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Numerical solution of stochastic differential equations by second order Runge–Kutta methods

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ABSTRACT

In this paper we propose the numerical solutions of stochastic initial value problems via random Runge–Kutta methods of the second order and mean square convergence of these methods is proved. A random mean value theorem is required and established. The concept of mean square modulus of continuity is also introduced. Expectation and variance of the approximating process are computed. Numerical examples show that the approximate solutions have a good degree of accuracy.

1. Introduction

Stochastic differential equations (SDEs) have many applications in economics, ecology and finance [1,2]. In recent years, the development of numerical methods for the approximation of SDEs has become a field of increasing interest, see e.g. [3,4] and references therein. For example in [5], a numerical solution of SDEs is given by the random Euler method and in this paper we obtain the expectation and variance of a numerical solution of these equations by the random Runge–Kutta method that have good accuracy, with respect to the Euler method [5].

Random differential equations of the form

$$\begin{cases} \dot{X}(t) = f(X(t), t), \quad t \in I = [t_0, T], \\ X(t_0) = X_0, \end{cases} \tag{1}$$

where X_0 is a random variable, and the unknown $X(t)$ as well as the right-hand side $f(X(t), t)$ are stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$, are powerful tools to model real problems with uncertainty. The authors of [6] treated the numerical solution of stochastic initial value problems based on a sample treatment of the right-hand side of the differential equations. The sample treatment approach developed in [6] has the advantage that conclusions remain true in the deterministic case, but in many situations the hypothesis assumed in [6] is not satisfied. This fact motivates research for alternative conditions under which good numerical approximations could be constructed. Here we do not assume any trajectorial condition but mean square change information of $f(X(t), t)$ is expressed in terms of its mean square modulus of continuity. Others numerical schemes for stochastic differential equations may be found in [3].

This paper is organized as follows,

Section 2 deals with some preliminaries addressed to clarify the presentation of concepts and results used later. A mean value theorem for stochastic processes is given in Section 3 and Random second order Runge–Kutta methods are given in Section 4. In Section 5 the mean square convergence method of the proposed scheme is established. In Section 6 some examples of [5], illustrate the accuracy of the presented results. Finally, Section 7 gives some brief conclusions.

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2. Preliminaries

We are interested in second order random variables X, having a density function f_X,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\mathrm{d}x < \infty$$

where E denotes the expectation operator, and it allows the introduction of the Banach space L_2 of all the second order random variables endowed with the norm

$$\|X\| = \sqrt{E[X^2]}.$$

A stochastic process $X(t)$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, is called a second order stochastic process if for each t, $X(t)$ is a second order random variable.

Hence the meaning of $\dot{X}(t)$ in (1) is the mean square limit in L_2 of the expression

$$\frac{X(t+\Delta t) - X(t)}{\Delta t}, \quad \text{as } \Delta t \to 0.$$

For the sake of clarity in the presentation we recall an important result [7, p. 88] used repeatedly in the following sections.

Lemma 1. *Let X_n and Y_n be two sequences of second order random variables mean square convergent to the second order random variable X, Y, respectively, i.e.,*

$$X_n \to X \quad \textit{and} \quad Y_n \to Y \quad \textit{as } n \to \infty$$

then

$$E[X_n Y_n] \to E[XY] \quad \textit{as} \quad n \to \infty$$

and so

$$\lim_{n\to\infty} E[X_n] = E[X] \quad \textit{and} \quad \lim_{n\to\infty} Var[X_n] = Var[X].$$

Let $g : I \longrightarrow L_2$ be a mean square bounded function and let $h > 0$, then the mean square modulus of continuity of g is the function

$$\omega(g, h) = \sup_{|t-t^*|\le h} \|g(t) - g(t^*)\|, \quad t, t^* \in I.$$

The function g is said to be mean square uniformly continuous in I, if

$$\lim_{h\to 0} \omega(g, h) = 0.$$

Let $f(X, t)$ be defined on $S \times I$ where S is a bounded set in L_2. We say that f is randomly bounded uniformly continuous in S, if

$$\lim_{h\to 0} \omega(f(X, .), h) = 0,$$

uniformly for $X \in S$.

And finally we have

$$\sup_{X\in S} \omega(f(X,), h) = \omega(h) \to 0.$$

Lemma 2. *For every positive integer number m and every $x \ge -1$, we have*

$$0 \le (1+x)^m \le \exp(mx).$$

Proof. See [8]. □

3. Random mean value theorem for stochastic processes

The purpose of this section is to establish a relationship between the increment $X(t) - X(t_0)$ of a second order stochastic process. And its mean square derivative $\dot{X}(\eta)$ for some η in $[t_0, t]$ for $t > t_0$. The result will be used to prove the convergence of the random Runge–Kutta method.

Lemma 3. *Let $X(t)$ be a second order stochastic process, mean square continuous on $I = [t_0, T]$. Then there exists $\eta \in I$ such that*

$$\int_{t_0}^{t} X(s)\mathrm{d}s = X(\eta)(t - t_0), \quad t_0 < t < T.$$

Proof. See [5]. □

Theorem 1. *Let $X(t)$ be a mean square differentiable second order stochastic process in $I = [t_0, T]$ and mean square continuous in it. Then there exists $\eta \in I$ such that*

$$X(t) - X(t_0) = \dot{X}(\eta)(t - t_0).$$

Proof. This result is a direct consequence of Lemma 3. □

4. Random second order Runge–Kutta methods

In this section we first introduce a lemma and by using it we attain the Random Runge–Kutta method of the second order.

Lemma 4. *Let $\phi(t)$ be a continuous function and $0 < \theta < 1$, then*

$$\int_0^1 \phi(t)\mathrm{d}t \simeq \left(1 - \frac{1}{2\theta}\right)\phi(0) + \frac{1}{2\theta}\phi(\theta).$$

Proof. This expression is easily proved with the second order Newton–Cots integration formula. □

Now, we consider the following SDE,

$$\begin{cases} \dot{X}(t) = f(X(t), t), & t \in I = [t_0, T], \\ X(t_0) = X_0, \end{cases}$$

where X_0 is a random variable, and the unknown $X(t)$ as well as the right-hand side $f(X(t), t)$ are stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$.

With attention to Lemma 4 we can write

$$\int_{t_{n-1}}^{t_n} \dot{X}(t)\mathrm{d}t = \int_{t_{n-1}}^{t_n} f(X(t), t)\mathrm{d}t \simeq \left(1 - \frac{1}{2\theta}\right) hf(X(t_{n-1}), t_{n-1}) + \frac{1}{2\theta} hf(X(t_{n-1} + \theta h), t_{n-1} + \theta h)$$

or

$$X(t_n) - X(t_{n-1}) \simeq \left(1 - \frac{1}{2\theta}\right) hf(X(t_{n-1}), t_{n-1}) + \frac{1}{2\theta} hf(X(t_{n-1} + \theta h), t_{n-1} + \theta h)$$

and by the Taylor expansion we have

$$X(t_{n-1} + \theta h) \simeq X(t_{n-1}) + \theta hf(X(t_{n-1}), t_{n-1})$$

so, random second order Runge–Kutta methods will have the following form

$$\begin{cases} Y_n = X_{n-1} + \theta hf(X_{n-1}, t_{n-1}), \\ X_n = X_{n-1} + \left(1 - \frac{1}{2\theta}\right) hf(X_{n-1}, t_{n-1}) + \frac{1}{2\theta} hf(Y_n, t_{n-1} + \theta h), & n \geq 1 \end{cases} \tag{2}$$

where $X_{n-1}, f(X_{n-1}, t_{n-1})$ are second order random variables, $h = t_i - t_{i-1}$ and $T = t_n = t_0 + nh$.

According to the value of θ, we have the following cases:

Case a. If $\theta = 1$, then our method is called the Modified-Euler method (RK2a).

Case b. If $\theta = \frac{1}{2}$, then our method is called the Mid-Point method (RK2b).

Case c. If $\theta = \frac{2}{3}$, then our method is called the Heun method (RK2c).

5. Convergence of random second order Runge–Kutta methods

In this section we are interested in the mean square convergence, in the fixed station sense, of our methods defined by (2).

Theorem 2. *Let $f(X(t), t)$ be defined on $S \times I$ to L_2, where S is a bounded set in L_2. If $f(X(t), t)$ satisfies the following conditions* C1 *and* C2*,*

C1: *$f(X, t)$ is randomly bounded uniformly continuous,*
C2: *$f(X, t)$ satisfies the mean square Lipschitz condition that is,*

$$\|f(X, t) - f(Y, t)\| \leq k(t)\|X - Y\| \tag{3}$$

where $\int_{t_0}^{T} k(t)\mathrm{d}t < \infty$,

then the random second order Runge–Kutta scheme (2) is mean square convergent.

Proof. Note that under the hypotheses C1 and C2, we are interested in the mean square convergence to zero of the error

$$e_n = X_n - X(t_n) \tag{4}$$

where $X(t)$ is the theoretical solution second order stochastic process of the problem (1).
Theorem 1, follows that,

$$X(t_{n+1}) = X(t_n) + hf(X(t_\eta), t_\eta) \quad t_\eta \in (t_n, t_{n+1}). \tag{5}$$

By relations (2), (4) and (5) if follows that

$$\|e_{n+1}\| \le \|e_n\| + \left(1 - \frac{1}{2\theta}\right) h\|f(X_n, t_n) - f(X(t_\eta), t_\eta)\| + \frac{1}{2\theta} h\|f(Y_{n+1}, t_n + \theta h) - f(X(t_\eta), t_\eta)\|. \tag{6}$$

By assumption

$$M = \sup_{t_0 \le t \le T} \|\dot{X}(t)\| \tag{7}$$

and using C1, C2 and Theorem 1 we have

$$\begin{aligned}\|f(X_n, t_n) - f(X(t_\eta), t_\eta)\| \le{}& \|f(X_n, t_n) - f(X(t_n), t_n)\| + \|f(X(t_n), t_n) - f(X(t_\eta), t_n)\| \\ &+ \|f(X(t_\eta), t_n) - f(X(t_\eta), t_\eta)\|\end{aligned}$$

and

$$\begin{aligned}\|f(Y_{n+1}, t_n + \theta h) - f(X(t_\eta), t_\eta)\| \le{}& \|f(Y_{n+1}, t_n + \theta h) - f(X(t_n), t_n + \theta h)\| + \|f(X(t_n), t_n + \theta h) \\ &- f(X(t_\eta), t_n + \theta h)\| + \|f(X(t_\eta), t_n + \theta h) - f(X(t_\eta), t_\eta)\|\end{aligned}$$

consequently,

$$\|f(X_n, t_n) - f(X(t_\eta), t_\eta)\| \le k(t_n)\|e_n\| + k(t_n)Mh + \omega(h) \tag{8}$$

$$\|f(Y_{n+1}, t_n + \theta h) - f(X(t_\eta), t_\eta)\| \le k(t_n + \theta h)\|e_n\| + k(t_n + \theta h)(1 + \theta)Mh + \omega(h). \tag{9}$$

So, from substituting (8) and (9) in (6), one gets

$$\begin{aligned}\|e_{n+1}\| \le{}& \left[1 + \left(1 - \frac{1}{2\theta}\right) hk(t_n) + \frac{1}{2\theta} hk(t_n + \theta h)\right] \|e_n\| \\ &+ \left(1 - \frac{1}{2\theta}\right) Mh^2 k(t_n) + \frac{1 + \theta}{2\theta} Mh^2 k(t_n + \theta h) + h\omega(h),\end{aligned} \tag{10}$$

and by setting

$$a_n = 1 + \left(1 - \frac{1}{2\theta}\right) hk(t_n) + \frac{1}{2\theta} hk(t_n + \theta h) \tag{11}$$

$$b_n = \left(1 - \frac{1}{2\theta}\right) Mh^2 k(t_n) + \frac{1 + \theta}{2\theta} Mh^2 k(t_n + \theta h) + h\omega(h), \tag{12}$$

the inequality (10) gets the following form

$$\|e_{n+1}\| \le a_n\|e_n\| + b_n, \quad n = 0, 1, 2, \dots. \tag{13}$$

By successive substitution, the relation (13) will become

$$\|e_{n+1}\| \le \left(\prod_{k=0}^{n} a_{n-k}\right) \|e_0\| + \sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} a_{n-k}\right) b_{n-i}, \quad n = 0, 1, 2, \dots. \tag{14}$$

By (11) and Lemma 2 we can write

$$\begin{aligned}\prod_{k=0}^{n} a_{n-k} &= \prod_{k=0}^{n} \left[1 + \left(1 - \frac{1}{2\theta}\right) hk(t_{n-k}) + \frac{1}{2\theta} hk(t_{n-k} + \theta h)\right] \\ &\le \prod_{k=0}^{n} \exp\left(\left(1 - \frac{1}{2\theta}\right) hk(t_{n-k}) + \frac{1}{2\theta} hk(t_{n-k} + \theta h)\right) \\ &\le \exp\left(\left(1 - \frac{1}{2\theta}\right)(n + 1)hk(t_n) + \frac{n + 1}{2\theta} hk(t_n + \theta h)\right).\end{aligned} \tag{15}$$

The relation (15) and geometrical progression and Lemma 2 will conclude

$$\sum_{i=0}^{n}\left(\prod_{k=0}^{i-1}a_{n-k}\right) \leq \frac{\exp\left((n+1)\left[\left(1-\frac{1}{2\theta}\right)hk(t_n)+\frac{1}{2\theta}hk(t_n+\theta h)\right]\right)-1}{\exp\left(\left(1-\frac{1}{2\theta}\right)hk(t_n)+\frac{1}{2\theta}hk(t_n+\theta h)\right)-1}$$
$$\leq \frac{\exp\left((n+1)\left[\left(1-\frac{1}{2\theta}\right)hk(t_n)+\frac{1}{2\theta}hk(t_n+\theta h)\right]\right)-1}{\left(1-\frac{1}{2\theta}\right)hk(t_n)+\frac{1}{2\theta}hk(t_n+\theta h)}. \tag{16}$$

Finally, from (12) and substituting (15) and (16) in (14), we obtain the following error bound

$$\|e_{n+1}\| \leq \exp\left((n+1)\left[\left(1-\frac{1}{2\theta}\right)hk(t_n)+\frac{1}{2\theta}hk(t_n+\theta h)\right]\right)\|e_0\|$$
$$+\frac{\exp\left((n+1)\left[\left(1-\frac{1}{2\theta}\right)hk(t_n)+\frac{1}{2\theta}hk(t_n+\theta h)\right]\right)-1}{\left(1-\frac{1}{2\theta}\right)k(t_n)+\frac{1}{2\theta}k(t_n+\theta h)}$$
$$\times\left[\left(1-\frac{1}{2\theta}\right)Mhk(t_n)+\frac{1+\theta}{2\theta}Mhk(t_n+\theta h)+\omega(h)\right]. \tag{17}$$

By presumption $e_0 = 0$ and $nh = T - t_0$, the above inequality can be written as

$$\|e_{n+1}\| \leq \frac{\exp\left((T-t_0+h)\left[\left(1-\frac{1}{2\theta}\right)k(T)+\frac{1}{2\theta}k(T+\theta h)\right]\right)-1}{\left(1-\frac{1}{2\theta}\right)k(T)+\frac{1}{2\theta}k(T+\theta h)}$$
$$\times\left[\left(1-\frac{1}{2\theta}\right)Mhk(T)+\frac{1+\theta}{\theta}Mhk(T+\theta h)+\omega(h)\right]. \tag{18}$$

Since $\omega(h) \to 0$ as $h \to 0$, by condition C1 and inequality (18) we can deduce that the sequence e_n is mean square convergent to zero as $h \to 0$. Thus we have established the theorem. □

6. Numerical examples

Here we present some examples. Since these examples can be found in [5], we can compare the results.

Example 1. Consider the following problem

$$\begin{cases}\dot{X}(t) = 2tX(t)+\exp(-t)+B(t), & t\in[0,1],\\ X(0)=X_0,\end{cases} \tag{19}$$

where $B(t)$ is a Brownian motion process and X_0 is a normal random variable, $X_0 \sim N(\frac{1}{2}, \frac{1}{12})$ independent of $B(t)$ for each $t \in [0, 1]$.

To compute the exact solution of the problem, by multiplying the equation by $\exp(-t^2)$ and using $W(t) = \frac{\mathrm{d}B(t)}{\mathrm{d}t}$, we have

$$-2t\exp(-t^2)X(t)\mathrm{d}t+\exp(-t^2)\mathrm{d}X(t) = \exp(-t^2)(\exp(-t)+B(t))\mathrm{d}t.$$

Using the Itô formula [9, p. 44], we deduce

$$d(\exp(-t^2X(t))) = -2t\exp(-t^2)X(t)\mathrm{d}t+\exp(-t^2)\mathrm{d}X(t) = \exp(-t^2)(\exp(-t)+B(t))\mathrm{d}t$$

and so

$$X(t) = \exp(t^2)\left\{X_0+\int_0^t \exp(-s^2)(\exp(-s)+B(s))\mathrm{d}s\right\}. \tag{20}$$

If $f(X(t), t) = 2tX(t)+\exp(-t)+B(t)$, we have

$$\|f(X,t)-f(X,t^*)\| \leq (2\|X\|+1)|t-t^*|+|t-t^*|^{\frac{1}{2}} \tag{21}$$

so $f(X, t)$ is randomly bounded uniformly continuous in any bounded set $S \subset L$.

Note that by the existence and uniqueness of Theorem 5.2.1 of [9, p. 66], the solution of (32) is mean square bounded in norm and by Theorem 2 and (21), the proposed scheme for (19) is mean square convergent.

Table 1
Absolute error of expectation of X_n with $h = \frac{1}{20}$ for Example 1.

t	RK2a	RK2b	RK2c
0	0.000000e+0	0.000000e+0	0.000000e+0
0.1	1.025581e−4	−5.315346e−5	−1.194541e−6
0.2	2.014752e−4	−1.173534e−4	−1.097171e−5
0.3	2.956809e−4	−2.074590e−4	−3.959382e−5
0.4	3.816618e−4	−3.433478e−4	−1.014838e−4
0.5	4.513887e−4	−5.541399e−4	−2.187327e−4
0.6	4.888795e−4	−8.845924e−4	−4.265091e−4
0.7	4.643258e−4	−1.405286e−3	−7.818092e−4
0.8	3.239383e−4	−2.229304e−3	−1.377962e−3
0.9	−2.772631e−5	−3.539988e−3	−2.369032e−3
1.0	−7.587040e−4	−5.637761e−3	−4.011347e−3

Table 2
Absolute error of variance of X_n with $h = \frac{1}{20}$ for Example 1.

t	RK2a	RK2b	RK2c
0	0.000000e+0	0.000000e+0	0.000000e+0
0.1	−2.229764e−5	3.914682e−5	8.170069e−6
0.2	−4.577274e−5	7.243662e−5	1.128838e−5
0.3	−7.227390e−5	9.568564e−5	4.959933e−6
0.4	−1.065760e−4	9.809547e−5	−2.090177e−5
0.5	−1.598814e−4	5.539461e−5	−8.807169e−5
0.6	−2.575246e−4	−8.438548e−5	−2.424795e−4
0.7	−4.559983e−4	−4.308204e−4	−5.805349e−4
0.8	−8.808950e−4	−1.214901e−3	−1.305573e−3
0.9	−1.812469e−3	−2.927431e−3	−2.850941e−3
1.0	−3.881467e−3	−6.624838e−3	−6.153477e−3

Now, from random second order Runge–Kutta methods we have,

$$\begin{aligned} X_n &= X_{n-1} + \left(1 - \frac{1}{2\theta}\right) hf(X_{n-1}, t_{n-1}) + \frac{1}{2\theta} hf(Y_n, t_{n-1} + \theta h) \\ &\quad + [1 + 2ht_{n-1} + h^2 + 2h^2(t_{n-1} + \theta h)t_{n-1}]X_{n-1} + \left(1 - \frac{1}{2\theta}\right) h(\exp(-t_{n-1}) + B(t_{n-1})) \\ &\quad + \frac{1}{2\theta} h(\exp(-t_{n-1} - \theta h) + B(t_{n-1} + \theta h)) + h^2(t_{n-1} + \theta h)(\exp(-t_{n-1}) + B(t_{n-1})), \end{aligned} \tag{22}$$

and by setting

$$\begin{aligned} c_{n-1} &= 1 + 2ht_{n-1} + h^2 + 2h^2(t_{n-1} + \theta h)t_{n-1} \\ d_{n-1} &= \left(1 - \frac{1}{2\theta}\right) h(\exp(-t_{n-1}) + B(t_{n-1})) + \frac{1}{2\theta} h(\exp(-t_{n-1} - \theta h) + B(t_{n-1} + \theta h)) \\ &\quad + h^2(t_{n-1} + \theta h)(\exp(-t_{n-1}) + B(t_{n-1})), \end{aligned}$$

we have

$$X_n = c_{n-1}X_{n-1} + d_{n-1}, \quad n = 1, 2, \ldots, \tag{23}$$

and so

$$X_n = \left(\prod_{i=1}^{n} c_{n-i}\right) X_0 + \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} c_{n-j}\right) d_{n-i}, \quad n = 1, 2, \ldots. \tag{24}$$

From (20) and (24), we obtain the expectations and variances of $X(t)$ and X_n (see Tables 1 and 2).

$$E[X(t)] = \exp(t^2) \left\{ \frac{1}{2} + \int_0^t \exp(-s^2 - s) ds \right\}. \tag{25}$$

$$\begin{aligned} E[X_n] &= \frac{1}{2} \prod_{i=1}^{n} c_{n-i} + \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} c_{n-j}\right) \left[\left(1 - \frac{1}{2\theta}\right) h(\exp(-t_{n-i})) \right. \\ &\quad \left. + \frac{1}{2\theta} h(\exp(-t_{n-i} - \theta h)) + h^2(t_{n-i} + \theta h)(\exp(-t_{n-i})) \right], \end{aligned} \tag{26}$$

$$
\begin{aligned}
Var[X(t)] &= \exp(2t^2)\left\{\frac{1}{12} + Var\left[\int_0^t \exp(-s^2)(\exp(-s) + B(s))\mathrm{d}s\right]\right\} \\
&= \exp(2t^2)\left\{\frac{1}{12} + \int_0^t \int_0^t \exp(-s^2 - r^2)\min(s, r)\mathrm{d}s\mathrm{d}r\right\}
\end{aligned} \tag{27}
$$

$$
Var[X_n] = \frac{1}{12}\left(\prod_{i=1}^{n} c_{n-i}\right)^2 + Var\left[\sum_{i=1}^{n}\left(\prod_{j=1}^{i-1} c_{n-j}\right) d_{n-i}\right], \quad n = 1, 2, \ldots. \tag{28}
$$

Now, we simplify (28) using the variance term of the right-hand side of it, i.e.

$$
Var\left[\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{i-1} c_{n-j}\right) d_{n-i}\right)\right] = E\left[\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{i-1} c_{n-j}\right) d_{n-i}\right)^2\right] - \left(E\left[\sum_{i=1}^{n}\left(\prod_{j=1}^{i-1} c_{n-j}\right) d_{n-i}\right]\right)^2. \tag{29}
$$

After some computations, we can conclude

$$
\begin{aligned}
Var[X_n] = &\ \frac{1}{12}\left(\prod_{i=1}^{n} c_{n-i}\right)^2 + \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\prod_{j=1}^{i-1} c_{n-j}\right)\left(\prod_{l=1}^{k-1} c_{n-l}\right)\Big[A_{i,k}\min(t_{n-i}, t_{n-k}) \\
&+ B_i \min(t_{n-i}, t_{n-k} + \theta h) + C_k \min(t_{n-i} + \theta h, t_{n-k}) + \frac{1}{4\theta^2} h^2 \min(t_{n-i} + \theta h, t_{n-k} + \theta h)\Big]
\end{aligned} \tag{30}
$$

where

$$
\begin{cases}
A_{i,k} = \left[\left(1 - \dfrac{1}{2\theta}\right)^2 h^2 + \left(1 - \dfrac{1}{2\theta}\right) h^3 (t_{n-k} + t_{n-i} + 2\theta h) + h^4 (t_{n-i} + \theta h)(t_{n-k} + \theta h)\right] \\
B_i = \left[\left(1 - \dfrac{1}{2\theta}\right)\dfrac{1}{2\theta} h^2 + \dfrac{1}{2\theta} h^3 (t_{n-i} + \theta h)\right] \\
C_k = \left[\left(1 - \dfrac{1}{2\theta}\right)\dfrac{1}{2\theta} h^2 + \dfrac{1}{2\theta} h^3 (t_{n-k} + \theta h)\right] \quad i, k = 1, 2, \ldots, n.
\end{cases} \tag{31}
$$

Example 2. Consider the following initial value problem

$$
\begin{cases}
\dot{X}(t) = t^2 X(t) + W(t), \quad t \in [0, 1], \\
X(0) = X_0,
\end{cases} \tag{32}
$$

where $W(t)$ is a Gaussian white noise process with mean zero and X_0 is an exponential random variable with parameter $\lambda = \frac{1}{2}$, independent of $W(t)$ for each $t \in [0, 1]$. Here $f(X(t), t)$ involves the white noise process with mean zero $W(t)$, i.e. $f(X(t), t) = t^2 X(t) + W(t)$. The covariance of $W(t)$,

$$
cov[W(t), W(s)] = \delta(t - s), \tag{33}
$$

where $\delta(t)$ is the delta generalized function. A convolution with the delta function always exists, see [10], and the delta function plays the same role for convolution as unity does for multiplication,

$$
\delta * g = g.
$$

So, taking $g(s) = h(s)\chi_{[0,t]}(s)$, where $h(s)$ is a C^∞ function and $\chi_{[0,t]}(s)$ denotes the characteristic function on the interval $[0, t]$, from (33) it follows that

$$
\int_{-\infty}^{\infty} g(s)\delta(s - r)\mathrm{d}s = \int_{-\infty}^{\infty} h(s)\chi_{[0,t]}(s)\delta(s - r)\mathrm{d}s = \int_0^t h(s)\delta(s - r)\mathrm{d}s = h(r).
$$

To compute the exact solution of the problem, by multiplying both sides of the Eq. (32) by $\exp(\frac{-t^3}{3})$, and using $W(t) = \frac{\mathrm{d}B(t)}{\mathrm{d}t}$, we have

$$
-t^2 \exp\left(\frac{-t^3}{3}\right) X(t) + \exp\left(\frac{-t^3}{3}\right) \mathrm{d}X(t) = \exp\left(\frac{-t^3}{3}\right) \mathrm{d}B(t),
$$

using the Itô formula, [9, p. 44], we conclude

$$
d\left(\exp\left(\frac{-t^3}{3}\right) X(t)\right) = -t^2 \exp\left(\frac{-t^3}{3}\right) X(t) + \exp\left(\frac{-t^3}{3}\right) \mathrm{d}X(t) = \exp\left(\frac{-t^3}{3}\right) \mathrm{d}B(t),
$$

Table 3
Absolute error of expectation of X_n with $h = \frac{1}{50}$ for Example 2.

t	RK2a	RK2b	RK2c
0	0.000000e+0	0.000000e+0	0.000000e+0
0.1	1.333518e−5	−6.678022e−6	−7.078801e−9
0.2	2.671648e−5	−1.352334e−5	−1.111624e−7
0.3	4.028535e−5	−2.098128e−5	−5.628999e−7
0.4	5.427839e−5	−2.983603e−5	−1.807212e−6
0.5	6.900342e−5	−4.130790e−5	−4.556191e−6
0.6	8.478776e−5	−5.725414e−5	−9.940396e−6
0.7	1.018738e−4	−8.054131e−5	−1.979209e−5
0.8	1.202124e−4	−1.157066e−4	−3.715539e−5
0.9	1.390543e−4	−1.701214e−4	−6.719874e−5
1.0	1.561413e−4	−2.560541e−4	−1.188604e−4

and so

$$X(t) = \exp\left(\frac{t^3}{3}\right)\left[X_0 + \int_0^t \exp\left(\frac{-s^3}{3}\right)\right]. \tag{34}$$

Now, we compute X_n from random second order Runge–Kutta methods,

$$\begin{aligned} X_n &= X_{n-1} + \left(1 - \frac{1}{2\theta}\right) hf(X_{n-1}, t_{n-1}) + \frac{1}{2\theta} hf(Y_n, t_{n-1} + \theta h) \\ &= X_{n-1} + \left(1 - \frac{1}{2\theta}\right) h[t_{n-1}^2 X_{n-1} + W(n-1)] + \frac{1}{2\theta} h[(t_{n-1} + \theta h)^2 Y_n + W(t_{n-1} + \theta h)] \\ &= \left[1 + \left(1 - \frac{1}{2\theta}\right) h t_{n-1}^2 + \frac{1}{2\theta} h(t_{n-1} + \theta h)^2 + \frac{h^2}{2}(t_{n-1} + \theta h)^2 t_{n-1}^2\right] X_{n-1} \\ &\quad + \left(1 - \frac{1}{2\theta}\right) hW(t_{n-1}) + \frac{1}{2\theta} hW(t_{n-1} + \theta h) + \frac{h^2}{2}(t_{n-1} + \theta h)^2 W(t_{n-1}) \end{aligned}$$

and by setting

$$\begin{aligned} a_{n-1} &= 1 + \left(1 - \frac{1}{2\theta}\right) h t_{n-1}^2 + \frac{1}{2\theta} h(t_{n-1} + \theta h)^2 + \frac{h^2}{2}(t_{n-1} + \theta h)^2 t_{n-1}^2 \\ b_{n-1} &= \left(1 - \frac{1}{2\theta}\right) hW(t_{n-1}) + \frac{1}{2\theta} hW(t_{n-1} + \theta h) + \frac{h^2}{2}(t_{n-1} + \theta h)^2 W(t_{n-1}) \end{aligned}$$

we have

$$X_n = a_{n-1} X_{n-1} + b_{n-1}$$

and so

$$X_n = \left(\prod_{k=1}^{n} a_{n-k}\right) X_0 + \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} a_{n-k}\right) b_{n-i}. \tag{35}$$

From (34) and (35) we obtain the expectation and variance of $X(t)$ and X_n (see Tables 3 and 4).

$$E[X_n] = \frac{1}{2} \prod_{k=1}^{n} a_{n-k} + \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} a_{n-k}\right) E[b_{n-i}] = \frac{1}{2} \prod_{k=1}^{n} a_{n-k}, \tag{36}$$

$$E[X(t)] = \frac{1}{2} \exp\left(\frac{t^3}{3}\right), \tag{37}$$

$$\begin{aligned} Var[X_n] &= \frac{1}{4}\left(\prod_{k=1}^{n} a_{n-k}\right)^2 + Var\left[\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1} a_{n-k}\right) b_{n-i}\right] \\ &= \frac{1}{4}\left(\prod_{k=1}^{n} a_{n-k}\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\prod_{k=1}^{i-1} a_{n-k}\right)\left(\prod_{l=1}^{j-1} a_{n-l}\right) E[b_{n-i} b_{n-j}]. \end{aligned} \tag{38}$$

Table 4
Absolute error of variance of X_n with $h = \frac{1}{50}$ for Example 2.

t	RK2a	RK2b	RK2c
0	0.000000e+0	0.000000e+0	0.000000e+0
0.1	−9.819568e−2	−9.807589e−2	−9.879947e−2
0.2	−1.972806e−1	−1.968412e−1	−1.982927e−1
0.3	−2.986479e−1	−2.980883e−1	−3.002841e−1
0.4	−4.057901e−1	−4.049073e−1	−4.078849e−1
0.5	−5.231501e−1	−5.219368e−1	−5.257652e−1
0.6	−6.572269e−1	−6.558761e−1	−6.606701e−1
0.7	−8.177162e−1	−8.164270e−1	−8.223665e−1
0.8	−1.019378e+0	−1.017956e+0	−1.025316e+0
0.9	−1.283943e+0	−1.282413e+0	−1.291614e+0
1.0	−1.646109e+0	−1.644533e+0	−1.656218e+0

In the right-hand side of relation (38), the expectation term is obtained as follows:

$$E[b_{n-i}b_{n-j}] = A_{i,j}\delta(t_{n-i} - t_{n-j}) + B_i\delta(t_{n-i} - t_{n-j} - \theta h) + C_j(t_{n-i} - t_{n-j} + \theta h) + \frac{1}{4\theta^2}h^2\delta(t_{n-i} - t_{n-j}),$$

where

$$\begin{cases} A_{i,j} = \left[\left(1 - \frac{1}{2\theta}\right)h + \frac{h^2}{2}(t_{n-i} + \theta h)^2\right]\left[\left(1 - \frac{1}{2\theta}\right)h + \frac{h^2}{2}(t_{n-j} + \theta h)^2\right] \\ B_i = \left[\left(1 - \frac{1}{2\theta}\right)h + \frac{h^2}{2}(t_{n-i} + \theta h)^2\right]\frac{1}{2\theta}h \\ C_j = \left[\left(1 - \frac{1}{2\theta}\right)h + \frac{h^2}{2}(t_{n-j} + \theta h)^2\right]\frac{1}{2\theta}h; \quad i, j = 1, 2, \ldots, n. \end{cases}$$

$$Var[X(t)] = \exp\left(\frac{2t^3}{3}\right)\frac{1}{4} + E\left[\left(\int_0^t \exp\left(\frac{-s^3}{3}\right)\mathrm{d}B_s\right)^2\right], \tag{39}$$

with Itô isometry [9, p. 29],

$$Var[X(t)] = \exp\left(\frac{2t^3}{3}\right)\frac{1}{4} + \int_0^t \exp\left(\frac{-2s^3}{3}\right)\mathrm{d}s. \tag{40}$$

Figs. 1–6 show that $E[X_n]$ and $Var[X_n]$ of the numerical solutions of stochastic initial value problems via random Runge–Kutta methods of the second order are close to $E[X(t)]$ and $Var[X(t)]$ respectively, as $h \to 0$.

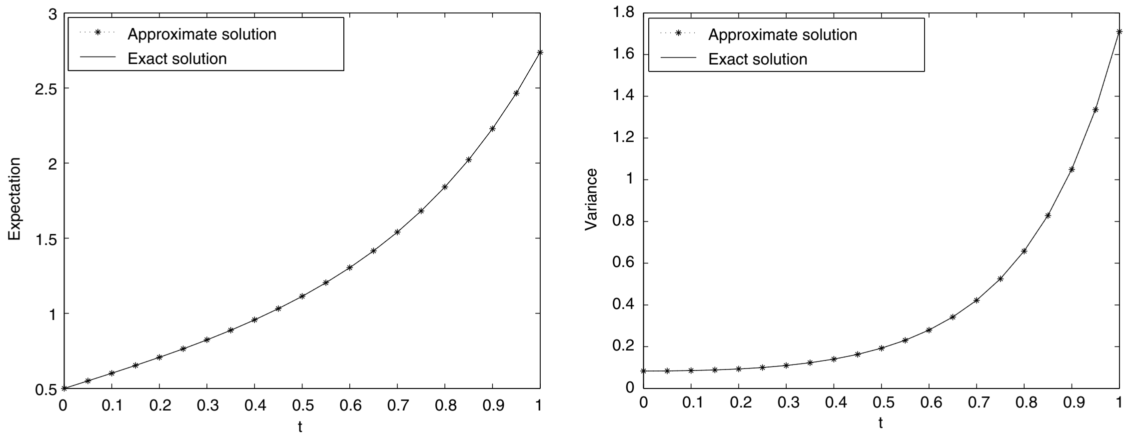

Fig. 1. Expectations and variances of $X(t)$ and X_n by SKR2a with $h = \frac{1}{20}$ for Example 1.

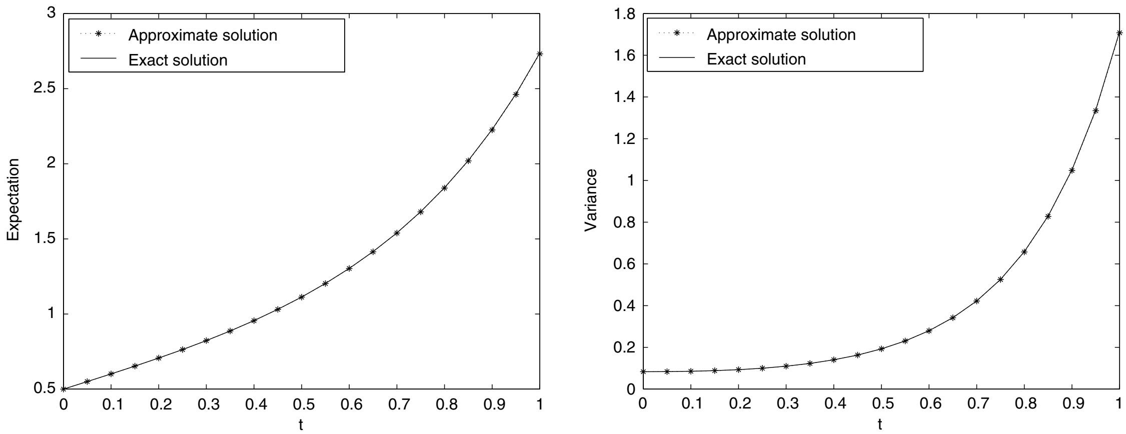

Fig. 2. Expectations and variances of $X(t)$ and X_n by SKR2b with $h = \frac{1}{20}$ for Example 1.

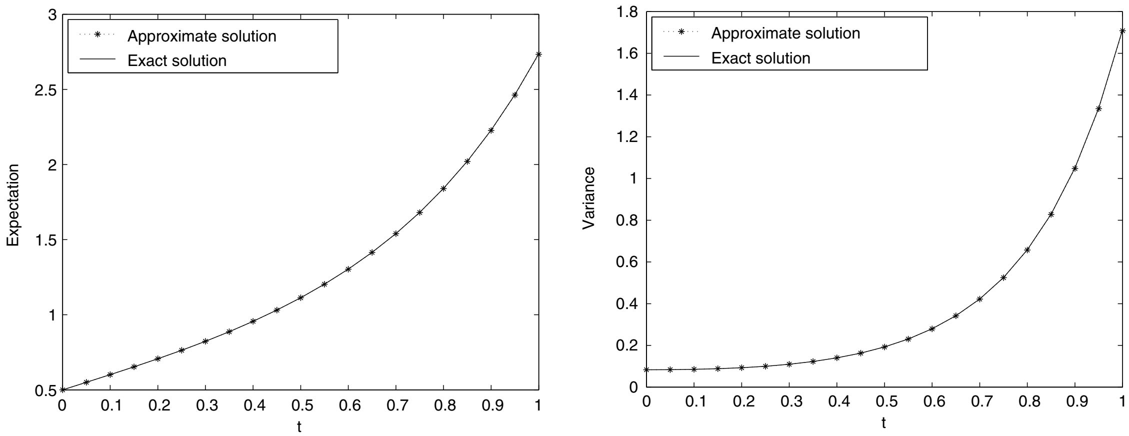

Fig. 3. Expectations and variances of $X(t)$ and X_n by SKR2c with $h = \frac{1}{50}$ for Example 1.

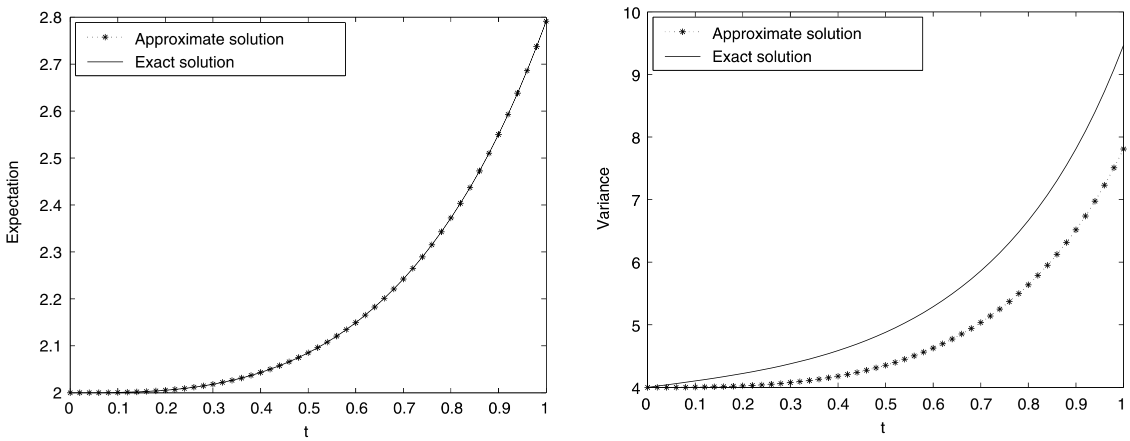

Fig. 4. Expectations and variances of $X(t)$ and X_n by SKR2a with $h = \frac{1}{50}$ for Example 2.

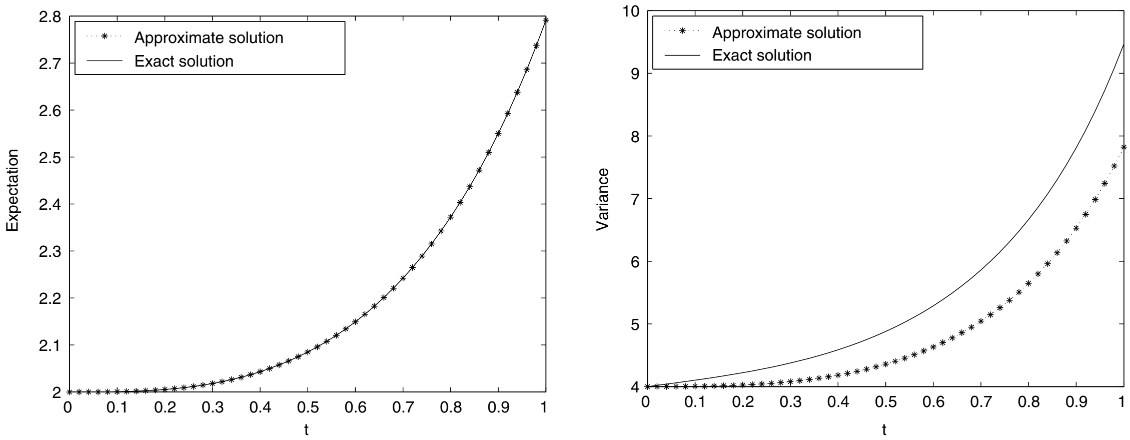

Fig. 5. Expectations and variances of $X(t)$ and X_n by SKR2b with $h = \frac{1}{50}$ for Example 2.

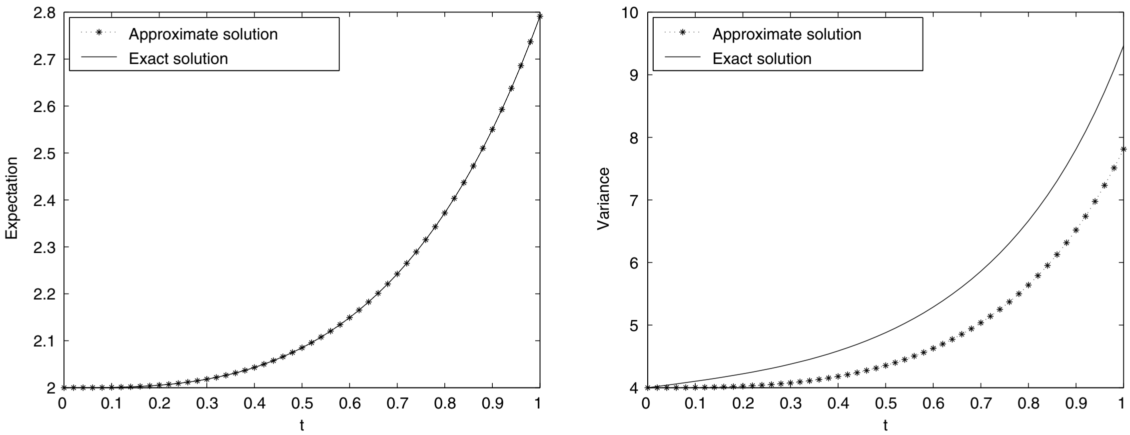

Fig. 6. Expectations and variances of $X(t)$ and X_n by SKR2c with $h = \frac{1}{50}$ for Example 2.

7. Conclusion

In this paper, the numerical solution of stochastic differential equations are discussed by second order Runge–Kutta methods with more details. The results can be compared with [5]. Our comparison showed that this method has more accurate than the Euler method in [5].

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