

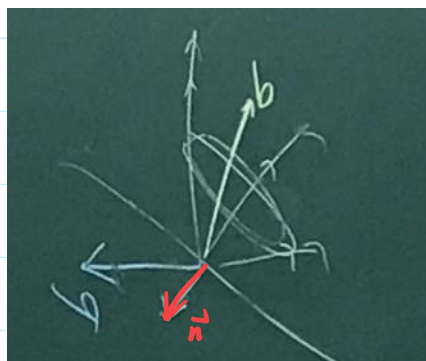
Class 11

2018年4月24日 星期二 10:08

* $xP = x$, 齐次 MC $\Rightarrow \perp P^n$ 不可约有极限的唯一性

$$x^T P = x^T \Rightarrow x^T (P - I) = 0$$

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$$\begin{cases} x^T A = b^T & x \geq 0, b \in \text{cone}(A) \\ Ay \leq 0 & b^T y > 0 \end{cases}$$



当有一个锥区域, 向量 b 或落于其中, 或置身其外
在后者发生情况下, 一定存在一个超平面将其分开
↓ 考虑法向量与其内积.

$$\text{令 } A = \begin{pmatrix} P-I & 1 \\ 1 & 1 \end{pmatrix}, \quad x^T A = (-0-1)$$

$$\Rightarrow \begin{pmatrix} P-I & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} \leq 0, \quad y_{n+1} > 0 \quad \text{用上述锥结论导出}$$

$$y_i = \min y_i, \quad \sum P_{ii} y_i < y_i \quad \text{矛盾}$$

* 带吸收壁的随机游走:

$$P(i, i+1) = P(i, i-1) = \frac{1}{2}, \quad i = 1, 2, \dots, n-1$$

$$\begin{cases} P(0, 1) = P(n, n-1) = 0 \\ P(0, 0) = P(n, n) = 1 \end{cases}$$

$$P^t(x, n) \leq P^t(y, n) \quad x \leq y \quad \text{直观理解为 } y \text{ 大}$$



耦合?

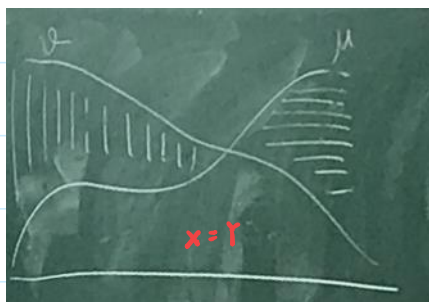
← 均符合 P 转移

* Total variation distance:

μ, ν probability distribution on Ω

Def-1 $\|\mu - \nu\|_{TV} = \max_{A \in \Omega} |\mu(A) - \nu(A)|$ Def-2 $= \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|$

$\|f - g\| \leq \|f - h\| + \|h - g\|$



落在左侧或右侧],

取全正或全负.

图中两块相等



Distance
Definition

Def-3 $\|\mu - \nu\|_{TV} = \frac{1}{2} \sup \left\{ \sum_{x \in \Omega} f(x) [\mu(x) - \nu(x)] : f \in \mathbb{R}^{\Omega}, \max |f(x)| \leq 1 \right\}$ HW

Def-4 $\|\mu - \nu\|_{TV} = \inf \left\{ P(X \neq Y) : (X, Y) \text{ is a coupling of } \mu \text{ and } \nu \right\}$

$$\gamma_{II}(x) = \frac{\min(\mu(x), \nu(x))}{P}$$

$$\gamma_I(x) = \frac{\max\{\mu(x) - \nu(x), 0\}}{1-P}$$

$$\gamma_{II}(x) = \frac{\max\{\nu(x) - \mu(x), 0\}}{1-P}$$

↑
概率分布函数

$x=y$ ← P的概率相等

$x \rightarrow x_I$

$y \rightarrow y_{II}$

1-P

$$\mu(x) = P \cdot \gamma_{II}(x) + (1-P) \gamma_I(x)$$

$$\nu(x) = P \gamma_{II}(x) + (1-P) \gamma_{II}(x)$$

* mixing time:

$\|P^t(x, \cdot) - \pi\|_{TV} \geq \|P^{t+1}(x, \cdot) - \pi\|_{TV}$ 引理: 由单调只需证明存在一点很小即可

π 为任-平稳分布, 证明其单调性

TIP: $P^{t+1} = P^t \cdot P$ HW

$\tau_x^n(\epsilon) = \min \left\{ t : \|P^t(x, \cdot) - \pi\|_{TV} \leq \epsilon \quad \forall s \geq t \right\}$ π 不可约唯一-
 $T(\epsilon) = \max_{x \in \Omega} \tau_x(\epsilon)$

A Markovian coupling of an MC with state space Ω and transition matrix P is an MC (X_t, Y_t) on $\Omega \times \Omega$ such that

$$\hat{P}(X_t = x' \mid X_{t-1} = x, Y_{t-1} = y) = P(x, x')$$

$$\hat{P}(Y_t = y' \mid X_{t-1} = x, Y_{t-1} = y) = P(y, y')$$

$$\hat{P}((x, y), (x', y')) = P(x, x') P(y, y')$$

Coupling Lemma Let (X_t, Y_t) be a Markovian coupling based on a ground MC (Z_t) on Ω . Suppose $t: [0, 1] \rightarrow \mathbb{N}$ is a function satisfying

$$P[X_{t(\varepsilon)} \neq Y_{t(\varepsilon)} \mid X_0 = x, Y_0 = y] \leq \varepsilon$$

for all $x, y \in \Omega$ and all $\varepsilon > 0$.

Then the mixing time $T(\varepsilon)$ of (Z_t) is bounded above by $t(\varepsilon)$

↪ 不可约: $\exists N \forall n > N, A^n > 0$

pf:

$$x_0 = x \in \Omega \text{ fixed}, Y_0 = \pi$$

$$\text{Pick } \|P^{t(\varepsilon)}(x, \cdot) - \pi\|_{TV} \leq \varepsilon$$

$$\max_{A \subseteq \Omega} (\pi(A) - P^{t(\varepsilon)}(x, A))$$

$$P^{t(\varepsilon)}(x, A) \geq \underbrace{P(Y_{t(\varepsilon)} \in A)}_{\parallel}$$

$$P^{t(\varepsilon)}(x, A) = P(X_{t(\varepsilon)} \in A) \geq \hat{P}(X_{t(\varepsilon)} = Y_{t(\varepsilon)}, Y_{t(\varepsilon)} \in A)$$

$$= 1 - \hat{P}(X_{t(\varepsilon)} \neq Y_{t(\varepsilon)} \text{ or } Y_{t(\varepsilon)} \notin A) \geq 1 - \hat{P}(X_{t(\varepsilon)} \neq Y_{t(\varepsilon)}) - P(Y_{t(\varepsilon)} \notin A)$$

$$= P(Y_{t(\varepsilon)} \in A) - \underbrace{\hat{P}(X_{t(\varepsilon)} \neq Y_{t(\varepsilon)})}_{\leq \varepsilon}$$

$$\geq P(Y_{t(\varepsilon)} \in A) - \varepsilon$$

证明上界后, 由引理继续

$$pf: P \text{ irreducible} + aperiodic \Rightarrow \lim_{n \rightarrow \infty} P^n$$

HW

$$\leadsto P \text{ irreducible} \Rightarrow \lim_{n \rightarrow \infty} \frac{1 + P + \dots + P^n}{n+1}$$