MTH5530 Introduction to Computational Finance and Monte Carlo Methods

Assignment Report

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ABSTRACT

In thie report, we introduce the Heston model, which is a special case of the constant elasticity of variance stochastic volatility (CEV-SV) models. When the feller condition fails, the volitility is no longer strictly positive. In this case, the Euler discretization with four schemes can be used to fix this issue. We investigate the errors and performance of these four schemes. Finally, a well-commented matlab implementation of the Heston model for pricing European call option is provided.

Keywords: Stochastic volatility, Heston, Euler discretization

Introduction

The Heston stochastic volatility models[1] has been widely applied by practitioners in today's financial market. It can be described as following:

$$
\begin{aligned}
dS(t) &= rS(t)dt + \lambda\sqrt{V(t)}S(t)dW_S(t),\\
dV(t) &= -\kappa(V(t)-\theta)dt + \omega\sqrt{V(t)}dW_V(t),\\
dW_S(t)dW_V(t) &= \rho dt,
\end{aligned}
$$

where (S) is the asset price process, (V) is the variance process, μ is the risk neutral deift of the asset price, $\kappa \geq 0$ is the speed of mean-reversion of the variance, $\theta \geq 0$ is the long-term average variance and $\omega \geq 0$ is so-called volatility of variance or volatility of volatility. Finally, λ is a scaling constant and W_S and W_V are correlated Brownian motions, with instantaneous correlation coefficient ρ.

In the context of Heston model, if the parameters satisfy the following condition (known as the Feller condition):

$$2\kappa\theta > \omega^2,$$

then the variance process (V) is strictly positive. However, this condition is not always satisfied in the real world. To fix this issue, the Euler discretization with four schemes can be applied.

Euler discretization and four schemes

The Euler discretization for the variance process V is:

$$V(t+\Delta t) = (1-\kappa\Delta t)V(t) + \kappa\theta\Delta t + \omega\sqrt{V(t)}\Delta W_V(t)$$

with $\Delta W_V(t) = W_V(t+\Delta t) - W_V(t)$. Because of the sqaure root term, we will be in trouble if V going negative. Practitioners have often opted for a quick 'fix' by either setting the process equal to zero whenever it attains a negative value, or by reflecting it in the origin, and continuing from there on. These fixes are often referred to as 'absorption' and 'reflection' respectively.

In[2], Lord, Koekkoek and Dijk introduced a new scheme, called 'full truncation' scheme, which is a modification of the Euler scheme introduce in[3], they referred to as the 'partial truncation' scheme. Their another contribution is unified these four Euler schemes into a framework. For our Heston model, the framework is:

$$\tilde{V}(t+\Delta t) = f_1(\tilde{V}(t)) - \kappa\Delta t(f_2(\tilde{(}t)) - \theta) + \omega\sqrt{f_3(\tilde{V}(t))}\Delta W_V(t),$$
$$V(t+\Delta t) = f_3(\tilde{V}(t+\Delta t)),$$

where $\tilde{V}(0) = V(0)$ and the function $f_i, i = 1,2,3$ is described in table (1).

Scheme	Paper	$f_1(x)$	$f_2(x)$	$f_3(x)$
Absorption	Unknown	x^+	x^+	x^+
Reflection	Diop[4], Bossy and Diop[5]	$\lvert x\rvert$	$\lvert x\rvert$	$\lvert x\rvert$
Partial truncation	Deelstra and Delbaen[3]	x	x	x^+
Full truncation	Lord, Koekkoek and Dijk[2]	x	x^+	x^+

Table 1. Overview of four Euler schemes

Then the asset price S can be simulated by:

$$\ln S(t+\Delta t) = \ln S(t) + (r - \frac{1}{2}V(t))\Delta t + \sqrt{(}V(t))\Delta W_S(t)$$

where the correlated Brownian motion $\Delta W_S(t) = \rho\Delta W_V(t) + \sqrt{1-\rho^2}\Delta Z(t)$ with $Z(t)$ independent of $W_V(t)$.

Numerical Results

In this section, we use the Heston model to price a European call option with the following parameters:

$$S_0 = 100, K = 100, T = 5, r = 0.05, V_0 = 0.09, \theta = 0.09, \kappa = 2, \omega = 1, \rho = -0.3,$$

where the Feller condition is violated. The true value for the call option is 34.9998. We will investigate the accuracy by calculating the following quantites for various number of paths and step sizes:

$$s(\hat{x}) = \sqrt{E[(\hat{x}-x)^2]}$$
$$bias(\hat{x}) = |E[\hat{x}] - x|$$
$$RMSE(\hat{x}) = \sqrt{bias(\hat{x})^2 + s(\hat{x})^2},$$

where $\hat{x}$ is the estimator of the true value x, $s(x)$ is the standard error and $RMSE(x)$ is called root mean square error which is a measure widely used for assessing pricing error. The numerical results are shown in Table (2).

As we can see, the Full truncation scheme has the best accuracy. Because although all schemes have similar standard error, the Full truncation scheme has be lowest bias, which makes it has the lowest RMSE. Since all four schemes, are unified into the one framework, so the implementations are quite similar, so there is no significant difference on run time.

Method	Paper Steps/year	10,000 20	40,000 40	160,000 80
Absorption	Bias	2.0560	1.6112	1.2000
	Std error	0.6341	0.3009	0.1544
	RMSE	2.1515	1.6391	1.2099
	Run Time	5.4250	38.621259	530.4107
Reflection	Bias	4.4746	3.1956	2.3847
	Std error	0.7690	0.3217	0.1548
	RMSE	4.5402	3.2117	2.3898
	Run Time	11.0912	38.8324	500.3686
Partial truncation	Bias	0.4021	0.1862	0.0732
	Std error	0.6110	0.3147	0.1444
	RMSE	0.7314	0.3657	0.1619
	Run Time	11.2164	37.9309	473.3707
Full truncation	Bias	0.1446	0.0091	0.0101
	Std error	0.5887	0.2615	0.1414
	RMSE	0.6062	0.2617	0.1417
	Run Time	11.0985	38.9902	481.2292

Table 2. Numerical results for pricing a European call option by Heston model with 4 Euler schemes (repeat 100 times)

Greeks: Delta and Gamma

Greeks estimate how the price V changes when parameters or initial states change, they can be very useful for calibration. To calculate the greeks, we use the central finite difference scheme:

$$\Delta_h = \frac{V(S_0+h,0) - V(S_0-h,0)}{2h},$$
$$\Gamma_h = \frac{V(S_0+h,0) - 2V(S_0,0) + V(S_0-h,0)}{h^2}.$$

Implementation could be tricky, using common random numbers are advised in practice.

Assuming increments $\Delta S = 0.01 S_0$, 100,000 simulations and 50 time steps/year, the Delta Δ and Gamma Γ calculated for the above European call option under Hestion model with full truncation scheme are:

$$\Delta_h^H = 0.7959, \quad \Gamma_h^H = 0.0047.$$

Comparing to the results generated by MATLAB blsdelta and blsgamma functions:

$$\Delta_h^B = 0.7606, \quad \Gamma_h^B = 0.0046,$$

our Δ_h^H and Γ_h^H are very close to the Greeks generated by Balck-Scholes model.

Another finding

Table (3) shows the numerical results with $\omega = 0.3$ and keep all other parameters unchanged. As we can see that, all four schemes has similar accuracy. That is because, in this case, the Feller condition is satisfied, so the variance V will not touch zero, therefore, all four schemes are equivalent.

Method	Paper Steps/year	10,000 20	40,000 40	160,000 80
Absorption	Bias	0.7320	0.8758	0.8467
	Std error	0.6695	0.7709	0.7703
	RMSE	0.9920	0.9298	0.8634
	Run Time	4.5737	38.3909	521.1152
Reflection	Bias	0.8053	0.8879	0.8710
	Std error	0.6103	0.3331	0.1550
	RMSE	1.0104	0.9483	0.8847
	Run Time	10.8535	38.3471	503.5099
Partial truncation	Bias	0.9111	0.8798	0.8763
	Std error	0.6352	0.3028	0.1493
	RMSE	1.1106	0.9304	0.8889
	Run Time	11.0649	38.6911	476.2894
Full truncation	Bias	0.8383	0.8479	0.8847
	Std error	0.5914	0.2954	0.1594
	RMSE	1.0259	0.8979	0.8989
	Run Time	10.8656	37.7323	501.5704

Table 3. Numerical results for pricing a European call option by Heston model with $\omega = 0.3$ (repeat 100 times)

Conclusion

In this report, we illustrate a simulation of pricing a European call option by the Heston stochastic volitality model with Euler discretization. We investigate an accuracy analysis of four difference schemes for solving the non-negativity issue arising when the Feller's condition fails. When the condition is not satisfied, the probability of variance V going to negative is greater than zero, the Full truncation scheme has the lowest bias and RMSE. However, when the Feller condition is met, all four schemes are equivalent. The numerical results show that they have similar accuracy as expected.

References

1. Heston, S. L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. financial studies* **6**, 327–343 (1993).
2. Lord, R., Koekkoek, R. & Dijk, D. V. A comparison of biased simulation schemes for stochastic volatility models. *Quant. Finance* **10**, 177–194 (2010).
3. Deelstra, G., Delbaen, F. *et al.* Convergence of discretized stochastic (interest rate) processes with stochastic drift term. *Appl. stochastic models data analysis* **14**, 77–84 (1998).
4. Diop, A., Guegan, D. *et al.* Extreme distribution for a generalized stochastic volatility model. *South Afr. Stat. J.* **37**, 127–127 (2003).
5. Diop, A. & Guegan, D. Tail behavior of a threshold autoregressive stochastic volatility model. *Extrem.* **7**, 367–375 (2004).