

A Solution for a question in Class 7

Zhanghao Wu

516030910593

1 Problem $h \in \mathbb{R}^{2^{[k]}}$, is an entropy function provided there are RVs, X_1, X_2, \dots, X_k , s.t. $h(A) = H(\{X_i\}_{i \in A})$, prove that h satisfies the properties of polymatroid:

1. $h(\emptyset) = 0$
2. $h(A) \leq h(B), A \subseteq B$
3. $h(A \cup B) + h(A \cap B) \leq h(A) + h(B)$

Solution

1. $h(\emptyset) = H(\emptyset) = 0$
2. As $A \subset B$, without loss of generality, let $A = \{1, \dots, s\}, B = \{1, \dots, s+t\}$

Then

$$\begin{aligned} h(A) &= H(X_1, \dots, X_s) \\ &= - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log p(x_1, \dots, x_s) \\ &= - \sum_{x_1, \dots, x_s} \left(\sum_{x_{s+1}, \dots, x_{s+t}} p(x_1, \dots, x_{s+t}) \right) \log \left(\sum_{x_{s+1}, \dots, x_{s+t}} p(x_1, \dots, x_{s+t}) \right) \\ &\leq - \sum_{x_1, \dots, x_s} \left(\sum_{x_{s+1}, \dots, x_{s+t}} p(x_1, \dots, x_{s+t}) \log p(x_1, \dots, x_{s+t}) \right) \\ &= - \sum_{x_1, \dots, x_{s+t}} p(x_1, \dots, x_{s+t}) \log p(x_1, \dots, x_{s+t}) \\ &= h(B) \end{aligned}$$

3. Without loss of generality, let $A \cup B = \{1, \dots, s\}, A \cap B = \{1, \dots, t\}$, and $A = \{1, \dots, m\}, B = \{1, \dots, t, m+1, \dots, s\}$, where $s \geq m \geq t$

According to the definition, we can have that:

$$h(A \cup B) = - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log p(x_1, \dots, x_s)$$

$$\begin{aligned}
h(A \cap B) &= - \sum_{x_1, \dots, x_t} p(x_1, \dots, x_t) \log p(x_1, \dots, x_t) \\
&= - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log p(x_1, \dots, x_t)
\end{aligned}$$

$$\begin{aligned}
h(A) &= - \sum_{x_1, \dots, x_m} p(x_1, \dots, x_m) \log p(x_1, \dots, x_m) \\
&= - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log p(x_1, \dots, x_m)
\end{aligned}$$

$$\begin{aligned}
h(B) &= - \sum_{x_1, \dots, x_t, x_{m+1}, \dots, x_s} p(x_1, \dots, x_t, x_{m+1}, \dots, x_s) \log p(x_1, \dots, x_t, x_{m+1}, \dots, x_s) \\
&= - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log p(x_1, \dots, x_t, x_{m+1}, \dots, x_s)
\end{aligned}$$

In order to prove $h(A \cup B) + h(A \cap B) \leq h(A) + h(B)$

$$\begin{aligned}
&\Leftrightarrow - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log(p(x_1, \dots, x_s)p(x_1, \dots, x_t)) \\
&\leq - \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log(p(x_1, \dots, x_m)p(x_1, \dots, x_t, x_{m+1}, \dots, x_s)) \\
&\Leftrightarrow \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log \frac{p(x_1, \dots, x_s)p(x_1, \dots, x_t)}{p(x_1, \dots, x_m)p(x_1, \dots, x_t, x_{m+1}, \dots, x_s)} \geq 0, \quad (1)
\end{aligned}$$

$$\text{Numerator} = p(x_1, \dots, x_m)p(x_{m+1}, \dots, x_s|x_1, \dots, x_m)p(x_1, \dots, x_t)$$

$$\text{Domain} = p(x_1, \dots, x_m)p(x_1, \dots, x_t)p(x_{m+1}, \dots, x_s|x_1, \dots, x_t)$$

Therefore, (1) equivalent to,

$$\sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log \frac{p(x_{m+1}, \dots, x_s|x_1, \dots, x_m)}{p(x_{m+1}, \dots, x_s|x_1, \dots, x_t)} \geq 0$$

For the left hand side we have,

$$\begin{aligned}
\text{L.H.S} &= \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log \frac{p(x_1, \dots, x_s)/p(x_1, \dots, x_m)}{p(x_{m+1}, \dots, x_s|x_1, \dots, x_t)} \\
&\geq \sum_{x_1, \dots, x_s} p(x_1, \dots, x_s) \log \frac{p(x_1, \dots, x_s)}{p(x_{m+1}, \dots, x_s|x_1, \dots, x_t)}
\end{aligned}$$

Assume that P, Q are two distributions, $P = p(X_1, \dots, X_s)$, $Q = p(X_{m+1}, \dots, X_s|X_1, \dots, X_t)$, then

$$\text{L.H.S} \geq \sum_{x_1, \dots, x_s} P \log \frac{P}{Q} = D_{KL}(P||Q) \geq 0 = \text{R.H.S}$$

Therefore, we proved the solution.