A Proof for Weierstrass Approximation Theorem

Xiao Yunxuan

March 23^{rd}, 2018

1 Weierstrass Approximation Theorem

Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial p such that for all x in $[a, b]$, we have $\sup_{a \le x \le b} |f(x) - p(x)| < \epsilon$.

Proof:

To Simplify the problem, we consider continuous function f is defined on interval $[0, 1]$.It's easy to transform $[0, 1]$ to $[a, b]$ with linear projection.

Firstly, we define a sequence of independent Bernoulli trials $X_1, X_2, \cdots, X_n$ with probability x of success and variance $\sigma^2 = \sqrt{x(1-x)} \le \frac{1}{2}$ on each trial.

S_n denotes the number of successes in n trials.

$$S_n = X_1 + X_2 + \cdots + X_n$$

Also, we define a random variable F_n in terms of trail sequence $\{X_i\}$, where

$$F_n = f\left(\frac{S_n}{n}\right)$$

When variable x is fixed, the expectation of F_n is identical to **Bernstein polynomial**

$$E[F_n] = E\left[f\left(\frac{S_n}{n}\right)\right] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} = B_{n,f}(x)$$

Since f is continuous, it is bounded on $[0, 1]$.Therefore there exists a finite positive number M such that $|f(x)| \le M/2$ on $[0, 1]$, which implies that $|f(x) - f(y)| \le M$.

Furthermore, the continuity of f on $[0, 1]$ indicates that for $\forall \epsilon > 0$, there exists $\eta > 0$ such that if $|x - y| < \eta$, then $|f(x) - f(y)| < \epsilon/2$.

$$\begin{aligned}
|B_{n,f}(x) - f(x)| &= \left| E\left[f\left(\frac{S_n}{n}\right)\right] - f(x) \right| \\
&= \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| P(S_n = k) \\
&= \sum_{k:|\frac{k}{n} - x| < \eta} \left| f\left(\frac{k}{n}\right) - f(x) \right| P(S_n = k) + \sum_{k:|\frac{k}{n} - x| \ge \eta} \left| f\left(\frac{k}{n}\right) - f(x) \right| P(S_n = k) \\
&< \frac{\epsilon}{2} \cdot \sum_{k:|\frac{k}{n} - x| < \eta} P(S_n = k) + M \cdot \sum_{k:|\frac{k}{n} - x| \ge \eta} P(S_n = k)
\end{aligned}$$

The first summand is less than $\frac{\epsilon}{2}$. As for the second summand, we apply The Weak Law of Large Number and yield:

$$M \cdot \sum_{|\frac{k}{n}-x|>\eta} P(S_n = k) = M \cdot P\left(|\frac{S_n}{n} - x| \geq \eta\right) \leq M \cdot \frac{\sigma^2}{n\eta^2} \leq \frac{M}{2n\eta^2}$$

When $n > [M/\epsilon\eta^2]$, the expression above is less than $\epsilon/2$.

Therefore, for any $x \in [0,1]$ and $n > [M/\epsilon\eta^2]$, the inequality $\sup\limits_{x\in[0,1]} |f(x) - B_{n,f}| < \epsilon$ holds. And thus

$$\lim_{n\to\infty} B_{n,f} = f$$

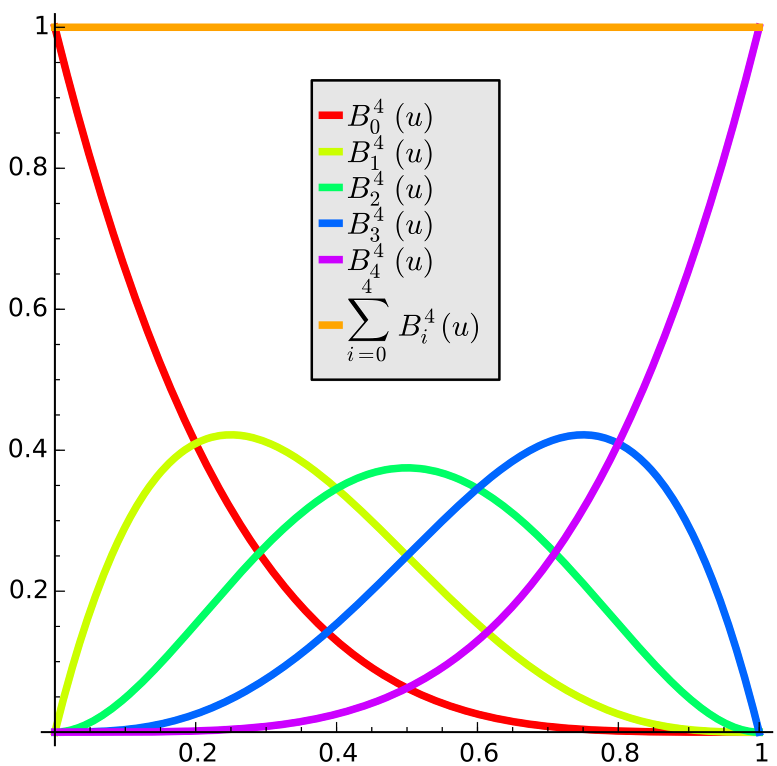

图 1: Bernstein Base Polynomials