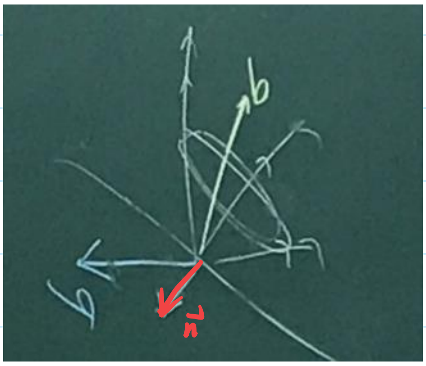
Class 12

2018年4月24日 星期二 10:08

* $xP = x$，齐次 MC $\Rightarrow$ 亡 P^n 不可约有极限的唯一性

$x^T P = x^T \Rightarrow x^T(P-I)=0$

$$\begin{cases} x^T A = b^T & x \geq 0,\ b \in cone(A) \\ Ay \leq 0 & b^T y > 0 \end{cases}$$



当有一个锥区域，向量b或落于其中，或置身其外

在后者发生情况下，一定存在一个超平面将其分开

考虑法向量与其内积.

令 $A = \begin{pmatrix} P-I & \begin{matrix}|\\1\\|\end{matrix} \end{pmatrix}$，$x^T A = (-0-\ 1)$

$\Rightarrow \begin{pmatrix} P-I & \begin{matrix}|\\1\\|\end{matrix} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} \leq 0$，$y_{n+1} > 0$ 用上述锥结论导出

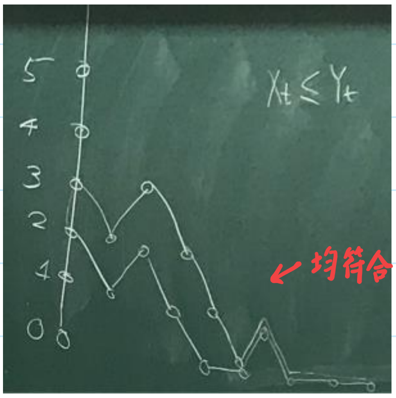
$y_1 = \min y_i$，$\sum P_{1i} \cdot y_i < y_1$ 矛盾

* 带吸收壁的随机游走：

$P(i, i+1) = P(i, i-1) = \frac{1}{2}$，$i = 1, 2, \cdots, n-1$

$$\begin{cases} P(0,1) = P(n, n-1) = 0 \\ P(0,0) = P(n,n) = 1 \end{cases}$$

$P^t(x, n) \leq P^t(y, n)$ $x \leq y$ 直观理解为y大



耦合？

均符合P转移

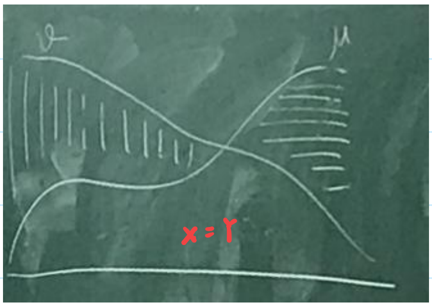
* Total variation distance :

μ, ν probability distribution on Ω

Def-1 $\|\mu-\nu\|_{TV} = \max_{A \in \Omega} |\mu(A)-\nu(A)|$ Def-2 $= \frac{1}{2}\sum_{x\in\Omega}|\mu(x)-\nu(x)|$

范式

$\|f-g\| \le \|f-h\| + \|h-g\|$



落在左侧或右侧,

取全正或全负,

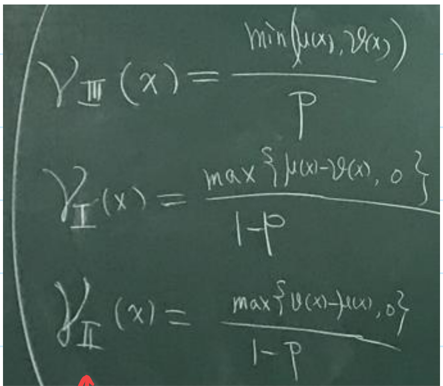
图中两块相等

?

Distance Definition

Def-3 $\|\mu-\nu\|_{TV} = \frac{1}{2}\sup\left\{\sum_{x\in\Omega} f(x)[\mu(x)-\nu(x)] : f\in\mathbb{R}^{\Omega},\ \max|f(x)|\le 1\right\}$ HW

Def-4 $\|\mu-\nu\|_{TV} = \mathrm{Inf}\{P(X\ne Y) : (X,Y) \text{ is a coupling of } \mu \text{ and } \nu\}$



概率分布函数

$X=Y$ P的概率相等

$X\to\gamma_{\text{I}}$ $\Rightarrow$

$Y\to\gamma_{\text{II}}$ 1-P

* mixing time :

$\|P^t(x,\cdot)-\pi\|_{TV} \ge \|P^{t+1}(x,\cdot)-\pi\|_{TV}$ 引理: 由单调只需证明存在一点很小即可

π 为任一平稳分布, 证明其单调性

TIP: $P^{t+1} = P^t \cdot P$ HW

$$\begin{cases} \tau_x^{\pi}(\varepsilon) = \min\{t : \|P^s(x,\cdot)-\pi\|_{TV} \le \varepsilon \ \ \forall s \ge t\} \\ \tau(\varepsilon) = \max_{x\in\Omega} \tau_x(\varepsilon) \end{cases}$$ π 不可约唯一

A $\boxed{\text{Markovian coupling}}$ of an MC with state space Ω and transition matrix P is an MC (X_t, Y_t) on $\Omega\times\Omega$ such that

$$\hat{P}(X_t = x' \mid X_{t-1}=x,\ Y_{t-1}=y) = P(x,x')$$

$$\hat{P}(Y_t = y' \mid X_{t-1}=x,\ Y_{t-1}=y) = P(y,y')$$

$$\hat{P}((x,y),(x',y')) = P(x,x')\,P(y,y')$$

$\boxed{\text{Couping Lemma}}$ Let (X_t, Y_t) be a Markovian coupling based on a ground MC (Z_t) on Ω. Suppose $t: [0,1]\to\mathbb{N}$ is a function satisfying

$$P[X_{t(\varepsilon)} \neq Y_{t(\varepsilon)} \mid X_0 = x,\ Y_0 = y] \le \varepsilon$$

for all $x,y\in\Omega$ and all $\varepsilon>0$

Then the mixing time $T(\varepsilon)$ of (Z_t) is bounded above by $t(\varepsilon)$

⇝ 不可约: $\exists N\ \forall n>N$, $A^n>0$

pf:

$X_0 = x\in\Omega$ fixed, $Y_0 = \pi$

即证 $\|P^{t(\varepsilon)}(x,\cdot)-\pi\|_{TV}\le\varepsilon$

$$= \max_{A\subseteq\Omega}\left(\pi(A) - P^{t(\varepsilon)}(x,A)\right)$$

$$P^t(x,A) \ge P(Y_t\in A) - \varepsilon$$

$$P^t(x,A) = P(X_t\in A) \ge \hat{P}(X_t = Y_t,\ Y_t\in A)$$

$$= 1-\hat{P}(X_t\neq Y_t \text{ or } Y_t\notin A) \ge 1-\hat{P}(X_t\neq Y_t) - P(Y_t\notin A)$$

$$= P(Y_t\in A) - \hat{P}(X_t\neq Y_t)$$

$$\ge P(Y_t\in A) - \varepsilon.$$

证明上界后，由引理继续

Pf: P irreducible + aperiodic $\Rightarrow \lim_{n\to\infty} P^n$

HW

$\rightsquigarrow$ P irreducible $\Rightarrow \lim \frac{I+P+\cdots+P^n}{n+1}$