

Class 14

2018年5月10日 星期四 10:00

Gamma distribution: $X \sim \gamma(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

↑ 归一化参数

$$\Gamma(\alpha) = (\alpha-1)! \quad \alpha = 1, 2, \dots$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

习题:

$$\left. \begin{array}{l} X \sim \text{gamma}(n, \lambda) \\ Y \sim \text{exponential}(\lambda) \end{array} \right\} \Rightarrow X+Y \sim \text{gamma}(n+1, \lambda)$$

习题:

Let α be an integer and suppose X has distribution Gamma(α, β). Then $P(X \leq \alpha) = P(Y \geq \alpha)$ where $Y \sim \text{Poisson}(\alpha\beta)$.

* Cauchy distribution: Cauchy(α, β) 人工反例分布, τ 与泊松更为自然

$$f(x) = \frac{1}{\beta\pi} \cdot \frac{1}{1 + (x-\alpha)^2/\beta^2}$$

$N(\mu, \sigma)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

* 特征函数:

此放缩未必好算好算, 加入 $f(x)$

$$P(X \geq a) = E(1_{X \geq a}) \leq E\left(\frac{X}{a}\right) = \frac{E X}{a}$$

↑ $f(x) \geq b$ ↑ $\frac{f(x)}{b}$ 寻找 $f(x)$ 更好算

定义: $X \in \mathbb{R}^n$, $\varphi_X(u) = E(e^{i\langle u, X \rangle}) \leftarrow E$ 即为积分

$$= \int e^{i\langle u, x \rangle} P(dx)$$

任意 \mathbb{R}^n 上函数, 如密度函数

$$\mu: \text{probability measure on } \mathbb{R}^n$$

$$\hat{\mu}: \hat{\mu}(u) = \int e^{i\langle u, x \rangle} \mu(dx)$$

* 母函数:

$$(b_0, b_1, \dots) \rightarrow \sum_{i=0}^{\infty} b_i t^i$$

错排问题:

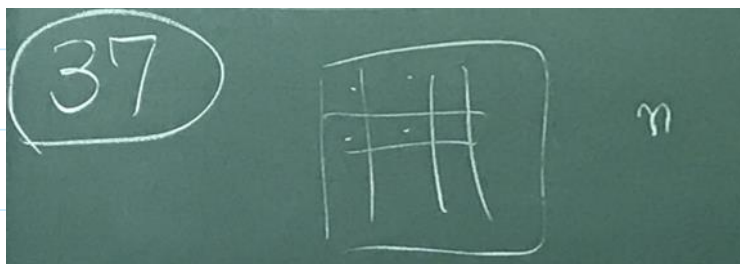
$$b_0 = 1, \quad b_n = P(X_n = 0) \quad n \geq 1 \quad \text{其中 } X_n = \text{Fix } \sigma, \quad \sigma \in S_n$$

$$B(t) = \sum_{i=0}^{\infty} b_i t^i \quad \frac{1}{1-t} = B(t) e^t$$

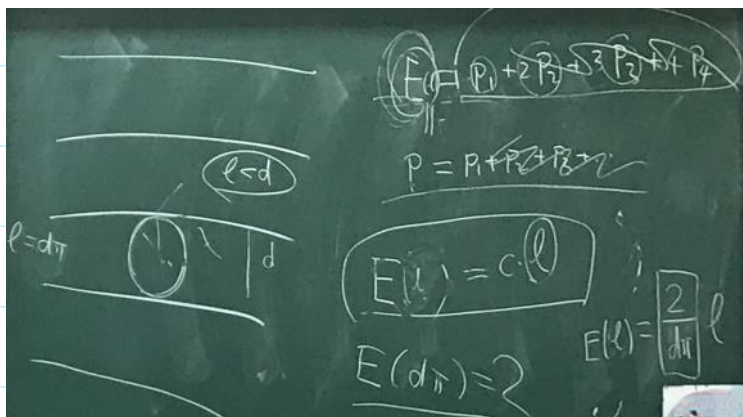
$$(1+t+t^2+\dots) = \left(\sum_{i=0}^{\infty} b_i t^i \right) \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \dots \right) \quad \left. \begin{array}{l} \text{pt: } 1 = \sum_k \frac{b_{n-k}}{k!} \end{array} \right\} \text{全概率公式}$$

$$1 = \sum_k \frac{b_{n-k}}{k!} \Leftrightarrow 1 = \sum_k P\{\sigma : |\text{Fix } \sigma| = k\}$$

$$\Rightarrow B(t) = \frac{e^{-t}}{1-t} \quad \lim_{n \rightarrow \infty} b_n = e^{-1} = 1 - 1 + \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \dots = 0.37$$



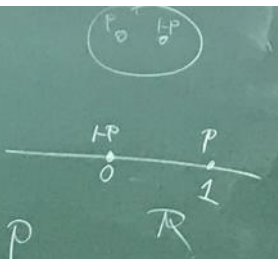
扔下 n 枚炸弹, 格子不被炸到概率为 0.37



布丰投针

期望线性性

* $X \sim \text{Bernoulli}(P)$:

$$\begin{aligned}\varphi_X(u) &= E\{e^{iuX}\} \\ &= e^{iu \cdot 0} (1-p) + e^{iu \cdot 1} p \\ &= (pe^{iu} + 1-p)\end{aligned}$$


Binomial (p, n) , $\varphi_X(u) = (pe^{iu} + 1-p)^n$

$$\begin{aligned}X &= \sum_{j=1}^n Y_j \\ Y_1, \dots, Y_n &\text{ independent} \\ \varphi_X(u) &= \prod_{j=1}^n \varphi_{Y_j}(u)\end{aligned}$$

习题

* $X \sim \text{Poisson}(\lambda)$:

$$\varphi_X(u) = e^{\lambda(e^{iu}-1)} \quad \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\begin{aligned}X &\text{ uniform on } (-a, a) \\ \varphi_X(u) &= \frac{1}{2a} \int_{-a}^a e^{iux} dx = \frac{e^{iua} - e^{-iua}}{2aiu} \\ &= \frac{\sin au}{au}\end{aligned}$$

* $X \sim N(0, 1)$:

$$\begin{aligned}\varphi_X(u) &= \int e^{iux} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= \int_{-\infty}^{\infty} \frac{\cos ux}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + i \cdot \int_{-\infty}^{\infty} \frac{\sin ux}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos ux e^{-\frac{x^2}{2}} dx\end{aligned}$$

$$\begin{aligned}\varphi_X'(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -x(\sin ux) e^{-\frac{x^2}{2}} dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\cos ux) e^{-\frac{x^2}{2}} dx \\ &= -u \varphi_X(u)\end{aligned}$$

$$\frac{\varphi_X'}{\varphi_X} = -u \rightarrow \ln |\varphi_X(u)| = -\frac{u^2}{2} + C$$

$$\varphi_X(0) = 1, \quad \varphi_X(u) = e^{-\frac{u^2}{2}}$$

* $X \sim \text{exponential}(\lambda)$:

$$\varphi_X(u) = \frac{\lambda}{\lambda - iu}$$

→ double exponential - Laplace distribution ($\alpha=0, \beta=1$)

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad \varphi_X(u) = \frac{1}{1+u^2}$$

*

Let $X = (X_1, \dots, X_n)$ be an \mathbb{R}^n -valued r.v. Then $(X_j)_{j=1}^n$ are independent iff

$$\varphi_X(u_1, \dots, u_n) = \prod_{j=1}^n \varphi_{X_j}(u_j)$$