Ergodicity 定理的表述和证明

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1 问题引入

Given an edge-weighted undirected graph, let w_{xy} denote the weight of the edge between nodes x and y, with$w_{xy} = 0$ if no such edge exist. Let $w_x = \sum w_{xy}$. The Markov chain has transition probabilites $p_{xy} = \frac{w_{xy}}{w_x}$. We assume the chain is connected. The we have the stationary distribution $\pi_x = \frac{w_x}{w_{total}}$.

Our problem is how fast the walk starts to reflect the stationary probability of the Markov process. In my proof, I use the idea of 'probabililty flow'.

2 Definitions

ϵ-mixing time

Fix $\epsilon > 0$, The ϵ-mixing time of a Markov chain is the minimum integer t such that for any starting distribution p, the 1-norm difference between the t-step running average probability distribution and the stationary distribution is at most ϵ.

normalized conductance

For a subset S of vertices, let $\pi(S)$ denote $\sum_{x \in S} \pi_x$. The normalized conductance $\Phi(S)$ of S is

$$\Phi(S) = \frac{\sum_{(x,y) \in (S,\bar{S})} \pi_x p_{xy}}{min(\pi(S), \pi(\bar{S}))})$$

Suppose without loss of generality that $\pi(S) \leq \pi(\bar{S})$. Then,

$$\Phi(S) = \sum_{x \in S} \frac{\pi_x}{\pi(S)} \sum_{y \in \bar{S}} p_{xy}$$

Thus, $\Phi(S)$ is the probability of moving from S to $\bar{S}$ in one step if we are in the stationary distribution restricted to S.

If wew started in the distribution $p_{0,x} = \frac{\pi_s}{\pi(S)}$ for x $\in$ S and $p_{0,x} = 0$ for

$x \in \bar{S}$. The expected number of steps before we step into $\bar{S}$ is

$$1\Phi(S) + 2(1-\Phi(S))\Phi(S) + 3(1-\Phi(S))^2\Phi(S) + \cdots = \frac{1}{\Phi(S)}$$

So, the mixing time is lower bounded by $\frac{1}{\Phi(S)}$

normalized conductance of the Markov chain

$$\Phi = \min_{S \subset V, S \neq \{\}} \Phi(S)$$

3 Bound of mixing time

Theorem

The ϵ-mixing time of a random walk on an undirected graph is

$$O(\frac{ln(\frac{1}{\pi_{min}})}{\Phi^2\epsilon^3})$$

Proof: Let $t = \frac{cln(\frac{1}{\pi_{min}})}{\Phi^2\epsilon^3}$, for a suitable constant c. Let

$$a = a(t) = \frac{1}{t}(p(0) + p(1) + \cdots + p(t-1))$$

be the running average distribution. We need to show that $||a - \pi||_1 \leq \epsilon$. Let

$$v_i = \frac{a_i}{\pi_i}$$

and renumber states so that $v_1 \geq v_2 \geq \cdots$.

We call a state i for which $v_i > 1$. heavy. Let i_0 be the maximum i such that $v_i > 1$.

$$||a - \pi||_1 = 2\sum_{i=1}^{i_0}(v_i - 1)\pi_i = 2\sum_{i \geq i_0+1}(1 - v_i)\pi_i$$

Let

$$\gamma_i = \pi_1 + \pi_2 + \cdots + \pi_i$$

Define a function f(x) $= v_i - 1$ for $x \in [\gamma_{i-1}, \gamma_i)$. Now

$$\sum_{i=1}^{i_0} (v_i - 1)\pi_i = \int_0^{\gamma_{i0}} f(x)dx$$

We divide 1,2,$\cdots, i_0$ into groups and let u_t be the maximum value in each group. Define a new function g(x) $= u(t) - 1$. So we have

$$\int_0^{\gamma_{i0}} f(x)dx \leq \int_0^{\gamma_{i0}} g(x)dx$$

Assert that,

$$\int_0^{\gamma_{i0}} g(x)dx = \sum_{t=1}^{r} \pi(G_1 \cup G_2 \cup \cdots \cup G_t)(u_t - u_{t+1})$$

We focus on proving:

$$\sum_{t=1}^{r} \pi(G_1 \cup G_2 \cup \cdots \cup G_t)(u_t - u_{t+1}) \leq \frac{\epsilon}{2}$$

through the 'probability flow' idea. We observe that for any subset A of heavy nodes,

$$Min(\pi(A), \pi(\bar{A})) \geq \frac{\epsilon}{2}\pi(A)$$

We define $G(1) = 1$, and the last element of $G(t)$ is the largest integer greater than or equal to k and at most i_0 so that

$$\sum_{j=k+1}^{l} \pi_j \leq \frac{\epsilon \Phi \gamma_k}{4}$$

As we have(By the following lemma)

$$\sum_{t=1}^{r} \pi(G_1 \cup G_2 \cup \cdots \cup G_t)(u_t - u_{t+1}) \leq \frac{8}{t\Phi\epsilon}$$

By definition of l, we have $\gamma_{l+1} \geq (1 + \frac{\epsilon\Phi}{2})\gamma_k$. So, $r \leq ln_{1+(\epsilon\Phi/2)}(\frac{1}{\pi_1}) + 2 \leq ln(\frac{1}{\pi_1})/(\epsilon\Phi/2) + 2$

Lemma

$$\sum_{t=1}^{r} \pi(G_1 \cup G_2 \cup \cdots \cup G_t)(u_t - u_{t+1}) \leq \frac{8}{t\Phi\epsilon}$$

Proof: We calculate the probability flow from heavy states to light states.

$$net - flow(i,j) = flow(i,j) - flow(j,i) = \pi_i p_{ij} v_i - \pi_j p_{ij} v_j = \pi_j p_{ji}(v_i - v_j)$$

Thus, for any two states i and j, with i heavier than j, i.e., i < j, there is a non-negative net flow from i to j. Since for $i \leq k$ and $j > l$, we have $v_i \geq v_k$ and $v_j \leq v_{l+1}$, the net loss from A, which is A - AP, is at least

$$\sum \pi_j p(ji)(v_i - v_j) \geq (v_k - v_(l+1)) \sum \pi_j p_{ji}$$

Thus,

$$(v_k - v_{l+1} \sum \pi_i p_{ji} \leq \frac{2}{t}) \tag{1}$$

Since

$$\sum_{i=1}^{k} \sum_{j=k+1}^{l} \pi_i p_{ij} \leq \sum_{j=k+1}^{l} \pi_j \leq \epsilon \Phi \pi(A/4)A$$

And by the definition of Φ

$$\sum_{i \leq k < j} \pi_j p_{ji} \geq \Phi min(\pi(A), \pi(\bar{A})) \geq \epsilon \Phi \frac{\gamma_k}{2}$$

We have $\sum \pi_j p_{ij} = \sum_{i \leq k < j} \pi_j p_{ji} = \sum_{i \leq k; j \leq l} \pi_j p_{ji} \geq \epsilon \Phi \frac{\gamma_k}{4}$. Subsititute this into the (1), we have

$$v_k - v_{l+1} \leq \frac{8}{t \epsilon \Phi \gamma_k}$$