Some Solutions

徐逸凡 516071910074

Problem 1. Let μ and ν be any two distributions on state space Ω. Proof that

$$\| \mu - \nu \|_{TV} = \frac{1}{2} \sup \left\{ \sum_{x \in \Omega} f(x) \left[\mu(x) - \nu(x) \right] : f \in \mathbf{R}^{\Omega}, \max_{x \in \Omega} |f(x)| \leq 1 \right\}.$$

Solution.
On the one hand, $\forall f \in \mathbf{R}^{\Omega}, \max_{x \in \Omega} |f(x)| \leq 1$,

$$\begin{aligned} \frac{1}{2} \sum_{x \in \Omega} f(x) \left[\mu(x) - \nu(x) \right] &\leq \frac{1}{2} \sum_{x \in \Omega} |f(x)| \left| \mu(x) - \nu(x) \right| \\ &\leq \frac{1}{2} \sum_{x \in \Omega} \left| \mu(x) - \nu(x) \right| = \| \mu - \nu \|_{TV} . \end{aligned}$$

On the other hand, define

$$g(x) = \begin{cases} 1 & \text{if } \mu(x) \geq \nu(x), \\ -1 & \text{if } \mu(x) < \nu(x). \end{cases}$$

Then

$$\begin{aligned} & \frac{1}{2} \sum_{x \in \Omega} g(x) \left[\mu(x) - \nu(x) \right] \\ = \; & \frac{1}{2} \left\{ \sum_{x \in \Omega, \mu(x) \geq \nu(x)} \left[\mu(x) - \nu(x) \right] + \sum_{x \in \Omega, \mu(x) < \nu(x)} \left[\nu(x) - \mu(x) \right] \right\} \\ = \; & \frac{1}{2} \sum_{x \in \Omega} \left| \mu(x) - \nu(x) \right| = \| \mu - \nu \|_{TV} . \end{aligned}$$

Therefore, we can conclude that

$$\| \mu - \nu \|_{TV} = \frac{1}{2} \sup \left\{ \sum_{x \in \Omega} f(x) \left[\mu(x) - \nu(x) \right] : f \in \mathbf{R}^{\Omega}, \max_{x \in \Omega} |f(x)| \leq 1 \right\}.$$

□

Problem 2. Let P be the transition matrix of a Markov chain with state space Ω and let π be a stationary distribution. $\forall x \in \Omega$,

$$\| P^t(x, \cdot) - \pi \|_{TV} \geq \| P^{t+1}(x, \cdot) - \pi \|_{TV} .$$

Solution.
Let μ and ν be any two distributions on Ω. We can prove that

$$\| \mu P - \nu P \|_{TV} \leq \| \mu - \nu \|_{TV} .$$

The proof is as follows:

$$\begin{aligned}
\| \mu P - \nu P \|_{TV} &= \frac{1}{2} \sum_{y \in \Omega} |\mu P(y) - \nu P(y)| \\
&= \frac{1}{2} \sum_{y \in \Omega} \left| \sum_{x \in \Omega} P(x, y) \mu(x) - \sum_{x \in \Omega} P(x, y) \nu(x) \right| \\
&\leq \frac{1}{2} \sum_{y \in \Omega} \sum_{x \in \Omega} P(x, y) \left| \mu(x) - \nu(x) \right| \\
&= \frac{1}{2} \sum_{x \in \Omega} \sum_{y \in \Omega} P(x, y) \left| \mu(x) - \nu(x) \right| \\
&= \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| \\
&= \| \mu - \nu \|_{TV} .
\end{aligned}$$

Using the above inequality, we get

$$\begin{aligned}
\| P^{t+1}(x, \cdot) - \pi \|_{TV} &= \| P^t(x, \cdot) P - \pi P \|_{TV} \\
&\leq \| P^t(x, \cdot) - \pi \|_{TV}
\end{aligned}$$

□