Hook Length Formula

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Proof. This is a problem generalize from the exercise 'Probabilistic Chain' in class.
I was trying to find a combinatoric proof but failed several times. I now give a probabilistic proof from [1].
Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ be a partition of a given n, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. A Young diagram is a left-justified array of square cells with m rows and λ_i elements one by one in each row. A standard Young tableau of λ is a Young diagram with all cells put a distinct integer in $[n]$ such that each row and column form increasing sequences. Define hook $h_{i,j}$ to be the number of cells (i', j') satisfying $(i' > i) \cap (j' = j) \cup (j' > j) \cap (i' = i)$. What I want to proof is that the total number of standard Young tableaux is $\frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$ where λ can also represent a Young diagram.
Let's begin!
Define $f_\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} h_{i,j}}$. Let $n = |\lambda|, g_\lambda$ be the total number of standard Young tableaux, then it is clear that n must be put in the corner(no right or down cell). Let $\lambda(\alpha)$ be $\lambda\backslash(\alpha, \lambda_\alpha)$ and $g_{\lambda(\alpha)}$ be 0 if $\lambda(\alpha)$ is not Young diagram. Considering the cell containing n must be a corner, we have

$$g_\lambda = \sum_\alpha g_{\lambda(\alpha)} \Longrightarrow \sum_\alpha \frac{g_{\lambda(\alpha)}}{g_\lambda} = 1$$

Let $f_\emptyset = 1$, then We only need to proof the following equation and by deduction we can proof the statement.

$$\sum_\alpha \frac{f_{\lambda(\alpha)}}{f_\lambda} = 1$$

Further, if $(\alpha, \beta = \lambda_\alpha)$ is a corner, then

$$\frac{f_{\lambda(\alpha)}}{f_\lambda} = \frac{1}{n} \prod_{0<i<\beta} (1 + \frac{1}{h_{\alpha,i} - 1}) \prod_{0<j<\alpha} (1 + \frac{1}{h_{j,\beta} - 1}) \tag{1}$$

Consider a probabilistic game: we randomly choose one cell in λ and then repeat uniformly choosing one next among the hook now until reaching the corner.
Consider one situation path $(a_1, b_1), (a_2, b_2), \cdots, (a_l, b_l) = (\alpha, \beta)$. Let the a projection and b projection be $A = \{a_1, a_2, \cdot, a_l\}$ and $B = \{b_1, b_2, \cdot, b_l\}$ respectively, where A and B are sets, not multi-sets.
I claim that $Prob(A, B|a_1, b_1) = \prod_{b \in B, b != \beta} \frac{1}{h_{\alpha,b} - 1} \prod_{a \in A, a != \alpha} \frac{1}{h_{a,\beta} - 1}$.
By deduction, we have

$$\begin{aligned}
Prob(A, B|a_1, b_1) &= \frac{1}{h_{a_1,b_1} - 1} \left[Prob(A \backslash a_1, B|a_2, b_1) + Prob(A, B \backslash b_1|a_1, b_2)\right] \\
&= \frac{1}{h_{a_1,b_1} - 1} \left[\Pi(h_{a_1,\beta} - 1) + \Pi(h_{\alpha,b_1} - 1)\right] \\
&= \Pi = \prod_{b \in B, b != \beta} \frac{1}{h_{\alpha,b} - 1} \prod_{a \in A, a != \alpha} \frac{1}{h_{a,\beta} - 1}
\end{aligned}$$

Using Law of Total Probability, we have the following, which is exactly (1).

$$\begin{aligned}
Prob(\alpha, \beta) &= \sum_{(a_1,b_1) \in \lambda} Prob(\alpha, \beta|a_1, b_1) \frac{1}{n} \\
&= \frac{1}{n} \sum_{A,B} Prob(A, B|a_1, b_1) \\
&= \frac{1}{n} \sum_{A,B} \prod_{b \in B, b != \beta} \frac{1}{h_{\alpha,b} - 1} \prod_{a \in A, a != \alpha} \frac{1}{h_{a,\beta} - 1} \\
&= \frac{1}{n} \prod_{0 < i < \beta} (1 + \frac{1}{h_{\alpha,i} - 1}) \prod_{0 < j < \alpha} (1 + \frac{1}{h_{j,\beta} - 1})
\end{aligned}$$

The last equation is similar to a Knapsack problem or a generating function problem.
Thus, the theorem is finally proved!

$\square$

References

[1] Herbert S. Wilf Curtis Greene, Albert Nijenhuis. A probabilistic proof of a formula for the number of young tableaux of a given shape. 1917.4.