Generalized Approximate Static Condensation Method for a Heterogeneous Multi-Material Diffusion Problem

Alexander Zhiliakov[1]

Collaborators: Daniil Svyatsky[2], Eugene Kikinzon[2], Mikhail Shashkov[2], Maxim Olshanskii[1]

[1]Department of Mathematics
University of Houston

[2]Los Alamos
National Laboratory

This work was performed under the auspices of the US Department of Energy

SIAM TX-LA Conference, Oct 5 – 7, 2018

Overview

1. The ASC(n) Method
   - Problem Setting
   - Description of the Method
   - ASC(0) and ASC(1)
2. Numerical Experiments
   - ASC(0) $\rightarrow$ ASC(1): Motivation
   - Piecewise Linear & Quadratic Benchmarks

Diffusion Problem

Our objective is to solve the diffusion problem in the mixed form

$$\begin{cases} \mathbf{K}^{-1}\,\mathbf{u} + \nabla\, p = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \nabla \cdot \mathbf{u} + c\, p \;= f & \text{in } \Omega, \end{cases}$$

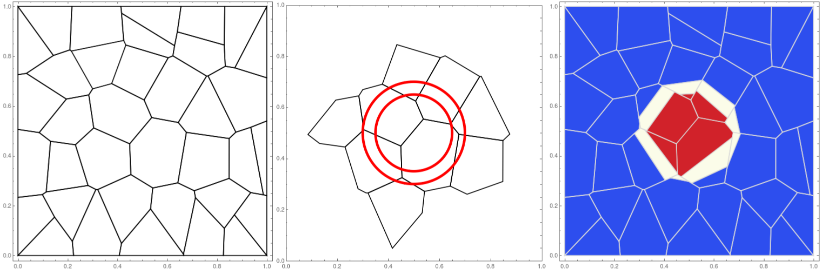
with boundary data

$$\begin{aligned} p &= g_D \quad \text{on } \partial\Omega_D, \\ \mathbf{u} \cdot \hat{\mathbf{n}} &= g_N \quad \text{on } \partial\Omega_N. \end{aligned}$$

Challenges:

- The diffusion tensor $\mathbf{K}$ may sharply vary in Ω and may be discontinuous
- We want to use general polygonal meshes, and
- being able to handle material interfaces not aligned with the mesh

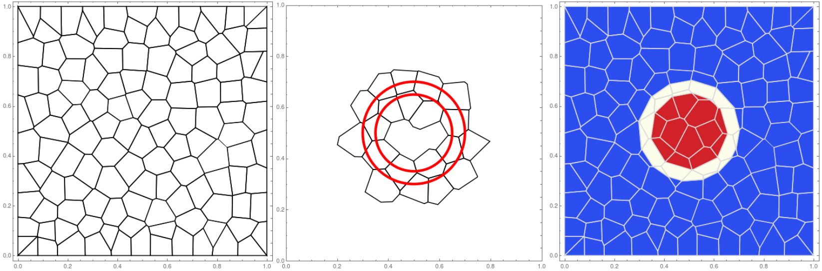
Mesh



(a) Base Mesh $\mathcal{T}_H$ (b) Multi-Material Cells (c) MOF

Moment-of-Fluid interface reconstruction $\Rightarrow$ reconstructed interface may be discontinuous

Mesh



(a) Base Mesh $\mathcal{T}_H$ (b) Multi-Material Cells (c) MOF

Moment-of-Fluid interface reconstruction $\Rightarrow$ reconstructed interface may be discontinuous

Local Problem

$$\text{Consider } T \in \mathcal{T}_H: \quad \begin{cases} \mathbf{K}^{-1}\,\mathbf{u} + \nabla\, p = 0 & \text{in } T, \\ \nabla \cdot \mathbf{u} + c\, p = f & \text{in } T, \\ p = \lambda & \text{on } \partial T \end{cases}$$

$$\Downarrow$$

Find trial functions $\langle \mathbf{u}, p \rangle \in \mathbb{H}_{\text{div}}(T) \times \mathbb{L}^2(T)$ such that

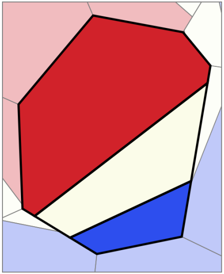
$$\begin{cases} \displaystyle\int_T \mathbf{K}^{-1}\,\mathbf{u} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} - \int_T p\,\nabla \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} = -\int_{\partial T} \lambda\,\mathbf{v} \cdot \hat{\mathbf{n}}\,\mathrm{d}l, \\ \displaystyle\int_T \nabla \cdot \mathbf{u}\, q\,\mathrm{d}\mathbf{x} + \int_T c\, p\, q\,\mathrm{d}\mathbf{x} = \int_T f\, q\,\mathrm{d}\mathbf{x} \end{cases}$$

holds for all test functions $\langle \mathbf{v}, q \rangle \in \mathbb{H}_{\text{div}}(T) \times \mathbb{L}^2(T)$

Local Problem

Consider $T \in \mathcal{T}_H$:
$$\begin{cases} \mathbf{K}^{-1}\mathbf{u} + \nabla p = 0 & \text{in } T, \\ \nabla \cdot \mathbf{u} + c\,p = f & \text{in } T, \\ p = \lambda & \text{on } \partial T \end{cases}$$

$$\Downarrow$$



Minimesh $\boldsymbol{\tau}_h$ of T

$\Rightarrow$

Discretization

Apply Mimetic Finite Difference Method*

$$\Downarrow$$

$$\begin{pmatrix} \mathbf{M}_{\boldsymbol{\tau}_h} & \mathbf{B}^T_{\boldsymbol{\tau}_h} \\ \mathbf{B}_{\boldsymbol{\tau}_h} & \boldsymbol{\Sigma}_{\boldsymbol{\tau}_h} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}}_{\boldsymbol{\tau}_h} \\ \bar{\mathbf{p}}_{\boldsymbol{\tau}_h} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\boldsymbol{\tau}_h} & \mathbf{C}_{\boldsymbol{\tau}_h} & \bar{\boldsymbol{\lambda}}_{\boldsymbol{\tau}_h} \\ & \bar{\mathbf{f}}_{\boldsymbol{\tau}_h} & \end{pmatrix}$$

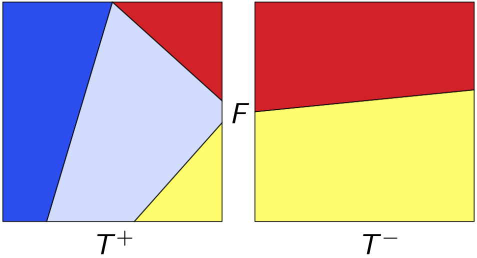
*L. Beirao da Veiga, K. Lipnikov, G. Manzini
The Mimetic Finite Difference Method for Elliptic Problems
Springer 2014

Approximate Static Condensation

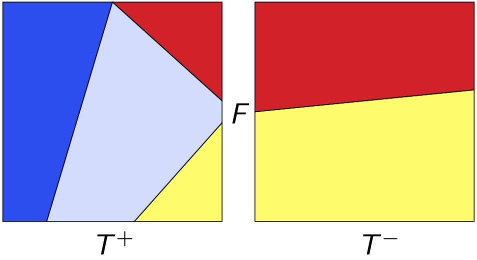
If one knows the pressure trace λ for each $T \in \mathcal{T}_H$, one can recover the solution in $\mathcal{T}_H$. The idea is **(i)** to express external flux DOFs in terms of trace DOFs (*static condensation*),

$$\bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h} := \mathbf{E}^T_{\boldsymbol{\tau}_h} \bar{\mathbf{u}}_{\boldsymbol{\tau}_h} = \mathbf{A}_{\boldsymbol{\tau}_h} \mathbf{C}_{\boldsymbol{\tau}_h} \bar{\boldsymbol{\lambda}}_{\boldsymbol{\tau}_h} - \bar{\mathbf{a}}_{\boldsymbol{\tau}_h},$$

and **(ii)** to get the system for trace DOFs by requiring weak continuity of fluxes. **Problem**: we may have different number of trace DOFs from T^+ and T^-



Approximate Static Condensation



Solution: approximate a pressure trace on F with a polynomial $\hat{\lambda} \in \mathbb{P}^n(F)$ described in terms of its $(n+1)$ moments

$$\frac{\int_F \hat{\lambda}\, s_i \,\mathrm{d}l}{|F|}, \quad i = 0, \ldots, n.$$

Here $s_i \in \mathbb{P}^i(F)$ is a fixed polynomial of degree i such that $s_i \perp_{\mathbb{L}^2} s_j$, $j < i$

ASC(n): DOFs and Constraints

DOFs: Now we express trace DOFs on mini faces of $\boldsymbol{\tau}_h$ via coarse trace DOFs := $(n+1)$ moments on each base face of T,

$$\bar{\boldsymbol{\lambda}}_{\boldsymbol{\tau}_h} = \mathbf{R}_{\boldsymbol{\tau}_h} \bar{\boldsymbol{\lambda}}_T \quad \Rightarrow$$

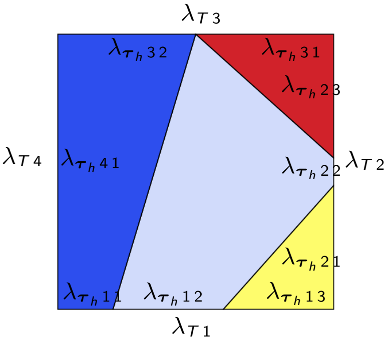
$$\bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h} = \mathbf{A}_{\boldsymbol{\tau}_h} \mathbf{C}_{\boldsymbol{\tau}_h} \mathbf{R}_{\boldsymbol{\tau}_h} \bar{\boldsymbol{\lambda}}_T - \bar{\mathbf{a}}_{\boldsymbol{\tau}_h},$$

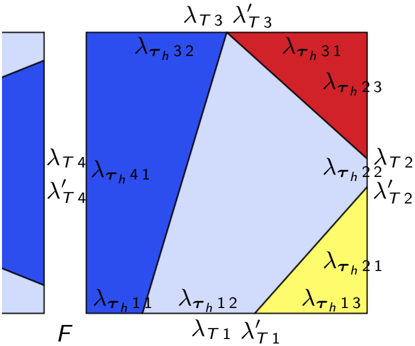
Constraints: and close the system by requiring weak continuity of normal fluxes on each base face

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \ldots, n \text{ for } F \in \mathcal{F}_{\text{int}}$$

Express fluxes in terms of traces $\Rightarrow$ get SLAE for coarse trace DOFs

ASC(0) and ASC(1) DOFs





$$\underbrace{\begin{pmatrix} \lambda_{\tau_h\,11} \\ \lambda_{\tau_h\,12} \\ \lambda_{\tau_h\,13} \\ \lambda_{\tau_h\,21} \\ \lambda_{\tau_h\,22} \\ \lambda_{\tau_h\,23} \\ \lambda_{\tau_h\,31} \\ \lambda_{\tau_h\,31} \\ \lambda_{\tau_h\,41} \end{pmatrix}}_{\bar{\lambda}_{\tau_h}=} = \underbrace{\begin{pmatrix} 1&0&0&0 \\ 1&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&1&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}}_{\mathbf{R}_{\tau_h}=} \underbrace{\begin{pmatrix} \lambda_{T\,1} \\ \lambda_{T\,2} \\ \lambda_{T\,3} \\ \lambda_{T\,4} \end{pmatrix}}_{\bar{\lambda}_T=}$$

$$\underbrace{\begin{pmatrix} \lambda_{\tau_h\,11} \\ \lambda_{\tau_h\,12} \\ \lambda_{\tau_h\,13} \\ \lambda_{\tau_h\,21} \\ \lambda_{\tau_h\,22} \\ \lambda_{\tau_h\,23} \\ \lambda_{\tau_h\,31} \\ \lambda_{\tau_h\,31} \\ \lambda_{\tau_h\,41} \end{pmatrix}}_{\bar{\lambda}_{\tau_h}=} = \underbrace{\begin{pmatrix} 1&\Delta s_{11}&0&0&0&0&0 \\ 1&\Delta s_{12}&0&0&0&0&0 \\ 1&\Delta s_{13}&0&0&0&0&0 \\ 0&0&1&\Delta s_{21}&0&0&0 \\ 0&0&1&\Delta s_{22}&0&0&0 \\ 0&0&1&\Delta s_{23}&0&0&0 \\ 0&0&0&0&1&\Delta s_{31}&0 \\ 0&0&0&0&1&\Delta s_{32}&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}}_{\mathbf{R}_{\tau_h}=} \underbrace{\begin{pmatrix} \lambda_{T\,1} \\ \lambda'_{T\,1} \\ \lambda_{T\,2} \\ \lambda'_{T\,2} \\ \lambda_{T\,3} \\ \lambda'_{T\,3} \\ \lambda_{T\,4} \end{pmatrix}}_{\bar{\lambda}_T=}$$

ASC(0) and ASC(1)

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \dots, n \text{ for } F \in \mathscr{F}_{\text{int}}$$

$$\Downarrow$$

$$n = 0: \quad \sum u^{\text{ext}}_{\tau_h^+\, i} \, |f^+_{F\, i}| + \sum u^{\text{ext}}_{\tau_h^-\, i} \, |f^-_{F\, i}| = 0,$$

$$n = 1: \quad \sum u^{\text{ext}}_{\tau_h^+\, i} \, \Delta s_i^+ \, |f^+_{F\, i}| + \sum u^{\text{ext}}_{\tau_h^-\, i} \, \Delta s_i^- \, |f^-_{F\, i}| = 0$$

$$\Downarrow$$

$$\left(\mathbf{R}^T_{\tau_h^+} \, \mathbf{C}_{\tau_h^+} \, \bar{\mathbf{u}}^{\text{ext}}_{\tau_h^+}\right)_i + \left(\mathbf{R}^T_{\tau_h^-} \, \mathbf{C}_{\tau_h^-} \, \bar{\mathbf{u}}^{\text{ext}}_{\tau_h^-}\right)_j = 0$$

$$\Downarrow$$

$$\Bigg(\underbrace{\left(\mathbf{R}^T_{\tau_h^+} \, \mathbf{C}_{\tau_h^+} \, \mathbf{A}_{\tau_h^+} \, \mathbf{C}_{\tau_h^+} \, \mathbf{R}_{\tau_h^+}\right)}_{\mathbf{S}_{T^+} :=} \bar{\boldsymbol{\lambda}}_{T^+}\Bigg)_i + \Bigg(\underbrace{\left(\mathbf{R}^T_{\tau_h^-} \, \mathbf{C}_{\tau_h^-} \, \mathbf{A}_{\tau_h^-} \, \mathbf{C}_{\tau_h^-} \, \mathbf{R}_{\tau_h^-}\right)}_{\mathbf{S}_{T^-} :=} \bar{\boldsymbol{\lambda}}_{T^-}\Bigg)_j =$$

$$\Big(\underbrace{\mathbf{R}^T_{\tau_h^+} \, \mathbf{C}_{\tau_h^+} \, \bar{\mathbf{a}}_{\tau_h^+}}_{\bar{\mathbf{s}}_{T^+}}\Big)_i + \Big(\underbrace{\mathbf{R}^T_{\tau_h^-} \, \mathbf{C}_{\tau_h^-} \, \bar{\mathbf{a}}_{\tau_h^-}}_{\bar{\mathbf{s}}_{T^-}}\Big)_j$$

ASC(0) and ASC(1)

$$\int_F \mathbf{u}|_{T^+} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l = \int_F \mathbf{u}|_{T^-} \cdot \hat{\mathbf{n}}\, s_i \,\mathrm{d}l, \; i = 0, \ldots, n \text{ for } F \in \mathcal{F}_{\text{int}}$$

$$\Downarrow$$

$$n = 0: \quad \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^+ \, i} \, |f^+_{F\,i}| + \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^- \, i} \, |f^-_{F\,i}| = 0,$$

$$n = 1: \quad \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^+ \, i} \, \Delta s_i^+ \, |f^+_{F\,i}| + \sum u^{\text{ext}}_{\boldsymbol{\tau}_h^- \, i} \, \Delta s_i^- \, |f^-_{F\,i}| = 0$$

$$\Downarrow$$

$$\left(\mathbf{R}^T_{\boldsymbol{\tau}_h^+} \, \mathbf{C}_{\boldsymbol{\tau}_h^+} \, \bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h^+}\right)_i + \left(\mathbf{R}^T_{\boldsymbol{\tau}_h^-} \, \mathbf{C}_{\boldsymbol{\tau}_h^-} \, \bar{\mathbf{u}}^{\text{ext}}_{\boldsymbol{\tau}_h^-}\right)_j = 0$$

$$\Downarrow$$

$$\mathbf{S}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \mathbf{S}_T \, \mathbf{N}_T, \qquad \bar{\mathbf{s}}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \bar{\mathbf{s}}_T,$$

Global system:

$$\mathbf{S}_{\mathcal{T}_H} \, \bar{\boldsymbol{\lambda}}_{\mathcal{T}_H} = \bar{\mathbf{s}}_{\mathcal{T}_H}$$

ASC(0) and ASC(1)

$$\mathbf{S}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \mathbf{S}_T \, \mathbf{N}_T, \qquad \bar{\mathbf{s}}_{\mathcal{T}_H} = \sum_{T \in \mathcal{T}_H} \mathbf{N}_T^T \, \bar{\mathbf{s}}_T,$$

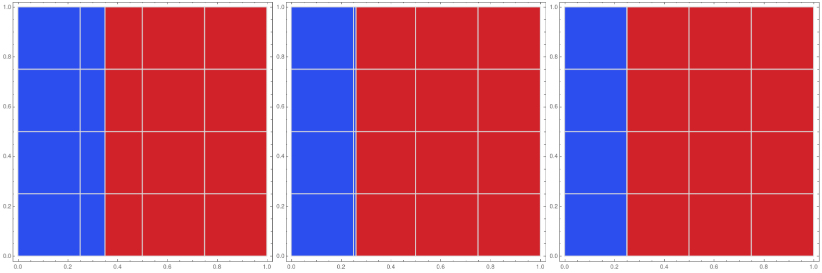
Global system:

$$\mathbf{S}_{\mathcal{T}_H} \bar{\boldsymbol{\lambda}}_{\mathcal{T}_H} = \bar{\mathbf{s}}_{\mathcal{T}_H}$$

- **Theorem**: system matrix $\mathbf{S}_{\mathcal{T}_H}$ is sparse and SPD for ASC(0) and ASC(1)
- Hence efficient solvers and preconditioners are available (e. g. CG + Algebraic Multigrid)
- Once we obtain $\bar{\boldsymbol{\lambda}}_{\mathcal{T}_H}$, we recover pressure and flux DOFs in each cell $T \in \mathcal{T}_H$ (this may be done in parallel)

ASC(1): Robustness (1 / 2)

Figure: $w :=$ width of the left minimesh cells



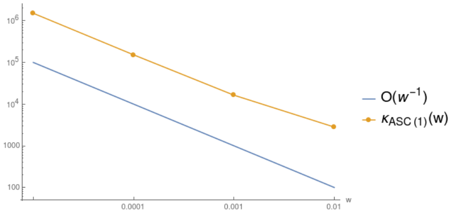
(a) $w = .1$ (b) $w = .01$ (c) $w = .001$

We solve the diffusion problem w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is piecewise linear

ASC(1): Robustness (2 / 2)

Figure: Condition Numbers of ASC(0) / ASC(1) Matrices

w	$\kappa_{\text{ASC}(0)}$	$\kappa_{\text{ASC}(1)}$
10^{-1}	41	1 730
10^{-2}	45	2 817
10^{-3}	48	16 391
10^{-4}	49	152 325
10^{-5}	49	1.5×10^6



$\kappa_{\text{ASC}(0)}$ does not depend on w, and $\kappa_{\text{ASC}(1)}$ is proportional to w^{-1}. However, if we remove 3 smallest eig values (corresponding to 3 int MM faces), **we will have $\tilde{\kappa}_{\text{ASC}(1)} = \kappa_{\text{ASC}(0)}$**. Starting from some iteration CG behaves like extreme eig values are not present; that is, several small eig values is not a problem

ℓ^2-Error

If the base mesh consists of triangles + we have no material interfaces, ASC(n) boils down to Mixed-Hybrid Raviart–Thomas FEM:

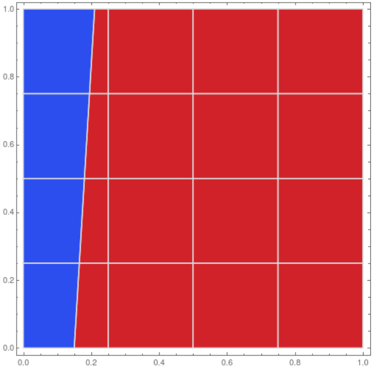
$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{L}^2(\Omega)} \le c\, h \,\|\mathbf{u}\|_{\mathbb{H}^1(\Omega)},$$
$$\|p - p_h\|_{\mathbb{L}^2(\Omega)} \le c \left(h \,\|p\|_{\mathbb{H}^1(\Omega)} + h^2 \,\|p\|_{\mathbb{H}^2(\Omega)} \right).$$

That is, we cannot expect ASC(n) convergence to be better than linear. We define **discrete $\mathbb{L}^2$-norm**

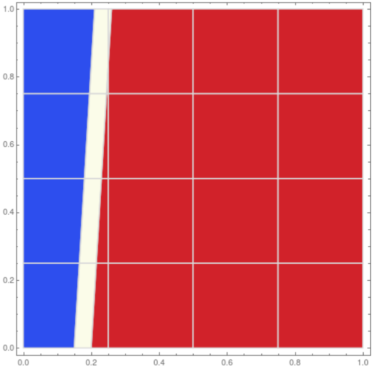
$$\|v\|_{\ell^2(\Omega)} := \|P_h\, v\|_{\mathbb{L}^2(\Omega)} \le \|v\|_{\mathbb{L}^2(\Omega)},$$

where $P_h :=$ $\mathbb{L}^2$-projection operator on the space of piecewise constant functions on each cell $T \in \mathcal{T}_H$ (or on each $\tau \in \boldsymbol{\tau}_h$ if T is a MMC)

$ASC(0) \to ASC(1)$: Motivation



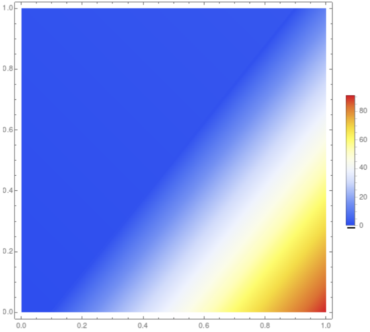
(a) $||p - p_h||_{\ell^2(\Omega)} = 6.38 \times 10^{-2}$



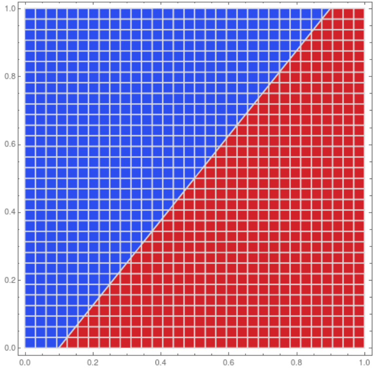
(b) $||p - p_h||_{\ell^2(\Omega)} = 6.41 \times 10^{-2}$

Here $\mathbf{K}_i = \mathbf{K}_j$ and the exact soln is linear. ASC(0) produces errors due to const trace approximation, and ACS(1) recovers the exact soln

We solve the diffusion problem on the sequence of square meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ on the left part and .1 on the right. Exact solution is piecewise linear



(a) Benchmark soln, p

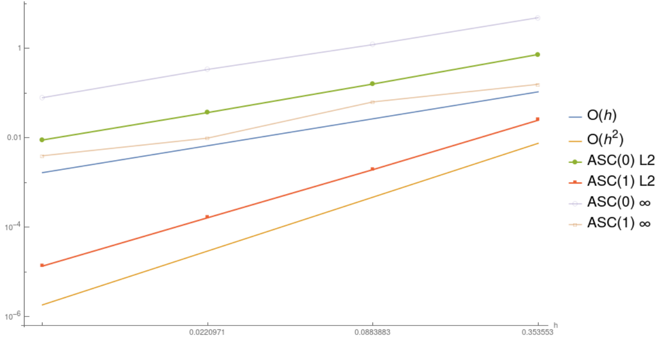


(b) Materials

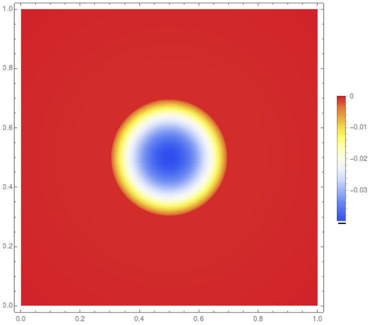
Piecewise Linear Benchmark (2 / 2)

	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(0)	3.5×10^{-1}	7.3×10^{-1}		4.8
	8.8×10^{-2}	1.6×10^{-1}	1.1	1.2
	2.2×10^{-2}	3.7×10^{-2}	1.1	3.4×10^{-1}
	5.5×10^{-3}	8.9×10^{-3}	1.0	7.9×10^{-2}
	h	$e_1^{\ell^2}$	p	e_1^{∞}
ASC(1)	3.5×10^{-1}	2.5×10^{-2}		1.6×10^{-1}
	8.8×10^{-2}	1.9×10^{-3}	1.84	6.3×10^{-2}
	2.2×10^{-2}	1.6×10^{-4}	1.79	9.8×10^{-3}
	5.5×10^{-3}	1.3×10^{-5}	1.80	4.0×10^{-3}

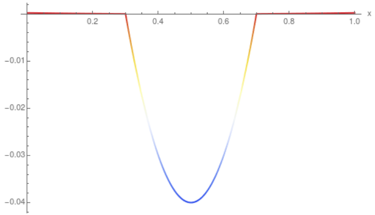
Piecewise Linear Benchmark (2 / 2)



We solve the diffusion problem on Voronoi meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the circle and .001 inside. Exact solution is pw quadratic. We compare convergence of ASC(0) and ASC(1)



(a) Benchmark soln, p

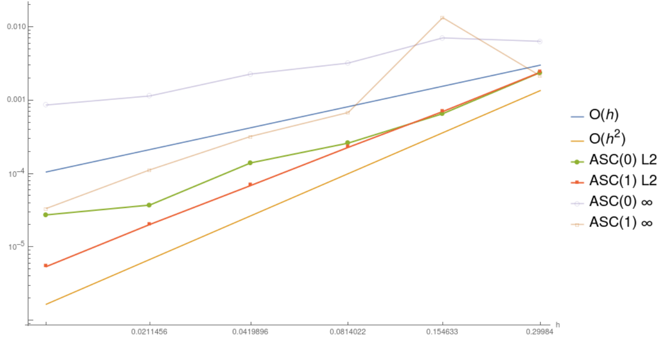


(b) $p(x, \frac{1}{2})$

	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(0)	3.0×10^{-1}	2.4×10^{-3}		6.3×10^{-1}
	1.5×10^{-1}	6.5×10^{-4}	2.0	7.0×10^{-3}
	8.1×10^{-2}	2.6×10^{-4}	1.4	3.2×10^{-3}
	4.2×10^{-2}	1.4×10^{-4}	9.4×10^{-1}	2.3×10^{-3}
	2.1×10^{-2}	3.7×10^{-5}	1.9	1.1×10^{-3}
	1.0×10^{-2}	2.7×10^{-5}	4.4×10^{-1}	8.6×10^{-4}

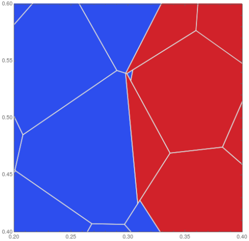
	h	$e_1^{\ell^2}$	p	e_1^{∞}
ASC(1)	3.0×10^{-1}	2.4×10^{-3}		2.1×10^{-3}
	1.5×10^{-1}	7.0×10^{-4}	1.9	1.3×10^{-2}
	8.1×10^{-2}	2.3×10^{-4}	1.8	6.8×10^{-4}
	4.2×10^{-2}	6.8×10^{-5}	1.8	3.2×10^{-4}
	2.1×10^{-2}	2.0×10^{-5}	1.8	1.1×10^{-4}
	1.0×10^{-2}	5.4×10^{-6}	1.9	3.3×10^{-5}

Piecewise Quadratic Benchmark w/ 2 Materials (2 / 3)



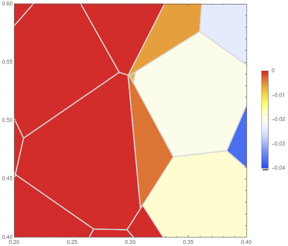
We observe a jump of ∞-error of ASC(1) at $h = 1.5 \times 10^{-1}$

Piecewise Quadratic Benchmark w/ 2 Materials (3 / 3)

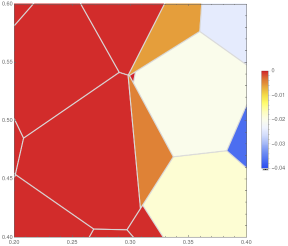


(a) Materials

$h = 1.5 \times 10^{-1}$: This example shows that ASC(1) ∞-norm may be sensitive to geometry errors. However, it does not affect ℓ^2-convergence

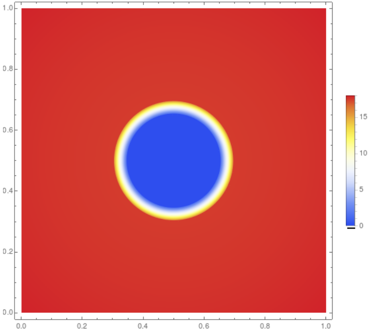


(b) ASC(0), p_h

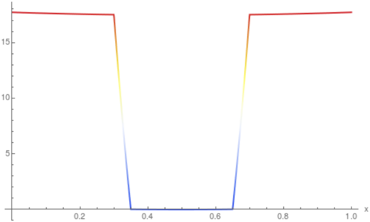


(c) ASC(1), p_h

We solve the diffusion problem on triangular meshes w/ $\mathbf{K} = k\,\mathbf{I}$, $k = 1$ outside the ring and .001 inside. Exact solution is piecewise quadratic



(a) Benchmark soln, p



(b) $p(x, \frac{1}{2})$

Piecewise Quadratic Benchmark w/ 3 Materials (2 / 2)

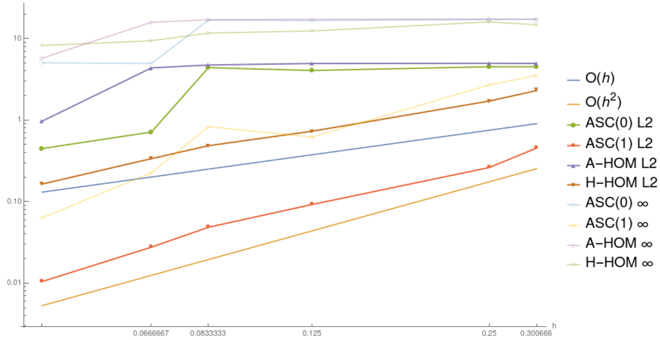
	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(0)	3.0×10^{-1}	4.5		17
	2.5×10^{-1}	4.5		17
	1.3×10^{-1}	4.0		17
	8.3×10^{-2}	4.4		17
	6.7×10^{-2}	7.1×10^{-1}		4.9
	4.3×10^{-2}	4.5×10^{-1}	1.2	5.0

	h	$e_0^{\ell^2}$	p	e_0^{∞}
ASC(1)	3.0×10^{-1}	4.5×10^{-1}		3.5
	2.5×10^{-1}	2.6×10^{-1}	3	2.7
	1.3×10^{-1}	9.2×10^{-2}	1.5	6.2×10^{-1}
	8.3×10^{-2}	4.8×10^{-2}	1.6	8.3×10^{-1}
	6.7×10^{-2}	2.8×10^{-2}	2.5	2.3×10^{-1}
	4.3×10^{-2}	1.0×10^{-2}	2.3	6.3×10^{-2}

Piecewise Quadratic Benchmark w/ 3 Materials (2 / 2)

Homogenization		h	$e_{\mathrm{AH}}^{\ell^2}$	p	e_{AH}^{∞}
	Arithmetic	3.0×10^{-1}	4.9		17
		2.5×10^{-1}	5.0		17
		1.3×10^{-1}	4.9		17
		8.3×10^{-2}	4.7		17
		6.7×10^{-2}	4.4		16
		4.3×10^{-2}	9.7×10^{-1}	3.5	5.7
		h	$e_{\mathrm{HH}}^{\ell^2}$	p	e_{HH}^{∞}
	Harmonic	3.0×10^{-1}	2.3		15
		2.5×10^{-1}	1.7	1.6	16
		1.3×10^{-1}	7.3×10^{-1}	1.2	12
		8.3×10^{-2}	4.8×10^{-1}	1.0	12
		6.7×10^{-2}	3.4×10^{-1}	1.6	9.4
		4.3×10^{-2}	1.6×10^{-1}	1.7	8.2

Piecewise Quadratic Benchmark w/ 3 Materials (2 / 2)



Before $h = 6.7 \times 10^{-2}$ we have cells / faces with 3 materials, and after this mesh level we have only 2 material MMCs

Summary

Results:

- ASC(n) is able to efficiently handle unfitted material interfaces
- 2nd order ℓ^2-convergence for ASC(1)
- Effective condition number seems to be uniformly bounded w.r.t. an interface position
- The underline matrix is SPD and sparse; its pattern does not depend on mini meshes

TODO List:

- Mixed formulation: convergence for fluxes?
- Time-dependent benchmarks
- Anisotropic diffusion: homogenization is not applicable; what about ASC(n)?