Problem 1. There is a game. One starts from $(0, 0)$ and walks to (n, n), allowed to walk towards right and walk towards up. Define the number of k upward walks above the straight line from $(0, 0)$ to (n, n) to be $f(n, k)$. We have that

$$f(n,k) = \frac{1}{n+1}\binom{2n}{n} = C_n$$

Proof. Proof by induction. We define the proposition $P(N)$ to be that the property described in the problem holds for all $0 \leq n \leq N$.
It is easy to check that $P(0)$ and $P(1)$ is true.
Assuming that $P(N)$ is true, we will prove that $P(N + 1)$ is also true.
We enumerate the last position (x, y) the person arrives that satisfies $x = y$. Assume the position is (r, r), where $0 \leq r \leq N$. We use the technique of series to express the schemes:

$$(f(r,0)x^0 + f(r,1)x^1 + f(r,2)x^2 + \cdots + f(r,r)x^r)(C_{N-r}x^0 + C_{N-r}x^{N-r})$$

where the coefficient of x^i equals to the number of schemes that there are i upward walks from $(0, 0)$ to $(N + 1, N + 1)$ above the straight line in such condition. The second part is the schemes that walks from (r, r) to $(N + 1, N + 1)$ and never arrives the positions on the straight line between (r, r) and $(N + 1, N + 1)$.
By the assumption, $f(r, i) = C_r$, this formula can also be expressed as

$$(C_rx^0 + C_rx^1 + C_rx^2 + \cdots + C_rx^r)(C_{N-r}x^0 + C_{N-r}x^{N+1-r})$$

We can sum them up when $0 \le r \le N$:

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\begin{aligned}
&\sum_{r=0}^{N}(C_r x^0 + C_r x^1 + C_r x^2 + \cdots + C_r x^r)(C_{N-r} x^0 + C_{N-r} x^{N+1-r}) \\
=&\sum_{r=0}^{N} C_r C_{N-r}(x^0 + x^1 + \cdots + x^r)(x^0 + x^{N+1-r}) \\
=&\sum_{r=0}^{N} C_r C_{N-r}(x^0 + x^1 + \cdots + x^r) + \sum_{r=0}^{N} C_r C_{N-r}(x^{N+1-r} + \cdots + x^{N+1}) \\
=&\sum_{r=0}^{N} C_r C_{N-r}(x^0 + x^1 + \cdots + x^r) + \sum_{r=0}^{N} C_r C_{N-r}(x^{r+1} + \cdots + x^{N+1}) \\
=&\sum_{r=0}^{N} C_r C_{N-r}(x^0 + x^1 + \cdots + x^{N+1}) \\
=&C_{N+1}(x^0 + x^1 + \cdots + x^{N+1})
\end{aligned}
$$

, which means that $f(N+1, r) = C_{N+1}$ for all $0 \le r \le N+1$ and $P(N+1)$ is true. $\square$

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