A Problem in Class 19 and Interesting Story behind

Yuwei Wu

1 Problem

Show that, for an ergodic Markov chain:

$$\sum_j m_{ij} w_j = \sum_j z_{jj} - 1$$

Solution

In class, we already have

$$m_{ij} = \frac{z_{jj} - z_{ij}}{w_j}$$

Then we get:

$$m_{ij} w_j = z_{jj} - z_{ij}$$

Sum the both sides:

$$\sum_j m_{ij} w_j = \sum_j z_{jj} - \sum_j z_{ij}$$

For a column vector of all 1' s c,we have:

$$c = (I - P + W)\, c$$

That is,

$$Zc = c$$

Then we have:

$$\sum_j z_{ij} = 1$$

Thus,

$$\sum_j m_{ij} w_j = \sum_j z_{jj} - 1$$

2 Insteresting Facts

From the equation above

$$\sum_j m_{ij} w_j = \sum_j z_{jj} - 1 = K$$

where the number K is independent of i.

That is the expected number of time steps required for a Markov chain to transition from a starting state i to a random destination state sampled from the Markov chain's stationary distribution does not depend on which starting state i is chosen.

The number K is called **Kemeny' s constant**. A prize was offered to the first person to give an intuitively plausible reason for the above sum to be independent of i.

In the course of a walk with Snell along Minnehaha Avenue in Minneapolis in the fall of 1983, Peter Doyle suggested the following explanation for the constancy of Kemeny's constant.

We are given an ergodic chain and do not know the starting state. However, we would like to start watching it at a time when it can be considered to be in equilibrium (i.e., as if we had started with the fixed vector w or as if we had waited a long time). However, we don' t know the starting state and we don' t want to wait a long time. Peter says to choose a state according to the fixed vector w. That is, choose state j with probability w_j using a spinner, for example. Then wait until the time T that this state occurs for the first time. We consider T as our starting time and observe the chain from this time on. Of course the probability that we start in state j is w_j , so we are starting in equilibrium. Kemeny' s constant is the expected value of T, and it is independent of the way in which the chain was started.

In other words, let's let $K_i = \sum_j P_j m_{ij} w_j$. Peter proposed to interpret K_i as what we could call the **expected time to equilibrium**: We pick a state j at random according to the equilibrium probability distribution w, and ask for the expected time to get to j from the starting state i.

So far we've seen nothing approaching an argument for why the expected time to equilibrium should be independent of the starting state i, so it's hard to see why Peter should have been given the prize.

Now after taking one step from i, the expected time to equilibrium will be $\sum_k P_{ik} K_k$, so the expected time K_i to equilibrium starting at i would seem to be $1 + \sum_k P_{ik} K_k$: That' s 1 for the first step, plus $\sum_k P_{ik} K_k$ to get to equilibrium from wherever we are after the first step. However, maybe **we were already in equilibrium when we started**! If the randomly selected target state j happened to coincide with the starting state i, then that first step was a mistake. The probability of making this mistake is w_i, and if we make it, the expected extra time it will cost is the mean recurrence time $r_i = \frac{1}{w_i}$. The expected cost attributable to the possibility of abandoning equilibrium is thus $w_i r_i = w_i \frac{1}{w_i} = 1$, which just cancels the benefit of taking one step of our journey:

$$K_i = 1 + \sum_k P_{ik} K_k - w_i r_i = \sum_k P_{ik} K_k$$

This equation says that the quantities K_i have **the averaging property**: The value at any state is the same as the average value at its neighbors, where the average is taken according to the matrix P. This means that the numbers K_i must all be the same, by **the maximum principle**.

To recapitulate, the expected time to equilbrium has the averaging property, because moving to a neighbor takes you one step closer to equilibrium—except if you were already in equilibrium, which happens with probability w_i and costs you $ri = \frac{1}{w_i}$. Since the expected time to equilibrium has the averaging property, it is constant.

The argument of Peter makes the constancy of K_i not just intuitively plausible, but intuitively obvious.

Sadly, the prize for explaining Kemeny' s constant has already been paid out. If you look carefully at the previous statement, you will observe that it states that 'a prize was offered' (emphasis added), rather than 'a prize is offered' . . . .

Another interesting fact is that When Laurie Snell mailed Peter Doyle the prize for Kemeny' s constant, he first made the mistake of trying to send a $50 bill by mail. That first letter never arrived!

Reference

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3. "Two exercises on Kemeny's constant" Retrieved 1 March 2013.
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