A Solution for a question in Class 7

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1 Problem $h \in \mathbb{R}^{2^{[k]}}$, is an entropy function provided there are RVs, $X_1, X_2, \ldots, X_k$, s.t.$h(A) = H(\{X_i\}_{i \in A})$, prove that h satisfies the proporties of polymatroid:

1. $h(\varnothing) = 0$
2. $h(A) \leq h(B), A \subseteq B$
3. $h(A \cup B) + h(A \cap B) \leq h(A) + h(B)$ (I fail to prove the result)

Solution

1. $h(\varnothing) = H(\varnothing) = 0$
2. As $A \subset B$, without loss of generality, let $A = \{1, \ldots, s\}, B = \{1, \ldots, s+t\}$

   Then

$$\begin{aligned}
h(A) &= H(X_1, \ldots, X_s) \\
&= -\sum_{x_1, \ldots, x_s} p(x_1, \ldots, x_s) logp(x_1, \ldots, x_s) \\
&= -\sum_{x_1, \ldots, x_s} (\sum_{x_{s+1}, \ldots, x_{s+t}} p(x_1, \ldots, x_{s+t})) log(\sum_{x_{s+1}, \ldots, x_{s+t}} p(x_1, \ldots, x_{s+t})) \\
&\leq -\sum_{x_1, \ldots, x_s} (\sum_{x_{s+1}, \ldots, x_{s+t}} p(x_1, \ldots, x_{s+t}) logp(x_1, \ldots, x_{s+t})) \\
&= -\sum_{x_{x_1, \ldots, x_{s+t}}} p(x_1, \ldots, x_{s+t}) logp(x_1, \ldots, x_{s+t}) \\
&= h(B)
\end{aligned}$$

3. Without loss of generality, let $A \cup B = \{1, \ldots, s\}, A \cap B = \{1, \ldots, t\}, andA = \{1, \ldots, m\}, B = \{1, \ldots, t, m+1, \ldots, s\}$, where $s \geq m \geq t$

   According to the definition, we can have that:

$$h(A \cup B) = -\sum_{x_1, \ldots, x_s} p(x_1, \ldots, x_s) logp(x_1, \ldots, x_s)$$

$$
\begin{aligned}
h(A \cap B) &= -\sum_{x_1,\dots,x_t} p(x_1,\dots,x_t) log p(x_1,\dots,x_t) \\
&= -\sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log p(x_1,\dots,x_t)
\end{aligned}
$$

$$
\begin{aligned}
h(A) &= -\sum_{x_1,\dots,x_m} p(x_1,\dots,x_m) log p(x_1,\dots,x_m) \\
&= -\sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log p(x_1,\dots,x_m)
\end{aligned}
$$

$$
\begin{aligned}
h(B) = &-\sum_{x_1,\dots,x_t,x_{m+1},\dots,x_s} p(x_1,\dots,x_t,x_{m+1},\dots,x_s) log p(x_1,\dots,x_t,x_{m+1},\dots,x_s) \\
&= -\sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log p(x_1,\dots,x_t,x_{m+1},\dots,x_s)
\end{aligned}
$$

In order to prove $h(A \cup B) + h(A \cap B) \leq h(A) + h(B)$

$$
\begin{aligned}
&\Leftrightarrow -\sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log(p(x_1,\dots,x_s)p(x_1,\dots,x_t)) \\
&\quad \leq -\sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log(p(x_1,\dots,x_m)p(x_1,\dots,x_t,x_{m+1},\dots,x_s))
\end{aligned}
$$

$$
\Leftrightarrow \sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log \frac{p(x_1,\dots,x_s)p(x_1,\dots,x_t)}{p(x_1,\dots,x_m)p(x_1,\dots,x_t,x_{m+1},\dots,x_s)} \geq 0, \qquad (1)
$$

$$
\text{Numerator} = p(x_1,\dots,x_m)p(x_{m+1},\dots,x_s|x_1,\dots,x_m)p(x_1,\dots,x_t)
$$

$$
\text{Domain} = p(x_1,\dots,x_m)p(x_1,\dots,x_t)p(x_{m+1},\dots,x_s|x_1,\dots,x_t)
$$

Therefore, (1) equivalent to,

$$
\sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log \frac{p(x_{m+1},\dots,x_s|x_1,\dots,x_m)}{p(x_{m+1},\dots,x_s|x_1,\dots,x_t)} \geq 0
$$

For the left hand side we have,

$$
\begin{aligned}
\text{L.H.S} &= \sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log \frac{p(x_1,\dots,x_s)/p(x_1,\dots,x_m)}{p(x_{m+1},\dots,x_s|x_1,\dots,x_t)} \\
&\geq \sum_{x_1,\dots,x_s} p(x_1,\dots,x_s) log \frac{p(x_1,\dots,x_s)}{p(x_{m+1},\dots,x_s|x_1,\dots,x_t)}
\end{aligned}
$$

Assume that P, Q are two distributions, $P = p(X_1,\dots,X_s)$, $Q = p(X_{m+1},\dots,X_s|X_1,\dots,X_t)$, then

$$
\text{L.H.S} \geq \sum_{x_1,\dots,x_s} P log \frac{P}{Q} = D_{KL}(P||Q) \geq 0 = \text{R.H.S}
$$

Therefore, we proved the solution.