Solutions to Some Problems in Lecture 4

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Remark There are multiple ways to define the independence of n discrete random variables, for cosistency we use the following definition which is used by Prof. Wu in this course.

1 Definition (Independent random variables) Consider n discrete random variables $X_1, X_2, \ldots, X_n$. We say that $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i), \quad \forall x_1, x_2, \ldots, x_n$$

Remark It's trivial that the following definition is equivalent to the first one. It's used as a property in one solution.

2 Definition (Equivalent definition) Consider n discrete random variables $X_1, X_2, \ldots, X_n$. We say that $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n) = \prod_{i=1}^{n} P(X_i \in A_i), \quad \forall A_1, A_2, \ldots, A_n$$

3 Problem Consider k random events $A_1, A_2, \ldots, A_k$, which satisfy the condition

$$\forall S \subset [k], \quad \prod_{i \in S} P(A_i) = P(\bigcap_{i \in S} A_i)$$

Prove that $1_{A_1}, 1_{A_2}, \ldots, 1_{A_k}$ are independent random variables, where random variable 1_{A_i} is defined as

$$1_{A_i}(x) = \begin{cases} 1, & x \in A_i \\ 0, & x \notin A_i \end{cases}$$

Proof From the definition of 1_{A_i}, we have

$$P(1_{A_i} = 1) = P(A)$$
$$P(1_{A_i} = 0) = P(A^c)$$

To prove that

$$P(1_{A_1} = x_1, 1_{A_2} = x_2, \dots, 1_{A_k} = x_k) = \prod_{i=1}^{k} P(1_{A_i} = x_i), \quad \forall x_1, x_2, \dots, x_k \in \{0, 1\}$$

is equivalent to prove that

$$P(1_{A_i} = 1, \forall i \in P; 1_{A_j} = 0, \forall j \in Q) = \prod_{i \in P} P(1_{A_i} = 1) \prod_{j \in Q} P(1_{A_j} = 0), \quad \forall P \subset [k], Q = [k] \backslash P$$

which is equivalent to prove that

$$P(A_i, \forall i \in P; {A_j}^c = 0, \forall j \in Q) = \prod_{i \in P} P(A_i) \prod_{j \in Q} (1 - P(A_j)), \quad \forall P \subset [k], Q = [k] \setminus P \quad (1)$$

From this condition

$$\forall S \subset [k], \quad \prod_{i \in S} P(A_i) = P(\bigcap_{i \in S} A_i)$$

$A_1, A_2, \dots, A_k$ are independet random events, thus the equation (1) holds. Therefore, $1_{A_1}, 1_{A_2}, \dots, 1_{A_k}$ are independent random variables.

4 Problem $(X_i)_{i \in I}$ are independent random variables. $J \subset I, K \subset I, J \cap K = \emptyset$, for random variables $(X_i)_{i \in J}, (X_i)_{i \in K}$ and functions $f \in \mathbb{R}^J, g \in \mathbb{R}^K$, whether random variables $Y = f((X_j)_{j \in J})$ and $Z = g((X_k)_{k \in K})$ are independent or not?

Solution Y and Z are independent random variables.

Proof For simplicity, let $M = (X_j)_{j \in J}, N = (X_k)_{k \in K}$. Then $Y = f(M), Z = g(N)$. Because $(X_i)_{i \in I}$ are independent random variables, $J \cap K = \emptyset$, M and N are independent random variables. With the equivalent definition 2 of independent random variables, we have

$$P(M \in A_M, N \in A_N) = P(M \in A_M) P(N \in A_N), \forall A_M, A_N$$

Then we have

$$\begin{aligned} P(Y = t_Y, Z = t_Z) &= P(M \in f^{-1}(t_Y), N \in g^{-1}(t_Z)) \\ &= P(M \in f^{-1}(t_Y)) P(N \in g^{-1}(t_Z)) \\ &= P(Y = t_Y) P(Z = t_Z) \\ &\forall t_Y, t_Z \end{aligned}$$

Therefore, Y and Z are independent random variables.

Remark The last part of the proof proves the following property of independent random variables, it's useful when solving similar problems.

5 Corollary If X and Y are two independent random variables, and $P = g(X)$ and $Q = h(Y)$ then P and Q are also independent to each other.

6 Problem X and Y are two independent discrete random variables, prove the following properties:

1. $E[XY] = E[X]E[Y]$
2. $Var(X+Y) = Var(X) + Var(Y)$

Proof [1]

$$\begin{aligned} E(XY) &= \sum_i \sum_j x_i y_j P(X = x_i, Y = y_j) \\ &= \sum_i \sum_j x_i y_j P(X = x_i) P(Y = y_j) \\ &= \big(\sum_i P(X = x_i)\big)\big(\sum_j x_i y_j P(Y = y_j)\big) \\ &= E(X)E(Y) \end{aligned}$$

Proof [2] Denote the covariance of X and Y as $Cov(X,Y)$, then

$$\begin{aligned} Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - 2E[Y]E[X] + E[x]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

X and Y are independent, thus

$$E[XY] = E[X]E[Y]$$

Then we have

$$Cov(X,Y) = 0$$

Therefore

$$Var(X+Y) = Var(X) + 2Cov(X,Y) + Var(Y) = Var(X) + Var(Y)$$

Remark These are two basic properties of independent discrete random variables.