \cdot\text{)} \\
& = a \cdot (1 + y) = a \cdot 1 = a && (1 + a = 1 \text{ proved in } \textbf{Problem 3} \text{ and } a \cdot 1 = a)
\end{aligned}
$$

Similarly, one can prove $\neg p_1 \wedge p_2$ by symmetry
Thus proved $a \leq x + y \impliedby a \leq x$ and $a \leq y$

Hence proved the equivalence $a \leq x + y \iff a \leq x$ and $a \leq y$

$\square$

1.10 Problem 10

If a is an atom, then $a \leq x \cdot y \iff a \leq x$ and $a \leq y$

Proof. Let $a \neq 0$ be an atom $\in \mathbb{B}$ s.t $a \leq x \cdot y$. Let A_x and A_y be the set of atoms s.t $x = \sum_{p_i \in A_x} p_i$ and $y = \sum_{q_i \in A_y} q_i$

$$a \cdot (x \cdot y) = a \qquad \text{(definition of } \leq)$$
$$x \cdot y = (\sum_{p_i \in A_x} p_i) \cdot (\sum_{q_i \in A_y} q_i)$$

$x \cdot y$ will only contain atoms $m \in A_x \cap A_y$ as for all the terms $p_i \neq q_j$ we have $p_i \cdot q_j = 0$. Also we know from set theory that $A_x \cap A_y \subseteq A_x$ and $A_x \cap A_y \subseteq A_y$. So the atoms $m \in A_x$ and $m \in A_y$

Thus $a \cdot (x \cdot y) = a \implies a \in m \implies a \in A_x$ and $a \in A_y$

$$a \cdot x = a \implies a \leq x \qquad (a \in A_x)$$
$$a \cdot y = a \implies a \leq y \qquad (a \in A_y)$$

Thus proved $a \leq x \cdot y \implies a \leq x$ and $a \leq y$

Suppose $a \leq x$ and $a \leq y$

$$a \leq x \implies a \cdot x = a \qquad \text{(definition of } \leq)$$
$$a \leq y \implies a \cdot y = a$$
$$a \cdot (x \cdot y) = (a \cdot x) \cdot y = a \cdot y = a \qquad \text{(commutativity of } \cdot)$$

Thus proved $a \leq x \cdot y \impliedby a \leq x$ and $a \leq y$

Hence proved the equivalence $a \leq x \cdot y \iff a \leq x$ and $a \leq y$

$\square$

1.11 Problem 11

In a finite Boolean algebra with at least two elements, for every $x \neq 0$ there is at least one atom $a \leq x$

Proof. Let $\mathbb{B}$ is a finite boolean algebra with n elements $n \geq 2$ and for the sake of contradiction, let $\exists\ x \neq 0 \in \mathbb{B}$ s.t there is no atom $a \leq x \implies x$ is not an atom

$$x \neq 0 \implies \exists\ y : y \neq 0, y \neq x, y \leq x \qquad \text{(otherwise x is an atom)}$$
$$y \neq 0 \implies \exists\ z_1 : z_1 \neq 0, z_1 \neq x, z_1 \neq y, z_1 \leq y \qquad \text{(otherwise y is an atom)}$$

We can keep finding distinct elements that are not atoms $z_2, z_3, z_4 \cdots : z_1 \geq z_2 \geq z_3 \ldots$. That would make the boolean algebra have infinite elements, which contradicts $\mathbb{B}$ being finite. For $\mathbb{B}$ to be finite $\exists\ z_n : z_n \neq 0, z_n \leq z_{n-1} \implies z_n = z_{n-1}$, which makes z_n an atom s.t $z_n \leq x$. Hence proved by contradiction. $\square$

1.12 Problem 12

If $a \leq b$ **and** $c \leq b$ **then** $(a+c) \leq b$

Proof. Let $a, b, c \in \mathbb{B}$ and $a \leq b,\ c \leq b$

$$
\begin{aligned}
&a \leq b \implies a \cdot b = a && \text{(definition of } \leq\text{)} \\
&c \leq b \implies c \cdot b = c && \text{(definition of } \leq\text{)} \\
&(a+c) \cdot b = a \cdot b + c \cdot b = a + c && \text{(distributivity of } \cdot\text{)}
\end{aligned}
$$

Hence $(a+c) \cdot b = a + c \implies (a+c) \leq b$ completes the proof

□

1.13 Problem 13

If $a \leq b$ **then** $\bar{b} \leq \bar{a}$

Proof. Let $a, b \in \mathbb{B}$ and $a \leq b$

$$
\begin{aligned}
&a \leq b \implies a \cdot b = a && \text{(definition of } \leq\text{)} \\
&\bar{b} \cdot \bar{a} = \bar{b} \cdot (\overline{a \cdot b}) = \bar{b} \cdot (\bar{a} + \bar{b}) && \text{(De Morgan's Law proved in } \textbf{Problem 5}\text{)} \\
&\implies \bar{b} \cdot \bar{a} + \bar{b} = \bar{b} \cdot (1 + \bar{a}) = \bar{b} && \text{(distributivity of } \cdot\text{)}
\end{aligned}
$$

Hence $\bar{b} \cdot \bar{a} = \bar{b} \implies \bar{b} \leq \bar{a}$ completes the proof

□

1.14 Problem 14

If $a \leq b$ **and** $a \leq c$ **then** $a \leq b \cdot c$

Proof. Let $a, b, c \in \mathbb{B}$ and $a \leq b,\ a \leq c$

$$
\begin{aligned}
&a \leq b \implies a \cdot b = a && \text{(definition of } \leq\text{)} \\
&a \leq c \implies a \cdot c = a && \text{(definition of } \leq\text{)} \\
&a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot c = a && \text{(commutativity of } \cdot\text{)}
\end{aligned}
$$

Hence $a \cdot (b \cdot c) = a \implies a \leq b \cdot c$ completes the proof

□

1.15 Problem 15

Prove that $\bar{a} \cdot b + a \cdot b = b$

Proof. Let $a, b \in \mathbb{B}$

$$
\begin{aligned}
&\bar{a} \cdot b + a \cdot b = b \cdot (a + \bar{a}) && \text{(distributivity of } \cdot\text{)} \\
&= b && (a + \bar{a} = 1)
\end{aligned}
$$

Hence this completes the proof

□

1.16 Problem 16

Prove that $a.\bar{b}.c + a.\bar{b}.\bar{c} + \bar{a}.\bar{b} = \bar{b}$

Proof. Let $a, b, c \in \mathbb{B}$

$$\begin{aligned}
a.\bar{b}.c + a.\bar{b}.\bar{c} + \bar{a}.\bar{b} &= a \cdot \bar{b} \cdot (c + \bar{c}) + \bar{a}.\bar{b} && \text{(distributivity of } \cdot\text{)} \\
&= a \cdot \bar{b} + \bar{a} \cdot \bar{b} && (a + \bar{a} = 1) \\
&= \bar{b} \cdot (a + \bar{a}) && \text{(distributivity of } \cdot\text{)} \\
&= \bar{b} && (a + \bar{a} = 1)
\end{aligned}$$

Hence this completes the proof

□