Class 9

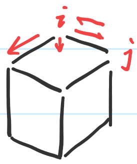
2018年4月10日 星期二 10:04

* 续 MC：← MC可理解为图上的概率随机游走 lumping

Ehrenfest urn:

$X_t \mid Y_t$ $\quad X_t + Y_t = n \quad \begin{cases} Y_{t+1} = Y_t \pm 1 \\ X_{t+1} = X_t \pm 1 \end{cases}$

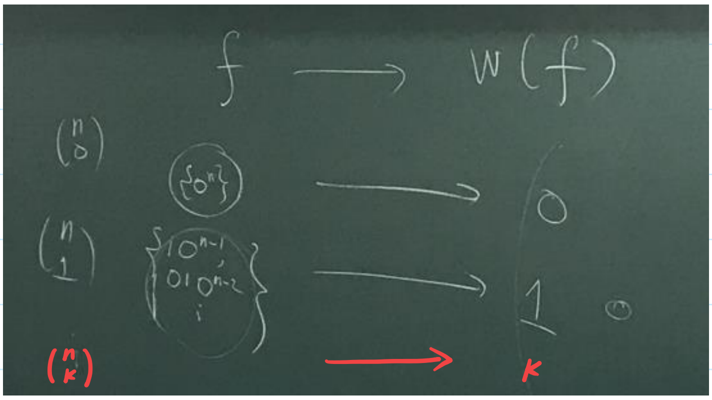
Reversible MC:



顶点集合：V ，$\pi \in \mathbb{R}^V$

$i \to j$ 转移概率：P_{ij}

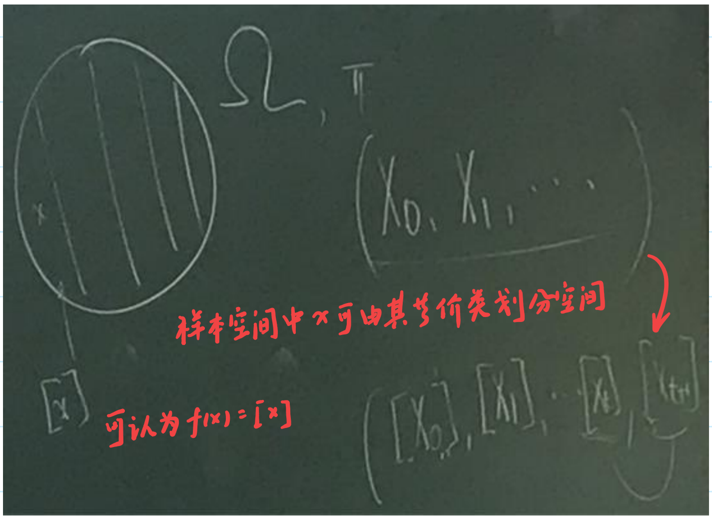
若 $\pi_i P_{ij} = \pi_j P_{ji}$ 为平稳分布



$X_i : S \to V$
$f : V \to W$
$\Big\}$ $f(X_i)$ $S \to W$

X_i 为MC，给定 f，$f(X_i)$ 是否为MC？

不一定是，给出反例以及为MC时所需条件



使 $[X_{t+1}]$ 只与 $[X_t]$ 相关

$P(x, [y]) = P(x', [y'])$ whenever $[x] = [x']$ $\quad \Omega^\# = \{[x] : x \in \Omega\}$

$\Rightarrow$ $P^\#([x], [y]) = P(x, [y])$ 若满足该条件，$f(x)$ 仍为MC

* $S = \{s_1, \dots, s_k\}$ 有限状态空间

$s_i \to s_j$: $\exists n$, $(P^n)_{ij} > 0$ $\quad$ i 有概率多次转移后到达 j

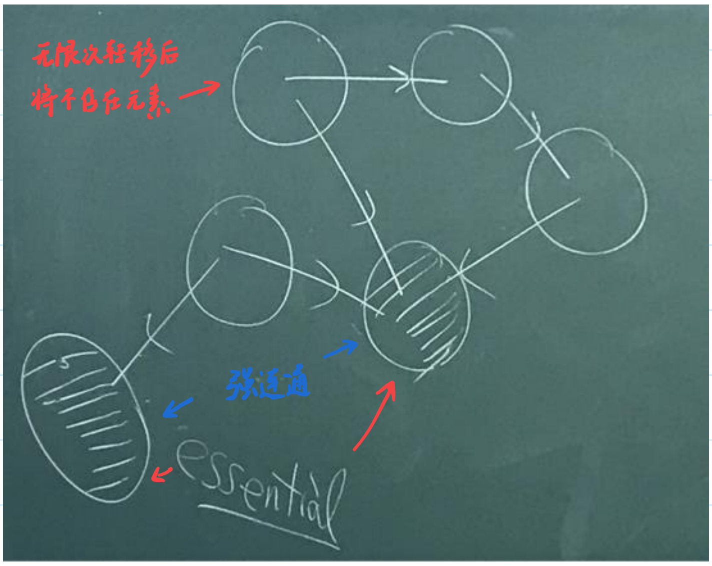
$s_i \to s_j$: $\exists n, (P^n)_{ij} > 0$ i有概率多次转移后到达j

* 强连通分量：

A MC is irreducible if for all $s_i, s_j \in S$ $|S| = n$, we have $s_i \to s_j$

⇒ $I + P + P^2 + \cdots > 0$

$I + P + P^2 + \cdots + P^{n-1} > 0$



* 定理:

For any finite state irreducible Markov chain, there exists at least one stationary distribution → $\pi P = \pi, \pi \geq 0$

* 若P不可约: $\pi_0 P_0 = \pi_0$

$$(0\ 0\ (\pi_0)) \begin{pmatrix} * & \\ 0 & \boxed{P_0} \end{pmatrix} = (0\ 0 \cdots 0\ \pi_0 P_0) = (0\ \pi_0)$$ ←矩阵论

* $\Delta_{n-1} \circlearrowleft P$ ←不动点定理

$f \in \Delta_{n-1}$, $f \to fP$

pf: $(X_0, X_1, X_2, \cdots)$ 时间转移矩阵

For $x \in \Omega$, hitting time for x, $\tau_x := \min\{t \geq 0;\ X_t = x\}$

$\tau_x^+ := \min\{t \geq 1;\ X_t = x\}$

Lem: For any states x, y of an irreducible chain,

$$E_x(\tau_y^+) < \infty$$

圣彼德堡悖论

pf: $E_x(\tau_y^+) = \sum_{t=1}^{\infty} t \cdot P_x(\tau_y^+ = t)$ 太过困难 ⇝ $P(X = 2^n) = (\frac{1}{2})^{n+1}$ $\begin{cases} P(X < \infty) = 1 \\ EX = \infty \end{cases}$

$P_x(\tau_y^+ < \infty) = 1$ 弱化条件

There $\exists\ \varepsilon > 0$ & $r > 0$ s.t.: For $\forall z, w \in \Omega$ $\exists j \leq r$ s.t. $P^j(z, w) > \varepsilon$ ←由强连通性质易理解

There $\exists\ \varepsilon>0$ & $r>0$ s.t.: For $\forall\ z,w\in\Omega\ \exists\ j\le r$ s.t. $P^j(z,w)>\varepsilon$ ← 由强连通性质易理解

$$P_x\{\tau_y^+>kr\}\le(1-\varepsilon)P_x\{\tau_y^+>(k-1)r\}\ \Rightarrow\ P_x\{\tau_y^+>kr\}\le(1-\varepsilon)^k$$

$$E_x(\tau_y^+)=\sum_{t=1}^{\infty}t\cdot P_x(\tau_y^+=t)=\sum_{t\ge0}P_x\{\tau_y^+>t\}$$ ← 期望面积换维度计算

$$\le\sum_{k\ge0}rP_x\{\tau_y^+>kr\}\le r\sum_{k\ge0}(1-\varepsilon)^k=\frac{r}{\varepsilon}<\infty$$

不可约: $\dim \mathrm{Ker}(P-I)=1$, $\mathrm{Ker}(P-I)=\{f: Pf=f\}$

$\sum P(i,j)f(j)=f(i)$

$f(i)$ max

⇒ 所有值均相等, f为一维

下找该唯一存在: $\pi(x)=\dfrac{1}{E_x(\tau_x^+)}$ $\quad\sum_{x\in\Omega}\pi(x)=1$

Pick $z\in\Omega$, $\tilde{\pi}(y)\overset{\text{def}}{=}E_z(z$ ↺ 过程中访问y次数$)\le E_z\tau_z^+$

Claim: $\tilde{\pi}P=\tilde{\pi}$

$$\sum_{x\in\Omega}\tilde{\pi}(x)P(x,y)=\sum_{x\in\Omega}\sum_{t=0}^{\infty}P_z\{X_t=x,\tau_z^+>t\}P(x,y)$$

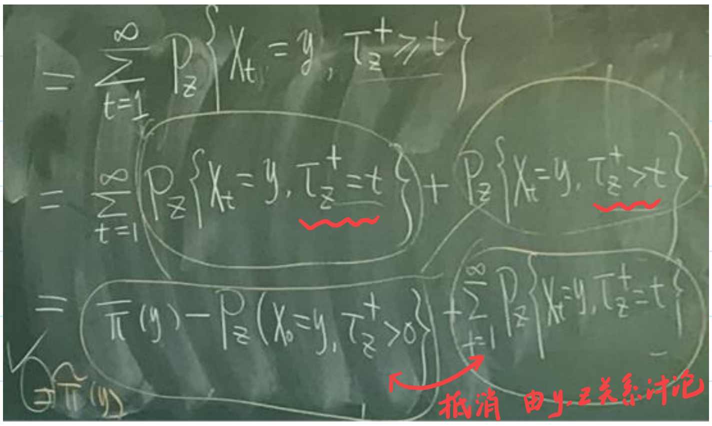
Σ交换次序

$$=\sum_{t=0}^{\infty}\sum_{x\in\Omega}P_z\{X_t=x,\tau_z^+>t\}P(x,y)$$

double-counting

$$=\sum_{t=0}^{\infty}\sum_{x\in\Omega}P_z(X_t=x,X_{t+1}=y,\tau_z^+\ge t+1)$$

$$=\sum_{t=0}^{\infty}P_z\{X_{t+1}=y,\tau_z^+\ge t+1\}$$



$$\frac{\tilde{\pi}^{(z)}}{E_z(\tau_z^+)} = \pi$$ $\longrightarrow$ z=x时为上述