Elementary Probability

Tuesday, March 6, 2018 10:11 AM

Tennis game

P: Alice wins her serve

q: Bob wins his serve

2种比赛办法:

Alice 第一局 $\begin{cases} \text{Alternating serve:} \\ \text{Winner serve:} \end{cases}$ 谁先赢n局胜利，考察两种规则?

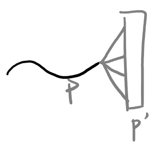
Solution: 两种赛制 A 胜的概率相同

实际这里结束.

AS' $\underbrace{\text{ABAB} \mid \cdots \text{BA}}_{2n-1}$

WS' $\underbrace{\text{AABAAAB} \mid \text{BB} \cdots \text{B}}_{2n-1}$

$\boxed{A: n \quad B: n-1 \quad \text{A赢的场数多于B的概率}}$



$\sum p' = p$ 全概率定理

Hat puzzle (Ebert's version)

有7个人分别戴着一顶帽子,帽子颜色只有两种

$x_1, x_2, \ldots, x_7 \in \{0,1\}^7$

每人能看到其他人的帽子,猜自己的帽子颜色,得到的猜测序列中不存在与真实序列相反的数,则这7人获胜 可猜{0,1,不猜},且至少有一人不猜*

例: 0110010 胜
0******

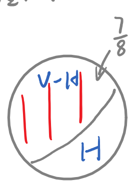
可使得获胜概率达到$\frac{7}{8}$

Solution:

$x \in \mathbb{F}_2^7$. 取 $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$ 为所有$\mathbb{F}_2^3$中非零向量构成的矩阵.

$H = \{x \in \mathbb{F}_2^7 \mid Bx = \vec{0}\}$ $|H| = 2^4$ $(n - r(B))$

$|\mathbb{F}_2^7 / H| = 2^3$,



构造 $e_0, e_1, \ldots, e_7 \in \mathbb{F}_2^7$ 为0和单位向量

$H+e_0, H+e_1, \ldots, H+e_7$ 8个空间互不相交, 为$\mathbb{F}_2^7$的一个分割.

(若相交有 $e_i - e_j \in H$, 而B各列不等且非零,矛盾)

因此, $\forall x \in \mathbb{F}_2^7$, ∃唯一t, s.t. $x \in H + e_t$

$\Rightarrow x + e_t \in H$ ①

$x + e_i \notin H$. $(i \neq t)$ ②

下证: 此时, 若 $x + e_i \in H \Rightarrow e_i + e_t \in H$

事实上, 若 $x + e_i \in H \Rightarrow e_i + e_t \in H$

$\Rightarrow B(e_i + e_t) = 0$ 但由B的取法知任两列和不为0

矛盾!

由①.②知
$\forall x \in H^c$，不妨设 $x \in H + e_t$，则有且仅有一个 x 的分量 x_t，
当 x_t 改变时，x 是否属于 H 的状态改变
那么对于原问题，采取如下策略．当一人发现，无论自己位置为 0 或 1，都无法改变 x 是否属于 H 的状态时，猜1，否则，猜使 $x \notin H$ 的值． $P(x \in H^c) = \frac{7}{8}$
那么 $P(\text{胜}) = P(\text{胜} \mid x \in H^c) \cdot P(x \in H^c) + P(\text{胜} \mid x \in H) P(x \in H) = 1 \times \frac{7}{8} + 0 = \frac{7}{8}$

Murphy's law: If sth. bad can happen, it eventually will.

Let $(A_n, n \geq 1)$ be any sequence of events in a probability space satisfying $A_n \subseteq A_{n+1}$ for all $n \geq 1$. Suppose that $P(\bigcup_{n=1}^{\infty} A_n \mid A_n^c) \geq \varepsilon$. Then $P(\bigcup_{n=1}^{\infty} A_n) = 1$

Pf: $\bigcup_{n=1}^{\infty} A_n = A \qquad P_n = P(A_n)$

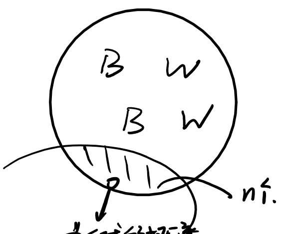
$$P(A) = P(A \mid A_n^c)(1 - P_n) + P(A \mid A_n) P_n$$
$$\geq \varepsilon(1 - P_n) + P_n \qquad ①$$

而 $P(\bigcup A_n) = P(\bigcup (A_n \setminus A_{n-1})) = \sum P(A_n \setminus A_{n-1})$
$$= \sum_{n \geq 1} [P(A_n) - P(A_{n-1})]$$
$$= \lim_{n \to \infty} P(A_n)$$

① 两侧取极限．
$$P(A) \geq \varepsilon(1 - P(A)) + P(A)$$
$$\Rightarrow P(A) \geq 1 \quad \#$$

Balls

\#White Balls $= a$
\#Black Balls $= b$
1° choose n balls
2° choose 1 from the n balls



为白球的概率

Let A be an event that the final chosen ball is white. $P(A)$?

solution 1: $\sum_{c=0}^{\infty} \frac{\binom{a}{c}\binom{b}{n-c}}{\binom{a+b}{n}} \cdot \frac{c}{n}$ ← 按所取n个球白球个数为序计算

Solution:
Let C be the number of white balls in the chosen n balls.

$$P(A \mid c) = \frac{c}{n}$$
$$P(A) = \sum_{c} P(A \mid C = c) P(C = c) \qquad \because E(C) = \frac{a}{a+b} n.$$
$$P(A) = E(P(A \mid C)) = E[E[A \mid C]] = E\left(\frac{C}{n}\right) = \frac{a}{a+b} \; ?$$