Approximate Static Condensation: Numerical Comparison of ASC(0) and ASC(1)

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Overview

1. Convergence
   - Piece-wise Linear Benchmark
   - Piece-wise Quadratic Benchmark
     - 2 Materials
     - 3 Materials

2. Robustness
   - Motivation
   - Numerical Example

Section 1

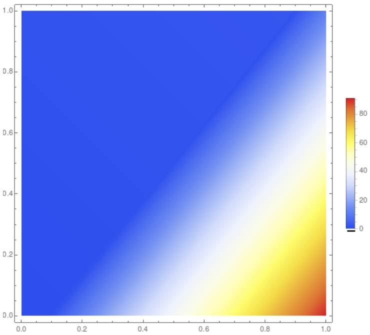
Convergence

Subsection 1

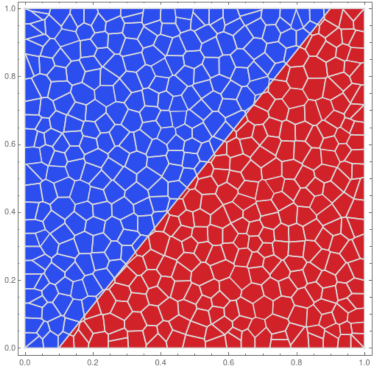
Piece-wise Linear Benchmark

Problem Description

We solve the diffusion problem w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is pw linear. We compare convergence of ASC(0) and ASC(1)

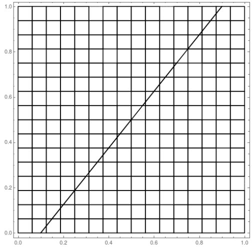


(a) Benchmark soln, p



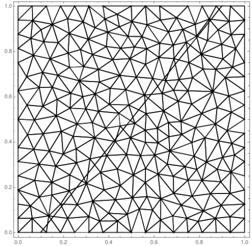
(b) Materials

Meshes

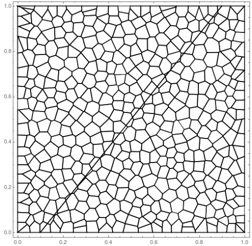


(a) Square

We solve the problem on a sequence of square, triangular, and voronoi meshes



(b) Triangular

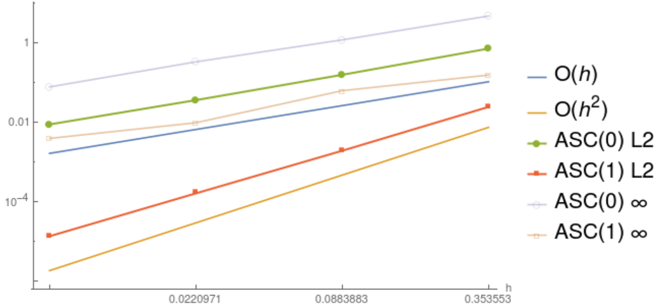


(c) Voronoi

Square Meshes

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_{\infty}$

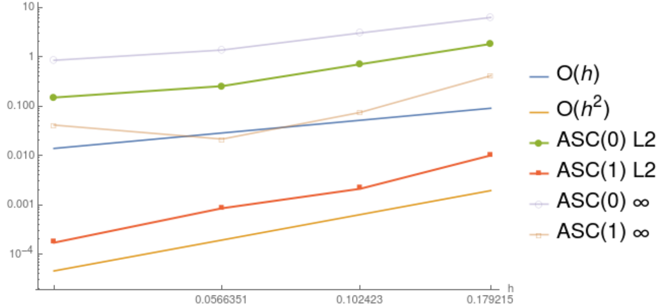
h	$e^{\mathbb{L}^2}_{\text{ASC}(0)}$	$e^{\infty}_{\text{ASC}(0)}$	$e^{\mathbb{L}^2}_{\text{ASC}(1)}$	$e^{\infty}_{\text{ASC}(1)}$
3.2×10^{-1}	7.3×10^{-1}	4.8	2.5×10^{-2}	1.6×10^{-1}
1.6×10^{-1}	3.6×10^{-1}	3.7	1.0×10^{-2}	3.9×10^{-1}
8.0×10^{-2}	1.6×10^{-1}	1.2	1.9×10^{-3}	6.3×10^{-2}
4.0×10^{-2}	7.6×10^{-2}	5.8×10^{-1}	6.3×10^{-4}	1.7×10^{-2}



Triangular Meshes

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_{\infty}$

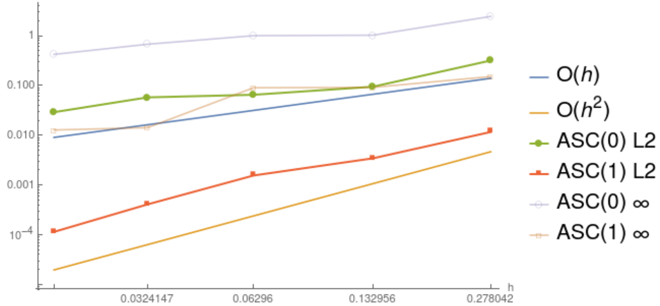
h	$e^{\mathbb{L}^2}_{\mathrm{ASC}(0)}$	$e^{\infty}_{\mathrm{ASC}(0)}$	$e^{\mathbb{L}^2}_{\mathrm{ASC}(1)}$	$e^{\infty}_{\mathrm{ASC}(1)}$
1.8×10^{-1}	1.8	6.2	9.9×10^{-3}	4.1×10^{-1}
1.0×10^{-1}	7.0×10^{-1}	3.0	2.1×10^{-3}	7.4×10^{-2}
5.7×10^{-2}	2.5×10^{-1}	1.4	8.5×10^{-4}	2.1×10^{-2}
2.8×10^{-2}	1.5×10^{-1}	8.4×10^{-1}	1.7×10^{-4}	4.1×10^{-2}



Voronoi Meshes

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_\infty$

h	$e^{\mathbb{L}^2}_{\text{ASC}(0)}$	$e^{\infty}_{\text{ASC}(0)}$	$e^{\mathbb{L}^2}_{\text{ASC}(1)}$	$e^{\infty}_{\text{ASC}(1)}$
2.8×10^{-1}	3.2×10^{-1}	2.4	1.2×10^{-2}	1.5×10^{-1}
1.3×10^{-1}	9.4×10^{-2}	1.0	3.4×10^{-3}	9.1×10^{-2}
6.3×10^{-2}	6.5×10^{-2}	1.0	1.5×10^{-3}	9.0×10^{-2}
3.2×10^{-2}	5.8×10^{-2}	6.7×10^{-1}	4.1×10^{-4}	1.4×10^{-2}
1.8×10^{-2}	2.9×10^{-2}	4.2×10^{-1}	1.1×10^{-4}	1.3×10^{-2}



Subsection 2

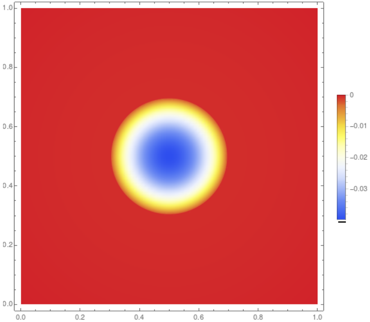
Piece-wise Quadratic Benchmark

Subsubsection 1

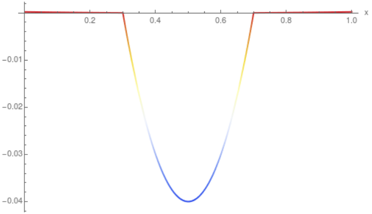
2 Materials

Problem Description

We solve the diffusion problem on voronoi meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the circle and .001 inside. Exact solution is pw quadratic. We compare convergence of ASC(0) and ASC(1)



(a) Benchmark soln, p

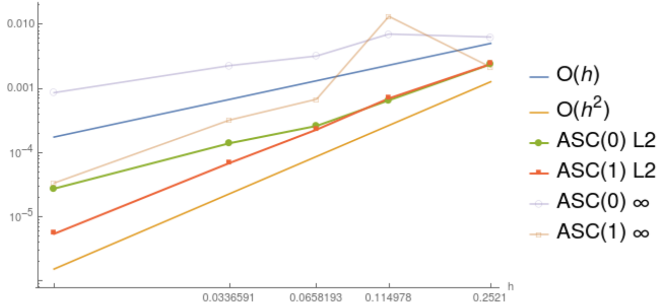


(b) $p(x, \frac{1}{2})$

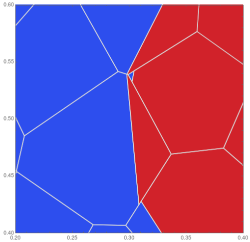
Convergence

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_{\infty}$

h	$e^{\mathbb{L}^2}_{\mathrm{ASC}(0)}$	$e^{\infty}_{\mathrm{ASC}(0)}$	$e^{\mathbb{L}^2}_{\mathrm{ASC}(1)}$	$e^{\infty}_{\mathrm{ASC}(1)}$
2.5×10^{-1}	2.4×10^{-3}	6.3×10^{-3}	2.4×10^{-3}	2.1×10^{-3}
1.1×10^{-1}	6.5×10^{-4}	7.0×10^{-3}	7.0×10^{-3}	1.3×10^{-2}
6.6×10^{-2}	2.6×10^{-4}	3.2×10^{-3}	2.3×10^{-4}	6.8×10^{-4}
3.4×10^{-2}	1.4×10^{-4}	2.3×10^{-3}	6.8×10^{-5}	3.2×10^{-4}
8.7×10^{-2}	2.7×10^{-5}	8.6×10^{-4}	5.4×10^{-6}	3.3×10^{-5}

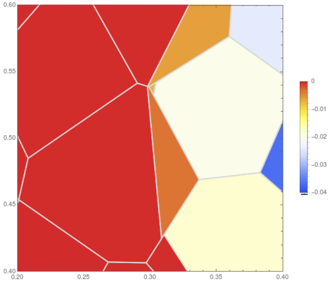


$h = 1.1 \times 10^{-1}$

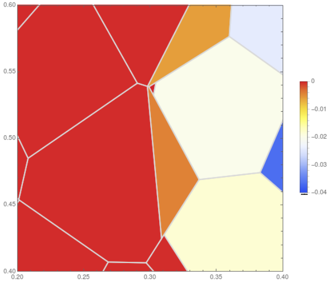


(a) Materials

$h = 1.1 \times 10^{-1}$: This example shows that ASC(1) ∞–norm may be sensitive to geometry errors. However, it does not affect $\mathbb{L}^2$–convergence



(b) ASC(0), p_h

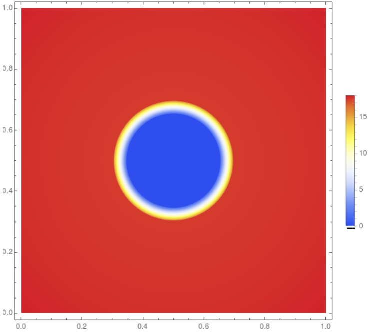


(c) ASC(1), p_h

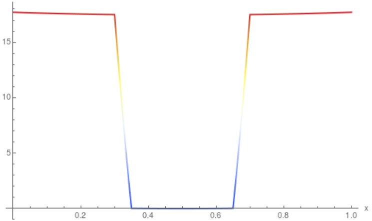
Subsubsection 2

3 Materials

We solve the diffusion problem on triangular meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the ring and .001 inside. Exact solution is pw quadratic. We compare convergence of ASC(0) and ASC(1)



(a) Benchmark soln, p

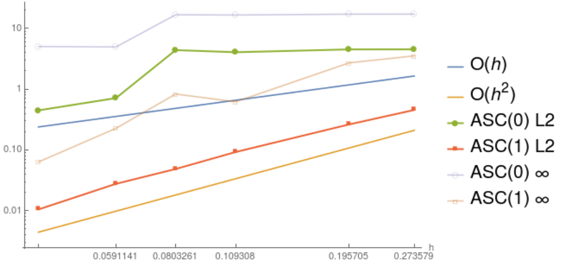


(b) $p(x, \frac{1}{2})$

Convergence

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_{\infty}$

h	$e^{\mathbb{L}^2}_{\mathrm{ASC}(0)}$	$e^{\infty}_{\mathrm{ASC}(0)}$	$e^{\mathbb{L}^2}_{\mathrm{ASC}(1)}$	$e^{\infty}_{\mathrm{ASC}(1)}$
2.7×10^{-1}	4.5	17	4.5×10^{-1}	3.5
2.0×10^{-1}	4.5	17	2.6×10^{-1}	2.7
1.1×10^{-1}	4.0	17	9.2×10^{-2}	6.2×10^{-1}
8.0×10^{-2}	4.4	17	4.8×10^{-2}	8.3×10^{-1}
5.9×10^{-2}	7.1×10^{-1}	4.9	2.8×10^{-2}	2.3×10^{-1}
4.0×10^{-2}	4.5×10^{-1}	5.0	1.0×10^{-2}	6.3×10^{-2}



Before $h = 8.0 \times 10^{-2}$ we have cells / faces with 3 materials, and after this mesh level we have only 2 material MMCs

Section 2

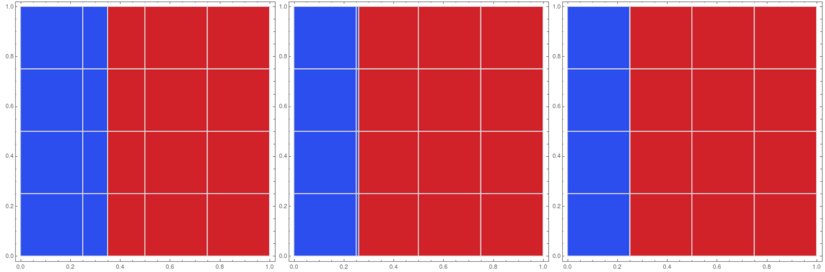
Robustness

Subsection 1

Motivation

Robustness Test: Geometry

Figure: $w :=$ width of the left minimesh cells



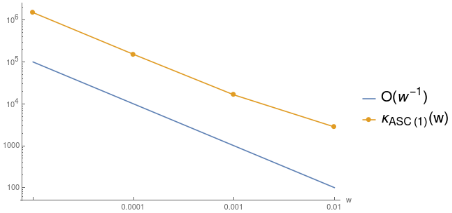
(a) $w = .1$ (b) $w = .01$ (c) $w = .001$

We solve the diffusion problem w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is pw linear

Robustness Test: Spectrum

Figure: Condition Numbers of ASC(0) / ASC(1) Matrices

w	$\kappa_{\mathrm{ASC}(0)}$	$\kappa_{\mathrm{ASC}(1)}$
10^{-1}	41	1 730
10^{-2}	45	2 817
10^{-3}	48	16 391
10^{-4}	49	152 325
10^{-5}	49	1.5×10^6

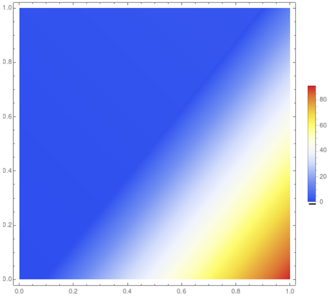


$\kappa_{\mathrm{ASC}(0)}$ does not depend on w, and $\kappa_{\mathrm{ASC}(1)}$ is proportional to w^{-1}. However, if we remove 3 smallest eig values (corresponding to 3 int MM faces), **we will have $\kappa_{\mathrm{ASC}(1)} = \kappa_{\mathrm{ASC}(0)}$**. Starting from some iteration CG behaves like extreme eig values are not present; that is, several small eig values is not a problem

Subsection 2

Numerical Example

Problem Description



(a) Benchmark soln, p

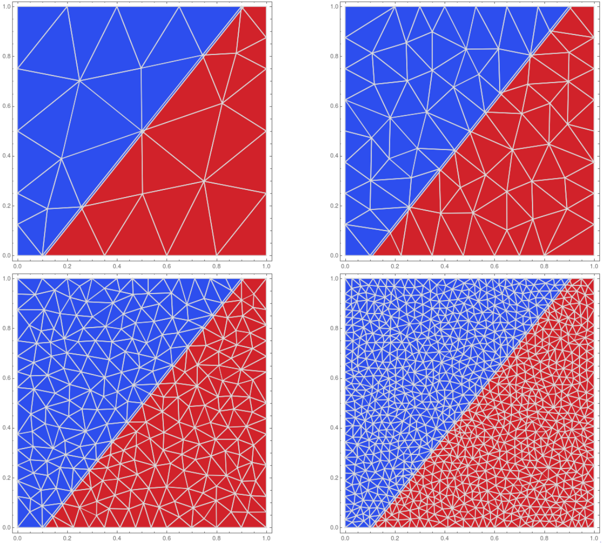
We solve the same diffusion problem as in the first section: $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right.

But here we use very specific triangular meshes built in such a way so that on each mesh level we have base vertices extremely close to the interface (that is, number of "degenerate" faces grows linearly with the mesh level).

Exact solution is pw linear. We compare convergence of ASC(0) and ASC(1)

Meshes

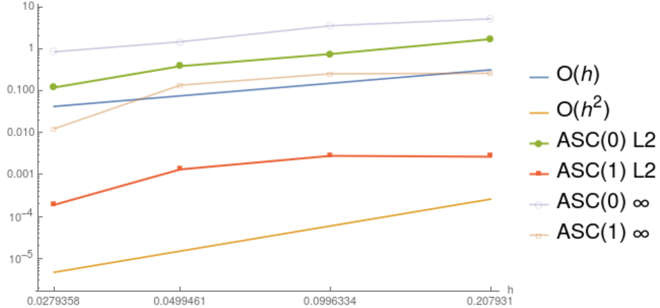
Figure: $w = .01$



Convergence, $w = .01$

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_{\infty}$

h	$e^{\mathbb{L}^2}_{\text{ASC}(0)}$	$e^{\infty}_{\text{ASC}(0)}$	$e^{\mathbb{L}^2}_{\text{ASC}(1)}$	$e^{\infty}_{\text{ASC}(1)}$
2.1×10^{-1}	1.7	5.1	2.7×10^{-3}	2.6×10^{-1}
1.0×10^{-1}	7.4×10^{-1}	3.5	2.8×10^{-3}	2.5×10^{-1}
5.0×10^{-2}	3.8×10^{-1}	1.5	1.3×10^{-3}	1.4×10^{-1}
2.8×10^{-2}	1.2×10^{-1}	8.4×10^{-1}	1.9×10^{-4}	1.2×10^{-2}



Convergence, $w = .001$

Figure: $e^{\mathbb{L}^2} := ||p - p_h||_{\mathbb{L}^2(\Omega)}$ and $e^{\infty} := ||p - p_h||_{\infty}$

h	$e^{\mathbb{L}^2}_{\text{ASC}(0)}$	$e^{\infty}_{\text{ASC}(0)}$	$e^{\mathbb{L}^2}_{\text{ASC}(1)}$	$e^{\infty}_{\text{ASC}(1)}$
2.2×10^{-1}	9.0×10^{-1}	8.1	4.0×10^{-5}	4.3×10^{-2}
1.1×10^{-1}	3.5×10^{-1}	4.8	5.6×10^{-5}	4.3×10^{-2}
5.0×10^{-2}	3.0×10^{-1}	2.3	5.7×10^{-5}	4.0×10^{-2}
2.8×10^{-2}	1.8×10^{-1}	8.9×10^{-1}	5.6×10^{-5}	3.1×10^{-2}