# Solutions to Some Problems in Lecture 4

范舟
516030910574
zhou.fan@sjtu.edu.cn
上海交通大学 ACM 班

## 1 Independent Random Variables

**Remark** There are multiple ways to define the independence of $n$ discrete random variables, for cosistency we use the following definition which is used by Prof. Wu in this course.

**Definition(Independent random variables)** Consider $n$ discrete random variables $X_1, X_2, \ldots, X_n$. We say that $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i), \quad \forall x_1, x_2, \ldots, x_n$$

**Remark** It's trivial that the following definition is equivalent to the first one. It's used as a property in one solution.

**Equivalent definition** Consider $n$ discrete random variables $X_1, X_2, \ldots, X_n$. We say that $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n) = \prod_{i=1}^{n} P(X_i \in A_i), \quad \forall A_1, A_2, \ldots, A_n$$

## 2 Problem 1

**Problem** Consider $k$ random events $A_1, A_2, \ldots, A_k$, which satisfy the condition

$$\forall S \subset [k], \quad \prod_{i \in S} P(A_i) = P(\bigcap_{i \in S} A_i)$$

Prove that $1_{A_1}, 1_{A_2}, \dots, 1_{A_k}$ are independent random variables, where random variable $1_{A_i}$ is defined as

$$1_{A_i}(x) = \begin{cases} 1, & x \in A_i \\ 0, & x \notin A_i \end{cases}$$

**Proof** From the definition of $1_{A_i}$, we have

$$P(1_{A_i} = 1) = P(A)$$
$$P(1_{A_i} = 0) = P(A^c)$$

To prove that

$$P(1_{A_1} = x_1, 1_{A_2} = x_2, \dots, 1_{A_k} = x_k) = \prod_{i=1}^{k} P(1_{A_i} = x_i), \quad \forall x_1, x_2, \dots, x_k \in \{0, 1\}$$

is equivalent to prove that

$$P(1_{A_i} = 1, \forall i \in P; 1_{A_j} = 0, \forall j \in Q) = \prod_{i\in P} P(1_{A_i} = 1) \prod_{j\in Q} P(1_{A_j} = 0), \quad \forall P \subset [k], Q = [k] \backslash P$$

which is equivalent to prove that

$$P(A_i, \forall i \in P; {A_j}^c, \forall j \in Q) = \prod_{i\in P} P(A_i) \prod_{j\in Q} (1 - P(A_j)), \quad \forall P \subset [k], Q = [k] \setminus P \qquad (1)$$

From this condition

$$\forall S \subset [k], \quad \prod_{i\in S} P(A_i) = P(\bigcap_{i\in S} A_i)$$

$A_1, A_2, \dots, A_k$ are independet random events, thus the equation (1) holds. Therefore, $1_{A_1}, 1_{A_2}, \dots, 1_{A_k}$ are independent random variables.

# 3 Problem 2

**Problem** $(X_i)_{i\in I}$ are independent random variables. $J \subset I, K \subset I, J \cap K = \emptyset$, for random variables $(X_i)_{i\in J}, (X_i)_{i\in K}$ and functions $f \in \mathbb{R}^J, g \in \mathbb{R}^K$, whether random variables $Y = f((X_j)_{j\in J})$ and $Z = g((X_k)_{k\in K})$ are independent or not?

**Solution** $Y$ and $Z$ are independent random variables.

**Proof** For simplicity, let $M = (X_j)_{j\in J}, N = (X_k)_{k\in K}$. Then $Y = f(M), Z = g(N)$. Because $(X_i)_{i\in I}$ are independent random variables, $J \cap K = \emptyset$, $M$ and $N$ are independent random variables. With the equivalent definition 1 of independent random variables, we have

$$P(M \in A_M, N \in A_N) = P(M \in A_M)P(N \in A_N), \forall A_M, A_N$$

Then we have

$$\begin{aligned}P(Y = t_Y, Z = t_Z) &= P(M \in f^{-1}(t_Y), N \in g^{-1}(t_Z)) \\ &= P(M \in f^{-1}(t_Y))P(N \in g^{-1}(t_Z)) \\ &= P(Y = t_Y)P(Z = t_Z) \\ &\forall t_Y, t_Z\end{aligned}$$

Therefore, $Y$ and $Z$ are independent random variables.

**Remark** The last part of the proof proves the following property of independent random variables, it's useful when solving similar problems.

**Corollary** If $X$ and $Y$ are two independent random variables, and $P = g(X)$ and $Q = h(Y)$ then $P$ and $Q$ are also independent to each other.

# 4 Problem 3

**Problem** $X$ and $Y$ are two independent discrete random variables, prove the following properties:

1. $E[XY] = E[X]E[Y]$
2. $Var(X + Y) = Var(X) + Var(Y)$

**Proof [1]**

$$\begin{aligned}E(XY) &= \sum_i \sum_j x_i y_j P(X = x_i, Y = y_j) \\ &= \sum_i \sum_j x_i y_j P(X = x_i)P(Y = y_j) \\ &= \big(\sum_i P(X = x_i)\big)\big(\sum_j x_i y_j P(Y = y_j)\big) \\ &= E(X)E(Y)\end{aligned}$$

**Proof [2]** Denote the covariance of $X$ and $Y$ as $Cov(X, Y)$, then

$$\begin{aligned}Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - 2E[Y]E[X] + E[x]E[Y] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

$X$ and $Y$ are independent, thus

$$E[XY] = E[X]E[Y]$$

Then we have

$$Cov(X, Y) = 0$$

Therefore

$$Var(X + Y) = Var(X) + 2Cov(X, Y) + Var(Y) = Var(X) + Var(Y)$$

**Remark** These are two basic properties of independent discrete random variables.