Class 6

2018年3月22日 星期四 10:03

Inequality

Markeov's Inequality:

$$P(X \geq a) \leq \frac{EX}{a}$$

Chebyshev's Inequality:

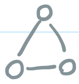
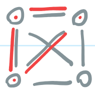
$$P(|X-EX| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$$

$$P((X-EX)^2 \geq \varepsilon^2)$$

Example 1

随机图:

$G(n,p)$, $V=\{1,2,\dots,n\}$ 每条边有 p 的概率加上，$(1-p)$ 的概率不加

图 G 固定如  求解 $\lim_{n\to\infty} P(G \leq G(n,p))$ 图中可发现同构子图 

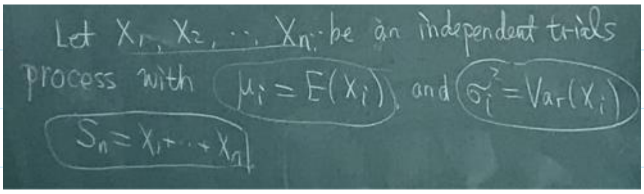
解: $P(X=0) \leq P(|X-EX| \geq EX) \leq \frac{VarX}{(EX)^2} = \frac{EX^2}{(EX)^2} - 1$

令 X_n 表示 $G(n,p)$ 中与 G 同构子图的个数 $\Rightarrow P(X_n=0) \leq \frac{VarX}{(EX)^2}$ ← 对方差与期望求解证明其小于一个高阶小量

若 $P_n \cdot n^{\frac{1}{m(G)}} \to \infty$, $P(X_n=0)=0$, $m(G)=\max\left\{\frac{|E(H)|}{|V(H)|} : H \leq G\right\}$

Example 2

Weak law of Large Numbers — J. Bernoulli 1713



Fix $\varepsilon > 0$

→ 与期望无关，只需方差一致有界

Claim: $P\left(\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \varepsilon\right) \to 1$ as $n \to \infty$

pf: $V(S_n) = \sigma_1^2 + \dots + \sigma_n^2$ ← 独立可加性

$$\begin{cases} Var\left(\frac{S_n}{n}\right) = \frac{\sigma_1^2 + \dots + \sigma_n^2}{n^2} \leq \frac{M}{n} & \text{← 设 } \sigma_i^2 \leq M \text{ 一致有界} \\ E\left(\frac{S_n}{n}\right) = \frac{\mu_1 + \dots + \mu_n}{n} \end{cases}$$

$\Rightarrow P\left(\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| > \varepsilon\right) \leq \frac{Var\left(\frac{S_n}{n}\right)}{\varepsilon^2} \to 0$ ← Chebyshov

Example 3

Wierstrass Approximation Theorem:

$[a,b] \subseteq \mathbb{R}$, f 为 $[a,b]$ 上的连续函数

$\Rightarrow \exists$ real polynomial function g s.t. $\sup_{a\leq x\leq b} |f(x) - g(x)| < \varepsilon$

→ Step 1: 令 $[a,b] = [0,1]$

Serge Bernstein (1912)

$$\sup_{0\leq x\leq 1} \left| f(x) - \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k \cdot (1-x)^{n-k} \right| \to 0 \text{ as } n \to \infty$$

Step 2: 令 $p = x \in [0,1]$ 固定 x

则 $\sum_{k=0}^{n} \binom{n}{k} f(\frac{k}{n})\, p^k \cdot (1-p)^{n-k} = Bernstein_{n,f}(p) = E[f(\frac{S_n}{n})]$

Step 3:

$|B_{n,f}(p) - f(p)| = |E[f(\frac{S_n}{n})] - f(p)| = |\sum_{k=0}^{n} (f(\frac{k}{n}) - f(p))\ P(S_n = k)|$

$\le \underbrace{\sum_{|\frac{k}{n}-p|\le\eta}}_{\varepsilon} + \sum_{|\frac{k}{n}-p|>\eta}$

#

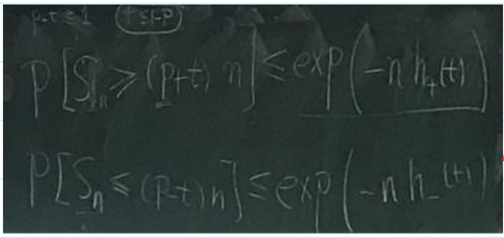
Bernstein:

$P(x \ge a) \le \frac{EX}{a}$ 可能等价于 $P(f(x) \ge a) \le \frac{EX}{a}$

Chebyshov 中 $f(x) = x^2$，现取 $f(x) = e^{tx}$ 强化 weak large number theorem，$t \in \mathbb{R}$

$P(|\frac{S_n}{n} - p| \ge t) \le \frac{p\cdot(1-p)}{nt^2}$

$P(|\frac{S_n}{n} - p| \ge t) \le e^{-nh_+(t)} + e^{-nh_-(t)}$

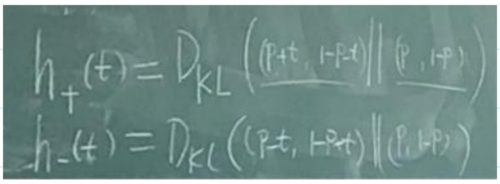


两者等价

Example 4

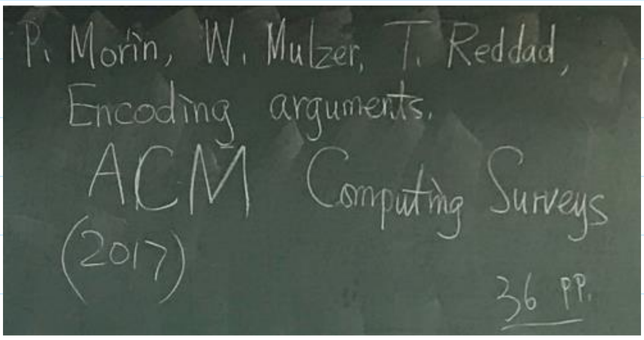
Kullback-Leibler divergence:

$D_{KL}(P \| Q) = \sum_i p_i \cdot \ln\frac{p_i}{q_i}$ → $D_{KL} \ge 0$ ⇒ HW



$P(S_n \ge (p+t)n) \le \left(\frac{pe^{\lambda} + 1 - p}{e^{\lambda(p+t)}}\right)^n$ 求导得解，也可不求导即为下一Example

Homework 1



⇒ Ref

Example 5

Let D be a probability distribution on $\{0,1\}^n$ that assigns to each $x \in \{0,1\}^n$ a probability P_x. Let w be a nonnegative weight function on $\{0,1\}^n$ such that $\sum_{x\in\{0,1\}^n} w(x) \le 1$.

For any $s \ge 1$,

$$\underset{x\sim D}{P}\left[w(x) \ge s P_x\right] \le \frac{1}{s}.$$

$$Z_s = \{x \in \{0,1\}^n : w(x) \ge s P_x\}.$$

$$\underset{x\sim D}{P}\left[w(x) \ge s P_x\right] \le \sum_{x\in Z_s} P_x \frac{w(x)}{s P_x} \le \frac{1}{s}\sum_x w(x) \le \frac{1}{s}$$

Example 5

$$X = \boxed{X_1 X_2 \cdots X_n}$$

$$P_x = p^{k_x}(1-p)^{n-k_x} \qquad k_x = \#1 \text{ in } x$$

$$w(x) = (p+t)^{k_x}(1-p-t)^{n-k_x}$$

$\Rightarrow$ HW

$$P(S_n \ge (p+t)n) = \underset{x\sim D}{P}\left[k_x \ge (p+t)n\right]$$

$$= \underset{x\sim D}{P}\left[w(x) \ge P_x \underbrace{e^{-n h_+(t)}}_{=s}\right] \le e^{-n h_+(t)}$$

Example 6

Homework 2