Generalized Approximate Static Condensation Method for a Heterogeneous Multi-Material Diffusion Problem

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Overview

1. The ASC(n) Method
   - Problem Setting
   - Description of the Method
   - ASC(0) and ASC(1)
2. Numerical Experiments
   - ASC(0) $\rightarrow$ ASC(1): Motivation
   - Piecewise Linear & Quadratic Benchmarks

Diffusion Problem

Our objective is to solve the diffusion problem in the mixed form

$$\begin{cases} \mathbf{K}^{-1}\,\mathbf{u} + \nabla\, p = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \nabla \cdot \mathbf{u} + c\, p \;= f & \text{in } \Omega, \end{cases}$$

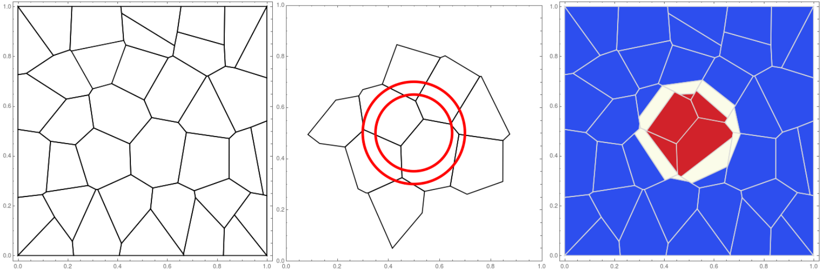
with boundary data

$$\begin{aligned} p &= g_D \quad \text{on } \partial\Omega_D, \\ \mathbf{u} \cdot \hat{\mathbf{n}} &= g_N \quad \text{on } \partial\Omega_N. \end{aligned}$$

Challenges:

- The diffusion tensor $\mathbf{K}$ may sharply vary in Ω and may be discontinuous
- We want to use general polygonal meshes, and
- being able to handle material interfaces not aligned with the mesh

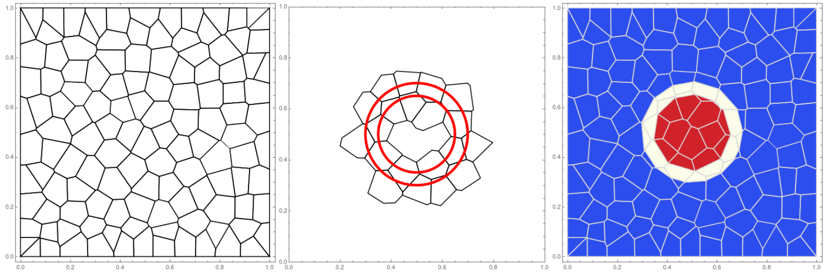
Mesh



(a) Base Mesh $\mathcal{T}_H$ (b) Multi-Material Cells (c) MOF

Moment-of-Fluid interface reconstruction $\Rightarrow$ reconstructed interface may be discontinuous

Mesh



(a) Base Mesh $\mathcal{T}_H$ (b) Multi-Material Cells (c) MOF

Moment-of-Fluid interface reconstruction $\Rightarrow$ reconstructed interface may be discontinuous

Consider $T \in \mathcal{T}_H$: $\begin{cases} \mathbf{K}^{-1}\,\mathbf{u} + \nabla\, p = 0 & \text{in } T, \\ \nabla\cdot\mathbf{u} + c\,p = f & \text{in } T, \\ p = \lambda & \text{on } \partial T \end{cases}$

$$\Downarrow$$

Find trial functions $\langle \mathbf{u}, p\rangle \in \mathbb{H}_{\text{div}}(T) \times \mathbb{L}^2(T)$ such that

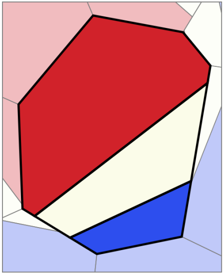
$$\begin{cases} \displaystyle\int_T \mathbf{K}^{-1}\,\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_T p\,\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = -\int_{\partial T} \lambda\,\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}l, \\ \displaystyle\int_T \nabla\cdot\mathbf{u}\,q\,\mathrm{d}\mathbf{x} + \int_T c\,p\,q\,\mathrm{d}\mathbf{x} = \int_T f\,q\,\mathrm{d}\mathbf{x} \end{cases}$$

holds for all test functions $\langle \mathbf{v}, q\rangle \in \mathbb{H}_{\text{div}}(T) \times \mathbb{L}^2(T)$

Local Problem

Consider $T \in \mathcal{T}_H$:
$$\begin{cases} \mathbf{K}^{-1}\mathbf{u} + \nabla p = 0 & \text{in } T, \\ \nabla \cdot \mathbf{u} + c\,p = f & \text{in } T, \\ p = \lambda & \text{on } \partial T \end{cases}$$

$\Downarrow$



Minimesh τ_h of T

$\Rightarrow$

Discretization

Apply Mimetic Finite Difference Method*

$\Downarrow$

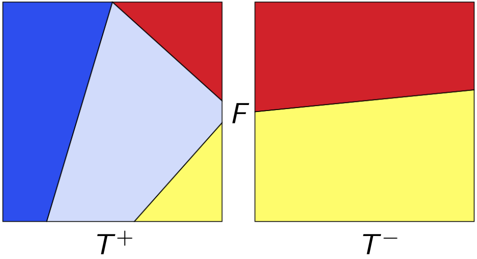
$$\begin{pmatrix} \mathbf{M}_{\tau_h} & \mathbf{B}_{\tau_h}^T \\ \mathbf{B}_{\tau_h} & \mathbf{\Sigma}_{\tau_h} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}}_{\tau_h} \\ \bar{\mathbf{p}}_{\tau_h} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\tau_h} & \mathbf{C}_{\tau_h} \\ & \bar{\mathbf{f}}_{\tau_h} \end{pmatrix} \bar{\boldsymbol{\lambda}}_{\tau_h}$$

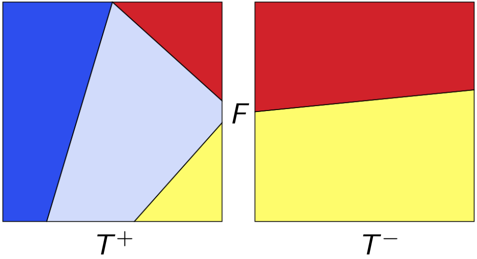
*L. Beirao da Veiga, K. Lipnikov, G. Manzini
The Mimetic Finite Difference Method for Elliptic Problems
Springer 2014

If one knows the pressure trace λ for each $T \in \mathcal{T}_H$, one can recover the solution in $\mathcal{T}_H$. The idea is **(i)** to express external flux DOFs in terms of trace DOFs (*static condensation*),

$$\bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h} := \mathbf{E}^T_{\boldsymbol{\tau}_h} \bar{\mathbf{u}}_{\boldsymbol{\tau}_h} = \mathbf{A}_{\boldsymbol{\tau}_h} \mathbf{C}_{\boldsymbol{\tau}_h} \bar{\boldsymbol{\lambda}}_{\boldsymbol{\tau}_h} - \bar{\mathbf{a}}_{\boldsymbol{\tau}_h},$$

and **(ii)** to get the system for trace DOFs by requiring weak continuity of fluxes. **Problem**: we may have different number of trace DOFs from T^+ and T^-





Solution: approximate a pressure trace on F with a polynomial $\hat{\lambda} \in \mathbb{P}^n(F)$ described in terms of its $(n+1)$ moments

$$\frac{\int_F \hat{\lambda}\, s_i \,\mathrm{d}l}{|F|}, \quad i = 0, \ldots, n.$$

Here $s_i \in \mathbb{P}^i(F)$ is a fixed polynomial of degree i such that $s_i \perp_{\mathbb{L}^2} s_j$, $j < i$

ASC(n): DOFs and Constraints

DOFs

Now we express trace DOFs on mini faces of $\boldsymbol{\tau}_h$ via coarse trace DOFs := $(n+1)$ moments on each base face of T,

$$\bar{\boldsymbol{\lambda}}_{\boldsymbol{\tau}_h} = \mathbf{R}_{\boldsymbol{\tau}_h} \bar{\boldsymbol{\lambda}}_T \quad \Rightarrow$$
$$\bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h} = \mathbf{A}_{\boldsymbol{\tau}_h} \mathbf{C}_{\boldsymbol{\tau}_h} \mathbf{R}_{\boldsymbol{\tau}_h} \bar{\boldsymbol{\lambda}}_T - \bar{\mathbf{a}}_{\boldsymbol{\tau}_h},$$

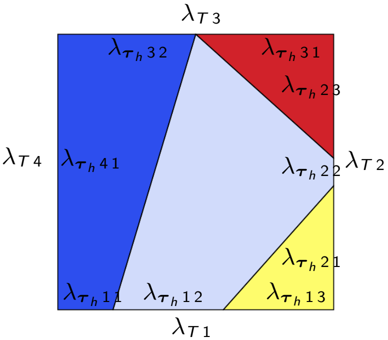
Constraints

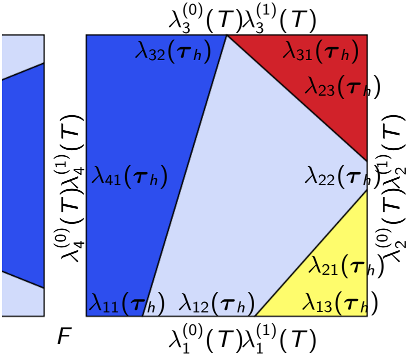
and close the system by requiring weak continuity of normal fluxes on each base face

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \ldots, n \text{ for } F \in \mathcal{F}_{\text{int}}$$

Express fluxes in terms of traces $\Rightarrow$ get SLAE for coarse trace DOFs

ASC(0) and ASC(1) DOFs





$$\underbrace{\begin{pmatrix}\lambda_{\tau_h\,11}\\ \lambda_{\tau_h\,12}\\ \lambda_{\tau_h\,13}\\ \lambda_{\tau_h\,21}\\ \lambda_{\tau_h\,22}\\ \lambda_{\tau_h\,23}\\ \lambda_{\tau_h\,31}\\ \lambda_{\tau_h\,31}\\ \lambda_{\tau_h\,41}\end{pmatrix}}_{\bar{\boldsymbol{\lambda}}_{\tau_h}=} = \underbrace{\begin{pmatrix}1&0&0&0\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}_{\mathbf{R}_{\tau_h}=} \underbrace{\begin{pmatrix}\lambda_{T\,1}\\ \lambda_{T\,2}\\ \lambda_{T\,3}\\ \lambda_{T\,4}\end{pmatrix}}_{\bar{\boldsymbol{\lambda}}_{T}=}$$

$$\underbrace{\begin{pmatrix}\lambda_{\tau_h\,11}\\ \lambda_{\tau_h\,12}\\ \lambda_{\tau_h\,13}\\ \lambda_{\tau_h\,21}\\ \lambda_{\tau_h\,22}\\ \lambda_{\tau_h\,23}\\ \lambda_{\tau_h\,31}\\ \lambda_{\tau_h\,31}\\ \lambda_{\tau_h\,41}\end{pmatrix}}_{\bar{\boldsymbol{\lambda}}_{\tau_h}=} = \underbrace{\begin{pmatrix}1&\Delta s_{11}&0&0&0&0&0\\1&\Delta s_{12}&0&0&0&0&0\\1&\Delta s_{13}&0&0&0&0&0\\0&0&1&\Delta s_{21}&0&0&0\\0&0&1&\Delta s_{22}&0&0&0\\0&0&1&\Delta s_{23}&0&0&0\\0&0&0&0&1&\Delta s_{31}&0\\0&0&0&0&1&\Delta s_{32}&0\\0&0&0&0&0&0&1\end{pmatrix}}_{\mathbf{R}_{\tau_h}=} \underbrace{\begin{pmatrix}\lambda_{T\,1}\\ \lambda'_{T\,1}\\ \lambda_{T\,2}\\ \lambda'_{T\,2}\\ \lambda_{T\,3}\\ \lambda'_{T\,3}\\ \lambda_{T\,4}\end{pmatrix}}_{\bar{\boldsymbol{\lambda}}_{T}=}$$

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l,\ i = 0, \dots, n \text{ for } F \in \mathscr{F}_{\text{int}}$$

$$\Downarrow$$

$$\begin{aligned} n = 0: &\quad \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^+\, i}\, |f^+_{F\, i}| + \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^-\, i}\, |f^-_{F\, i}| = 0, \\ n = 1: &\quad \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^+\, i}\, \Delta s_i^+\, |f^+_{F\, i}| + \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^-\, i}\, \Delta s_i^-\, |f^-_{F\, i}| = 0 \end{aligned}$$

$$\Downarrow$$

$$\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^+}\, \mathbf{C}_{\boldsymbol{\tau}_h^+}\, \bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h^+}\right)_i + \left(\mathbf{R}^T_{\boldsymbol{\tau}_h^-}\, \mathbf{C}_{\boldsymbol{\tau}_h^-}\, \bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h^-}\right)_j = 0$$

$$\Downarrow$$

$$\left(\underbrace{\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^+}\, \mathbf{C}_{\boldsymbol{\tau}_h^+}\, \mathbf{A}_{\boldsymbol{\tau}_h^+}\, \mathbf{C}_{\boldsymbol{\tau}_h^+}\, \mathbf{R}_{\boldsymbol{\tau}_h^+}\right)}_{\mathbf{S}_{T^+} :=} \bar{\boldsymbol{\lambda}}_{T^+}\right)_i + \left(\underbrace{\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^-}\, \mathbf{C}_{\boldsymbol{\tau}_h^-}\, \mathbf{A}_{\boldsymbol{\tau}_h^-}\, \mathbf{C}_{\boldsymbol{\tau}_h^-}\, \mathbf{R}_{\boldsymbol{\tau}_h^-}\right)}_{\mathbf{S}_{T^-} :=} \bar{\boldsymbol{\lambda}}_{T^-}\right)_j =$$

$$\underbrace{\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^+}\, \mathbf{C}_{\boldsymbol{\tau}_h^+}\, \bar{\mathbf{a}}_{\boldsymbol{\tau}_h^+}\right)}_{\bar{\mathbf{s}}_{T^+}}{}_i + \underbrace{\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^-}\, \mathbf{C}_{\boldsymbol{\tau}_h^-}\, \bar{\mathbf{a}}_{\boldsymbol{\tau}_h^-}\right)}_{\bar{\mathbf{s}}_{T^-}}{}_j$$

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \ldots, n \text{ for } F \in \mathcal{F}_{\text{int}}$$

$$\Downarrow$$

$$n = 0: \quad \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^+\, i} \, |f^+_{F\, i}| + \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^-\, i} \, |f^-_{F\, i}| = 0,$$

$$n = 1: \quad \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^+\, i} \, \Delta s_i^+ \, |f^+_{F\, i}| + \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^-\, i} \, \Delta s_i^- \, |f^-_{F\, i}| = 0$$

$$\Downarrow$$

$$\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^+} \, \mathbf{C}_{\boldsymbol{\tau}_h^+} \, \bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h^+}\right)_i + \left(\mathbf{R}^T_{\boldsymbol{\tau}_h^-} \, \mathbf{C}_{\boldsymbol{\tau}_h^-} \, \bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h^-}\right)_j = 0$$

$$\Downarrow$$

$$\mathbf{S}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \mathbf{S}_T \, \mathbf{N}_T, \qquad \bar{\mathbf{s}}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \bar{\mathbf{s}}_T,$$

Global system:

$$\mathbf{S}_{\mathcal{T}_H} \, \bar{\boldsymbol{\lambda}}_{\mathcal{T}_H} = \bar{\mathbf{s}}_{\mathcal{T}_H}$$

ASC(0) and ASC(1)

$$\mathbf{S}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \mathbf{S}_T \, \mathbf{N}_T, \qquad \bar{\mathbf{s}}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \bar{\mathbf{s}}_T,$$

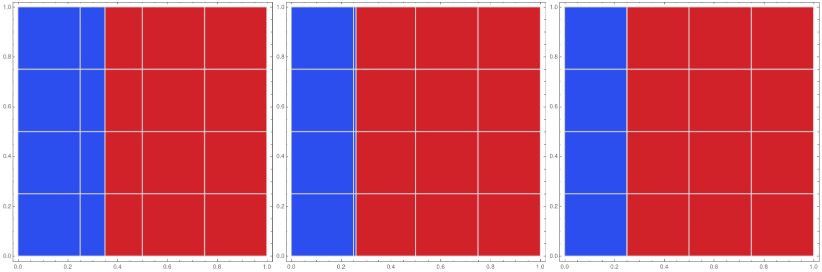
Global system:

$$\mathbf{S}_{\mathcal{T}_H} \bar{\boldsymbol{\lambda}}_{\mathcal{T}_H} = \bar{\mathbf{s}}_{\mathcal{T}_H}$$

- **Theorem**: system matrix $\mathbf{S}_{\mathcal{T}_H}$ is sparse and SPD for ASC(0) and ASC(1)
- Hence efficient solvers and preconditioners are available (e. g. CG + Algebraic Multigrid)
- Once we obtain $\bar{\boldsymbol{\lambda}}_{\mathcal{T}_H}$, we recover pressure and flux DOFs in each cell $T \in \mathcal{T}_H$ (this may be done in parallel)

ASC(1): Robustness (1 / 2)

Figure: $w :=$ width of the left minimesh cells

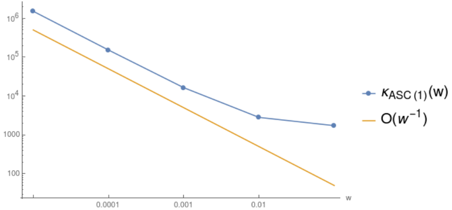


(a) $w = .1$ (b) $w = .01$ (c) $w = .001$

We solve the diffusion problem w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is piecewise linear

Figure: Condition Numbers of ASC(0) / ASC(1) Matrices

w	$\kappa_{\text{ASC}(0)}$	$\kappa_{\text{ASC}(1)}$
10^{-1}	41	1 730
10^{-2}	45	2 817
10^{-3}	48	16 391
10^{-4}	49	152 325
10^{-5}	49	1.5×10^6



$\kappa_{\text{ASC}(0)}$ does not depend on w, and $\kappa_{\text{ASC}(1)}$ is proportional to w^{-1}. However, if we remove 3 smallest eig values (corresponding to 3 int MM faces), **we will have $\tilde{\kappa}_{\text{ASC}(1)} = \kappa_{\text{ASC}(0)}$**. Starting from some iteration CG behaves like extreme eig values are not present; that is, several small eig values is not a problem

ℓ^2-Error

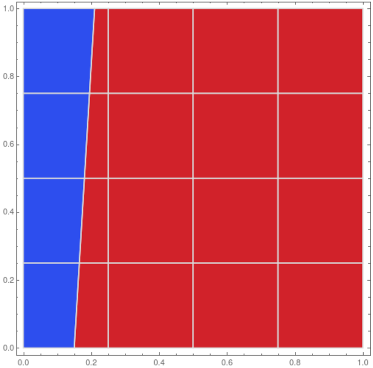
If the base mesh consists of triangles + we have no material interfaces, ASC(n) boils down to Mixed-Hybrid Raviart – Thomas FEM:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{L}^2(\Omega)} \le c\, h\, \|\mathbf{u}\|_{\mathbb{H}^1(\Omega)},$$
$$\|p - p_h\|_{\mathbb{L}^2(\Omega)} \le c\left(h\,\|p\|_{\mathbb{H}^1(\Omega)} + h^2\,\|p\|_{\mathbb{H}^2(\Omega)}\right).$$

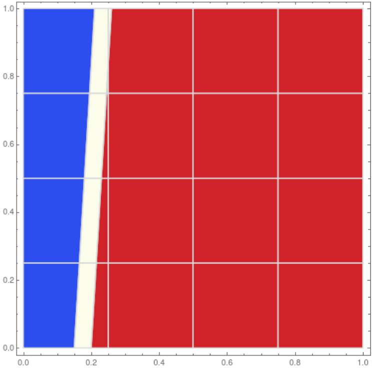
That is, we cannot expect ASC(n) convergence to be better than linear. We define **discrete $\mathbb{L}^2$-norm**

$$\|v\|_{\ell^2(\Omega)} := \|P_h\, v\|_{\mathbb{L}^2(\Omega)} \le \|v\|_{\mathbb{L}^2(\Omega)},$$

where $P_h :=$ $\mathbb{L}^2$-projection operator on the space of piecewise constant functions on each cell $T \in \mathcal{T}_H$ (or on each $\tau \in \boldsymbol{\tau}_h$ if T is a MMC)



(a) $||p - p_h||_{\ell^2(\Omega)} = 6.38 \times 10^{-2}$

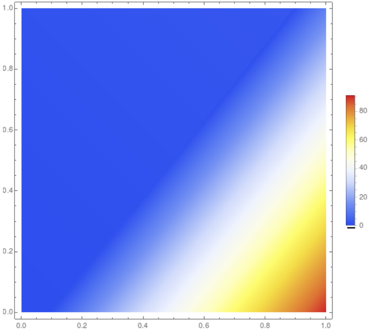


(b) $||p - p_h||_{\ell^2(\Omega)} = 6.41 \times 10^{-2}$

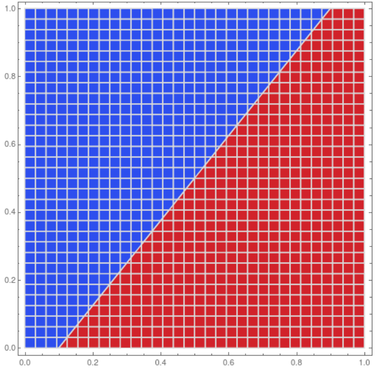
Here $\mathbf{K}_i = \mathbf{K}_j$ and the exact soln is linear. ASC(0) produces errors due to const trace approximation, and ACS(1) recovers the exact soln

Piecewise Linear Benchmark (1 / 2)

We solve the diffusion problem on the sequence of square meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is piecewise linear



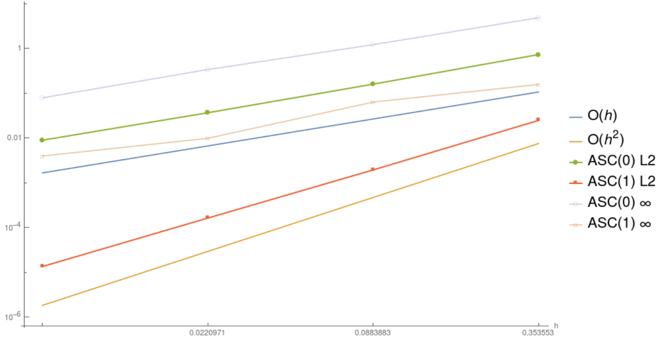
(a) Benchmark soln, p



(b) Materials

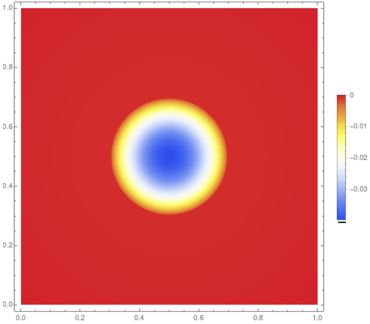
Piecewise Linear Benchmark (2 / 2)

	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(0)	3.5×10^{-1}	7.3×10^{-1}		4.8
	8.8×10^{-2}	1.6×10^{-1}	1.1	1.2
	2.2×10^{-2}	3.7×10^{-2}	1.1	3.4×10^{-1}
	5.5×10^{-3}	8.9×10^{-3}	1.0	7.9×10^{-2}
	h	$e_1^{\ell^2}$	p	e_1^{∞}
ASC(1)	3.5×10^{-1}	2.5×10^{-2}		1.6×10^{-1}
	8.8×10^{-2}	1.9×10^{-3}	1.84	6.3×10^{-2}
	2.2×10^{-2}	1.6×10^{-4}	1.79	9.8×10^{-3}
	5.5×10^{-3}	1.3×10^{-5}	1.80	4.0×10^{-3}

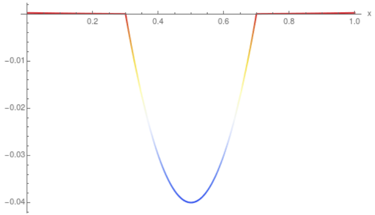


1
0.01
10^{-4}
10^{-6}
0.0220971
0.0883883
0.353553
h
O(h)
O(h^2)
ASC(0) L2
ASC(1) L2
ASC(0) ∞
ASC(1) ∞

We solve the diffusion problem on Voronoi meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the circle and .001 inside. Exact solution is pw quadratic. We compare convergence of ASC(0) and ASC(1)



(a) Benchmark soln, p

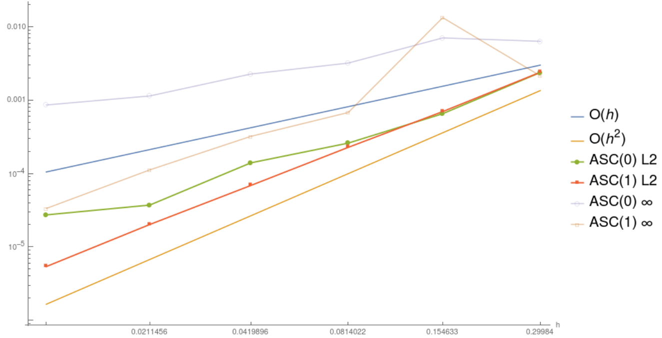


(b) $p(x, \frac{1}{2})$

Piecewise Quadratic Benchmark w/ 2 Materials (2 / 3)

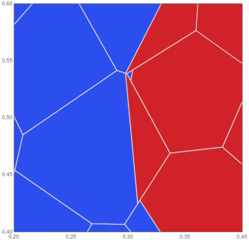
	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(0)	3.0×10^{-1}	2.4×10^{-3}		6.3×10^{-1}
	1.5×10^{-1}	6.5×10^{-4}	2.0	7.0×10^{-3}
	8.1×10^{-2}	2.6×10^{-4}	1.4	3.2×10^{-3}
	4.2×10^{-2}	1.4×10^{-4}	9.4×10^{-1}	2.3×10^{-3}
	2.1×10^{-2}	3.7×10^{-5}	1.9	1.1×10^{-3}
	1.0×10^{-2}	2.7×10^{-5}	4.4×10^{-1}	8.6×10^{-4}

	h	$e_1^{\ell^2}$	p	e_1^{∞}
ASC(1)	3.0×10^{-1}	2.4×10^{-3}		2.1×10^{-3}
	1.5×10^{-1}	7.0×10^{-4}	1.9	1.3×10^{-2}
	8.1×10^{-2}	2.3×10^{-4}	1.8	6.8×10^{-4}
	4.2×10^{-2}	6.8×10^{-5}	1.8	3.2×10^{-4}
	2.1×10^{-2}	2.0×10^{-5}	1.8	1.1×10^{-4}
	1.0×10^{-2}	5.4×10^{-6}	1.9	3.3×10^{-5}



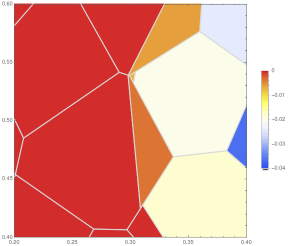
We observe a jump of ∞-error of ASC(1) at $h = 1.5 \times 10^{-1}$

Piecewise Quadratic Benchmark w/ 2 Materials (3 / 3)

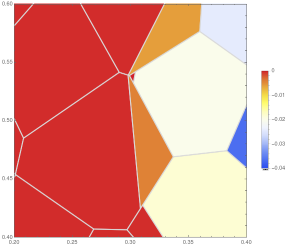


(a) Materials

$h = 1.5 \times 10^{-1}$: This example shows that ASC(1) ∞-norm may be sensitive to geometry errors. However, it does not affect ℓ^2-convergence

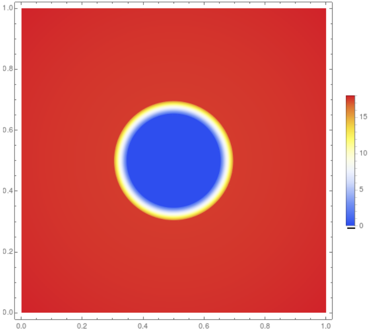


(b) ASC(0), p_h

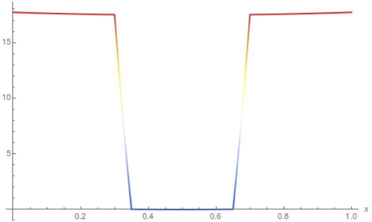


(c) ASC(1), p_h

We solve the diffusion problem on triangular meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the ring and .001 inside. Exact solution is piecewise quadratic



(a) Benchmark soln, p



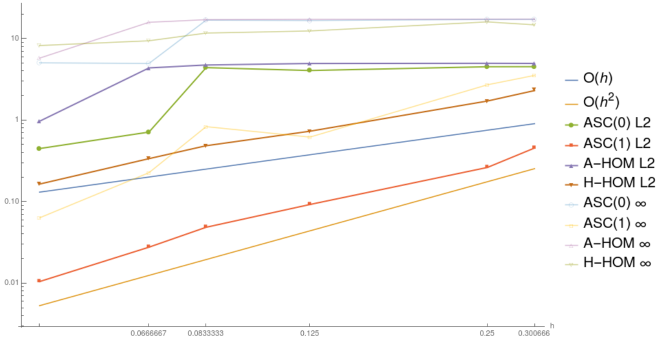
(b) $p(x, \frac{1}{2})$

	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(0)	3.0×10^{-1}	4.5		17
	2.5×10^{-1}	4.5		17
	1.3×10^{-1}	4.0		17
	8.3×10^{-2}	4.4		17
	6.7×10^{-2}	7.1×10^{-1}		4.9
	4.3×10^{-2}	4.5×10^{-1}	1.2	5.0

	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(1)	3.0×10^{-1}	4.5×10^{-1}		3.5
	2.5×10^{-1}	2.6×10^{-1}	3	2.7
	1.3×10^{-1}	9.2×10^{-2}	1.5	6.2×10^{-1}
	8.3×10^{-2}	4.8×10^{-2}	1.6	8.3×10^{-1}
	6.7×10^{-2}	2.8×10^{-2}	2.5	2.3×10^{-1}
	4.3×10^{-2}	1.0×10^{-2}	2.3	6.3×10^{-2}

Homogenization		h	$e^{\ell^2}_{\mathrm{AH}}$	p	e^{∞}_{AH}
	Arithmetic	3.0×10^{-1}	4.9		17
		2.5×10^{-1}	5.0		17
		1.3×10^{-1}	4.9		17
		8.3×10^{-2}	4.7		17
		6.7×10^{-2}	4.4		16
		4.3×10^{-2}	9.7×10^{-1}	3.5	5.7
		h	$e^{\ell^2}_{\mathrm{HH}}$	p	e^{∞}_{HH}
	Harmonic	3.0×10^{-1}	2.3		15
		2.5×10^{-1}	1.7	1.6	16
		1.3×10^{-1}	7.3×10^{-1}	1.2	12
		8.3×10^{-2}	4.8×10^{-1}	1.0	12
		6.7×10^{-2}	3.4×10^{-1}	1.6	9.4
		4.3×10^{-2}	1.6×10^{-1}	1.7	8.2

Piecewise Quadratic Benchmark w/ 3 Materials (2 / 2)



Before $h = 6.7 \times 10^{-2}$ we have cells / faces with 3 materials, and after this mesh level we have only 2 material MMCs

Results:

- ASC(n) is able to efficiently handle unfitted material interfaces
- 2nd order ℓ^2-convergence for ASC(1)
- Effective condition number seems to be uniformly bounded w.r.t. an interface position
- The underline matrix is SPD and sparse; its pattern does not depend on mini meshes

TODO List:

- Mixed formulation: convergence for fluxes?
- Time-dependent benchmarks
- Anisotropic diffusion: homogenization is not applicable; what about ASC(n)?