Nonlinear Legendre Spectral Finite Elements for Wind Turbine Blade Dynamics

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This paper presents a numerical implementation and examination of new wind turbine blade finite element model based on Geometrically Exact Beam Theory (GEBT) and a high-order spectral finite element method. The displacement-based GEBT is presented, which includes the coupling effects that exist in composite structures and geometric nonlinearity. Legendre spectral finite elements (LSFEs) are high-rder finite elements with nodes located at the Gauss-Legendre-Lobatto points. LSFEs can be an order of magnitude more efficient that low-order finite elements for a given accuracy level. Interpolation of the three-dimensional rotation, a major technical barrier in large-deformation simulation, is discussed in the context of LSFEs. It is shown, by numerical example, that the high-order LSFEs, where weak forms are evaluated with nodal quadrature, do not suffer from a drawback that exists in low-order finite elements where the tangent-stiffness matrix is calculated at the Gauss points. Finally, the new LSFE code is implemented in the new FAST Modularization Framework for dynamic simulation of highly flexible composite-material wind turbine blades. The framework allows for fully interactive simulations of turbine blades in operating conditions. Numerical examples showing validation and LSFE performance will be provided in the final paper.

I. Introduction

Wind power is becoming one of the most important renewable energy sources in the United States as demonstrated by the fact that the electricity produced from wind amounted to 3.56% of all generated electrical energy for the 12 months until March 2013[1]. In recent years, the size of wind turbines has been increasing immensely to lower the cost, which also leads to highly flexible turbine blades. This huge electro-mechanical system poses a significant challenge for engineering design and analysis. Although possible with modern super computers, direct three-dimensional (3D) structural analysis is so computationally expensive that engineers are always seeking for efficient high-fidelity simplified models.

Beam models are widely used to represent and analyze engineering structures that have one of its dimensions much larger than the other two. Many engineering components can be idealized as beams: bridges in civil engineering, joists and lever arms in heavy-machine industries, and helicopter rotor blades. The blades, tower, and shaft in a wind turbine system can be considered as beams. In the weight-critical applications of beam structures, like high-aspect-ratio wings in aerospace and wind energy, composite materials are attractive due to their superior weight-to-strength and weight-to stiffness ratios. However, analysis of structures made of composite materials is more difficult than their isotropic counterparts due to the coupling effects. The Geometrically Exact Beam Theory (GEBT), which was first proposed by Reissner in 1973[2], is a method that has proven powerful for analysis of highly flexible composite beams in the helicopter engineering community. During the past several decades, much effort has been invested in this area. Simo[3] and Simo and Vu-Quoc[4] extended Reissner's work to deal with three-dimensional (3D) dynamic problems. Jelenić and Crisfield[5] implemented this theory using the finite element method where a new approach for interpolating the rotation field was proposed that preserves the geometric exactness. Betsch and Steinmann[6] circumvented the interpolation of rotation by introducing a re-parameterization of the weak form corresponding to the equations of motion of GEBT. It is noted that Ibrahimbegović and his colleagues implemented this theory for static[7] and dynamic[8] analysis. In contrast to the displacement-based implementations, the geometric exact beam theory has also been formulated by mixed finite elements where both the primary and dual field are independently interpolated[9]. In the mixed formulation, all of the necessary ingredients, including Hamilton's principle and kinematic equations, are combined in a single variational formulation statement; Lagrange multipliers, motion variables, generalized strains, forces and moments, linear and angular momenta, and displacement and rotation variables are considered as independent quantities. Yu et al.[10,11] presented the implementation of GEBT in a mixed formulation; various rotation parameters were investigated and the code was validated against analytical and numerical solutions. The readers are referred to a textbook by Hodges[12], where comprehensive derivations and discussions on nonlinear composite beam theories can be found.

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Legendre spectral finite elements[13,14] are p-type finite elements whose shape functions are Lagrangian interpolants with node locations at the Gauss-Lobatto-Legendre points. It combines the accuracy of global spectral methods with geometric flexibility of h-type FEs. The spectral FEs have seen extensive, highly successful use in the simulation of fluid dynamics[13–15], two-dimensional elastic wave propagation in solid media in geophysics[16], elastodynamics[17], and acoustic wave propagation[18]. However, it has seen limited application to dynamic analysis of beam[19–22] and plate elements[23?,24].

In this paper, we present a displacement-based implementation of geometrically exact beam theory using Legendre spectral finite elements (LSFEs). This work builds on a previous effort which showed the implementation of three-dimensional rotation parameters[11] and a demonstration example of two-dimensional nonlinear spectral beam elements[25]. The theoretical foundation, the geometrically exact beam theory, is introduced first. Then the interpolation of the 3D rotation field by LSFEs is discussed along with a simple numerical example. Finally, validation examples are provided to show the accuracy and efficiency of the present model for realistic wind turbine blades. The code implemented in this work is in accordance to FAST Modularization Framework[26], which allows simulation of a whole turbine under realistic operating conditions.

II. Geometrically Exact Beam Theory

For completeness, this section reviews the geometric exact beam theory and linearization process of the governing equations. The content of this section can be found in many other papers and textbooks. Figure 1 shows a beam in its initial undeformed and deformed states. A reference frame $\mathbf{b}_i$ is introduced along the beam axis for the undeformed state; a frame $\mathbf{B}_i$ is introduced along each point of the deformed beam axis. Curvilinear coordinate x_1 defines the intrinsic parameterization of the reference line. In this paper, we use matrix notation to denote vectorial or vectorial-

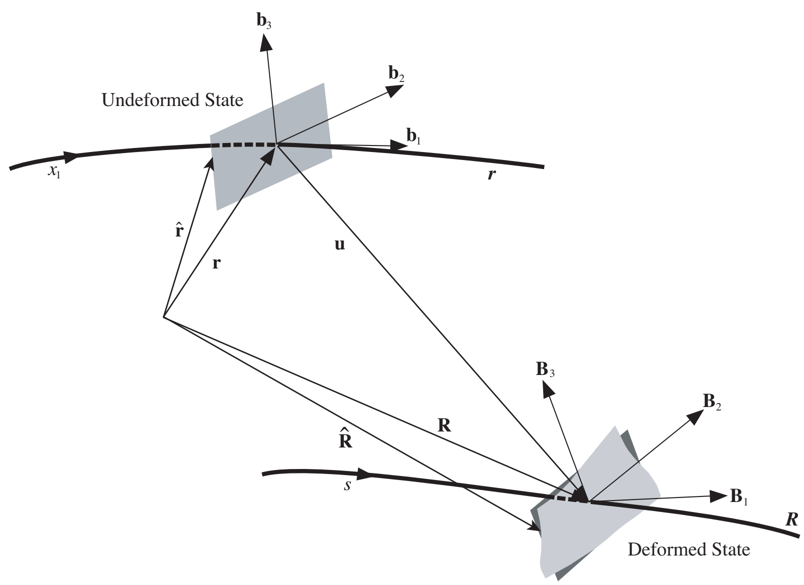


Figure 1: Schematic of beam deformation

like quantities. For example, we use a underline to denote a vector $\underline{u}$, a bar to denote unit vector $\bar{n}$, and double underline to denote a tensor $\underline{\underline{\Delta}}$. Note that sometimes the underlines only denote the dimension of the corresponding matrix. The governing equations of motion for geometric exact beam theory can be written as[27]

$$\dot{\underline{h}} - \underline{F}' = \underline{f} \tag{1}$$

$$\dot{\underline{g}} + \dot{\tilde{u}}\underline{h} - \underline{M}' - (\tilde{x}_0' + \tilde{u}')\underline{F} = \underline{m} \tag{2}$$

where $\underline{h}$ and $\underline{g}$ are the linear and angular momenta resolved in the inertial coordinate system, respectively; $\underline{F}$ and $\underline{M}$ are the beam's sectional forces and moments, respectively; $\underline{u}$ is the 1D displacement of the reference line; $\underline{x}_0$ is the position vector of a point along the beam's reference line; $\underline{f}$ and $\underline{m}$ are the distributed force and moment applied to the beam structure. Notation $(\bullet)'$ indicates a derivative with respect to the beam axis x_1 and $(\dot{\bullet})$ indicates a derivative with respect to time. The tilde operator $(\widetilde{\bullet})$ defines a second-order, skew-symmetric tensor corresponding to the given vector. In the literature, it is also termed as "cross-product matrix". For example,

$$\widetilde{n} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

The constitutive equations relate the velocities to the momenta and the one-dimensional strain measures to the sectional resultants as

$$\begin{Bmatrix} \underline{h} \\ \underline{g} \end{Bmatrix} = \underline{\underline{\mathcal{M}}} \begin{Bmatrix} \underline{\dot{u}} \\ \underline{\omega} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} \underline{F} \\ \underline{M} \end{Bmatrix} = \underline{\underline{\mathcal{C}}} \begin{Bmatrix} \underline{\epsilon} \\ \underline{\kappa} \end{Bmatrix} \tag{4}$$

where $\underline{\underline{\mathcal{M}}}$ and $\underline{\underline{\mathcal{C}}}$ are the 6×6 sectional mass and stiffness matrices,respectively, note that they are not really tensors; $\underline{\epsilon}$ and $\underline{\kappa}$ are the 1D strains and curvatures, respectively. $\underline{\omega}$ is the angular velocity vector that is defined by the rotation tensor $\underline{\underline{R}}$ as $\underline{\omega} = \mathrm{axial}(\underline{\underline{\dot{R}}}\,\underline{\underline{R}})$.

For a displacement-based finite element implementation, there are six degree-of-freedoms(DoFs) at each node: 3 displacement components and 3 rotation components. Here we use $\underline{q}$ to denote the elemental displacement array as $\underline{q} = \begin{bmatrix} \underline{u}^T & \underline{p}^T \end{bmatrix}$ where $\underline{u}$ is the 1D displacement and $\underline{p}$ is the rotation parameter vector. The acceleration array can thus be defined as $\underline{a} = \begin{bmatrix} \underline{\ddot{u}}^T & \underline{\dot{\omega}}^T \end{bmatrix}$. For nonlinear finite element analysis, the discretized and incremental forms of displacement, velocity, and acceleration array are written as

$$\underline{q}(x_1) = \underline{\underline{N}}\,\underline{\hat{q}} \quad \Delta\underline{q}^T = \begin{bmatrix} \Delta\underline{u}^T & \Delta\underline{p}^T \end{bmatrix} \tag{5}$$

$$\underline{v}(x_1) = \underline{\underline{N}}\,\underline{\hat{v}} \quad \Delta\underline{\dot{v}}^T = \begin{bmatrix} \Delta\underline{\dot{u}}^T & \Delta\underline{\omega}^T \end{bmatrix} \tag{6}$$

$$\underline{a}(x_1) = \underline{\underline{N}}\,\underline{\hat{a}} \quad \Delta\underline{a}^T = \begin{bmatrix} \Delta\underline{\ddot{u}}^T & \Delta\underline{\dot{\omega}}^T \end{bmatrix} \tag{7}$$

where $\underline{\underline{N}}$ is the shape function matrix and $(\hat{\bullet})$ denotes a column matrix of nodal values. The governing equations for beams are highly nonlinear so that a linearization process is needed. According to Bauchau[27], the linearized governing equations in Eq. (1) and (2) are in the form of

$$\underline{\underline{\hat{M}}}\Delta\underline{\hat{a}} + \underline{\underline{\hat{G}}}\Delta\underline{\hat{v}} + \underline{\underline{\hat{K}}}\Delta\underline{\hat{q}} = \underline{\hat{F}}^{ext} - \underline{\hat{F}} \tag{8}$$

where the $\underline{\underline{\hat{M}}}$, $\underline{\underline{\hat{G}}}$, and $\underline{\underline{\hat{K}}}$ are the elemental mass, gyroscopic, and stiffness matrices, respectively; $\underline{\hat{F}}$ and $\underline{\hat{F}}^{ext}$ are the elemental forces and externally applied loads, respectively. They are defined as follows

$$\underline{\underline{\hat{M}}} = \int_0^l \underline{\underline{N}}^T \underline{\underline{\mathcal{M}}}\,\underline{\underline{N}} dx_1 \tag{9}$$

$$\underline{\underline{\hat{G}}} = \int_0^l \underline{\underline{N}}^T \underline{\underline{\mathcal{G}}}^I\,\underline{\underline{N}} dx_1 \tag{10}$$

$$\underline{\underline{\hat{K}}} = \int_0^l \left[\underline{\underline{N}}^T (\underline{\underline{\mathcal{K}}}^I + \underline{\underline{\mathcal{Q}}})\,\underline{\underline{N}} + \underline{\underline{N}}^T \underline{\underline{\mathcal{P}}}\,\underline{\underline{N}}' + \underline{\underline{N}}'^T \underline{\underline{\mathcal{C}}}\,\underline{\underline{N}}' + \underline{\underline{N}}'^T \underline{\underline{\mathcal{O}}}\,\underline{\underline{N}} \right] dx_1 \tag{11}$$

$$\underline{\hat{F}} = \int_0^l (\underline{\underline{N}}^T \underline{\mathcal{F}}^I + \underline{\underline{N}}^T \underline{\mathcal{F}}^D + \underline{\underline{N}}'^T \underline{\mathcal{F}}^C) dx_1 \tag{12}$$

$$\underline{\hat{F}}^{ext} = \int_0^l \underline{\underline{N}}^T \underline{\mathcal{F}}^{ext} dx_1 \tag{13}$$

The new matrix notations in Eq. (9) to (13) are briefly introduced here. $\underline{\underline{\mathcal{M}}}$ is the sectional mass matrix resolved in inertial system; $\underline{\mathcal{F}}^C$ and $\underline{\mathcal{F}}^D$ are elastic forces obtained from Eq. (1) and (2) as

$$\underline{\mathcal{F}}^C = \begin{Bmatrix} \underline{F} \\ \underline{M} \end{Bmatrix} = \underline{\underline{\mathcal{C}}} \begin{Bmatrix} \underline{\epsilon} \\ \underline{\kappa} \end{Bmatrix} \tag{14}$$

$$\underline{\mathcal{F}}^D = \begin{bmatrix} \underline{0} & \underline{\underline{0}} \\ (\tilde{x}_0' + \tilde{u}')^T & \underline{\underline{0}} \end{bmatrix} \underline{\mathcal{F}}^C \equiv \underline{\underline{\Upsilon}}\,\underline{\mathcal{F}}^C \tag{15}$$

where $\underline{\underline{0}}$ denotes a 3×3 null matrix. The $\underline{\underline{\mathcal{G}}}^I$, $\underline{\underline{\mathcal{K}}}^I$, $\underline{\underline{\mathcal{O}}}$, $\underline{\underline{\mathcal{P}}}$, $\underline{\underline{\mathcal{Q}}}$, and $\underline{\mathcal{F}}^I$ in Eq. (10), Eq. (11), and Eq. (12) are defined as

$$\underline{\underline{\mathcal{G}}}^I = \begin{bmatrix} \underline{\underline{0}} & (\tilde{\omega} m \underline{\eta})^T + \tilde{\omega} m \tilde{\eta}^T \\ \underline{\underline{0}} & \tilde{\omega} \underline{\underline{\varrho}} - \widetilde{\underline{\underline{\varrho}} \omega} \end{bmatrix} \tag{16}$$

$$\underline{\underline{\mathcal{K}}}^I = \begin{bmatrix} \underline{\underline{0}} & \dot{\tilde{\omega}} m \tilde{\eta}^T + \tilde{\omega} \tilde{\omega} m \tilde{\eta}^T \\ \underline{\underline{0}} & \ddot{\tilde{u}} m \tilde{\eta} + \underline{\underline{\varrho}} \dot{\tilde{\omega}} - \widetilde{\underline{\underline{\varrho}} \dot{\omega}} + \tilde{\omega} \underline{\underline{\varrho}} \tilde{\omega} - \tilde{\omega} \widetilde{\underline{\underline{\varrho}} \omega} \end{bmatrix} \tag{17}$$

$$\underline{\underline{\mathcal{O}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{C}}_{11} \tilde{E}_1 - \tilde{F} \\ \underline{\underline{0}} & \underline{\underline{C}}_{21} \tilde{E}_1 - \tilde{M} \end{bmatrix} \tag{18}$$

$$\underline{\underline{\mathcal{P}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \tilde{F} + (\underline{\underline{C}}_{11} \tilde{E}_1)^T & (\underline{\underline{C}}_{21} \tilde{E}_1)^T \end{bmatrix} \tag{19}$$

$$\underline{\underline{\mathcal{Q}}} = \underline{\underline{\Upsilon}}\, \underline{\underline{\mathcal{O}}} \tag{20}$$

$$\underline{\mathcal{F}}^I = \begin{Bmatrix} m \underline{\ddot{u}} + (\dot{\tilde{\omega}} + \tilde{\omega} \tilde{\omega}) m \underline{\eta} \\ m \tilde{\eta} \underline{\ddot{u}} + \underline{\underline{\varrho}} \underline{\dot{\omega}} + \tilde{\omega} \underline{\underline{\varrho}} \underline{\omega} \end{Bmatrix} \tag{21}$$

The following notations were introduced to simply the writing of the above expressions

$$\underline{E}_1 = \underline{x}_0' + \underline{u}' \tag{22}$$

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{C}}_{11} & \underline{\underline{C}}_{12} \\ \underline{\underline{C}}_{21} & \underline{\underline{C}}_{22} \end{bmatrix} \tag{23}$$

The derivation and linearization of governing equations of geometrically exact beam theory can be found in Bauchau[27].

III. Interpolation of Rotation Field by Spectral Finite Element

The displacement fields in a element can be interpolated as

$$\underline{u}(s) = h^k(s) \underline{\hat{u}}^k \tag{24}$$

$$\underline{u}'(s) = h^{k\prime}(s) \underline{\hat{u}}^k \tag{25}$$

where $\underline{u}$ is the displacement field, $h^k(s), k = 1, 2, ...n$ are the shape functions from first to the n^{th} node; $\underline{\hat{u}}$ is the nodal values of the displacement field. However, as discussed in Bauchau[27], the three-dimensional rotation field cannot be simply interpolated as the displacement field in the form of

$$\underline{c}(s) = h^k(s) \underline{\hat{c}}^k \tag{26}$$

$$\underline{c}'(s) = h^{k\prime}(s) \underline{\hat{c}}^k \tag{27}$$

where $\underline{c}$ is the rotation field in a element and $\underline{\hat{c}}^k$ is the nodal value at the k^{th} node, for three reasons: 1) rotations do not form a linear space so that they have to be composed instead of added; 2) rescaling operation is needed to eliminate the singularity existing in the vectorial rotation parameters; 3) it is lack of objectivity, which is defined by Crisfield and Jelenić[5] refers to the invariance of strain measures computed through interpolation to the addition of a rigid body motion. Therefore, we adopt a more robust interpolation approach proposed by Crisfield and Jelenić[5] to deal with the finite rotations.

Step 1: Compute the nodal relative rotations, $\underline{\hat{r}}^k$ by removing the rigid body rotation, $\underline{\hat{c}}^1$, from the finite rotation at each node, $\underline{\hat{r}}^k = \underline{\hat{c}}^{1-} \oplus \underline{\hat{c}}^k$.

Step 2: Interpolate the relative rotation field: $\underline{r}(s) = h^k(s) \underline{\hat{r}}^k$ and $\underline{r}'(s) = h^{k\prime}(s) \underline{\hat{r}}^k$. Find the curvature field $\underline{\kappa}(s) = \underline{\underline{R}}(\underline{\hat{c}}^1) \underline{\underline{H}}(\underline{r}) \underline{r}'$.

Step 3: Restore the rigid body rotation removed in Step 1: $\underline{c}(s) = \underline{\hat{c}}^1 \oplus \underline{r}(s)$.

where H is the tangent tensor that relates the curvature vector k and rotation vector p as

$$\underline{k} = \underline{\underline{H}}\, \underline{p}' \tag{28}$$

In the LSFE approach, shape functions (e.g., those composing $\underline{\underline{N}}$) are n^{th}-order Lagrangian interpolants, where nodes are located at the $n + 1$ GLL-quadrature points in the $[-1, 1]$ element natural-coordinate domain.

Table 1: Axial displacement u_1 of a cantilever beam subject to a constant moment (in inches).

λ	Analytical	BeamDyn	% Error
0.4	-2.4317	-2.4317	0.00
0.8	-7.6613	-7.6613	0.00
1.2	-11.5591	-11.5591	0.00
1.6	-11.8921	-11.8921	0.00
2.0	-10.0000	-10.0000	0.00

IV. Numerical Examples

A. Example 1: Static bending of a cantilever beam

The first example is a benchmark problem for geometrically nonlinear analysis of beams[3?]. We calculate the static deflection of a cantilever beam that is subjected at its free end to a constant moment M. The length of the beam L is $10\,in$ and the cross-sectional stiffness matrix is given below:

$$C = 10^3 \times \begin{bmatrix} 1770 & 0 & 0 & 0 & 0 & 0 \\ & 1770 & 0 & 0 & 0 & 0 \\ & & 1770 & 0 & 0 & 0 \\ & & & 8.16 & 0 & 0 \\ & & & & 86.9 & 0 \\ & & & & & 215 \end{bmatrix} \tag{29}$$

The load is given by the following equation:

$$M_2 = \lambda \bar{M}_2 \tag{30}$$

where $\bar{M}_2 = \pi \frac{EI_2}{L}$. The parameter λ will vary between 0 and 2. In this case, the beam is discretized with two 5^{th} order elements. The deformations of the beam are shown in Figure 2. The calculated results are compared with the

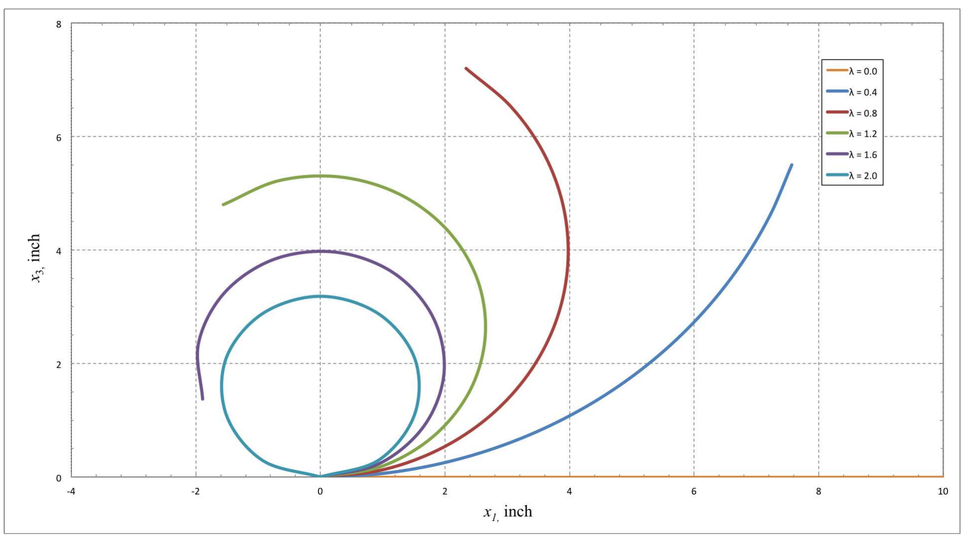


Figure 2: Deformations of a cantilever beam under several constant bending moments.

analytical solution, which can be found in Ref.[?] as

$$u_1 = \rho sin\left(\frac{x_1}{\rho}\right) - x_1 \qquad u_3 = \rho\left(1 - cos\left(\frac{x_1}{\rho}\right)\right) \tag{31}$$

The results can be found in Table 1 and 2, respectively. Good agreement can be observed between these two sets of results.

The rotation parameters at each node along beam axis x_1 obtained from BeamDyn are plotted in Figure 3 for $\lambda = 0.8$ and $\lambda = 2.0$, respectively. It is noted that the three-dimensional rotations are represented by Wiener-Milenković parameter defined in the following equation:

$$\underline{p} = 4tan\frac{\phi}{4}\bar{n} \tag{32}$$

Table 2: Vertical displacement u_3 of a cantilever beam subject to a constant moment (in inches).

λ	Analytical	BeamDyn	% Error
0.4	5.4987	5.4987	0.00
0.8	7.1978	7.1979	0.0013
1.2	4.7986	4.7986	0.00
1.6	1.3747	1.3747	0.00
2.0	0.0000	0.0000	0.00

where ϕ is the rotation angle and $\bar{n}$ is the unit vector of rotation axis. The singularity exists in the above definition can be removed by a rescaling operation, which can be observed in Figure 3. Figure 4 shows the normalized error

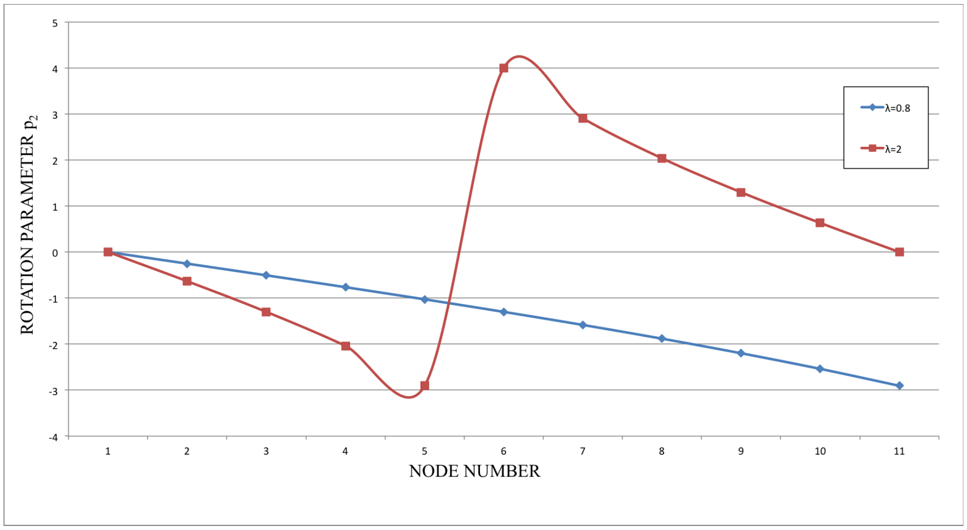
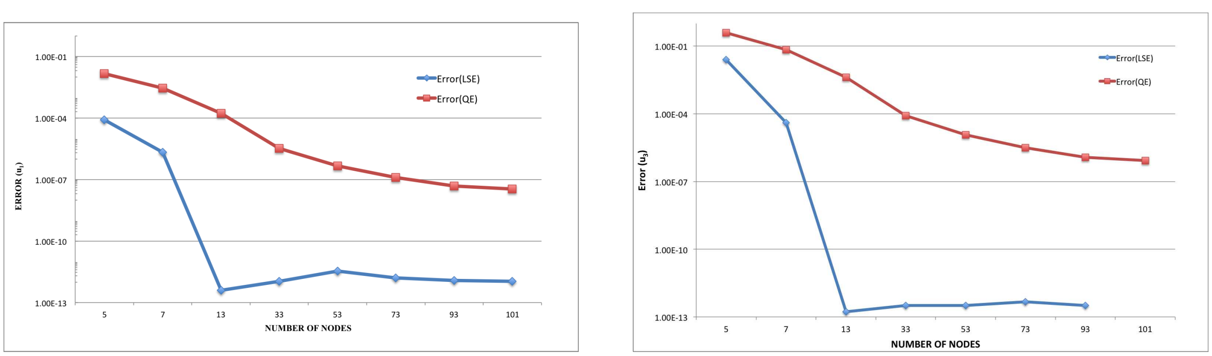


Figure 3: Wiener-Milenković rotation parameters along beam axis x_1 .

$\epsilon(u)$, where u is the tip displacement (at $x = L$), as a function of the number of model nodes for the calculation with Dymore4 quadratic elements (QE) and a single Legendre spectral element (LSE), where

$$\epsilon(u) = \left| \frac{u - u^a}{u^a} \right| \tag{33}$$

and u is the test solution and u^a is the analytical solution. The parameter λ is set to 1.0 for this case. The Legendre spectral elements (with p-refinement) exhibit highly desirable exponential convergence to machine precision error.



(a) u_1

(b) u_3

Figure 4: Normalized error of the (a) u_1 and (b) u_3 displacements as a function of the total number of nodes

B. Example 2: Static analysis of a composite beam

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