Question

If $X_1, X_2. X_3, ...$ are independant random variables,

(1) $Y_i = X_i X_{i+1} X_{i+2}$, are the random variables, are they independant?

(2) S_1, S_2 are any subsets of $\mathbf{N}^*$, $Y_S = \prod_{i \in S} X_i$. When are S_1 and S_2 independant?

Solution

(1) No.

As $X_1, X_2. X_3, ...$ are independant random variables,

$$E[Y_1] = E[X_1 X_2 X_3] = E[X_1]E[X_2]E[X_3]$$

Similarly, we have

$$E[Y_2] = E[X_2 X_3 X_4] = E[X_2]E[X_3]E[X_4]$$

So

$$E[Y_1]E[Y_2] = E[X_1]E[X_2]^2 E[X_3]^2 E[X_4]$$

But

$$E[Y_1 Y_2] = E[X_1 X_2^2 X_3^2 X_4] = E[X_1]E[X_2^2]E[X_3^2]E[X_4]$$

Since we could not derive $E[X_2^2] = E[X_2]^2$, so

$$E[Y_1 Y_2] = E[Y_1]E[Y2]$$

So they are not independant.

(2) With the condition

$$E[X_i^2] = E[X_i]^2, \quad \forall i \in \mathbf{N}^*$$

We can show that S_1 and S_2 are independant. It is also easy to show that the condition is also necessary.