Extension for Ergodicity Theorem

Zhanghao Wu
516030910593

Ergodicity Theorem

If P is irreducible and aperiodic, then there is a unique stationary distribution π such that

$$\forall x, lim_{t\to\infty} P^t(x,\cdot) = \pi$$

Thoughts This theorem can be proved by using coupling, which is shown in another note by Boyu Tian. Now we introduce a extension of the this theorem in which we will prove the long-term average probability distribution $a^t(x,\cdot) = \frac{1}{t}(P^0(x,\cdot) + P^1(x,\cdot) + \cdots + P^{t-1}(x,\cdot))$ will also converge to π.

Extension

Let $a^t(x,\cdot)$ be the long term average probability distribution,

$$a^t(x,\cdot) = \frac{1}{t}(P^0(x,\cdot) + P^1(x,\cdot) + \cdots + P^{t-1}(x,\cdot))$$

For a irreducible and aperiodic markov chain, there is a unique stationary distribution π subject to

$$\forall x, lim_{t\to\infty} a(x,\cdot) = \pi$$

Proof 1:

Lemma Let P be the transition probability matrix for a connected Markov chain. The $n \times (n+1)$ matrix $A = [P - I, 1]$ obtained by augmenting the matrix $P - I$ with an additional column of ones has rank n.

Proof: We prove the lemma by contradiction. We assume that the rank of $rank(A) < n$. Then the subspace of solutions to $Ax = 0$ is at least $n+1-(n-1) \geq 2$ dimensions. Each row in P sums to one, so each row in $P - I$ sums to 0. Thus $x_0 = (\mathbf{1}, 0)$ whose entries are all 1 but the last one is 0, is a solution to $Ax = 0$. Assume there was a second solution (x, α) different with $(\mathbf{1}, 0)$. Then $(P-I)x+\alpha 1 = 0$ and $\forall i, x_i = \sum_j p_{ij}x_j+\alpha$. Let S_{max} be the set of $i = argmax_i x_i$. As x differ from $\mathbf{1}$, $\bar{S} \neq \varnothing$. Irreducible implies that for some $k \in S$, x_k is adjacent to some xl of lower value. Thus, $x_k > \sum_j p_{kj}x_j \Rightarrow \alpha > 0$.
On the other hand, for the set S_{min} of $i = argmin_i x_i$, we can prove that $\alpha < 0$. A contradiction!

Proof for theorem: Similar to the lemma above, let $A = [P-I, \mathbf{1}]$. And let $B = A(:, 2 : end)$ (a notation from matlab, which means that construct a matrix consisting the second column to the last of A). As the sum of each row of $P - I$ is 0, the first n column must be relavant. According to the lemma, $rank(A) = n$, and the first n column of A is relavant, so $rank(B) = n$.
Consider,

$$\begin{aligned}
a^t(x,\cdot)P - a^t(x,\cdot) &= \frac{1}{t}(P^0(x,\cdot)P + P^1(x,\cdot)P + \cdots + P^{t-1}(x,\cdot)P) \\
&- \frac{1}{t}(P^0(x,\cdot) + P^1(x,\cdot) + \cdots + P^{t-1}(x,\cdot)) \\
&= \frac{1}{t}(P^t(x,\cdot) - P^0(x,\cdot))
\end{aligned}$$

Let $b^t(x,\cdot) = a^t(x,\cdot)P - a^t(x,\cdot)$. We have that $|b^t(x,\cdot)| \leq \frac{2}{t} \to 0$, as $t \to \infty$. Let $c^t(x,\cdot)$ be a vector containing all of the entries of $b^t(x,\cdot)$ except the first entry. Therefore,

$$\begin{aligned}
a^t(x,\cdot)B &= [a^t(x,\cdot)P - a^t(x,\cdot), 1](:, 2 : end) \\
&= [b^t(x,\cdot), 1](:, 2 : end) \\
&= [c^t(x,\cdot), 1] \to [\mathbf{0}, 1]
\end{aligned}$$

As proved before, B is invertible. Therefore,

$$a^t(x,\cdot) \to [\mathbf{0}, 1]B^{-1}, \text{ as } t \to \infty$$

Because, $a^t(x,\cdot)P - a^t(x,\cdot) = b^t(x,\cdot) \to 0$, we have,

$$[\mathbf{0}, 1]B^{-1}P = [\mathbf{0}, 1]B^{-1}, \text{ i.e. } \pi = [\mathbf{0}, 1]B^{-1}$$

Remark: Using the lemma above, we can give another proof for the uniqueness of π, $\pi P = \pi$, for a irreducible Markov chain transition matrix P. In fact, according to the lemma, $A = [P - I, 1]$ has rank n, and as each column of $P - I$ sums to $\mathbf{0}$, we have that $rank(P - I) = n - 1$. Therefore, $v = vP$ has a one-dimensional space of solutions. This space only contains one vector whose entries sum to 1.

Proof 2:

[1] Here is another proof for the *Extension*. Let $x_t = a^t(x, \cdot)$. Similar to the former proof, we can prove that,

$$\|x_t(I - P)\| = \frac{\|x(I - P^t)\|}{t} \leq \frac{2}{t}, \forall x$$

so any subsequential limit point π of the sequence $\{x_t\}_{t=1}^{\infty}$ satisfies $\pi P = \pi$. Since π satisfies $\pi P = \pi$ for any non-negative integer t, i.e. $\pi_y = \sum_x \pi_x P_{xy}^t$. Thus, if $\pi_x > 0$ and $P_{xy}^t > 0$, then $\pi_y > 0$. As P is irreducible and there exists x with $\pi_x > 0$, then $\forall y$ satisfy $\pi_y > 0$. One such x exists because $\sum_x \pi_x = 1$.
Now, we prove the sequence $\{x_t\}$ converges.
If $z = y(I - P)$ satisfies $z = zP$, then $z = a^t(z, \cdot) = \frac{1}{t} y(I - P^t)$ must satisfy $\|z\| \leq \frac{2\|y\|}{t}, \forall t$. Therefore, $z = 0$. Since the dimensions of $Im(I - P)$ and $Ker(I - P)$ add up to n, it follows that any vector $v \in \mathbb{R}^n$ has a unique representation

$$x = u + w, \text{with } u \in Im(I - P) \text{ and } w \in Ker(I - P)$$

Therefore, $x_t = a^t(x, \cdot) = a^t(u, \cdot) + w$, so writing $u = y(I - P)$ we conclude that $\|x_t - \pi\| \leq \frac{2\|y\|}{t}$. If x is a distribution, then also w is a distribution due to w being the limit of x_t. Thus we can take $\pi = w$

[1]This proof is shown in *Markov Chains and Mixing Times, second edition-David.A.Levin Yuval Peres*