

Class 15

2018年5月15日 星期二 10:02

{ generating function X.r.v. $g_X(z) = E(z^X)$
Laplace transform
moment generating function
characteristic function (Fourier transform)

* 母函数 互相唯一-决定

$$(a_n)_{n=0}^{\infty} \rightarrow \sum_{n=0}^{\infty} a_n \cdot z^n = g(z)$$

$$a_n = P(X=n) \begin{cases} E(X) = \sum n \cdot a_n = g'(1) \rightarrow E(X^2) - E(X) = g''(1) \\ \text{Var}(X) = E(X^2) - (E(X))^2 = g''(1) - g'(1)^2 + g'(1) \end{cases} \Rightarrow a_j = \frac{1}{j!} g^{(j)}(0)$$

* 唯一性定理:

$$z=0 \quad \sum_{j=1}^{\infty} a_j z^j = \sum_{j=1}^{\infty} b_j z^j \quad \text{不同概率分布对应不同母函数}$$

$$z=0 \rightarrow a_0 = b_0 \rightarrow /z \rightarrow z=0 \rightarrow a_1 = b_1 \dots$$

* 卷积:

$$(a_j) * (b_k) \Rightarrow (c_l) \quad \star \text{当 } a, b \text{ 独立且 } c = a+b, \quad g_a(z) \cdot g_b(z) = g_c(z)$$
$$c_l = \sum_{j=0}^l a_j b_{l-j}$$

* eg 1:

$$P(T=j) = P_j = q^{j-1} p, \quad j \geq 1$$

$$g(z) = \sum_{j=1}^{\infty} P_j \cdot z^j = \frac{p}{q} \sum_{j=1}^{\infty} (qz)^j = \frac{pz}{1-qz}$$

抛硬币第一次看到n次向上:

$$S_n = T_1 + T_2 + \dots + T_n$$

$$g_{S_n}(z) = (g_T(z))^n = \left(\frac{pz}{1-qz} \right)^n$$

$$= \sum_{k=n}^{\infty} \binom{k-1}{n-1} p^n q^{k-n} z^k$$

$$P(S_n = n+j) = \binom{n+j-1}{j} p^n q^j = \binom{n}{j} p^n (-q)^j$$

* Laplace 变换:

$$X: r.v. \rightarrow E(e^{-\lambda X}), \lambda \in [0, \infty)$$

$$\int_{-\infty}^{\infty} e^{-\lambda u} f(u) du, \sum_{j=0}^{\infty} a_j e^{-j\lambda}$$

$$E(z^{X_1+X_2}) = E(z^{X_1}) \cdot E(z^{X_2})$$

* 局母函数: (见课本)

$X: r.v.$

$$M_X(t) = E(e^{tx}) \xrightarrow{\text{Taylor 展开}} E\left(1 + tx + \frac{t^2 x^2}{2!} + \dots\right) \xrightarrow{\text{期望线性性}} 1 + tE_x + \frac{t^2 E_{x^2}}{2!} \dots$$

唯一性
独立性质

* 特征函数:

$$\varphi_X(\theta) = E(e^{i\theta x})$$

	$M_X(t)$	$\varphi(t)$
Bernoulli	$1-p+pe^t$	$1-p+pe^{it}$
Geometric	$\frac{pe^t}{1-(1-p)e^t}$ ($t < -\ln(1-p)$)	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson(λ)	$e^{\lambda(e^t-1)}$	$e^{\lambda(e^{it}-1)}$
exponential	$(1-t\lambda)^{-1}$ ($t < 1$)	$(1-it\lambda)^{-1}$
Normal $N(\mu, \sigma^2)$	$e^{t\mu + \frac{1}{2}\sigma^2 t^2}$	$e^{it\mu - \frac{1}{2}\sigma^2 t^2}$

* P226 - 42, P227 - 43 可能会考!

* $X \sim N(0, 1)$

$$\begin{aligned} M_X(\theta) &= \int_{-\infty}^{\infty} e^{\theta x} p(x) dx, \quad p(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\theta x - \frac{x^2}{2}\right) dx \\ &= e^{\frac{\theta^2}{2}} \int_{-\infty}^{\infty} p(x - \theta) dx = e^{\frac{\theta^2}{2}} \end{aligned}$$

$$1 + \frac{\theta^2}{2} + \frac{1}{2!} \left(\frac{\theta^2}{2}\right)^2 + \frac{1}{3!} \left(\frac{\theta^2}{2}\right)^3 + \dots = \sum_{n=0}^{\infty} \frac{\theta^{2n}}{n!} \quad (M_n)$$

$$\begin{cases} M_{2n+1} = 0 \\ M_{2n} = \frac{(2n)!}{n! 2^n} \end{cases} \quad \frac{1}{n! 2^n} = \frac{1}{2n!} \quad (M_{2n})$$

(Independent) $\begin{cases} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{cases} \Rightarrow X_1 + X_2 \sim N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right)$

Pf: $M_{X_1+X_2}(\theta) = M_{X_1}(\theta) M_{X_2}(\theta)$

$$= e^{\mu_1 \theta + \frac{\sigma_1^2 \theta^2}{2}} e^{\mu_2 \theta + \frac{\sigma_2^2 \theta^2}{2}} = e^{(\mu_1 + \mu_2) \theta + \frac{(\sigma_1^2 + \sigma_2^2) \theta^2}{2}} = M_X(\theta)$$

* **Central Limit Theorem:** P.37

$$X_1, X_2, \dots \quad i.i.d \quad \text{独立同分布} \quad X_i \sim X \quad \begin{cases} E X = \mu \\ \text{Var } X = \sigma^2 \end{cases}$$

$$Z_n = \frac{X_1 + \dots + X_n - n\mu}{\sqrt{n}\sigma}$$

$$EX = \mu$$

$$\text{Var} X = \sigma^2$$

$$\lim_{n \rightarrow \infty} P(a \leq Z_n \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = F(b) - F(a)$$

$$\varphi_{X-\mu}(0) = \varphi(0)$$

$$\varphi_{Z_n}(0) = \prod_{j=1}^n \varphi\left(\frac{0}{\sigma\sqrt{n}}\right) = \left(\varphi\left(\frac{0}{\sigma\sqrt{n}}\right)\right)^n = e^{n \log\left(1 - \frac{\theta}{2n} + \frac{\theta^2}{n\sigma^2} h\left(\frac{\theta}{\sigma\sqrt{n}}\right)\right)}$$

*

Levy's Continuity Theorem

$$\begin{cases} \varphi'(\theta) = i E((X-\mu) e^{i\theta(X-\mu)}) \\ \varphi''(\theta) = -E((X-\mu)^2 e^{i\theta(X-\mu)}) \end{cases}$$

$$\varphi(\theta) = 1 + 0 - \frac{\sigma^2 \theta^2}{2} + \theta^2 h(\theta), \quad h(\theta) \rightarrow 0 \text{ when } \theta \rightarrow 0$$

$$\lim_{n \rightarrow \infty} \varphi_{Z_n}(\theta) = e^{-\frac{\theta^2}{2}}$$