05月17日 Class 16

- Poisson Distribution vs. Binomial Distribution

Poisson(λ) $\quad f(k)=\frac{\lambda^k}{k!}e^{-\lambda}$ 固定时间长度，发生次数

Exponential(λ) $\quad f(x)=\lambda e^{-\lambda x}$ 直到发生的等待时间

- Lucien de Cam 1960:

$X_r \sim \text{Bernoulli}(p_r)$, $r=1,\dots,n$ independent

$P=\text{Poisson}\left(\sum_{r=1}^n p_r\right)$, $S=\sum_{r=1}^n X_r$ 估计P与S相差多少

当 p_r 为常数时，$S\sim \text{Binomial}(n,p)$

$$d_{TV}(P,S)\le 2\sum_{r=1}^n p_r^2$$

Proof: coupling (X_r,Y_r) $r=1,\dots,n$ $\quad Y_r\sim\text{Poisson}(p_r)$

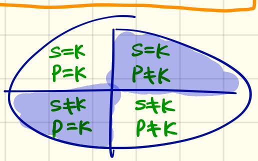
构造使得 $\sum\mathbb{P}=1$，边际分布分别为 Poisson, Bernoulli

$$\mathbb{P}(X_r=x,Y_r=y)=\begin{cases}1-p_r, & x=y=0\\ e^{-p_r}-(1-p_r), & x=1,y=0\\ \frac{p_r^y}{y!}e^{-p_r} & x=1,y\ge 1\end{cases}$$

$\sum_{r=1}^n (X_r,Y_r)$

边际投影 $\swarrow\ \searrow$

$S=\sum X_r \quad P=\sum Y_r$



两块的差小于等于两块的和

$$d_{TV}(S,P)=\frac12\sum_{k=0}^{\infty}\left|\mathbb{P}(S=k)-\mathbb{P}(P=k)\right|$$

$$\le\frac12\Big[\underbrace{\sum_k\mathbb{P}(S=k,P\ne S)}_{=\mathbb{P}(S\ne P)}+\underbrace{\sum_k\mathbb{P}(P=k,S\ne P)}_{=\mathbb{P}(S\ne P)}\Big]$$

$$=\mathbb{P}(S\ne P)\le\sum_{r=1}^n\mathbb{P}(X_r\ne Y_r)$$

$\bigcup_r\{X_r\ne Y_r\}\supseteq S\ne P$

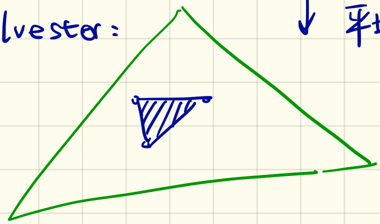
$$d_{TV}(P,S) \le \sum_{r=1}^{n} \mathbb{P}(X_r \ne Y_r)$$
$$= \sum_{r=1}^{n} e^{-p_r} - (1-p_r) + \underbrace{\mathbb{P}(Y_r \ge 2)}_{1-e^{-p_r}-p_r e^{-p_r}}$$
$$= \sum_{r=1}^{n} p_r(1-e^{-p_r}) \le \sum_{r=1}^{n} p_r^2 \quad \#$$

HW: 4(花色) x 13(张数) $\longrightarrow$ 13张牌之型？

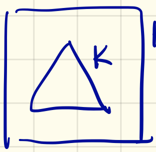
- Triangle：随机扔3个点，构成钝角三角形概率

$\downarrow$ 平均的面积

- Sylvester：

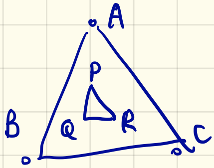


猜想：$\mathbb{R}^d$上某个凸体中扔$d+1$个点
使得平均面积max的形状是单纯形



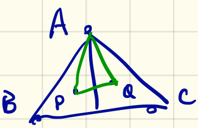
$H \supseteq K$ 在低维$(d=1,2)$ $\overline{\text{Area}}(H) \ge \overline{\text{Area}}(K)$
高维有反例

- Croften Method：



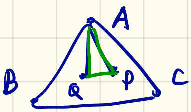
$E|PQR| = \frac{1}{12}$

- Lem 1:



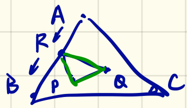
$E|APQ| = \frac{2}{9}$

Lem 2:

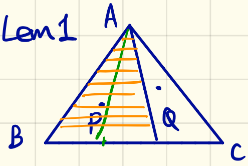


$E|APQ| = \frac{4}{27}$

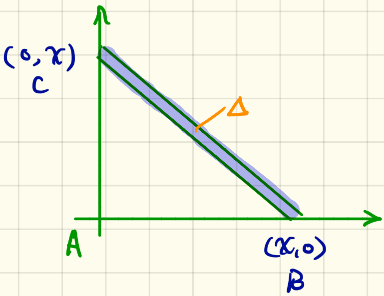
Lem 3:



$E|PQR| = \frac{1}{9}$

Lem1 

$E|APQ| = \text{Area}$ (P,Q分割取重心)



Lem 2: 全概率公式

Lem3: ......

Proof: $E|PQR| = \frac{1}{12}|ABC|$

$$E_x(PQR) = A(x) \overset{?}{=} \frac{1}{24}x^2$$

$$\frac{dA}{dx} = \lim_{\delta \to 0} \frac{A(x+\delta x) - A(x)}{\delta x}$$

Taylor展开

$$\mathbb{P}_{x+\delta x}(P,Q,R \in ABC) = \frac{|ABC|}{|AB'C'|} = \left[\frac{x^2}{(x+\delta x)^2}\right]^3 = 1 - \frac{6\delta x}{x} + o(\delta x)$$

$$\mathbb{P}_{x+\delta x}(P,Q \in ABC \cap R \in \Delta) = \left[\frac{x^2}{(x+\delta x)^2}\right]^2 \frac{(\delta x)^2 + 2x\delta x}{(x+\delta x)^2} = \frac{2\delta x}{x} + o(\delta x)$$

$\because$ $3\mathbb{P}_{x+\delta}(P,Q \in ABC \cap R \in \Delta) + \mathbb{P}_{x+\delta x}(P,Q,R \in ABC) \approx 1$

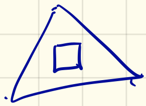
$\therefore$ 认为最多只有1个点落入Δ

Lem3

$$A(x+\delta x) = A(x)\left(1 - \frac{6\delta x}{x}\right) + \boxed{\frac{1}{9} \cdot \frac{x^2}{2}} \cdot \frac{6\delta x}{x} + o(\delta x)$$

$$\frac{dA}{dx} = \frac{A(x+\delta x) - A(x)}{\delta x} = \frac{-6A(x)}{x} + \frac{x}{3}$$

边界条件 $A(0) = 0 \Rightarrow A(x) = \frac{x^2}{24}$ HW: 升高维度的平均体积.

HW:  扔4个点凸包是四边形的概率