EE 709 Assignment 2
Switching Algebras

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1 Solutions to the provided problem set

1.1 Problem 1

Any element x in a Boolean Algebra, can be written as $x = \prod_{a \leq \bar{x}} \bar{a}$

Proof. Suppose $x \in \mathbb{B}$, A be the set of atoms in $\mathbb{B}$. We know that $\exists\ A_x \subseteq A$ s.t

$$x = \sum_{a_i \in A_x, a_i \leq x} a_i$$

Similarly $\bar{x}$ can be represented as a sum of atoms $b_i \in A - A_x$

$$\begin{aligned}
\bar{x} &= \sum_{b_i \in A - A_x, b_i \leq \bar{x}} b_i \\
\bar{x} &= (b_1 + b_2 + b_3 + b_4 \dots) \\
x &= (\overline{b_1 + b_2 + b_3 + b_4 \dots}) && (\bar{\bar{x}} = x) \\
x &= \bar{b_1} \cdot \bar{b_2} \cdot \bar{b_3} \dots && \text{(De Morgan's Law)} \\
x &= \prod_{b_i \in A - A_x, b_i \leq \bar{x}} \bar{b_i}
\end{aligned}$$

Hence proved the above claim. □

1.2 Problem 2(a)

To prove $\exists_x(f + g) = (\exists_x f) + (\exists_x g)$

Proof. Let f_x denote the cofactor of the boolean function f wrt to the variable x

$$\begin{aligned}
\exists_x(f) &= f_x + f_{\bar{x}} && \text{(from definition)} \\
\exists_x(g) &= g_x + g_{\bar{x}} && \text{(from definition)} \\
\exists_x(f) + \exists_x(g) &= f_x + f_{\bar{x}} + g_x + g_{\bar{x}} && -(1)
\end{aligned}$$

Also from Shannon's Decomposition, we complete the proof as

$$\begin{aligned}
& f = x \cdot f_x + \bar{x} \cdot f_{\bar{x}} \\
& g = x \cdot g_x + \bar{x} \cdot g_{\bar{x}} \\
& f + g = x \cdot \underbrace{(f_x + g_x)}_{h_x} + \bar{x} \cdot \underbrace{(f_{\bar{x}} + g_{\bar{x}})}_{h_{\bar{x}}} \\
& \exists_x(f + g) = h_x + h_{\bar{x}} = f_x + g_x + f_{\bar{x}} + g_{\bar{x}} = \exists_x(f) + \exists_x(g) \qquad \text{from (1)}
\end{aligned}$$

□

1.3 Problem 2(b)

To prove $\exists_x(f \cdot g) = (\exists_x f) \cdot (\exists_x g)$

This is an **incorrect claim**, disproved in the following way with a counter example.

$$
\begin{aligned}
&\text{Suppose we have } f = x \text{ and } g = \bar{x} \\
&f \cdot g = 0 && (x \cdot \bar{x} = 0) \\
&\implies \exists_x(f \cdot g) = 0 \\
&f = x \cdot 1 + \bar{x} \cdot 0 \implies f_x = 1, f_{\bar{x}} = 0 \implies \exists_x(f) = 1 \\
&g = x \cdot 0 + \bar{x} \cdot 1 \implies g_x = 0, g_{\bar{x}} = 1 \implies \exists_x(g) = 1 \\
&\therefore \exists_x(f \cdot g) \neq \exists_x(f) \cdot \exists_x(g)
\end{aligned}
$$

Hence we have disproved the above claim, which was supposed to be proved, with the use of a counter example.

1.4 Problem 2(c)

1.4.1 To prove $\exists_x f = \neg(\forall_x(\neg f))$

Proof. We know from Shannon's Decomposition Theorem that

$$
\begin{aligned}
&f = x \cdot f_x + \bar{x} \cdot f_{\bar{x}} \\
&\bar{f} = (\bar{x} + \neg(f_x)) \cdot (x + \neg(f_{\bar{x}})) \\
&\neg f = \bar{x} \cdot \neg(f_{\bar{x}}) + x \cdot \neg(f_x) + \neg(f_x) \cdot \neg(f_{\bar{x}}) \\
&\neg f = x \cdot g_x + \bar{x} \cdot g_{\bar{x}} \\
&g_x = \neg(f_x) + \neg(f_x) \cdot \neg(f_{\bar{x}}) = \neg(f_x) && x = 1 \\
&g_{\bar{x}} = \neg(f_{\bar{x}}) + \neg(f_x) \cdot \neg(f_{\bar{x}}) = \neg(f_{\bar{x}}) && x = 0 \\
&\therefore \forall_x(\neg f) = g_x \cdot g_{\bar{x}} = \neg(f_x) \cdot \neg(f_{\bar{x}}) = \neg(f_x + f_{\bar{x}}) && \text{(from De Morgan's Law)} \\
&\therefore \neg(\forall_x(\neg f)) = f_x + f_{\bar{x}} = \exists_x(f)
\end{aligned}
$$

□

1.5 Problem 3

1.5.1 To prove $\exists_x \exists_y f = \exists_y \exists_x f$

Proof. Suppose $f(x_1, x_2, \ldots x_n) \in \mathbb{B}_2^n$ is a boolean function of n variables. Without loss of generality suppose $x = x_1$ and $y = x_2$ arbitrarily

$$
\begin{aligned}
&\exists_{x_1} f = f(1, x_2, x_3 \ldots) + f(0, x_2, x_3 \ldots) && \text{(from definition)} \\
&\exists_{x_2}\exists_{x_1} f = (f(1, 1, x_3 \ldots) + f(0, 1, x_3 \ldots)) + (f(1, 0, x_3 \ldots) + f(0, 0, x_3 \ldots))
\end{aligned}
$$

Similarly we have the following

$$
\begin{aligned}
&\exists_{x_2} f = f(x_1, 1, x_3 \ldots) + f(x_1, 0, x_3 \ldots) && \text{(from definition)} \\
&\exists_{x_1}\exists_{x_2} f = (f(1, 1, x_3 \ldots) + f(1, 0, x_3 \ldots)) + (f(0, 1, x_3 \ldots) + f(0, 0, x_3 \ldots))
\end{aligned}
$$

Thus from commutativity of + operation, for arbitrary x, y we have

$$\exists_x \exists_y f = \exists_y \exists_x f$$

□

1.6 Problem 4

1.6.1 To prove $\forall_x \forall_y f = \forall_y \forall_x f$

Proof. Suppose $f(x_1, x_2, \ldots x_n) \in \mathbb{B}_2^n$ is a boolean function of n variables. Without loss of generality suppose $x = x_1$ and $y = x_2$ arbitrarily

$$\forall_{x_1} f = f(1, x_2, x_3 \ldots) \cdot f(0, x_2, x_3 \ldots) \qquad \text{(from definition)}$$
$$\forall_{x_2} \forall_{x_1} f = (f(1, 1, x_3 \ldots) \cdot f(0, 1, x_3 \ldots)) \cdot (f(1, 0, x_3 \ldots) \cdot f(0, 0, x_3 \ldots))$$

Similarly we have the following

$$\forall_{x_2} f = f(x_1, 1, x_3 \ldots) \cdot f(x_1, 0, x_3 \ldots) \qquad \text{(from definition)}$$
$$\forall_{x_1} \forall_{x_2} f = (f(1, 1, x_3 \ldots) \cdot f(1, 0, x_3 \ldots)) \cdot (f(0, 1, x_3 \ldots) \cdot f(0, 0, x_3 \ldots))$$

Thus from commutativity of $\cdot$ operation, for arbitrary x, y we have

$$\forall_x \forall_y f = \forall_y \forall_x f$$

□

1.7 Problem 5(a)

On expanding the terms where literals are missing a variable out of x_1, x_2, x_3, x_4 using $(1 = x + \bar{x})$, and distributivity of $\cdot$ we can get the **DNF** representation

$$f_{DNF} = x_1 \cdot \bar{x_2} \cdot x_3 \cdot x_4 + x_1 \cdot \bar{x_2} \cdot x_3 \cdot \bar{x_4} + x_1 \cdot \bar{x_2} \cdot \bar{x_3} \cdot x_4 + x_1 \cdot \bar{x_2} \cdot \bar{x_3} \cdot \bar{x_4}$$
$$+ \bar{x_1} \cdot x_2 \cdot x_3 \cdot \bar{x_4} + \bar{x_1} \cdot x_2 \cdot \bar{x_3} \cdot \bar{x_4} + \bar{x_1} \cdot \bar{x_2} \cdot x_3 \cdot \bar{x_4}$$

For the **CNF** representation we use the following result.

$$f_{CNF} = \overline{\bar{f}_{DNF}} \qquad \text{(From De Morgan's Law)}$$
$$\bar{f}_{DNF} = \bar{x_1} \cdot \bar{x_2} \cdot \bar{x_3} \cdot \bar{x_4} + \bar{x_1} \cdot \bar{x_2} \cdot x_3 \cdot x_4 + \bar{x_1} \cdot \bar{x_2} \cdot \bar{x_3} \cdot x_4$$
$$+ x_1 \cdot x_2 \cdot x_3 \cdot x_4 + x_1 \cdot x_2 \cdot \bar{x_3} \cdot x_4 + x_1 \cdot x_2 \cdot x_3 \cdot \bar{x_4} + x_1 \cdot x_2 \cdot \bar{x_3} \cdot \bar{x_4}$$
$$+ \bar{x_1} \cdot x_2 \cdot x_3 \cdot x_4 + \bar{x_1} \cdot x_2 \cdot \bar{x_3} \cdot x_4 \qquad \text{(from } f + \bar{f} = 1\text{)}$$

Thus the **CNF** representation is given by

$$\therefore f_{CNF} = (x_1 + x_2 + x_3 + x_4) \cdot (x_1 + x_2 + \bar{x_3} + \bar{x_4}) \cdot (x_1 + x_2 + x_3 + \bar{x_4})$$
$$\cdot (\bar{x_1} + \bar{x_2} + \bar{x_3} + \bar{x_4}) \cdot (\bar{x_1} + \bar{x_2} + x_3 + \bar{x_4}) \cdot (\bar{x_1} + \bar{x_2} + \bar{x_3} + x_4) \cdot (\bar{x_1} + \bar{x_2} + x_3 + x_4)$$
$$\cdot (x_1 + \bar{x_2} + \bar{x_3} + \bar{x_4}) \cdot (x_1 + \bar{x_2} + x_3 + \bar{x_4})$$

1.8 Problem 5(b)

We use the initial formula provided and make use of this result

$$\exists_{x_2} f = f(x_1, 1, x_3 \ldots) + f(x_1, 0, x_3 \ldots) \qquad \text{(from definition)}$$

On putting $x_2 = 1$ for f_{x_2} and $x_2 = 0$ for $f_{\bar{x_2}}$

$$f_{x_2} = \bar{x_1} \cdot \bar{x_4}, \;\; f_{\bar{x_2}} = x_1 + \bar{x_1} \cdot \bar{x_4} \cdot (x_3 + x_4)$$
$$\exists_{x_2} f = f_{x_2} + f_{\bar{x_2}} = \bar{x_1} \cdot \bar{x_4} + x_1 + \bar{x_1} \cdot \bar{x_4} \cdot (x_3 + x_4) \equiv x_1 + \bar{x_4}$$

1.9 Problem 5(c)

We use the initial formula provided and make use of this result

$$\forall_{x_3} f = f(x_1, x_2, 1 \ldots) \cdot f(x_1, x_2, 0 \ldots) \qquad \text{(from definition)}$$

On putting $x_3 = 1$ for f_{x_3} and $x_3 = 0$ for $f_{\bar{x_3}}$

$$f_{x_3} = x_1 \cdot \bar{x_2} + \bar{x_1} \cdot \bar{x_4},\ f_{\bar{x_3}} = x_1 \cdot \bar{x_2} + \bar{x_1} \cdot \bar{x_4} \cdot x_2$$
$$\forall_{x_3} f = f_{x_3} \cdot f_{\bar{x_3}} = (x_1 \cdot \bar{x_2} + \bar{x_1} \cdot \bar{x_4}) \cdot (x_1 \cdot \bar{x_2} + \bar{x_1} \cdot \bar{x_4} \cdot x_2)$$