15 Orbital Mechanics

We will now use our newfound knowledge about gravitation to study one of the core topics of astrophysics: the mechanics of planetary orbits, also known as **orbital mechanics**. Parts of this section will be somewhat more challenging and mathematically involved than others (for instances, the proofs of various laws) so feel free to skip over them.

15.1 Kepler's Laws

Kepler's three laws give some basic properties of planetary motion. Specifically, Kepler's laws state the following:

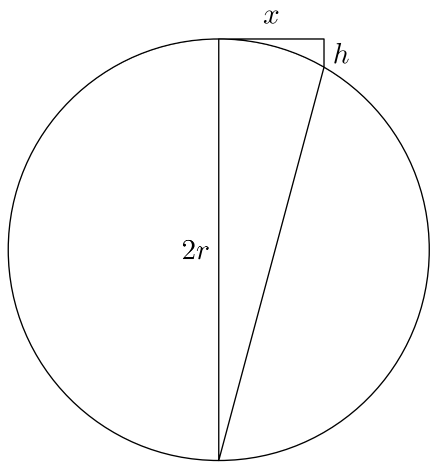
Theorem 15.1 (Kepler's Laws)**.**

1. ***Law of Orbits:*** *The orbit of every planet is an ellipse with the Sun being one of the foci.*
2. ***Law of Areas:*** *As planets move, they sweep through elliptical arcs of equal area in equal amounts of time.*
3. ***Law of Periods:*** *As the radius of orbit changes, the period of orbit changes according to the proportion*
$$T^2 \propto a^3.$$
where a is the semi-major axis.

Note that Kepler's second law tells us is two things. First, consider the small area increment ΔA covered in a time interval Δt. The area of this approximately triangle wedge is given by its basic formula $A = \frac{1}{2}bh$ where the base is $r\Delta\theta$ and the height is r. The rate at which this area is swept out is
$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\frac{(r\Delta\theta)(r)}{\Delta t}.$$
In the instantaneous limit, this in turn becomes
$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}r^2\frac{\Delta\theta}{\Delta t} = \frac{1}{2}r^2\omega$$
Second, assuming that the more massive body M is practically at rest. The angular momentum of the orbiting body m relative to the origin at the central body is, when choosing the z-axis to be perpendicular to the plane of origin $L_z = I\omega = mr^2\omega$. And therefore,
$$\frac{dA}{dt} = \frac{L_z}{2m}.$$

We will also now give a proof of Kepler's Third Law. The proof is rather long, and as such is optional to look over. However, the proof utilizes many nice things that we have been learning before and therefore, it is advised to look over it.

Proof. Let's first look at how to relate speed and acceleration (or gravitational forces). Let the body move in a circle for an infinitesimally short time so that the displacement in the horizontal direction (see figure) would be x and h in the vertical direction.



From the equality of boundary angles we get similar triangles

$$\frac{h}{x} = \frac{x}{2r-h} \approx \frac{x}{2r}, \quad h = \frac{x^2}{2r}.$$

Let this movement take place during the period Δt. Recall the formula for centripetal acceleration:

$$a_c = \frac{F}{m} = \frac{v^2}{r}.$$

Now move the object to the perigee of its elliptical orbit (closest point to a focus). The corresponding displacements are denoted by x' and h'. If we compress the ellipse in the direction of the semi-major axis by a coefficient $k = a/b$, we would get a circle with radius $r = b$. Doing that and combining it with our earlier result, we can see that:

$$\frac{h'}{k} = \frac{x'^2}{2b}, \qquad h' = \frac{x'^2 a}{2b^2}, \qquad \frac{F}{m} = \frac{v^2 a}{b^2},$$

where we have switched to acceleration and speed. The distance of the perigee from the focus is $a - c$, hence

$$F = \frac{GMm}{(a-c)^2}, \qquad v^2 = \frac{Fb^2}{ma} = \frac{GMmb^2}{ma(a-c)^2} = \frac{GM(a^2-c^2)}{a(a-c)^2} = \frac{GM(a+c)}{a(a-c)},$$

where we proceeded from the relation $a^2 = b^2 + c^2$. To determine the orbital period, we use Kepler's Second law (i.e. the time-area equality):

$$\frac{T}{S} = \frac{\Delta t}{\Delta S}, \qquad T = \pi ab \frac{\Delta t}{\frac{(a-c)v\Delta t}{2}} = \frac{2\pi ab}{(a-c)v},$$

where the area of the ellipse is $S = \pi ab$ and the area of the smaller section was from an isosceles triangle with base $v\Delta t$ and height $a - c$. Since we already have an expression for v^2, we can find that

$$T^2 = \frac{(2\pi)^2 a^2 b^2}{(a-c)^2 v^2} = \frac{(2\pi)^2 a^2 b^2 a(a-c)}{(a-c)^2 GM(a+c)} = \frac{(2\pi)^2 a^3 b^2}{GM(a^2-c^2)} = \frac{(2\pi)^2 a^3}{GM},$$

proving Kepler's Third Law (the square of the period is proportional to the cube of the semi-major axis). Note that the constant of proportionality is given by

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}.$$

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Kepler's Third Law is very important in many celestial mechanics problems.

Example 15.2. An object of mass m is separated a distance R away from a large mass M. Calculate the total amount of time required for m to reach M. Assume that $m \ll M$.

Solution. There are, in fact, two different ways we can solve this problem. One way is by conserving energy and then integrating velocity to find the total amount of time. This approach leads to some very nasty results but will lead to an elegant answer at the end. However, usually if there is an elegant answer, there will be an elegant solution. Note that the linear path of the $m \to M$ is like a degenerate ellipse. So we can use Kepler's laws!

As an ellipse gets very thin, its foci go towards the ends. So, the sun, which has to be at the focus of orbit, is at the very end of this elongated ellipse. From Kepler's third law,

$$T^2 \propto a^3 \implies T \propto a^{3/2}.$$

From the definition of the degenerate ellipse, the semi-major axis is just $a = R/2$. This means that the total period will be $\frac{T}{2^{3/2}}$ where T is the time taken for m to orbit M in a circular orbit. To find T, we again note from Kepler's laws that

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \implies T = 2\pi\sqrt{\frac{R^3}{GM}}.$$

Since the time taken to reach the large mass is just half of this period, the answer is

$$t = \frac{1}{2} \cdot \frac{1}{2^{3/2}} 2\pi\sqrt{\frac{R^3}{GM}} = \boxed{\frac{1}{2\sqrt{2}}\pi\sqrt{\frac{R^3}{GM}}}$$

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There will be a similar problem using this idea of degenerate ellipses in the problems section of this handout. Make sure to try that problem to test your understanding of this concept.

Exercise. Show that in a binary star system, that two stars of masses M_1 and M_2 move in elliptical orbits one of whose foci (for each orbit) is at the center of mass of the system and show that the orbital period of the system is given by

$$T^2 = \frac{4\pi^2 (a_1 + a_2)^3}{G(M_1 + M_2)},$$

where a_1 and a_2 are the semi-major axes of the elliptical orbits.

15.2 Energy

Definition 15.3. The **total energy** of an object in orbital motion is given by

$$E = -\frac{GMm}{2a}.$$

Proof. Since the total energy of the object in orbit does not change, we write energy expressions at the apogee and perigee of the object's orbit:

$$E = \frac{mv_1^2}{2} - \frac{GMm}{r_1}, \quad E = \frac{mv_2^2}{2} - \frac{GMm}{r_2}.$$

We consider using conservation of angular momentum, $r_1 v_1 = r_2 v_2$. We multiply the first (energy) equation by r_1^2 and the second by r_2^2 and subtract, noting that $r_1 = a - c$ and $r_2 = a + c$:

$$E\left(r_1^2 - r_2^2\right) = -GMm(r_1 - r_2),$$

$$E = -GMm\frac{r_1 - r_2}{r_1^2 - r_2^2} = -\frac{GMm}{r_1 + r_2} = -\frac{GMm}{2a}.$$

■

If the total energy is negative, then the object is unable to escape the gravitational field of the star to the point at infinity since infinitely far away, the potential energy is zero and the total energy would be equal to the kinetic energy, which cannot be negative. Sometimes problems give an object enough speed to escape the gravitational field, in which case the total energy would be positive.

> **Example 15.4.** An explosion takes place near a star, causing many small objects to fly outwards with speed v. The objects begin to move in elliptical orbits, with one of the foci at the star. Determine the locus (set of possible positions) of the second focus.

Solution. For this problem, one only needs to know the properties of an ellipse and the fact that the total energy, which is the same for all pieces, depends on the longer half-axis of the ellipse, which thus becomes the same for all pieces. Additionally, we know that the sum of the distances to the foci from each point on the ellipse is equal to twice the semi-major axis. We see that the total energy is given by

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{2a}$$

where R is the distance from the star to the explosion and a is the semi-major axis of the resulting orbit. Since this value is constant, we can find that

$$a = \frac{GMR}{2GM - v^2 R}.$$

We know that the sum of the distances to the foci is $2a$; one of these distances is already known to be R. Thus, the locus of the second foci of the ellipse is a circle (with center at the explosion) of radius

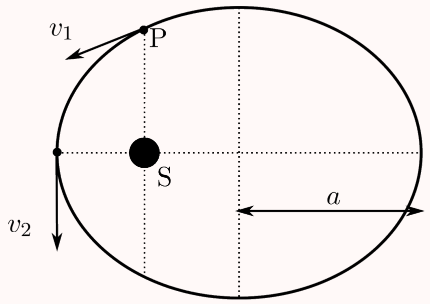
$$r = 2a - R = \frac{2GMR}{2GM - v^2 R} - R = \boxed{\frac{v^2 R^2}{2GM - v^2 R}}.$$

■

15.2.1 The Vis-Viva Equation

The Vis-Viva equation is essentially a killer formula that can destroy several gravitation problems. We can first start with a small example.

> **Example 15.5** ($F = ma$). A planet orbits around a star S, as shown in the figure. The semi-major axis of the orbit is a. The perigee, namely the shortest distance between the planet and the star is $0.5a$. When the planet passes point P (on the line through the star and perpendicular to the major axis), its speed is v_1. What is its speed v_2 when it passes the perigee?



Solution. Because the sum of the distances from a point on the ellipse to the two foci is constant, we can solve for the length SP as

$$SP + \sqrt{SP^2 + a^2} = 2a \implies SP = \frac{3}{4}a.$$

Since we know the length of SP, the total energy at P is

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{3a/4}.$$

Since the length of the planet at periastron is $a/2$, the energy at periastron is

$$E = \frac{1}{2}mv_2^2 - \frac{GMm}{a/2}.$$

Lastly, we know that $E = -\frac{GMm}{2a}$ thus by replacing E into our conservation of energy equation at P, we have

$$E = \frac{1}{2}mv_1^2 + \frac{8}{3}E \implies v_1^2 = -\frac{10}{3}\frac{E}{m}.$$

The equation for energy at periastron is

$$v_2^2 = -6\frac{E}{m}$$

Finally, we have that

$$\frac{E}{m} = -\frac{3}{10}v_1^2 \implies v_2^2 = -6 \cdot \left(-\frac{3}{10}v_1^2\right) \implies v_2 = \frac{3}{\sqrt{5}}v_1$$

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What we have just done, is used the vis-viva equation without knowing it!

Theorem 15.6 (Vis-Viva Equation). *In an orbit,*

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

Proof. Let a mass m be in orbit about a mass M. Let the velocity of m be v when it is a distance r from M and let a be the semi-major axis of the orbit. We then use conservation of energy to find that

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$
$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

■

We can now write a much shorter and quicker solution to this problem.

Solution. Because the sum of the distances from a point on the ellipse to the two foci is constant, we can solve for the length SP as

$$SP + \sqrt{SP^2 + a^2} = 2a \implies SP = \frac{3}{4}a.$$

The vis-viva equation then tells us

$$\frac{v_2^2}{v_1^2} = \frac{\frac{2}{a/2} - \frac{1}{a}}{\frac{2}{3a/4} - \frac{1}{a}} = \frac{4-1}{8/3-1} = \frac{9}{5}.$$

Thus, $v_2 = \frac{3}{\sqrt{5}}v_1$. ■

15.2.2 The Virial Theorem

Theorem 15.7 (Virial Theorem). *The relationship between the average kinetic energy $\langle T \rangle$ and potential energy $\langle U \rangle$ in an orbit is given by*

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle.$$

While this theorem is not as useful as the vis-viva equation, it is still very important as we can express the total energy soley in terms of kinetic or potential energy.

Example 15.8 ($F = ma$). A satellite is in a circular orbit about the Earth. Over a long period of time, the effects of air resistance decrease the satellite's total energy by 1 J. What happens to the kinetic energy of the satellite?

Solution. By the virial theorem, the total energy in an orbit is given by

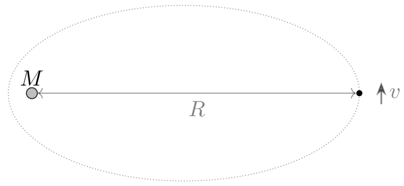
$$E = \langle T \rangle + \langle U \rangle = -\langle T \rangle$$

Because the satellites total energy decreases by 1 J, this means the total kinetic energy increases by 1 J. ■

15.3 Homework Problems

Problem 15.1 (NAO). [4] Imagine that our Sun was suddenly replaced by an M-dwarf with a mass half that of the Sun (By Ashmit of course). If our Earth kept the same semi-major axis during this change, what would Earth's new orbital period be around the M-dwarf? Report $(T'/T)^2$

Problem 15.2 ($F = ma$). [4] A mote of dust is initially located at distance R from the sun, which has mass M. At this point, the mote has a small tangential velocity v. Which of the following is a good approximation for the distance of closest subsequent approach between the mote and the sun? If it is of the format $\alpha R^a v^b G^c M^d$, then report $\alpha + a + 2b + c + d$



Problem 15.3 ($F = ma$)**.** [5] An astronaut standing on the exterior of the international space station wants to dispose of three pieces of trash. They face the station's direction of travel with the Earth to their left. From the astronaut's perspective, the three pieces are thrown (I) left, (II) right, and (III) up. To the astronaut's frustration, some of the pieces of trash return to the space station after several hours. In which ways, (I), (II), and/or (III) can the astronaut do this? Report sum of numbers of pieces which come back.

Problem 15.4 (Mock $F = ma$)**.** [5] Rick Astley is on a new planet with a grandfather clock calibrated on Earth.A person from earth asks him a question about time. Now, Rick has promised to never give you up and never let you down, so he has to answer, but the time scale on the new planet is different. This new solar system is an exact copy of our solar system except all distances have been doubled. For example, the distance from the planet to the star will be doubled, and the radius will be doubled as well, among other lengths. The density stays the same. Find out Earth years will Rick Astley measure to be one year on this new planet?

Problem 15.5 (Irodov)**.** [6] A planet moves around the Sun (mass M) along an ellipse so that its minimum distance to the Sun is r_1, and its maximum distance to the sun is r_2. Find the period of its revolution. Use $M = 2 \times 10^{30} kg$, $r_1 = 1.47 \times 10^{11} m$, $r_2 = 1.52 \times 10^{11} m$ and $\pi = \pi$ (What did you even think I was gonna give lmao). Report in days rounded to nearest day.

Problem 15.6 ($F = ma$)**.** [7] A body of mass M and a body of mass $m \ll M$ are in circular orbits about their center of mass under the influence of their mutual gravitational attraction to each other. The distance between the bodies is R, which is much larger than the size of either body.

A small amount of matter $\delta m \ll m$ is removed from the body of mass m and transferred to the body of mass M. The transfer is done in such a way so that the orbits of the two bodies remain circular, and remain separated by a distance R. Which of the following statements is correct?

(A) The gravitational force between the two bodies increases.

(B) The gravitational force between the two bodies remains constant.

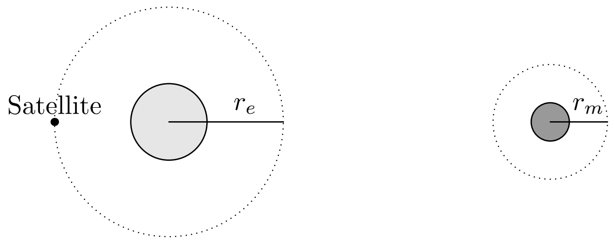
(C) The total angular momentum of the system increases.

(D) The total angular momentum of the system remains constant.

(E) The period of the orbit of two bodies remains constant.

Problem 15.7 (Pathfinder)**.** [8] A small moon of radius r and mass m is orbiting around a planet of mass M in a circular orbit of radius R ($R \gg r$) always keeping the same face towards the planet. If an object on the moon closest to the planet is in weightlessness, find a suitable expression for the radius of the orbit. Use $r = 1m$, $m = 1kg$ (yes, the moon has 1kg mass, problem?) , and $M = 109503kg$

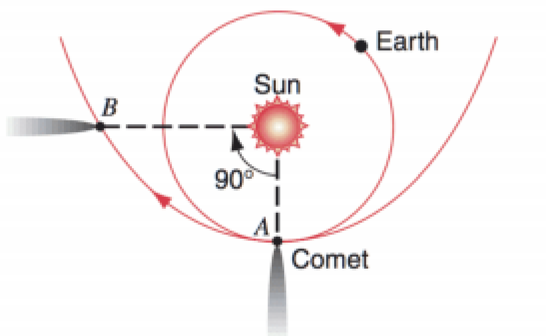
Problem 15.8 (Mock $F = ma$)**.** [8] NASA wishes to bring a small rocket of mass m to the moon. First, it starts off in a circular orbit at a semi-major axis r_e around Earth. The engines fire in a negligible amount of time and bring the rocket to the moon, a distance d away. You may assume that $d \gg r_e$. Once the rocket reaches the moon, the engines fire once again and bring the rocket into a circular orbit at a semi-major axis r_m. The mass of the Earth and the moon are M_e and M_m respectively.



What is the total impulse the engines must impart on the rocket? Use $m = 419kg$, $r_e = 10m$, $M = 10^11kg$, ignore relativistic effects due to unrealistic values please and thank you. Report 3 significant digits

Problem 15.9 (Kalda)**.** [9] An object is thrown vertically from the ground and reaches a distance $R = R_M$ above the surface of the Earth before returning. If R_M is the radius of the earth, determine the time of flight of the object. Use $R = 10m$, $M = 10^11kg$, report to nearest second

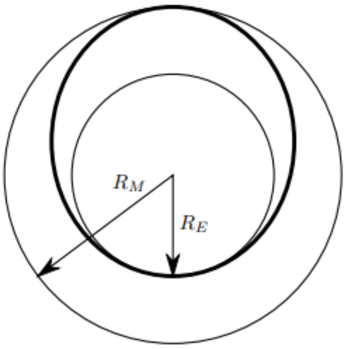
Problem 15.10 (HRK)**.** [11] A comet passes by the Sun as shown, in a parabolic path.



How long, in years, does the comet take to get from point A to point B? Report only 1 significant digit. **Hint:** If you apply Kepler's laws and properties of conics, this problem can be done with almost no computation.

Problem 15.11 (USAPhO)**.** [15] A ship starts out in a circular orbit around the sun very near the Earth and has a goal of moving to a circular orbit around the Sun that is very close to Mars. It will make this transfer in an elliptical orbit as shown in bold in the diagram below. This is accomplished with an initial velocity boost near the Earth Δv_1 and then a second velocity boost near Mars Δv_2.

Assume that both of these boosts are from instantaneous impulses, and ignore mass changes in the rocket as well as gravitational attraction to either Earth or Mars. Don't ignore the Sun! Assume that the Earth and Mars are both in circular orbits around the Sun of radii R_E and $R_M = R_E/\alpha$ respectively. The orbital speeds are v_E and v_M respectively. Use $\alpha = 0.25$



(a) [5] Derive an expression for the velocity boost Δv_1 to change the orbit from circular to elliptical. Express your answer in terms of v_E and α.Report $\Delta v_1/v_E$ in 3 significant figures.

(b) [5] Derive an expression for the velocity boost Δv_2 to change the orbit from elliptical to circular. Express your answer in terms of v_E and α. Report $\Delta v_2/v_M$ in 3 significant figures.

(c) [5] What is the angular separation between Earth and Mars, as measured from the Sun, at the time of launch so that the rocket will start from Earth and arrive at Mars when it reaches the orbit of Mars? Express your answer in terms of α.Report θ/π

15.3.1 Written Solutions

Problem 15.12 (HRK)**.** [18] A pair of stars revolve about their common center of mass. On of the stars has a mass M that is twice the mass M of the other; that is, $M = 2m$. Their centers are a distance d apart, d being large compared to the size of either star.

(a) [6] Derive an expression for the period of revolution of the stars about their common center of mass in terms of d, m and G. Report $T = \alpha d^a G^b M^c$, then report $\alpha + a + b + c$ to 3 significant figures

(b) [6] Compare the angular momenta of the two stars about their common center of mass by calculating the ratio L_m/L_M.

(c) [6] Compare the kinetic energies of the two stars by calculating the ratio K_m/K_M.