Homework 5: Photometric Stereo

16-820 Advanced Computer Vision (Fall 2023)
https://16820advancedcv.github.io/

OUT: November 10th, 2023
DUE: November 27th, 2023
Instructor: David Held
TAs: Vineet Tambe, Bart Duisterhof, Ronit Hire, Dishani Lahiri

Instructions/Hints

- Please refer to the course logistics page for information on the **Collaboration Policy** and **Late Submission Policy**.
- **Submitting your work:** There will be two submission slots for this homework on **Gradescope**: Written and Programming.
  - **Write-up.** For written problems such as short answers, multiple choice, derivations, proofs, or plots, we will be using the written submission slot. Please use this provided template. **We don't accept handwritten submissions.** Each answer should be completed in the boxes provided below the question. You are allowed to adjust the size of these boxes, but **make sure to link your answer to each question when submitting to Gradescope**. Otherwise, your submission will not be graded. To use the provided template - upload the template .zip file directly to Overleaf.
  - **Code.** You are also required to upload your code, which you wrote to solve this homework, to the Programming submission slot. Your code may be run by TAs so please make sure it is in a workable state. The assignment must be completed using Python 3.10.12. We recommend setting up python virtual environment (conda or venv) for the assignment.
  - Regrade requests can be made after the homework grades are released, however, this gives the TA the opportunity to regrade your entire paper, meaning if additional mistakes are found then points will be deducted.
- **Start early!** This homework is difficult and may take a long time to complete.
- **Verify your implementation as you proceed.** If you don't verify that your implementation is correct on toy examples, you will risk having a huge mess when you put everything

together.

- **Q&A.** If you have any questions or need clarifications, please post in Slack or visit the TAs during office hours. Additionally, we provide a **FAQ** (**??**) with questions from previous semesters. Make sure you read it prior to starting your implementations.
- **Write-up**: Please type out your answers to theory questions and include experimental results and discussions into a document `<andrew-id>.pdf` (preferably in LaTeX). Please note that we *DO NOT* accept handwritten scans for your write-up in this homework.
- **Submission**: Submit the pdf of your writeup to Gradescope. Your writeup should contain a copy of your code. Do not forget to select the relevant writeup and code pages in Gradescope for each problem. Also submit the `src` folder containing your code to Gradescope.

Overview

As we discussed in class, photometric stereo is a computational imaging method to determine the shape of an object from its appearance under a set of lighting directions. In the course of this homework, we will solve a simplified version of the photometric stereo problem. We will make certain assumptions: that the object is Lambertian and is imaged with an orthographic camera under a set of directional lights. We will first solve calibrated photometric stereo, in which lighting directions are given; then uncalibrated photometric stereo, where we have to infer lighting directions *and* the shape of the object jointly from the given images. In this process, we will learn about an ambiguity in the resultant shape introduced because of this joint estimation.

Even though the homework is fairly self-contained, we strongly recommend going over the papers in the references before starting. As always, start early.

1 Calibrated photometric stereo (80 points)

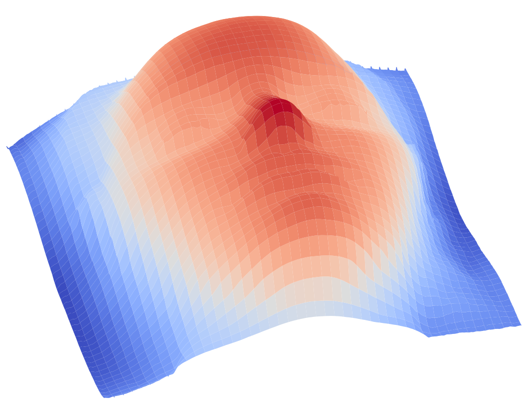
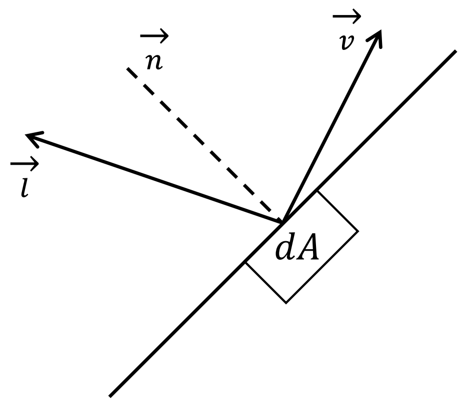


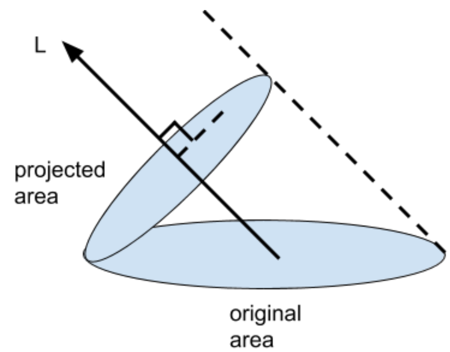
Figure 1: 3D reconstruction result from calibrated photometric stereo

All code in this part goes in `src/q1.py`. The code to pipeline these functions and generate the results are provided for you at the end of the file.

Image formation. We will now look at image formation under our specific assumptions. We will understand and implement the process that occurs in a scene when we take a picture with an orthographic camera.



(a) Geometry of photometric stereo



(b) Projected area

Figure 2: Geometry of photometric stereo

(a, 5 points) Understanding n-dot-l lighting. In your write-up, explain the geometry of the n-dot-l lighting model from Fig. 2a. Where does the dot product come from? Where does projected area (Fig. 2b) come into the equation? Why does the viewing direction not matter?

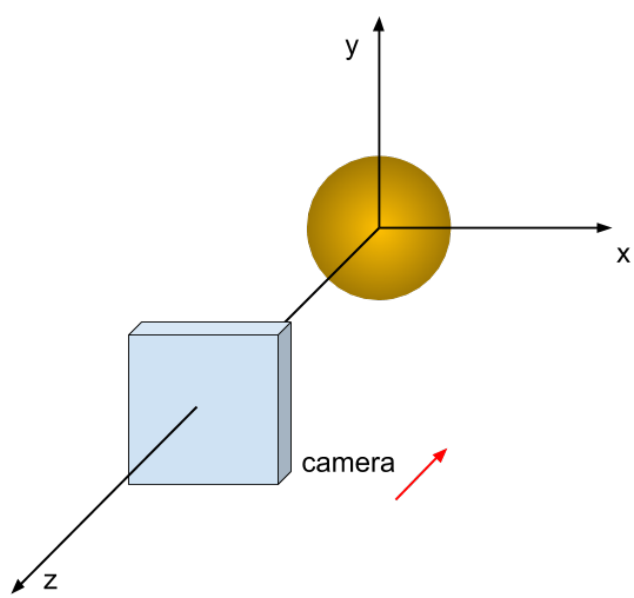


Figure 3: Setup for Problem 1b

(b, 10 points) Rendering n-dot-l lighting. Consider a uniform fully reflective Lambertian sphere with its center at the origin and a radius of 0.75 cm (Fig 3). An orthographic camera located at (0, 0, 10) cm looks towards the negative z-axis with its sensor axes aligned and centered on the x- and y-axes. The pixels on the camera are squares 7 μm in size, and the resolution of the camera is 3840 $\times$ 2160 pixels. Simulate the appearance of the sphere under the n-dot-l model with directional light sources with incoming lighting directions (1, 1, 1)/$\sqrt{3}$, (1, -1, 1)/$\sqrt{3}$ and (-1, -1, 1)/ $\sqrt{3}$ in the function `renderNDotLSphere` (individual images for all three lighting directions). Note that your rendering isn't required to be absolutely radiometrically accurate: we need only evaluate the n-dot-l model. Include the 3 renderings in your write-up. Include a snippet of the code you wrote for this part in your write-up.

Inverting the image formation model. Armed with this knowledge, we will now invert the image formation process given the lighting directions. Seven images of a face lit from different directions are given to us, along with the ground-truth directions of light sources. These images are taken at lighting directions such that there is not a lot of occlusion or normals being directed opposite to the lighting, so we will ignore these effects. The images are in the `data/` directory, named as `input_n.tif` for the n^{th} image. The source directions are given in the file `data/sources.npy`.

(c, 5 points) Loading data. In the function `loadData`, read the images into Python. Convert the RGB images into the XYZ color space and extract the luminance channel. Vectorize these luminance images and stack them in a $7 \times P$ matrix, where P is the number of pixels in each image. This is the matrix $\mathbf{I}$, which is given to us by the camera. Next, load the sources file and convert it to a 3×7 matrix L. For this question, include a screenshot of your function `loadData`.

These images are *16-bit* `.tifs`, and it's important to preserve that bit depth while reading the images for good results. Therefore, make sure that the datatype of your images is `uint16` after loading them.

(d, 10 points) Initials. Recall that in general, we also need to consider the reflectance, or albedo, of the surface we're reconstructing. We find it convenient to group the normals and albedos (both of which are a property of only the surface) into a pseudonormal $\mathbf{b} = a \cdot \mathbf{n}$, where a is the scalar albedo. We then have the relation $\mathbf{I} = \mathbf{L}^T\mathbf{B}$, where the $3 \times P$ matrix $\mathbf{B}$ is the set of pseudonormals in the images. With this model, explain why the rank of $\mathbf{I}$ should be 3. Perform a singular value decomposition of $\mathbf{I}$ and report the singular values in your write-up. Do the singular values agree with the rank-3 requirement? Explain this result and include the code snippet in the write-up.

(e, 20 points) Estimating pseudonormals. Since we have more measurements (7 per pixel) than variables (3 per pixel), we will estimate the pseudonormals in a least-squares sense. Note that there is a linear relation between $\mathbf{I}$ and $\mathbf{B}$ through $\mathbf{L}$: therefore, we can write a linear system of the form $\mathbf{A}\mathbf{x} = \mathbf{y}$ out of the relation $\mathbf{I} = \mathbf{L}^T\mathbf{B}$ and solve it to get $\mathbf{B}$. Solve this linear system in the function `estimatePseudonormalsCalibrated`. In your write-up, mention how you constructed the matrix $\mathbf{A}$ and the vector $\mathbf{y}$ along with the code for the function you wrote.

Note that here matrices you end up creating might be huge and might not fit in your computer's memory. In that case, to prevent your computer from freezing or crashing completely, make sure to use the `sparse` module from `scipy`. You might also want to use the sparse linear system solver and kronecker product in the same module.

(f, 10 points) Albedos and normals. Estimate per-pixel albedos and normals from matrix $\mathbf{B}$ in `estimateAlbedosNormals`. Note that the albedos are the magnitudes of the pseudonormals by definition. Calculate the albedos, reshape them into the original size of the images and display the resulting image in the function `displayAlbedosNormals`. Include the image in your write-up and comment on any unusual or unnatural features you may find in the albedo image, and on why they might be happening. Make sure to display in the `gray` colormap.

The per-pixel normals can be viewed as an RGB image. Reshape the estimated normals into an image with 3 channels and display it in the function `displayAlbedoNormals`. Note that the components of these normals will have values in the range $[-1, 1]$. You will need to rescale them so that they lie in $[0, 1]$ to display them properly as RGB images. Include this image in the write-up. Do the normals match your expectation of the curvature of the face? Make sure to display in the `rainbow` colormap. Your results should look like that in Fig. 4. Include the code snippets.

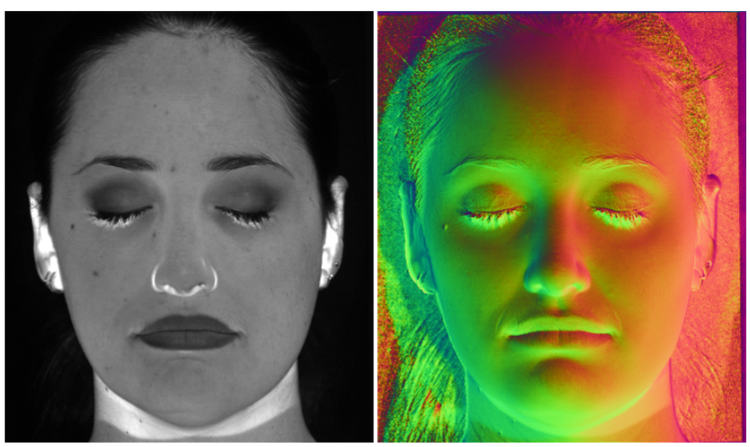


Figure 4: Calibrated photometric stereo results. Estimated albedo ($\times 10$) and normals.

(g, 5 points) Normals and depth. We will now estimate from the normals the actual shape of the face. Represent the shape of the face as a 3D depth map given by a function $z = f(x, y)$. Let the normal at the point (x, y) be $\mathbf{n} = (n_1, n_2, n_3)$. Explain, in your write-up, why $\mathbf{n}$ is related to the *partial derivatives* of f at (x, y): $f_x = \partial f(x, y)/\partial x = -n_1/n_3$ and $f_y = \partial f(x, y)/\partial y = -n_2/n_3$. You may consider the 1D case where $z = f(x)$.

Normal integration. Given that the normals represent the derivatives of the depth map, we will integrate the normals obtained in (g). We will use a special case of the Frankot-Chellappa algorithm for normal integration, as given in [1]. You can read the paper for the general version of the normal integration algorithm.

(h, 5 points) Understanding integrability of gradients. Consider the 2D, discrete function g on space given by the matrix below. Find its x and y gradient, given that gradients are calculated as $g_x(x_i, y_j) = g(x_{i+1}, y_j) - g(x_i, y_j)$ for all i, j (and similarly for y). Let us define (0, 0) as the top left, with x going in the horizontal direction and y in the vertical.

$$g = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \tag{1}$$

Note that we can reconstruct the entire of g given the values at its boundaries using g_x and g_y. Given that $g(0, 0) = 1$, perform these two procedures:

1. Use g_x to construct the first row of g, then use g_y to construct the rest of g;
2. Use g_y to construct the first column of g, then use g_x to construct the rest of g.

Are these the same?

Note that these were two ways to reconstruct g from its gradients. Given arbitrary g_x and g_y, these two procedures will not give the same answer, and therefore this pair of gradients does not correspond to a true surface. Integrability implies that the value of g estimated in both these ways (or any other way you can think of) is the same. How can we modify the gradients you calculated above to make g_x and g_y non-integrable? Why may the gradients estimated in the way of (g) be non-integrable? Note all this down in your write-up.

The Frankot-Chellappa algorithm for estimating shapes from their gradient first projects the (possibly non-integrable) gradients obtained in (h) above onto the set of all integrable gradients. The resulting (integrable) projection can then be used to solve for the depth map. The function `integrateFrankot` in `utils.py` implements this algorithm given the two surface gradients.

(i, 10 points) Shape estimation. Write a function `estimateShape` to apply the Frankot-Chellappa algorithm to your estimated normals. Once you have the function $f(x, y)$, plot it as a surface in the function `plotSurface` and include some significant viewpoints in your write-up. The 3D projection from `mpl_toolkits.mplot3D.Axes3D` along with the function `plot_surface` might be of help. Make sure to plot this in the `coolwarm` colormap. The result we expect of you is shown in Fig. 1. Include a snippet of the code you wrote for this part in your write-up.

2 Uncalibrated photometric stereo (50 points)

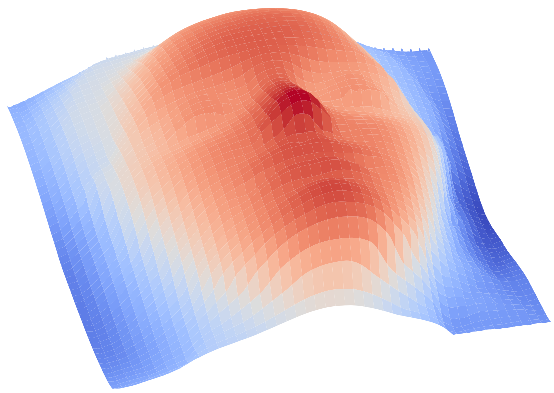


Figure 5: 3D reconstruction result from uncalibrated photometric stereo

We will now put aside the lighting direction and estimate shape directly from the images of the face. As before, load the lights into $\mathbf{L}_0$ and images in the matrix $\mathbf{I}$. We will not use the matrix $\mathbf{L}_0$ for anything except as a comparison against our estimate in this part. All code for this question goes in `src/q2.py`.

(a, 10 points) Uncalibrated normal estimation. Recall the relation $\mathbf{I} = \mathbf{L}^T\mathbf{B}$. Here, we know neither $\mathbf{L}$ nor $\mathbf{B}$. Therefore, this is a matrix factorization problem with the constraint that with the estimated $\hat{\mathbf{L}}$ and $\hat{\mathbf{B}}$, the rank of $\hat{\mathbf{I}} = \hat{\mathbf{L}}^T\hat{\mathbf{B}}$ be 3 (as you answered in the previous question), and the estimated $\hat{\mathbf{I}}$ and $\hat{\mathbf{L}}$ have appropriate dimensions.

It is well-known that the best rank-k approximation to a $m \times n$ matrix $\mathbf{M}$, where $k \leq \min\{m, n\}$ is calculated as the following: perform a singular value decomposition SVD $\mathbf{M} = \mathbf{U\Sigma V}^T$, set all singular values except the top k from $\mathbf{\Sigma}$ to 0 to get the matrix $\hat{\mathbf{\Sigma}}$, and reconstitute $\hat{\mathbf{M}} = \mathbf{U}\hat{\mathbf{\Sigma}}\mathbf{V}^T$. Explain in your write-up how this can be used to construct a factorization of the form detailed above following the required constraints.

(b, 10 points) Calculation and visualization. With your method, estimate the pseudonormals $\hat{\mathbf{B}}$ in `estimatePseudonormalsUncalibrated`, include a snippet of this function in your write-up and visualize the resultant albedos and normals in the `gray` and `rainbow` colormaps respectively. Include these too in the write-up. A sample result has been shown in Fig. 6.

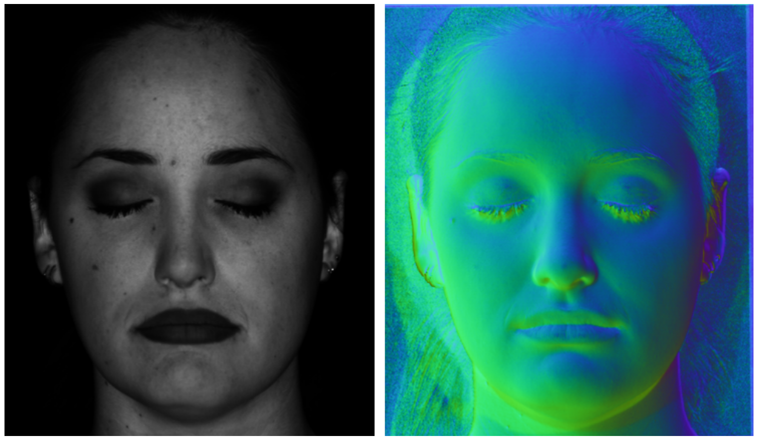


Figure 6: Uncalibrated photometric stereo. Estimated albedo and normals.

(c, 5 points) Comparing to ground truth lighting. In your write-up, compare the $\hat{\mathbf{L}}$ estimated by the factorization above to the ground truth lighting directions given in $\mathbf{L}_0$. Are they similar? Unless a special choice of factorization is made, they will be different. Describe a simple change to the procedure in (a) that changes the $\hat{\mathbf{L}}$ and $\hat{\mathbf{B}}$, but keeps the images rendered using them the same using only the matrices you calculated during the singular value decomposition (U/S/V). (No need to code anything for this part, just describe the change in the write-up)

(d, 5 points) Reconstructing the shape, attempt 1. Use the given implementation of the Frankot-Chellappa algorithm from the previous question to reconstruct a 3D depth map and visualize it as a surface in the 'coolwarm' colormap as in the previous question. Does this look like a face?

Enforcing integrability explicitly. The ambiguity you demonstrated in (c) can be (partially) resolved by explicitly imposing integrability on the pseudonormals recovered from the factorization. We will follow the approach in [2] to transform our estimated pseudonormals into a set of pseudonormals that are integrable. The function `enforceIntegrability` in `utils.py` implements this process.

(e, 5 points) Reconstructing the shape, attempt 2. Input your pseudonormals into the `enforceIntegrability` function, use the output pseudonormals to estimate the shape with the Frankot-Chellappa algorithm and plot a surface as in the previous questions (results shown in Fig. 5). Does this surface look like the one output by calibrated photometric stereo? Include at least three viewpoints of the surface and your answers in your write-up.

Hint After integration, your surface may look 'inside-out'. You can deal with this by applying the following GBR (generalized bas-relief) transform matrix to your pseudo-normals.

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Check below for what $\mathbf{G}$ is and once this transformation is applied, make sure to recompute the normals and reintegrate the surface afterwards.

The generalized bas-relief ambiguity. Unfortunately, the procedure in (e) resolves the ambiguity only up to a matrix of the form

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & \nu & \lambda \end{bmatrix} \tag{2}$$

where $\lambda > 0$, as shown in [3]. This means that for any μ, ν and $\lambda > 0$, if $\mathbf{B}$ is a set of integrable pseudonormals producing a given appearance, $\mathbf{G}^{-T}\mathbf{B}$ is another set of integrable pseudonormals producing the same appearance. The ambiguity introduced in uncalibrated photometric stereo by matrices of this kind is called the *generalized bas-relief ambiguity* (French, translates to 'low-relief', pronounced 'bah ree-leef').

(f, 5 points) Why low relief? Vary the parameters μ, ν and λ in the bas-relief transformation and visualize the corresponding surfaces. Include at least six (two with each parameter varied) of the significant ones in your write-up. Looking at these, what is your guess for why the bas-relief ambiguity is so named? In your write-up, describe how the three parameters affect the surface.

(g, 5 points) Flattest surface possible. With the bas-relief ambiguity, in Eq. 2, how would you design a transformation that makes the estimated surface as flat as possible?

(h, 5 points) More measurements. We solved the problem with 7 pictures of the face. Will acquiring more pictures from more lighting directions help resolve the ambiguity?

Credits This homework, in a large part, was inspired from the photometric stereo homework in Computational Photography (15-463) offered by Prof. Ioannis Gkioulekas at CMU. The data comes from the course on Computer Vision offered by Prof. Todd Zickler at Harvard.

References

[1] R. T. Frankot and R. Chellappa. A method for enforcing integrability in shape from shading algorithms. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 10(4):439–451, July 1988.

[2] Alan Yuille and Daniel Snow. Shape and albedo from multiple images using integrability. In *Proceedings of IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, pages 158–164. IEEE, 1997.

[3] P. N. Belhumeur, D. J. Kriegman, and A. L. Yuille. The bas-relief ambiguity. In *Proceedings of IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, pages 1060–1066, June 1997.