9 Celestial Coordinate Conversions

Coordinate conversion is very common in Olympiad questions and real academic research. This chapter will test your understanding of the previous chapters on spherical geometry and celestial coordinate systems. Fortunately, we only need two formulas - the cosine formula and the sine formula. You should also understand why the following simplifications are true:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\cos(360^\circ + \theta) = \cos\theta$$
$$\sin(360^\circ + \theta) = \sin\theta$$
$$\cos(360^\circ - \theta) = \cos\theta$$
$$\sin(360^\circ - \theta) = -\sin\theta$$
$$\cos(90^\circ - \theta) = \sin\theta$$
$$\sin(90^\circ - \theta) = \cos\theta$$
$$\cos(90^\circ + \theta) = -\sin\theta$$
$$\sin(90^\circ + \theta) = \cos\theta$$

If you are skeptical about any of the equations above, draw a triangle in a unit circle to experience the joy of discovery.

9.1 Horizontal coordinates to Equatorial coordinates and vice versa

The coordinate conversion formulas can be easily derived from a celestial sphere diagram. Here we derive the formulas for a star X for an observer at latitude ϕ. Note that we use P for North Celestial Pole for conciseness.

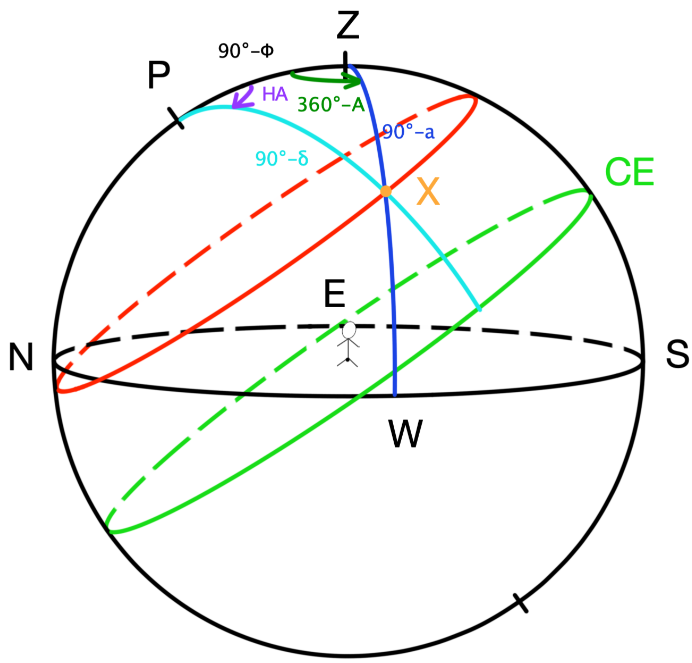

Figure 9.1: Drawing a celestial sphere to derive conversion formulas

From Fig 9.1, we note the following:

- $PX = 90° - \delta$
- $ZX = 90° - a$
- $PZ = 90° - \phi$
- $ZPX = HA$
- $PZX = 360° - A$

Remember that HA is measured westwards along CE from the observer's meridian while A is measured eastwards along the horizon from the north. In Fig 9.1, the most important part is the spherical triangle PZX. If we wish to convert from horizontal coordinates to equatorial coordinates, we must know the altitude and azimuth of the star, and the latitude of the observer. We can then fully solve the spherical triangle PZX since we already have 3 known variables!

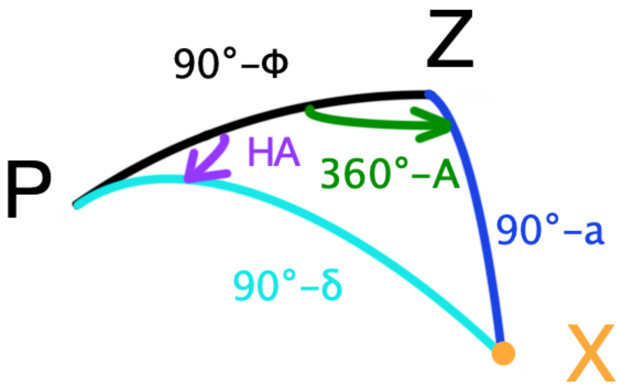

Figure 9.2: Spherical triangle PZX to derive conversion formulas

Let's begin by finding an expression for declination. We can use the cosine formula with $\cos(90° - \delta)$ as the subject.

$$\begin{aligned}\cos PX &= \cos PZ \cos XZ + \sin PZ \sin XZ \cos PZX \\ \cos(90° - \delta) &= \cos(90° - \phi)\cos(90° - a) + \sin(90° - \phi)\sin(90° - a)\cos(360° - A)\end{aligned}$$

Simplifying we arrive at:

$$\sin\delta = \sin\phi\sin a + \cos\phi\cos a\cos A$$

Note that since ϕ, a, A is given, we can calculate δ directly.

We then proceed to find HA (angle ZPX) by using the cosine formula with its opposite side $\cos ZX$ as the subject. This is to orientate the spherical triangle so that HA is involved in the calculation (as the cosine formula involves one angle only, the rest being 3 sides).

$$\begin{aligned}\cos XZ &= \cos XP \cos ZP + \sin XP \sin ZP \cos ZPX \\ \cos(90° - a) &= \cos(90° - \delta)\cos(90° - \phi) + \sin(90° - \delta)\sin(90° - \phi)\cos HA \\ \sin a &= \sin\delta\sin\phi + \cos\delta\cos\phi\cos HA\end{aligned}$$

Rearranging we arrive at:

$$\cos HA = \frac{\sin a - \sin\delta \sin\phi}{\cos\delta\cos\phi}$$

An easier way is to use the sine formula:

$$\begin{aligned}
\frac{\sin ZPX}{\sin ZX} &= \frac{\sin PZX}{\sin PX} \\
\frac{\sin HA}{\sin(90^\circ - a)} &= \frac{\sin(360^\circ - A)}{\sin(90^\circ - \delta)} \\
\frac{\sin HA}{\cos a} &= \frac{-\sin A}{\cos\delta} \\
\sin HA &= -\frac{\sin A \cos a}{\cos\delta}
\end{aligned}$$

However, we recommend using the equation derived from the cosine formula because the angle returned by $\cos^{-1}$ is in the correct quadrant while $\sin^{-1}$ does not differentiate between angles of quadrant 1 and quadrant 2.

The general strategy for coordinate conversion is:

1. Draw the celestial sphere and related constructions of both coordinate systems.
2. Label the angles and lines with variables.
3. Isolate the spherical triangle with all related variables. This triangle must have the side subtended by two poles (here it is the side PZ).
4. Use the cosine formula to find the height-related variable (here it is declination).
5. Use the new variable (declination) and cosine formula to find direction-related variable (here it is hour angle). Use the opposite side of the desired angle as the subject (ZX). Then since $\cos^{-1} HA$ only returns the basic angle $0 \le HA \le 180^\circ$, check the diagram you have drawn in step 1 to choose the correct quadrant of HA.

Congratulations! Now you also know how to convert most coordinate systems too; just apply the general strategy outlined above.

To see the effectiveness of the strategy, let's convert back from equatorial coordinates to horizontal coordinates.

- Step 1: We draw Fig 9.1.
- Step 2: We label Fig 9.1 with δ, ϕ, a, A, HA.
- Step 3: We isolate the triangle into Fig 9.2.
- Step 4: A quick glance on Fig 9.2 revealed that we can use the cosine formula to solve directly for a (height-related), by treating $\cos(90^\circ - a)$ as the subject.

$$\begin{aligned}
\cos XZ &= \cos PZ \cos PX + \sin PZ \sin PX \cos ZPX \\
\cos(90^\circ - a) &= \cos(90^\circ - \delta)\cos(90^\circ - \phi) + \sin(90^\circ - \delta)\sin(90^\circ - \phi)\cos HA
\end{aligned}$$

Simplifying we arrive at

$$\sin a = \sin\delta\sin\phi + \cos\delta\cos\phi\cos HA$$

- Step 5: We can then solve for A by using its opposite side PX, thus $\cos PX$ as the subject.

$$\begin{aligned}
\cos PX &= \cos PZ \cos XZ + \sin PZ \sin XZ \cos PZX \\
\cos(90^\circ - \delta) &= \cos(90^\circ - \phi)\cos(90^\circ - a) + \sin(90^\circ - \phi)\sin(90^\circ - a)\cos(360^\circ - A) \\
\sin\delta &= \sin\phi\sin a + \cos\phi\cos a\cos A
\end{aligned}$$

In case you haven't realised, the above equations are exactly those we used to solve for declination. Rearranging we arrive at

$$\cos A = \frac{\sin\delta - \sin\phi\sin a}{\cos\phi\cos a}$$

Remember to check the quadrant using the diagram in step 1. You can also use the sine formula instead to convert the longitude-related variable.

Example. Ali observes at star with $A = 50°$ and $a = 46°$ at $\phi = 32°$. What is the declination and hour angle of the star? Do not blindly apply the formulas, you must derive them using a celestial sphere diagram in order to get marks.

Solution. You should draw the celestial sphere with the NCP pointing right so that the star is facing you. Then, we find $\delta = 49°27'$ and $HA = 54.94° = 3^h40^m$. Since the star is at east, the HA is actually negative for such value. So, $HA = 24^h - 3^h40^m = 20^h20^m$. ■

Example 9.1 (USAAAO NAO 2017). You are observing a star with declination $\delta = 42°21'$ N and hour angle $H = 8^h16^m42^s$. If you are in a place with latitude $\varphi = 60°$, compute the star's azimuth angle (A) and its height above the horizon (h) at the moment of observation.

Solution. This is an equatorial-horizontal conversion question. Now we directly proceed to computation of altitude, after you have followed the general strategy to derive the necessary equations. First we convert HA from hours to degrees.

$$\begin{aligned} \text{8h 16m 42s} &= \text{8h 16m 42s} \times \frac{360°}{24\text{h}} \\ &= 124°10'30'' \end{aligned}$$

Then altitude is

$$\begin{aligned} \sin a &= \sin\delta\sin\phi + \cos\delta\cos\phi\cos HA \\ &= \sin 42°21'\sin 60° + \cos 42°21'\cos 60°\cos 124°10'30'' \\ a &= 22°4'34'' \end{aligned}$$

We then find azimuth

$$\begin{aligned} \cos A &= \frac{\sin\delta - \sin\phi\sin a}{\cos\phi\cos a} \\ &= \frac{\sin 42°21' - \sin 60°\sin 22°4'34''}{\cos 60°\cos 22°4'34''} \\ A &= 41°17'7'' \end{aligned}$$

■

Exercise. Derive an equation for azimuth using the sine formula, when converting from equatorial to horizontal coordinates.

9.2 Equatorial coordinates to Ecliptic coordinates

The ecliptic plane is the orbital plane of the Earth around the Sun. Recall that the planets and the Sun roughly lies on the ecliptic plane, which is the defining feature of the plane. Here we use right ascension instead of hour angle since both coordinate systems are time-independent.

1. The **vernal equinox** Υ is the point of intersection between the ecliptic plane and the equatorial plane (celestial equator).
2. The **right ascension** α is the angle of the object measured counterclockwise from the vernal equinox along the equatorial plane.
3. The **ecliptic longitude** λ is the angle of the object measured counterclockwise from the vernal equinox along the ecliptic plane.
4. The **ecliptic latitude** β is the angle of the object measured from ecliptic plane towards the ecliptic north pole.

Let's proceed to convert equatorial coordinates to ecliptic coordinates using the general strategy.

1. **Draw the celestial sphere and related constructions of both coordinate systems**, which is Fig 9.3. Note that K is the (celestial) ecliptic pole and P is the equatorial pole.

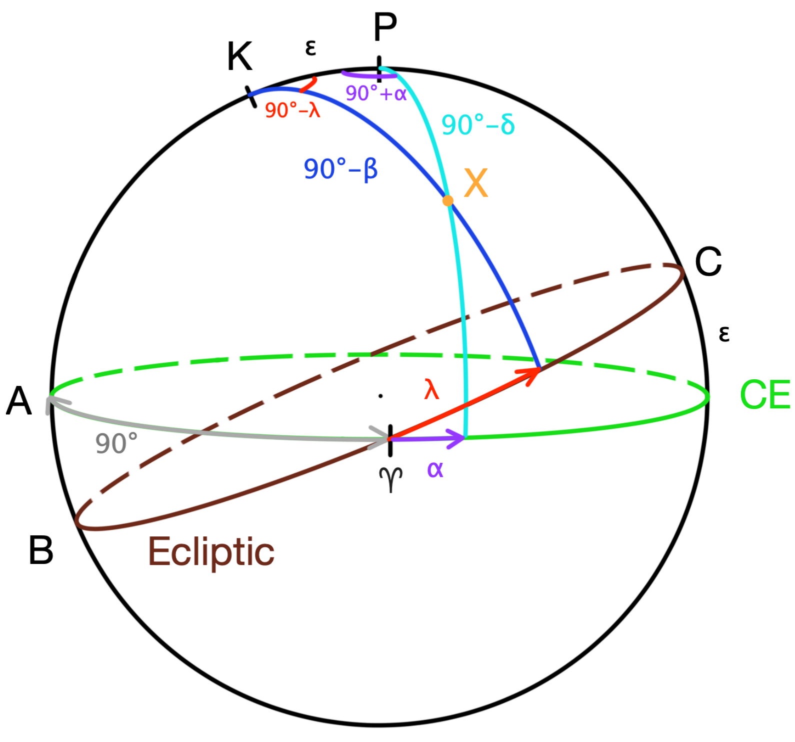

Figure 9.3: Drawing celestial sphere to derive conversion formulas

2. **Label the angles and lines with variables.** From Fig 9.3, we note the following:

- $A\Upsilon = 90°$
- $\Upsilon C = 90°$
- $PK = \epsilon$
- $PX = 90° - \delta$
- $KX = 90° - \beta$
- $KPX = 90° + \alpha$
- $PKX = 90° - \lambda$

3. **Isolate the spherical triangle PKX with all related variables.**

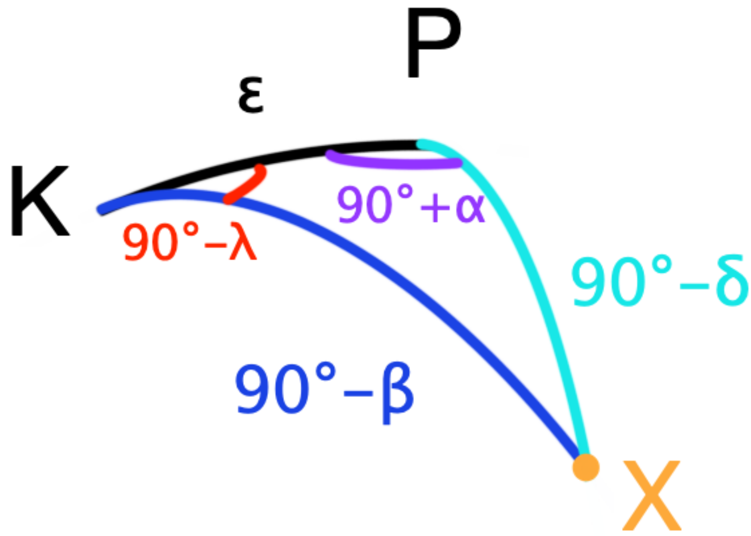

Figure 9.4: Spherical triangle PKX

4. **Use the cosine formula to find height-related variable** which is ecliptic latitude β.

$$\cos KX = \cos PK \cos PX + \sin PK \sin PX \cos KPX$$
$$\cos(90^\circ - \beta) = \cos\epsilon \cos(90^\circ - \delta) + \sin\epsilon \sin(90^\circ - \delta)\cos(90^\circ + \alpha)$$

Simplifying we arrive at

$$\sin\beta = \cos\epsilon \sin\delta - \sin\epsilon \cos\delta \sin\alpha$$

Remember that $\epsilon = 23^\circ 26'$.

5. **Use the new variable** (ecliptic latitude) **and cosine formula to find** ecliptic longitude. **Use the opposite side of the desired angle as the subject.** PX is opposite of PKX, so we treat PX as the subject.

$$\cos PX = \cos KX \cos KP + \sin KX \sin KP \cos XKP$$
$$\cos(90^\circ - \delta) = \cos(90^\circ - \beta)\cos\epsilon + \sin(90^\circ - \beta)\sin\epsilon \cos(90^\circ - \lambda)$$
$$\sin\delta = \sin\beta\cos\epsilon + \cos\beta\sin\epsilon\sin\lambda$$

Rearranging we arrive at

$$\sin\lambda = \frac{\sin\delta - \sin\beta\cos\epsilon}{\cos\beta\sin\epsilon}$$

Remember to check the quadrant of ecliptic longitude. Note that you can solve for ecliptic longitude first by substituting the expression of ecliptic latitude in Step 4 into the equation in Step 5 above. However, you should generally avoid doing so. Following the general strategy instead will serve you faithfully well.

> **Example.** On a certain day, the Everaise satellite measured the coordinates of Saturn to be $\alpha = 20^h 13^m 53^s$, $\delta = -20°0'49''$. What is the ecliptic coordinates of Saturn at that time?

Solution. Draw the celestial sphere with Saturn in the southern hemisphere. The calculated coordinates are $\beta = -0°8', \lambda = 301°13'$. If you get $\lambda = -58.788°$, add 360° to it. λ is in the fourth quadrant. ■

> **Example 9.2** (Roy and Clarke: Astronomy Principles and Practice)**.** The pole of the galactic equator has coordinates RA = 12h 51m, Dec = 27°08′ N. Calculate its ecliptic longitude and latitude.

Solution. This is an equatorial-ecliptic question. Remember the general strategy, after we derived the equations for equatorial-ecliptic conversion. First we convert HA (also denoted as α) from hours into degrees.

$$\begin{aligned} 12\text{h } 51\text{m} &= 12\text{h } 51\text{m} \times \frac{360°}{24\text{h}} \\ &= 192°45' \end{aligned}$$

We proceed with the ecliptic latitude first, remembering $\epsilon = 23°26'$.

$$\begin{aligned} \sin\beta &= \cos\epsilon \sin\delta - \sin\epsilon\cos\delta\sin\alpha \\ &= \cos 23°26' \sin 27°08' - \sin 23°26' \cos 27°08' \sin 192°45' \\ \beta &= 29°46' \end{aligned}$$

We continue to find ecliptic longitude

$$\begin{aligned} \sin\lambda &= \frac{\sin\delta - \sin\beta\cos\epsilon}{\cos\beta\sin\epsilon} \\ &= \frac{\sin 27°08' - \sin 29°46' \cos 23°26'}{\cos 29°46' \sin 23°26'} \\ &= 4'35'' \end{aligned}$$

Inspecting the celestial sphere diagram you have drawn for this question, you will easily note that the quadrant is 2. So

$$\begin{aligned} \lambda &= 180° - 4'35'' \\ &= 179°55' \\ &\approx 180° \end{aligned}$$

■

Exercise. Derive an equation for ecliptic longitude using the sine formula, when converting from equatorial to ecliptic coordinates.

Exercise. By drawing a celestial sphere with a diameter of at least 10cm, derive the conversion formulas from ecliptic coordinates to equatorial coordinates.

9.3 Equatorial coordinates to Galactic coordinates

Lastly, we derive the formulas for conversion between equatorial and galactic coordinates. For galactic coordinates, L is the direction of the galactic centre, the **position angle** $PGL = \theta$, and the equatorial coordinates of the North Galactic Pole G is described as α_G and δ_G.

1. The **galactic longitude l** is the angle of the object measured counterclockwise from the galactic centre L along the galactic plane.
2. The **galactic latitude b** is the angle of the object measured from galactic plane towards the galactic north pole.

Now we follow the general strategy. When converting to galactic coordinates, θ, α_G and δ_G should be given.

1. **Draw the celestial sphere and related constructions of both coordinate systems**, which is Fig 9.5. Note that GE is the galactic equator.

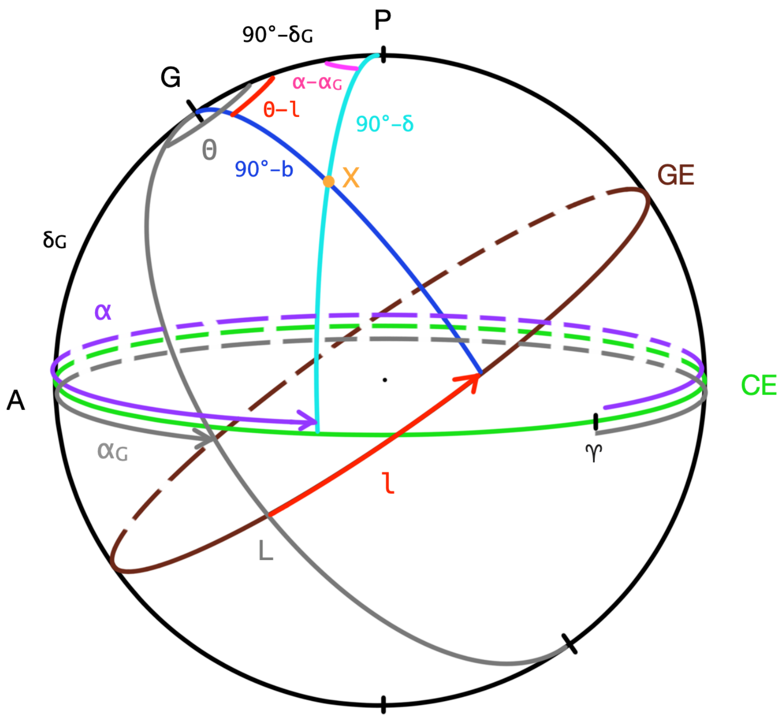

Figure 9.5: Drawing celestial sphere to derive conversion formulas

2. **Label the angles and lines with variables.** From Fig 9.5, we note the following:

- $PG = 90° - \delta_G$
- $PX = 90° - \delta$
- $GX = 90° - b$
- $GPX = \alpha + \alpha_G$
- $PGX = \theta - l$

3. **Isolate the spherical triangle PGX with all related variables.**
4. **Use the cosine formula to find height-related variable** which is galactic latitude b.

$$\cos GX = \cos PG \cos PX + \sin PG \sin PX \cos GPX$$
$$\cos(90° - b) = \cos(90° - \delta_G)\cos(90° - \delta) + \sin(90° - \delta_G)\sin(90° - \delta)\cos(\alpha - \alpha_G)$$

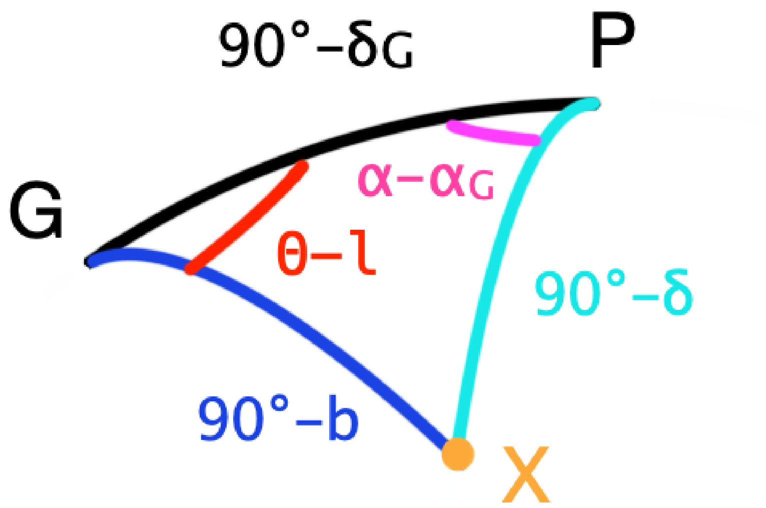

Figure 9.6: Spherical triangle PGX

Simplifying we arrive at

$$\sin b = \sin \delta_G \sin \delta - \cos \delta_G \cos \delta \cos (\alpha - \alpha_G)$$

5. **Use the new variable** (ecliptic latitude) **and cosine formula to find** ecliptic longitude l. **Use the opposite side** PX **of the desired angle** PGX **as the subject.**

$$\begin{aligned}
\cos PX &= \cos GX \cos GP + \sin GX \sin GP \cos XGP \\
\cos (90^\circ - \delta) &= \cos (90^\circ - b) \cos (90^\circ - \delta_G) + \sin (90^\circ - b) \sin (90^\circ - \delta_G) \cos (\theta - l) \\
\sin \delta &= \sin b \sin \delta_G + \cos b \cos \delta_G \cos (\theta - l)
\end{aligned}$$

Rearranging we arrive at

$$\cos (\theta - l) = \frac{\sin \delta - \sin b \sin \delta_G}{\cos b \cos \delta_G}$$

Remember that $\theta = 123^\circ, \alpha_G = 12^h 51.4^m, \delta_G = 27.13^\circ$. Remember to check the quadrant of $\theta - l$.

Example. The brightest star Sirius has coordinates $\delta = -16^\circ 43', \alpha = 6^h 45^m$. Calculate its galactic coordinates.

Solution. $b = -8.9^\circ$, $l = 227.3^\circ$ ■

Exercise. Derive an equation for galactic longitude using the sine formula, when converting from equatorial to galactic coordinates.

Now you can convert from whichever coordinate system to any other coordinate system using the techniques you learnt in this chapter, especially the general strategy. You can follow the conversion "chain" in Fig 9.7 when you need to convert between different systems. When converting "through" the equatorial system, convert HA to RA using the relation $LST = HA + RA$ (You will learn LST in later chapters). Remember, draw a celestial sphere and derive

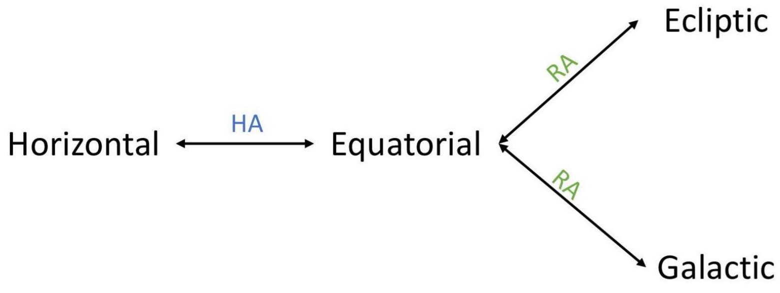

Figure 9.7: Coordinate conversion chain

these equations; we should leave memorisation only to equations where derivation is too long or hard.

Exercise. Derive the equations to convert from galactic coordinates to equatorial coordinates.
Hint: Use the general strategy and draw Fig 9.6.

9.4 Homework Problems

Problem 9.1 (Roy and Clarke: Astronomy Principles and Practice)**.** Calculate the hour angle of Vega (declination $38°44'$ N) when on the prime vertical west in latitude $50°$ N. For what latitudes is Vega circumpolar?

Problem 9.2 (IOAA 2013 Theory)**.** Find the equatorial coordinates (hour angle and declination) of a star at Madrid, geographic latitude $\varphi = 40°$, when the star has zenith angle $z = 30°$ and azimuth $A = 50°$ (azimuth as measured from the South clockwise)

Problem 9.3 (USAAAO NAO 2019)**.** You are in the northern hemisphere and are observing rise of star A with declination $\delta = -8°$, and at the same time a star B with declination $\delta = +16°$ is setting. What will happen first: next setting of star A or rising of star B?