Class 9

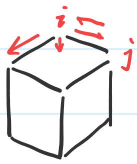
2018年4月10日 星期二 10:04

* 继 MC: ← MC可理解为图上的概率随机游走 lumping

Ehrenfest urn:

$X_t \mid Y_t$ $\quad X_t + Y_t = n$ $\quad \begin{cases} Y_{t+1} = Y_t \pm 1 \\ X_{t+1} = X_t \pm 1 \end{cases}$

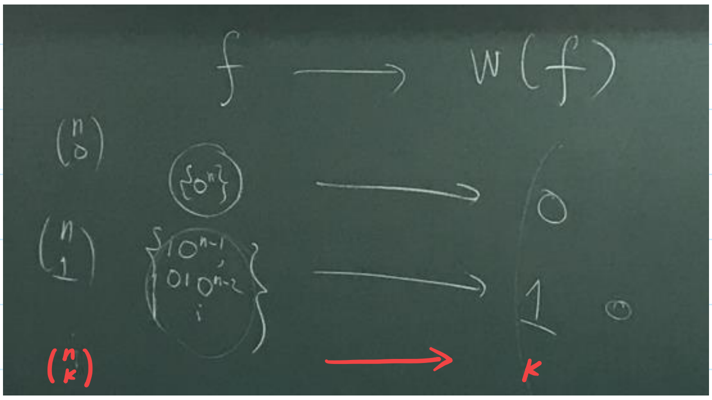
Reversible MC:



顶点集合: V , $\pi \in \mathbb{R}^V$

$i \to j$ 转移概率: P_{ij}

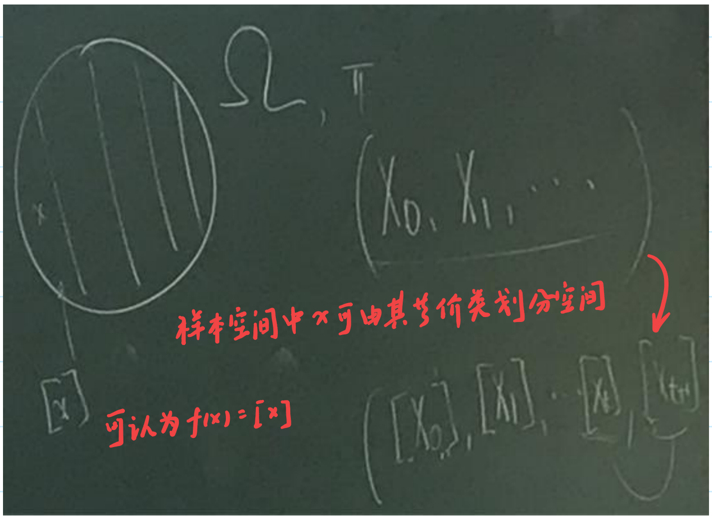
若 $\pi_i P_{ij} = \pi_j P_{ji}$ 为平稳分布



$\left.\begin{array}{l} X_i: S \to V \\ f: V \to W \end{array}\right\} \begin{array}{l} f(X_i) \\ S \to W \end{array}$

X_i为MC，给定f，$f(X_i)$是否为MC？

不一定是，给出反例以及为MC时所需条件



使$[X_{t+1}]$只与$[X_t]$相关

$P(x,[y]) = P(x',[y'])$ whenever $[x]=[x']$ $\quad \Omega^{\#} = \{[x] : x \in \Omega\}$

$\Rightarrow$ $P^{\#}([x],[y]) = P(x,[y])$ 若满足该条件，$f(x)$仍为MC

* $S = \{s_1, \dots, s_k\}$ 有限状态空间

$s_i \to s_j$: $\exists n, (P^n)_{ij} > 0$ i有概率多次转移后到达j

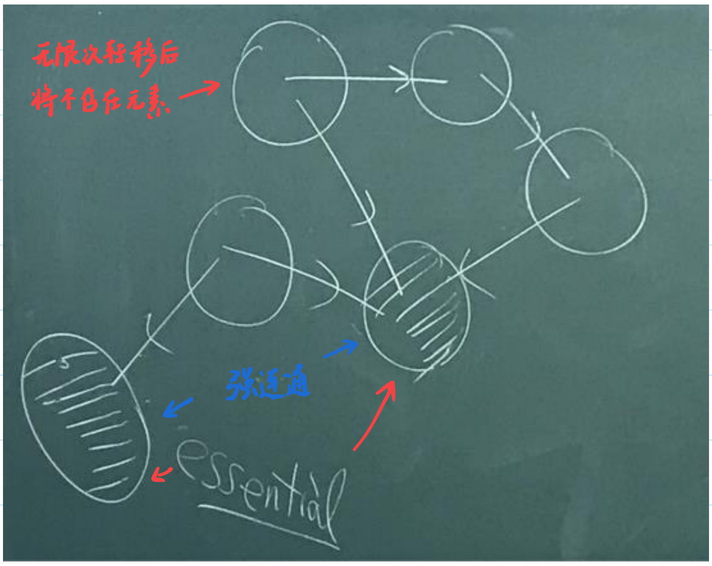
$s_i \to s_j$: $\exists\, n \cdot (P^n)_{ij} > 0$ i有概率多次转移后到达j

* 强连通分量：

A MC is irreducible if for all $s_i, s_j \in S$ $|S| = n$, we have $s_i \to s_j$

$\Rightarrow I + P + P^2 + \cdots > 0$

$I + P + P^2 + \cdots + P^{n-1} > 0$



* 定理：

For any finite state irreducible Markov chain, there exists at least one stationary distribution → $\pi P = \pi$, $\pi \geq 0$

* 若 P 不可约：$\pi_0 P_0 = \pi_0$

$$(0\ 0\ (\pi_0)) \begin{pmatrix} * & \\ 0 & \boxed{P_0} \end{pmatrix} = (0\ 0 \cdots 0\ \pi_0 P_0) = (0\ \pi_0)$$ ← 矩阵论

* $\Delta_{n-1} \circlearrowleft P$ ← 不动点定理

$f \in \Delta_{n-1}$, $f \to fP$

pf: $(X_0, X_1, X_2, \cdots)$ 时间转移矩阵

For $x \in \Omega$, hitting time for x, $\tau_x := \min\{t \geq 0 ; X_t = x\}$

$\tau_x^+ := \min\{t \geq 1 ; X_t = x\}$

Lem: For any states x, y of an irreducible chain,

$$E_x(\tau_y^+) < \infty$$

圣彼德堡悖论

pf: $E_x(\tau_y^+) = \sum_{t=1}^{\infty} t \cdot P_x(\tau_y^+ = t)$ 太过困难 ⇝ $P(X = 2^n) = (\frac{1}{2})^{n+1}$ $\begin{cases} P(X < \infty) = 1 \\ EX = \infty \end{cases}$

$P_x(\tau_y^+ < \infty) = 1$ 弱化条件

There $\exists\ \varepsilon>0$ & $r>0$ s.t.: For $\forall z,w\in\Omega\ \exists j\le r$ s.t. $P^{j}(z,w)>\varepsilon$ ← 由强连通性质易理解

$P_x\{\tau_y^+>kr\}<(1-\varepsilon)P_x\{\tau_y^+>(k-1)r\}\Rightarrow P_x\{\tau_y^+>kr\}<(1-\varepsilon)^k$

$E_x(\tau_y^+)=\sum_{t=1}^{\infty}t\cdot P_x(\tau_y^+=t)=\sum_{t\ge0}P_x\{\tau_y^+>t\}$ ← 期望面积换维度计算

$\le\sum_{k\ge0}rP_x\{\tau_y^+>kr\}<r\sum_{k\ge0}(1-\varepsilon)^k=\frac{r}{\varepsilon}<\infty$

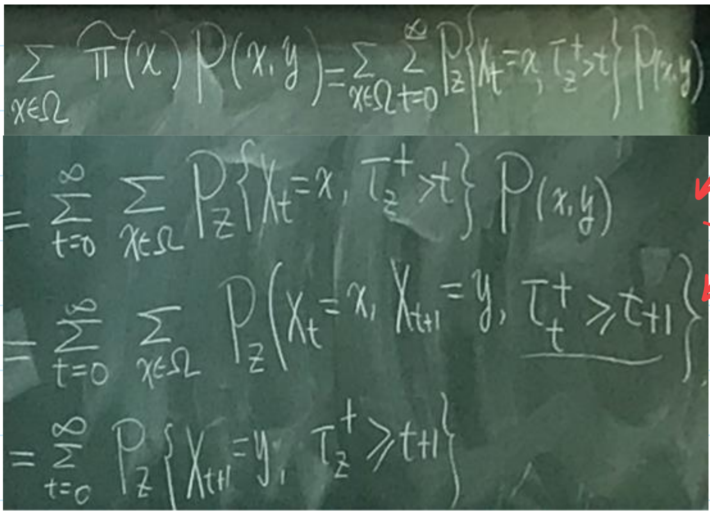
不可约: $\dim Ker(P-I)=1$, $Ker(P-I)=\{f: Pf=f\}$

$\left.\begin{array}{l}\sum P(i,j)f(j)=f(i)\\ f(i)\ \text{max}\end{array}\right\}\Rightarrow$ 所有值均相等, f为一维

下找该唯一存在: $\pi(x)=\frac{1}{E_x(\tau_x^+)}$ $\sum_{x\in\Omega}\pi(x)=1$

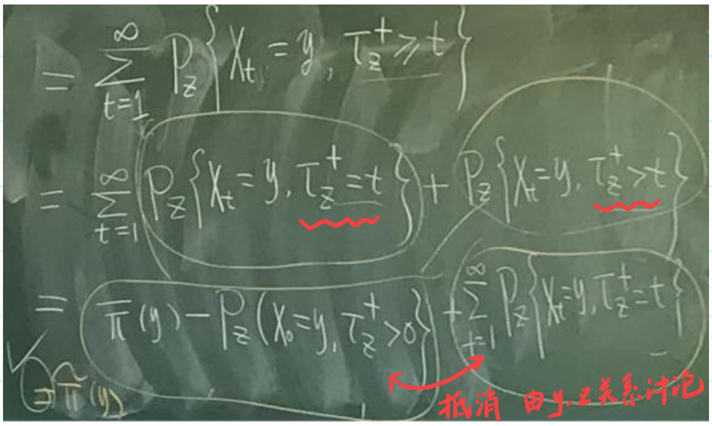
Pick $z\in\Omega$, $\tilde{\pi}(y)\overset{def}{=}E_z($ z ↺ 过程中访问y次数$)\le E_z\tau_z^+$

Claim: $\tilde{\pi}P=\tilde{\pi}$



Σ交换次序

double-counting



$$\frac{\tilde{\pi}^{(z)}}{E_z(\tau_z^+)} = \pi$$ → z=x时为上述